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Accurate Evaluation of Potential Integrals with Gauss Quadrature Formulas for Rational Functions.

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Introduction

In Moment Method (MoM) applications, self-term integrals are obtained for coincident source and testing domains. These integrals contain a singular kernel and require special numerical treatment. Near-self integrals are obtained whenever the source and the testing sub-domains are very close to each other, but do not overlap. In the near-self case, the integral kernel is not singular, although the accurate evaluation of these integrals is often more difficult than in the self-term case, see for instance [1]. Different integration techniques have been studied for evaluation of self- and near-self integrals; these techniques are based on the singularity subtraction method, or on the singularity cancellation method. The superiority of the cancellation method with respect to the subtraction method has been recently demonstrated [1]. Details on the subtraction method are available, for example, in [2], [3]. The cancellation method is based on variable transformations whose Jacobian cancels out the singularity of the kernel of the potential integral; the numerical integration is then performed by using quadrature schemes obtained by variable transformations. The results of [1] clearly show the advantages in using the cancellation technique in terms of precision, however, the technique presented in [1] does not permit to anticipate the precision of the numerical results, even in the simpler case of static potential integrals.

This paper describes a new numerical technique to compute singular and nearly singular potential integrals with machine precision by working directly in the *parent* reference-frame [4]; the rules to establish the quadrature weights/points (including their number) to guarantee machine precision are reported in [4]. Our integration scheme is based on a new rational expression of the singular and nearly singular integrals obtained by special variable transformations and quadratures: Gauss quadrature for rational functions [5], together with classical Gauss-Legendre quadrature. The technique can deal with static and dynamic potentials on surface and volume elements. In particular, in the static case of polynomial source distributions the new cancellation procedure allows for the exact integration of the potential integrals. Several numerical results for potential integrals will be presented at the conference.

The new cancellation technique

Potential integrals on a given element are normally evaluated by subdividing the object-space element region into sub-domains obtained by joining with a line each vertex of the entire domain to the given observation point $\boldsymbol{r}$. Subdivision requires element sub-meshing driven by the observation point, and re-parameterization of each sub-domain with a parent sub-mapping using parent coordinates. We assume that the expansion functions are given in terms of the parent coordinates of the entire element. The variable transformation formulas required for singularity cancellation are easily obtained by applying, in the parent space, a general four-step procedure (details will be supplied at the conference): 1) Duplication of the parent set; 2) Introduction of the pseudo-radial variable; 3) Radial-binding at observation point $\boldsymbol{\xi}^\circ$; 4) Sub-domain selection via zero-blocking and variable transformation formulas.

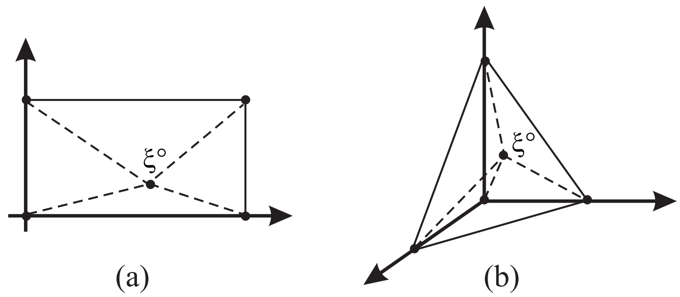

Figure 1: A domain of size σ is broken into σ sub-domains by joining the point $\boldsymbol{\xi}^\circ = (\xi_1^\circ, \xi_2^\circ, \dots, \xi_\sigma^\circ)$ to each domain vertex. The figure illustrates the case for elements of size four: a) the two-dimensional quadrilateral element is subdivided about the point $\boldsymbol{\xi}^\circ$ into four triangular sub-domains; b) the three-dimensional tetrahedral element is broken about the point $\boldsymbol{\xi}^\circ$ into four tetrahedral sub-domains.

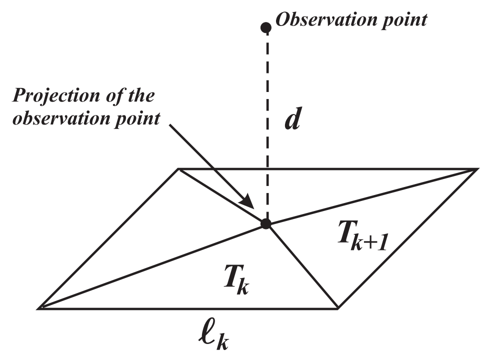

Figure 2: A four-sided planar patch of the object space is broken into four triangular subdomains about the normal projection $\boldsymbol{r}_p$ of the observation point $\boldsymbol{r}$ onto the plane of the patch; the distance from $\boldsymbol{r}$ to the plane of the patch is d.

Evaluation of the potential integral

The singularity cancellation procedure is applied to potential integrals of the form

$$\mathcal{I}_S \;=\; \int_S \boldsymbol{\Lambda}(\boldsymbol{r}')\,\frac{\exp\left(-jkR\right)}{4\pi R}\,\mathrm{d}S' \qquad (1)$$

where $\boldsymbol{\Lambda}(\boldsymbol{r})$ is a vector or scalar basis function, $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ is the vector distance from observation to integration point, and $R = |\boldsymbol{R}|$. By using successive variable transformations into parent coordinates and then into pseudo-radial coordinates one gets

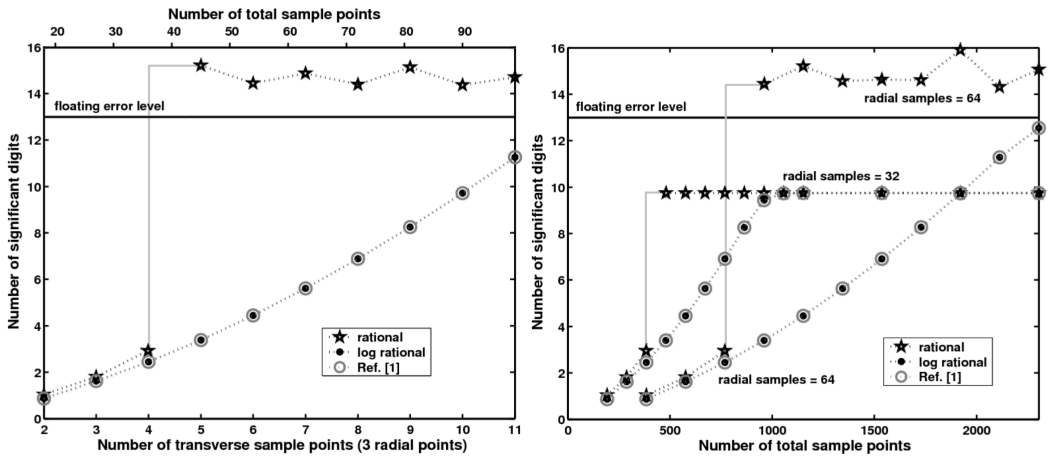

Figure 3: Convergence comparison for the static potential of the source distribution $\mathbf{\Lambda}(\boldsymbol{r}) = x_T^4$ over the right-triangle T, with observation point $(x, y, z) = (0.1, 0.1, d)$ [m]. The results for $d = 0$ (at left) and for $d = 0.01$ [m] (at right) prove that convergence with 5 transverse samples is assured only by Gautschi's integration of (3); this further guarantees machine precision results with 3 and 64 radial samples in the case of $d = 0$ and $d = 0.01$ [m], respectively.

$$\mathcal{I}_S = \int_{\mathcal{E}_\xi} \mathbf{\Lambda}(\boldsymbol{\xi}) \frac{\exp(-jkR)}{4\pi R} \mathcal{J} \mathrm{d}\boldsymbol{\xi} = \frac{\mathcal{J}}{4\pi} \sum_k \xi_k^\circ \int_0^1 \int_0^1 \mathbf{\Lambda}(\rho, \Upsilon) \exp(-jkR) \rho \frac{\mathrm{d}\Upsilon}{R} \mathrm{d}\rho \quad (2)$$

where $\mathcal{J}$ is the Jacobian of the transformation between global and parametric ξ-coordinates. The pseudo-radial transformation originates a ρ factor in the integrand that cancels out the singularity of (2) at $R = 0$ in the *non-displaced* case of $d = 0$ (see Fig. 2). The modified Euler's substitution given in [4] reduces (2) to

$$\mathcal{I}_S = \frac{\mathcal{J}}{4\pi} \sum_k \frac{\xi_k^\circ}{\ell_k} \int_0^1 \mathrm{d}\rho \int_0^1 \mathbf{\Lambda}(\rho, \varphi) \frac{\exp(-jkR)}{\varphi - \widetilde{\varphi}} \mathrm{d}\varphi \quad (3)$$

where φ is the new integration variable that has substituted Υ, and with $\widetilde{\varphi} = -C_k/\rho$. C_k is a real function of ρ that does not depend on φ, with $C_k > 0$ and $\widetilde{\varphi} < 0$ for all ρ in the integration interval $[0, 1]$. The integral (3) is evaluated numerically by integrating first along φ (using Gauss quadrature for rational functions [5]), that is for $\rho =$ *const.*, and then on ρ (using Gauss-Legendre quadrature). This integral simplifies considerably when the observation points lies on the patch-surface (self-element integration) or on its extension, that is for $d = 0$. It is interesting to observe that the static form of (3) in case of a constant basis function $\mathbf{\Lambda}$ immediately yields

$$\mathcal{I}_S = \mathbf{\Lambda} \frac{\mathcal{J}}{4\pi} \sum_k \frac{\xi_k^\circ}{\ell_k} \int_0^1 \ln\left(1 + \frac{\rho}{C_k}\right) \mathrm{d}\rho \quad (4)$$

For $d = 0$, the free-space static potential of a constant source distribution is exactly integrated over a triangular element by using only three sampling points; as pointed out in [1], this is a typical feature of numerical integration schemes based on the cancellation method.

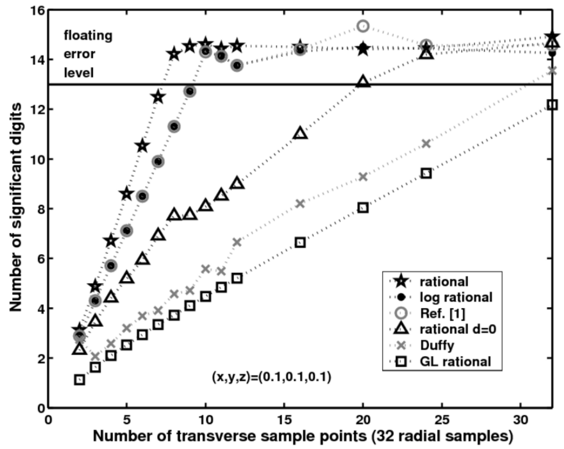

Figure 4: Convergence comparison for a nearly singular integral over T, with observation point $(x, y, z) = (0.01\lambda,\, 0.01\lambda,\, 0.01\lambda)$, and $\mathbf{\Lambda}(\boldsymbol{r}) = 1$. In the body of the figure the global coordinates of the observation point are reported in meters.

Preliminary results for a right triangle T, with catheti of 1[m] in length are shown in Figs. 3, 4. The vertices of T are at $(0, 0, 0)$ $(1., 0, 0)$, $(0, 1., 0)$; the results of Fig. 4 were obtained for $\lambda = 10$ [m].

Acknowledgment

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