On an assertion in Riemann's Habilitationvortrag

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Abstract

We study an assertion in Riemann's Habilitation Lecture of 1854. Namely, the determination of the metric given $n\frac{n-1}{2}$ sectional curvatures.

1   Introduction

Modern differential geometry was born with the Riemann's Habilitation Lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" (On the Hypotheses which lie at the Foundations of Geometry) of 1854 at Göttingen [R], [We]. In this lecture Riemann defines the curvature tensor R. One says that M is flat if M is locally isometric to $\mathbb{R}^n$ with the usual metric; the tensor R vanishes if and only if the metric is flat. M. Spivak [Sp1] translates Riemann's Lecture and explains it in modern terms. Let

$$Q(X,Y) := \frac{\langle R(X,Y)Y, X \rangle}{\mid X \wedge Y \mid^2}$$

be the sectional curvature. Spivak [Sp1, pp. 4B-25], [Sp2, pp. 176] makes the following:

Assertion 1.1 *If M is n-dimensional and if $Q=0$ for $n\frac{n-1}{2}$ independent 2-dimensional subspaces of each M_q, then M is flat.*

It is well known that the metric is flat if and only if the sectional curvature Q vanishes identically. The number $n\frac{n-1}{2}$ of Assertion 1.1 is "deduced" from the following "counting argument" given by Riemann: the metric $ds^2 = \sum g_{ij}dx_idx_j$ contains $\frac{n(n+1)}{2}$ functions while a new coordinate system involves only n functions, so that we can change only n of the g_{ij}, leaving $\frac{n(n-1)}{2}$ other functions which depend on the metric; thus there should be some set of $\frac{n(n-1)}{2}$ functions which will determine the metric completely (see [Di, pp.198], [Sp1, pp. 4B-4]). We quote from the original text as follows [We], [R]:

"... wenn also das Krümmungsmass in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen gegeben wird, so werden daraus die Massverhältnisse der Mannigfaltigkeit sich bestimmen lassen, wofern nur zwischen diesen Werthen keine identischen Relationen stattfinden, was in der That, allgemein zu reden, nicht der Fall ist."
"... es reicht aber nach der frühern Untersuchung, um die Massverhältnisse zu bestimmen, hin zu wissen, dass es in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen, deren Krümmungsmasse von einander unabhängig sind, Null sei. "

We remark that this text is omitted by Hermann Weyl in his discussion of Riemann's ideas. Relating the curvature tensor to the metric is a very classical subject and we refer to [Ku, Ya, B] for further details.

In this note, we construct several families of counter-examples to Assertion 1.1. In §2, we discuss the space of algebraic curvature tensors and construct an algebraic curvature tensor in dimension 3 which has vanishing sectional curvature on 3 independent 2 planes; this shows that Assertion 1.1 is not an algebraic consequence of the identities of the curvature tensor. Let H^2, S^2 and T^k denote the hyperbolic plane, the sphere and the torus with the metrics of constant curvature -1, 1, and 0. Give $M = S^2 \times H^2 \times T^k$ the product metric; this manifold is not flat. In §3, we construct local orthonormal frames $\{e_i\}$ and local coordinate frames ∂_i for the tangent bundle so that the sectional curvatures $Q(e_i, e_j)$ and $Q(\partial_i, \partial_j)$ vanish for $i \neq j$. Again, this shows Assertion 1.1 is false. Finally, in §4, we use warped products to construct still other examples of non-flat metrics which are counter-examples to Assertion 1.1. It is a pleasant task to thank Professors V. Cortez and P. Gilkey for helpful discussions concerning these matters.

2   An algebraic example

Let V be an n-dimensional real vector space and let $\langle, \rangle$ be a positive definite inner product defined on V. A bilinear $R : V \times V \rightarrow End(V)$ is called an algebraic curvature tensor if it has the following three properties:

$$\langle R(x,y)z, w \rangle = -\langle R(y,x)z, w \rangle \tag{1}$$

$$\langle R(x,y)z, w \rangle = -\langle R(x,y)w, z \rangle \tag{2}$$

$$\langle R(x,y)z, w \rangle + \langle R(y,z)x, w \rangle + \langle R(z,x)y, w \rangle = 0 \tag{3}$$

These three properties then imply the following symmetry property

$$\langle R(x,y)z, w \rangle = \langle R(z,w)x, y \rangle$$

see [KN, pp. 198] or [Sp1, pp. 4D-17]) for details. We can also identify the space of algebraic curvature tensors with the space K of symmetric endomorphisms of the second exterior product $\Lambda^2(V)$ such that:

$$\langle K(x \wedge y), z \wedge w \rangle + \langle K(y \wedge z), x \wedge w \rangle + \langle K(z \wedge x), y \wedge w \rangle = 0 \tag{4}$$

Here the inner product on $\Lambda^2(V)$ is induced from the inner product on V. We say that a collection of 2-dimensional subspaces are linearly independent if the associated elements of $\Lambda^2(V)$ are linearly independent in $\Lambda^2(V)$. For example, let $\{e_1, ..., e_n\}$ be a basis of V. Then the 2-subspaces spanned by $\{e_i, e_j\}_{i \neq j}$ are independent. The bi-quadratic tensor $\langle R(x, y)y, x \rangle$ determines R, we refer to [KN, pp. 198] for the proof of the following result:

Proposition 2.1 *Let R an algebraic curvature tensor such that $\langle R(x, y)y, x \rangle = 0$ for all x, y. Then $R = 0$.*

The space of curvature tensors has dimension $\frac{n^2(n^2-1)}{12}$, see for example M. Berger [B, pp. 63]. Thus if $n = 3$, then equations (3) and (4) follow from equations (1) and (2). Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for V. We define a symmetric endomorphism K of $\Lambda^2(V)$ by:

$$K(e_1 \wedge e_2) = e_3 \wedge e_1, \ \ K(e_2 \wedge e_3) = 0, \ \ K(e_3 \wedge e_1) = e_1 \wedge e_2$$

Note that K is a non-trivial algebraic curvature tensor with the following three vanishing sectional curvatures:

$$Q_K(e_1 \wedge e_2) = Q_K(e_2 \wedge e_3) = Q_K(e_3 \wedge e_1) = 0.$$

More generally let $n \geq 3$ and let $\{e_1, ..., e_n\}$ be an orthonormal basis for V. If we impose the condition that $Q_K(e_i \wedge e_j) = 0$ with $i < j$, then we have imposed $\frac{n(n-1)}{2}$ conditions. Since the dimension of the space of algebraic curvature tensors is $\frac{n^2(n^2-1)}{12} > \frac{n(n-1)}{2}$, then a simple counting argument shows there are non-trivial algebraic curvatures with $Q_K(e_i \wedge e_j) = 0$ for $i < j$; thus Assertion 1.1 fails in the algebraic setting.

3 Curvature zero 2 planes in $S^a \times H^a \times T^b$

In this section we discuss two examples showing Assertion 1.1 is false. Let H^a, S^a, and T^b be spaces of constant sectional curvature -1, $+1$, and 0 where $a \geq 2$. We begin by studying orthonormal frame fields.

Proposition 3.1 *Let $M(a, b) := S^a \times H^a \times T^b$ with the product metric where $a \geq 2$. There exists a local orthonormal frame $\{e_i\}$ for the tangent bundle of $M(a, b)$ so that $Q(e_i \wedge e_j) = 0$ for $1 \leq i < j \leq 2a + b$.*

Proof. Let $\{u_i\}$ and $\{v_i\}$ be local orthonormal frames for the tangent bundles of S^a and H^a for $1 \leq i \leq a$. Let $\{w_j\}$ be a local orthonormal frame for the tangent bundle of T^b for $1 \leq j \leq b$. Define

$$e_{2i-1} := \frac{u_i + v_i}{\sqrt{2}} \text{ for } 1 \leq i \leq a, e_{2i} := \frac{u_i - v_i}{\sqrt{2}} \text{ for } 1 \leq i \leq a,$$
$$e_{2a+j} := w_j \text{ for } 1 \leq j \leq b.$$

The $\{e_k\}$ for $1 \le k \le 2a + b$ is a local orthonormal frame for the tangent space of $M(a,b) := S^a \times H^a \times T^b$. We have $\langle R(u_i, w_j)w_j, u_i \rangle = 0$, $\langle R(v_i, w_j)w_j, v_i \rangle = 0$, and $\langle R(v_i, w_j)w_j, v_i \rangle = 0$. Thus $Q(e_i \wedge e_j) = 0$ if either $i > 2a$ or $j > 2a$. We also have $\langle R(u_{i_1}, u_{i_2})u_{i_2}, u_{i_1} \rangle = +1$ and $\langle R(v_{i_1}, v_{i_2})v_{i_1}, v_{i_2} \rangle = -1$ for $i_1 < i_2$. We can show that $Q(e_i \wedge e_j) = 0$ for $i \le 2a$ and $j \le 2a$ by computing:

$$\langle R(e_1, e_2)e_2, e_1 \rangle = 0$$
$$\langle R(e_1, e_3)e_3, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)u_2, u_1 \rangle + \langle R(v_1, v_2)v_2, v_1 \rangle\} = 0$$
$$\langle R(e_1, e_4)e_4, e_1 \rangle = \tfrac{1}{4}\{\langle R(u_1, u_2)u_2, u_1 \rangle + (-1)^2\langle R(v_1, v_2)v_2, v_1 \rangle\} = 0 \text{ etc. } \square$$

Proposition 3.1 deals with orthonormal frames. We now turn to coordinate frames. If $(x_1, ..., x_n)$ is a system of local coordinates, set $\partial_i^x := \frac{\partial}{\partial x_j}$.

Proposition 3.2 *Let $M(2,b) := S^2 \times H^2 \times T^b$. There exist local coordinates $(u_1, ..., u_{4+b})$ on $M(2,b)$ so that $Q(\partial_i^u \wedge \partial_j^u) = 0$ for $1 \le i < j \le 4 + b$.*

Let ω be the volume form. Before beginning the proof of Proposition 3.2, we recall the following technical result and refer to see [K, pp. 6] for details:

Lemma 3.3 *Let M^n be an orientable Riemannian manifold. Then around each point there exists a coordinate system $\{x_1, ..., x_n\}$ such that $\omega(\partial_1^x, ..., \partial_n^x) = 1$.*

Proof of Proposition 3.2. We use lemma 3.3 to find local coordinates (x_1, x_2) and (y_1, y_2) on S^2 and H^2 so that $\omega(\partial_1^x, \partial_2^x) = 1$ and $\omega(\partial_1^y, \partial_2^y) = 1$. Let $(z_1, ..., z_b)$ be the usual flat coordinates on T^b. Define local coordinates on $S^2 \times H^2 \times T^b$ by:

$$u_1 := x_1 + y_1, \quad u_2 := x_1 - y_1, \quad u_3 := x_2 + y_2, u_4 := x_2 - y_2,$$

and $u_{k+4} = w_k$ for $1 \le k \le b$. We then have

$$\partial_1^u = \partial_1^x + \partial_1^y, \quad \partial_2^u = \partial_1^x - \partial_1^y, \quad \partial_3^u = \partial_2^x + \partial_2^y, \partial_4^u = \partial_2^x + \partial_2^y,$$

and $\partial_{4+k}^u = \partial_k^w$ for $k > 0$. If N is a Riemann surface with constant sectional curvature ϵ, then $\langle R(x,y)y, x \rangle = \epsilon\omega(x,y)$. Thus, the calculations performed in the proof of Proposition 3.1 show $Q(\partial_i^u \wedge \partial_i^v) = 0$. $\boxdot$

4   Curvature zero 2 planes in warped products

We can use warped products to construct additional examples where Assertion 1.1 fails. We adopt the notation of [O, pp. 210].

Proposition 4.1 *Let $M = B \times_f F$ be a warped product, where B is a small open ball around $(0,0)$ in $\mathbb{R}^2$, where $f(x,y) = x + y + xy + 1$, is positive and where $F = \mathbb{R}$. Then M is not flat. Furthermore $Q(\partial_x \wedge \partial_y) = 0$, $Q(\partial_x \wedge \partial_z) = 0$, and $Q(\partial_y \wedge \partial_z) = 0$.*

Proof. We use [O, pp. 210, Proposition 42], to compute:

$$\langle R(\partial_x, \partial_y)\partial_x, \partial_z \rangle = 0, \qquad \langle R(\partial_x, \partial_z)\partial_x, \partial_z \rangle = 0,$$
$$\langle R(\partial_y, \partial_z)\partial_y, \partial_z \rangle = 0, \qquad \langle R(\partial_x, \partial_z)\partial_z, \partial_y \rangle = f. \quad \square$$

Proposition 4.1 generalizes to higher dimensions by taking products with flat tori.

5 Concluding comments

In order to solve the local equivalence problem (i.e. when two metrics $\mathbf{g_1}, \mathbf{g_2}$ on a differentiable manifold M^n differ (locally) by a diffeomorphism.) Riemann tried to compute $n\frac{n-1}{2}$ $Diff(M^n)$-equivariant functions (i.e. $K(\mathbf{g_2})(p) = K(\mathbf{g_1})(f(p))$ for all $f \in Diff(M^n)$, $p \in M^n$, $\mathbf{g_2} = f^*\mathbf{g_1}$). The Gaussian curvature K is such a function when $n = 2$. To do this, Riemann expanded the metric in normal coordinates and defined a map Q from $\mathcal{M}_n$, the space of Riemannian metrics on M^n, to $C^\infty(G_2(M^n))$, where $G_2(M^n)$ is the two Grassmannian bundle over M^n. In other words, $Q(\mathbf{g})(\pi_p)$ is the sectional curvature of the 2-plane π_p at $p \in M^n$ with respect to the metric $\mathbf{g}$. Then, he said that "... if the curvature is given in $n\frac{n-1}{2}$ surface directions at every point, then the metric relations of the manifold may be determined ..." [Sp2, pp. 144]. More precisely, Riemann took $n\frac{n-1}{2}$ independent sections π_{ij} of the bundle $G_2(M^n)$ and he defined the $n\frac{n-1}{2}$ functions by composing with Q (i.e. a map from $\mathcal{M}_n$ to $\{C^\infty(M^n)\}^{n\frac{n-1}{2}}$). Perhaps the expression of Q in coordinates, the two dimensional flat case and the counting argument led Riemann to the wrong conclusion that Q can be recovered from evaluation in $n\frac{n-1}{2}$ independent 2-planes. It is hard to believe that he did not observe that this map is not actually a $Diff(M^n)$-equivariant morphism, as follows from the fact that a generic diffeomorphism does not preserve the π_{ij} (i.e. $f^*\pi_{ij} \neq \pi_{ij}$) when $n > 2$.

Remark 5.1 *A way of defining $n\frac{n-1}{2}$ $Diff(M^n)$-equivariant functions from $\mathcal{M}_n$ to $C^\infty(M^n)$ such that:*

(i) If $n = 2$ then the function is the Gauss curvature K.

(ii) If the $n\frac{n-1}{2}$ functions vanish identically then the metric $\mathbf{g}$ is flat.

is as follows. Regarding the curvature tensor R as a symmetric endomorphism of the second exterior product bundle $\Lambda^2(M^n)$ one can take the characteristic polynomial $\chi_R(X)$ of R. Then, the coefficients of $\chi_R(X)$ are the required $n\frac{n-1}{2}$ functions.

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