



POLITECNICO DI TORINO
Repository ISTITUZIONALE

Minimal homogeneous submanifolds in euclidean spaces

Original

Minimal homogeneous submanifolds in euclidean spaces / DI SCALA, A.J.. - In: ANNALS OF GLOBAL ANALYSIS AND GEOMETRY. - ISSN 0232-704X. - STAMPA. - 21(2002), pp. 15-18. [10.1023/A:1014260931008]

Availability:

This version is available at: 11583/1660774 since: 2015-12-01T10:36:10Z

Publisher:

Kluwer

Published

DOI:10.1023/A:1014260931008

Terms of use:

openAccess

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

Minimal homogeneous submanifolds in euclidean spaces

Antonio J. Di Scala *

Post print (i.e. final draft post-refereeing) version of an article published on *Ann. Global Anal. Geom.* 21 (2002), no. 1, 1518 .

Beyond the journal formatting, please note that there could be minor changes from this document to the final published version. The final published version is accessible from here:

<http://link.springer.com/article/10.1023%2FA%3A1014260931008>

This document has made accessible through PORTO, the Open Access Repository of Politecnico di Torino (<http://porto.polito.it>), in compliance with the Publisher's copyright policy as reported in the SHERPA-ROMEO website: <http://www.sherpa.ac.uk/romeo/search.php?issn=0232-704X>

Abstract

We prove that minimal (extrinsically) homogeneous submanifolds of the euclidean space are totally geodesic. As an application, we obtain that a complex homogeneous submanifold of \mathbb{C}^N must be totally geodesic.

Mathematics Subject Classification(2000): 53C40, 53C42

Key Words: minimal submanifolds, orbits of isometry groups, homogeneous spaces, homogeneous submanifolds.

1 Introduction

The theory of minimal immersions into spheres is very well developed [L], [C2], [S], [DW]. There is a beautiful method, using eigenfunctions of the Laplacian, for constructing minimal equivariant immersions of compact homogeneous spaces into spheres [T], [W]. In particular, Hsiang [H] has constructed orbits of subgroups of isometries of the sphere which are minimal

*Supported by a CONICET fellowship, partially supported by CONICOR and Secyt-UNC

(see also [H-L]). In this paper we consider the analogous problem for the euclidean space.

A (extrinsically) homogeneous submanifold of the euclidean space is a submanifold which is an orbit of a Lie subgroup of isometries of the euclidean space. The following theorem shows that in the euclidean spaces there are only trivial minimal homogeneous submanifolds.

Theorem 1.1 *A (extrinsically) homogeneous minimal submanifold of the euclidean space must be totally geodesic.*

We remark that the homogeneity hypothesis cannot be weakened, since there exist minimal submanifolds of the euclidean space with cohomogeneity 1 and they are not totally geodesic. For instance, we can take a minimal surface of revolution, or the complex submanifold of \mathbb{C}^2 defined by the equation $z^2 + w^2 = 1$. We also remark that in the case that the submanifold is (extrinsically) symmetric (i.e. has parallel second fundamental form) the result is due to D. Ferus [F, Lemma 4].

It is a well known result that a complex immersed submanifold of \mathbb{C}^N is minimal [S, Th. 3.1.2], [KN, pp. 380]. On the other hand, Calabi [C1] has shown that complex isometric immersions are rigid. A simple consequence of these two facts is the following corollary, which was in fact the starting question of this paper [D].

Corollary 1.2 *A complex isometric immersion from a complex homogeneous space into \mathbb{C}^N must be totally geodesic.*

In other words, such an isometric immersion can not exist unless the immersed manifold is an affine space. A special case of this corollary, for symmetric bounded domains, is contained in [B, Th. 13].

As another application of our theorem we obtain the following improvement of the corollary in [O2, pp. 2928] (see also [O1])

Corollary 1.3 *Let M^n ($n \geq 2$) be a homogeneous irreducible submanifold of the euclidean space with parallel mean curvature vector H . Then $H \neq 0$ and M is contained in a sphere, where it is either minimal or it is an orbit of the isotropy representation of a simple symmetric space.*

It is interesting to note that our result plays an important role in the proof of the same result in the hyperbolic space (i.e. a minimal homogeneous submanifold of the hyperbolic space must be totally geodesic, see [DO]). On the other hand, there exist nontrivial homogeneous minimal hypersurfaces in complex hyperbolic spaces or in more general symmetric spaces of negative curvature see [Be].

2 Homogeneous submanifolds of the euclidean space

We say that an orbit $G.v$ of \mathbb{R}^N is *reducible* if $G.v = M_1 \times M_2$ (Riemannian product) where M_1, M_2 are nontrivial factors and $i = i_1 \times i_2$ where i is the natural inclusion of $G.v$ in \mathbb{R}^N and $i_1 : M_1 \rightarrow \mathbb{R}^{N_1}, i_2 : M_2 \rightarrow \mathbb{R}^{N_2}$ are isometric immersions and $N = N_1 + N_2$. If $G.v$ is a reducible submanifold, then each factor is also a homogeneous submanifold of the corresponding euclidean space.

We need the following stronger version of the theorem in [O2, appendix] (see also [V]). Roughly speaking, it says that (non compact) homogeneous submanifolds of the euclidean space are generalized helicoids.

Theorem 2.1 *Let $M = G.v$ be a homogeneous irreducible submanifold of \mathbb{R}^N , where G is a Lie subgroup of the isometry group $I(\mathbb{R}^N)$ of \mathbb{R}^N . Then, the universal cover \tilde{G} of G splits as $K \times \mathbb{R}^k$, where K is a compact simply connected Lie group. Moreover, the representation ρ of $K \times \mathbb{R}^k$ into $I(\mathbb{R}^N)$ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of $K \times \mathbb{R}^k$ into $SO(\mathbb{R}^d)$ and ρ_2 is a linear map of \mathbb{R}^k into \mathbb{R}^e , ($N=d+e$), regarding \mathbb{R}^e as its group of translations.*

Proof. By the theorem in [O2, Appendix], we just need to show that any representation $\rho : \mathbb{R}^k \rightarrow I(\mathbb{R}^N)$ is equivalent to a direct sum $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of \mathbb{R}^k into $SO(\mathbb{R}^d)$ and ρ_2 is a linear map of \mathbb{R}^k into \mathbb{R}^e ($N = d + e$). The Lie algebra $\mathcal{L}(I(\mathbb{R}^N))$ is the semidirect product $\mathcal{L}(SO(N)) \ltimes \mathbb{R}^N$, where the bracket is defined by $[(A, v), (B, u)] = ([A, B], A(u) - B(v))$, and the exponential is given by $exp(t.(A, v))(p) = e^{t.A}.(p - c) + c + t.d$ for $d \in \ker(A)$ and $v = d - A(c)$.

We are going to show that there exists a common c for the “rotational” part of the Lie algebra $\mathcal{L}(\rho(\mathbb{R}^N))$. Let \mathcal{R} be the projection of $\mathcal{L}(\rho(\mathbb{R}^N))$

in $\mathcal{L}(SO(\mathbb{R}^N))$. The abelian family \mathcal{R} of skew symmetric endomorphisms can be diagonalized simultaneously in \mathbb{C} . Now let λ_i ($i = 1, 2, \dots, n$) be the different non zero linear functionals associated to each eigenspace. The set $\mathcal{O} = \{R \in \mathcal{R} : \lambda_i(R) \neq 0 \text{ for all } i\}$ is open and dense. It is not hard to show that there exists a basis $w_1 = (R_1, d_1 - R_1(c_1)), \dots, w_r = (R_r, d_r - R_r(c_r))$ of $\mathcal{L}(\rho(\mathbb{R}^N))$ such that R_i belongs to \mathcal{O} for all $i = 1, \dots, r$. This implies that $d_i \in V = \ker(R_j) = \bigcap_{j=1, \dots, r} \ker(R_j)$ ($i, j = 1, \dots, r$). By the bracket formula we obtain that $R_i(R_j(c_i - c_j)) = 0$ ($i, j = 1, \dots, r$) and this implies in turn $c_i = c_j$ for $i, j = 1, \dots, r$. By fixing the origin at c_1 we deduce that ρ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of \mathbb{R}^k into $SO(V^\perp)$ and ρ_2 is a linear map of \mathbb{R}^k into V .

Now we can prove our principal result.

Proof of Theorem 1.1. Without loss of generality we may assume that the homogeneous submanifold $G.p$ is irreducible (see [O2, section 1]). By Theorem 2.1 we can choose a basis of $\mathcal{L}(G)$ of the form $(A_1, d_1), \dots, (A_n, d_n)$ where $d_i \in \ker(A_i) = V$ ($i = 1, \dots, n$). Moreover, we can choose this basis in such a way that $(A_1, d_1), \dots, (A_r, d_r)$ belong to the isotropy subalgebra of G at p (and so $d_1, \dots, d_r = 0$) and $(A_{r+1}, d_{r+1}).p = A_{r+1}(p) + d_{r+1}, \dots, (A_n, d_n).p = A_n(p) + d_n$ form an orthonormal basis of $T_p(G.p)$. Let us decompose $p = p_1 + p_2$, with $p_1 \in V^\perp$ and $p_2 \in V$. Set $\gamma_i(t) = e^{tA_i}.p + t.d_i$ $i = 1, \dots, n$. We observe that p_1 is a normal vector to $G.p$ at p . We then claim that p_1 must be zero. In fact, if α is the second fundamental form, then $0 = \sum_{i=1}^r \langle \alpha(\dot{\gamma}_i, \dot{\gamma}_i), p_1 \rangle = \sum_{i=1}^r \langle A_i^2.(p_1), p_1 \rangle = \sum_{i=1}^r -\langle A_i(p_1), A_i(p_1) \rangle$. This implies $p_1 = 0$, as $A_i(p_1) = 0$ for all i and $p \in V$, and we conclude that the orbit is totally geodesic.

Acknowledgment

I thank Carlos Olmos for many helpful discussions and suggestions.

References

- [B] BOCHNER, S.: *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. 53 (1947), pp. 179-195.
- [Be] BERNDT, J.: *Homogeneous hypersurfaces in hyperbolic spaces*, Math. Z. 229 (1998), pp. 589-600.

- [C1] CALABI, E.: *Isometric imbedding of complex manifolds*, Ann. of Math. 58 (1953), pp. 1-23.
- [C2] CALABI, E.: *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry 1 (1967), pp. 111-126.
- [D] DI SCALA, A.J.: *Reducibility of complex submanifolds of the euclidean space*, To appear in Math. Z.
- [DO] DI SCALA, A.J. AND OLMOS, C.: *The geometry of homogeneous submanifolds of hyperbolic space*, To appear in Math. Z.
- [DW] DO CARMO, M. AND WALLACH, N.: *Minimal immersions of spheres into spheres*, Ann. of Math. 93 (1971), pp. 43-62.
- [F] FERUS, D.: *Produkt-Zerlegung von Immersionen mit paralleler zweiter Fundamentalform*, Math. Ann. 211, pp. 1-5 (1974).
- [H] HSIANG, W.Y.: *On the compact homogeneous minimal submanifolds*, Proc. Nat. Acad. Sci. USA 56 (1966), pp. 5-6.
- [H-L] HSIANG, W.Y. AND LAWSON, B.H.: *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry 5 (1971) pp. 1-38.
- [KN] KOBAYASHI, S. AND NOMIZU, K.: *Foundations of differential geometry*, Vol II, Interscience Publishers, (1969).
- [L] LAWSON, B.H.: *Lectures on minimal submanifolds*, Publish and Perish Inc. Vol. I (1980).
- [O1] OLMOS, C.: *Homogeneous submanifolds of higher rank and parallel mean curvature*, J. Differential Geometry 39 (1994), pp. 605-627.
- [O2] OLMOS, C.: *Orbits of rank one and parallel mean curvature*, Trans. Amer. Math. Soc., Vol 347, No 8 ,(1995), pp. 2927-2939.
- [S] SIMONS, J.: *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968), pp. 62-105.
- [T] TOTH, G.: *Harmonic maps and minimal immersions through representation theory*, Acad. Press , Perspectives in Math. Vol. 12 (1990).
- [V] VARGAS, J.: *A symmetric space of noncompact type has no equivariant isometric immersions into the Euclidean space*, Proc. Amer. Math. Soc., 81 (1981), pp. 149-150.

- [W] WALLACH, N.R.: *Minimal immersions of Symmetric Spaces into Spheres*, Symmetric Spaces, Marcel Dekker, Inc. New York 1972, pp. 1-40.

Fa.M.A.F., Universidad Nacional de Córdoba, Ciudad Universitaria, 5000
Córdoba, Argentina
e-mail address: discala@mate.uncor.edu