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# A note on primes in short intervals

Danilo Bazzanella

**Abstract.** This paper is concerned with the number of primes in short intervals. We present a method to use mean value estimates for the number of primes in  $(x, x+x^\theta]$  to obtain the asymptotic behavior of  $\psi(x+x^\theta)-\psi(x)$ . The main idea is to use the properties of the exceptional set for the distribution of primes in short intervals.

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## 1. Introduction

This paper is concerned with the asymptotic formula

$$\psi(x+x^\theta)-\psi(x)\sim x^\theta \quad x\rightarrow\infty, \quad (1.1)$$

which estimates the number of primes in the interval  $(x, x+x^\theta]$ . The prime number theorem implies that (1.1) holds with  $\theta\geq 1$ . An interval  $(x, x+x^\theta]$  with  $\theta<1$  is called a short interval. The best known unconditional result about the constant  $\theta$  is due to Huxley [4] and asserts that (1.1) holds for  $\theta>7/12$ , which was slightly by Heath-Brown [3] to  $7/12-o(1)$ . Assuming some well-known hypotheses we can handle smaller  $\theta$ . In particular under the assumption of the Lindelöf hypothesis, which states that the Riemann Zeta-function satisfies

$$\zeta(\sigma+it)\ll t^\eta \quad (\sigma\geq\frac{1}{2}, t\geq 2),$$

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for any  $\eta > 0$ , Ingham proved that (1.1) holds for  $\theta > 1/2$ , see [5].

We can relax our request and investigate if (1.1) holds for “almost all”  $x$ . By this we mean that the measure of  $x \in [X, 2X]$  for which (1.1) does not hold is  $o(X)$ . Huxley’s zero density estimate [4], in conjunction with the method of Selberg [7], shows that (1.1) holds for almost all  $x$  with  $\theta > 1/6$ , which was slightly by Zaccagnini [9] to  $1/6 - o(1)$ .

We observe that a suitable mean value estimate is sufficient to get results for almost all  $x$ . Moreover we can use mean value estimates to provide a bound for the exceptional set for the distribution of primes in short intervals but it is never sufficient to prove directly that (1.1) holds for all values of  $x$ .

The aim of this paper is to present a method to use a mean value estimate for the number of primes in  $(x, x + x^\theta]$  to obtain the asymptotic behavior of  $\psi(x + x^\theta) - \psi(x)$ .

Our results will depend upon the following hypothesis about a four-power mean value for the Chebyshev’s function  $\psi(x)$  in short intervals.

**Hypothesis.** *There exist a constant  $X_0$  and a function  $\Delta(y, T)$  such that, for every  $\beta < 1/2$  and  $\varepsilon > 0$ , we have*

$$\int_X^{2X} \left| \psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T) \right|^4 dy \ll X^{4+\varepsilon} T^{-3} \quad (1.2)$$

and

$$\Delta(y, T) \ll \frac{y}{T \ln y} \quad (1.3)$$

uniformly for  $X \geq X_0$ ,  $X^{5/12} \leq T \leq X^\beta$  and  $X \leq y \leq 2X$ .

As noted above it is known that the asymptotic formula (1.1) holds for  $\theta \geq 7/12$ .

Our hypothesis essentially says that there are not too many exceptions to the asymptotic formula (1.1), with  $1/2 < \theta < 7/12$ . Our result is the following.

**Theorem 1.** *Assume the above hypothesis. Then for every  $\theta > 1/2$  the intervals  $[x, x + x^\theta]$  contain the expected number of primes for  $x \rightarrow \infty$ .*

We remark that our hypothesis is weaker than the Lindelöf hypothesis, see Lemma 2, and then we get the following result of Ingham as a corollary.

**Corollary.** *Assume the Lindelöf hypothesis and let  $\theta > 1/2$ . The intervals  $[x, x + x^\theta]$  contain the expected number of primes for  $x \rightarrow \infty$ .*

## 2. The basic lemmas

The first lemma is a result about the structure of the exceptional set for the asymptotic formula (1.1). Let  $X$  be a large positive number,  $\delta > 0$  and let  $|\cdot|$  denote the modulus of a complex number or the Lebesgue measure of a set. We define

$$E_\delta(X, \theta) = \{X \leq x \leq 2X : |\psi(x + x^\theta) - \psi(x) - x^\theta| \geq \delta x^\theta\}.$$

It is clear that (1.1) holds if and only if for every  $\delta > 0$  there exists  $X_0(\delta)$  such that  $E_\delta(X, \theta) = \emptyset$  for  $X \geq X_0(\delta)$ . Hence for small  $\delta > 0$ ,  $X$  tending to  $\infty$ , the set  $E_\delta(X, \theta)$  contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, \theta) \subset E_{\delta'}(X, \theta) \quad \text{if } 0 < \delta' < \delta.$$

The following lemma provides the basic structure of the exceptional set  $E_\delta(X, \theta)$ .

**Lemma 1.** *Let  $0 < \theta < 1$ ,  $X$  be sufficiently large,  $0 < \delta' < \delta$  with  $\delta - \delta' \geq \exp(-\sqrt{\log X})$ . If  $x_0 \in E_\delta(X, \theta)$  then  $E_{\delta'}(X, \theta)$  contains the interval  $[x_0 - cX^\theta, x_0 + cX^\theta] \cap [X, 2X]$ , where  $c = (\delta - \delta')\theta/5$ . In particular, if  $E_\delta(X, \theta) \neq \emptyset$  then*

$$|E_{\delta'}(X, \theta)| \gg_\theta (\delta - \delta')X^\theta.$$

This first lemma essentially says that if we have a single exception in  $E_\delta(X, \theta)$ , with a fixed  $\delta$ , then we necessarily have an interval of exceptions in  $E_{\delta'}(X, \theta)$ , with  $\delta'$  little smaller than  $\delta$ . The interesting consequence of this lemma is that we can use a suitable bound for the exceptional set to prove the non-existence of the exceptions.

The second lemma concerns the conditional estimate for the four-power mean value of the function  $\psi(y)$ .

**Lemma 2.** *Assume the Lindelöf hypothesis and let  $\varepsilon > 0$ . Then there exists a function  $\Delta(y, T)$  such that for every  $\varepsilon > 0$  we have*

$$\int_X^{2X} \left| \psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T) \right|^4 dy \ll X^{4+\varepsilon} T^{-3}$$

and

$$\Delta(y, T) \ll \frac{y}{T \ln y}$$

uniformly for  $X \geq 2, 1 \leq T \leq X$  and  $X \leq y \leq 2X$ .

The Lemma 2 implies that our hypothesis is weaker than the Lindelöf hypothesis.

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [2], and Lemma 2 is Lemma B of Yu, see [8].

### 3. Proof of the Theorem

Our theorem asserts that (1.1) holds with  $\theta > 1/2$ . For  $\theta > 7/12$  the result follows unconditionally by Huxley, see [4], and then we consider only  $1/2 < \theta \leq 7/12$ . In order to prove the theorem we assume that (1.1) does not hold. Then there exists  $\delta_0 > 0$  and a sequence  $X_n \rightarrow \infty$  such that

$$|\psi(X_n + X_n^\theta) - \psi(X_n) - X_n^\theta| \geq \delta_0 X_n^\theta.$$

Using the above definition of the exceptional set we have then  $X_n \in E_{\delta_0}(X_n, \theta)$ .

The use of Lemma 1 with  $\delta' = \delta_0/2$  leads to

$$|E_{\delta'}(X_n, \theta)| \gg X_n^\theta. \quad (3.1)$$

On the other hand, assuming our hypothesis, we can get a bound for  $|E_{\delta'}(X_n, \theta)|$ . To perform this, given any  $\varepsilon > 0$ , we subdivide the interval  $[X, 2X]$  into  $\ll X^\varepsilon$  intervals of type  $I_j = [X_j, X_j + Y]$  with  $X \leq X_j < 2X$  and  $Y \ll X^{1-\varepsilon}$ . For every  $y \in E_{\delta'}(X, \theta)$  we have

$$|\psi(y + y^\theta) - \psi(y) - y^\theta| \gg X^\theta,$$

and then

$$\begin{aligned} |E_{\delta'}(X, \theta)| X^{4\theta} &\ll \int_{E_{\delta'}(N, \theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta|^4 dy \\ &= \sum_j \int_{E_{\delta'}^j(N, \theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta|^4 dy, \end{aligned} \quad (3.2)$$

where  $E_{\delta'}^j(X, \theta) = E_{\delta'}(X, \theta) \cap [X_j, X_j + Y]$ . Our hypothesis asserts that for  $X$  sufficiently large and suitable values of  $T$  there exists a function  $\Delta(y, T)$  which satisfies (1.2) and (1.3).

Let  $T_j = X_j^{1-\theta}$  and let  $\Delta_j(y, T_j)$  the functions which satisfy the conditions (1.2) and (1.3) for every  $j$ . Applying the Brunn-Titchmarsh inequality we can deduce

$$\left( \psi(y + y^\theta) - \psi(y) - y^\theta \right) - \left( \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right) \ll \frac{y}{T_j \log X},$$

for every  $j$  and  $y \in E_{\delta'}^j(X, \theta)$ , and then from (3.2) it follows that

$$\begin{aligned} |E_{\delta'}(X, \theta)| X^{4\theta} &\ll \sum_j \int_{E_{\delta'}^j(X, \theta)} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy \\ &\leq \sum_j \int_X^{2X} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy. \end{aligned}$$

Moreover our hypothesis implies that for every  $\varepsilon > 0$  we have

$$|E_{\delta'}(X, \theta)| \ll X^{-4\theta} \sum_j \int_X^{2X} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy \ll X^{1-\theta+\varepsilon},$$

and this leads to

$$|E_{\delta'}(X_n, \theta)| \ll X_n^{1-\theta+\varepsilon}, \quad (3.3)$$

for  $n$  sufficiently large and for every  $1/2 < \theta \leq 7/12$ .

For  $X_n$  sufficiently large, we have a contradiction between (3.1) and (3.3), and this completes the proof of the theorem.

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