The geometry of homogeneous submanifolds of hyperbolic space

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Antonio J. Di Scala and Carlos Olmos *

Abstract

We prove, in a purely geometric way, that there are no connected irreducible proper subgroups of $SO(N,1)$. Moreover, a weakly irreducible subgroup of $SO(N,1)$ must either act transitively on the hyperbolic space or on a horosphere. This has obvious consequences for Lorentzian holonomy and in particular explains clasification results of Marcel Berger (e.g. the fact that an irreducible Lorentzian locally symmetric space has constant curvatures). We also prove that a minimal homogeneous submanifold of hyperbolic space must be totally-geodesic.

1 Introduction

In this article we study homogeneous submanifolds of hyperbolic space from a purely geometric point of view and we obtain very general results for representations in $SO(N,1)$. This results explains the classification list of M. Berger for Lorentzian holonomy and irreducible Lorentzian locally symmetric spaces [B1], [B2] (in particular, the fact that an irreducible Lorentzian locally

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symmetric space has constant curvatures). This answers a question posed by L. Berard Bergery and A. Ikemakhen in [BI, pp. 31].

For studying homogeneous submanifolds it seems to be more suitable and natural to use classical tools of Riemannian geometry than representation theory (see e.g. [OS], [O1], [O2], [O3], [T], [D]). So, we adopt in this article the geometric approach, for investigating orbits in hyperbolic space.

Our main results are the following theorems.

Theorem 1.1 *Let G be a connected (non nec. closed) Lie subgroup of $SO(N,1)$ and assume that the action of G on the Lorentzian space $\mathbb{R}^{N,1}$ is irreducible. Then $G = SO_0(N,1)$.*

Theorem 1.2 *Let G be a connected (non nec. closed) Lie subgroup of $SO(N,1)$ and assume that the action of G on the Lorentzian space $\mathbb{R}^{N,1}$ is weakly irreducible. Then either G acts transitively on H^N or G acts transitively on a horosphere of hyperbolic space.*

The main tool for proving the above theorems is the following geometric characterization of homogeneous submanifolds of hyperbolic space.

Theorem 1.3 *Let G be a connected (non nec. closed) Lie subgroup of the isometries of hyperbolic space H^N. Then one of the following assertions holds:*

i) G has a fixed point.

ii) G has a unique non trivial totally geodesic orbit (eventually the full space).

iii) All orbits are included in horospheres centered at the same point at the infinity.

As a corollary, using the uniqueness result of Lemma 3.1 we obtain the following result (which is also true in Euclidean space [D]).

Corollary 1.4 *A minimal (extrinsically) homogeneous submanifold of hyperbolic space must be totally geodesic.*

The above corollary does not extend to arbitrary symmetric spaces of the noncompact type. In fact, J. Berndt [BJ] constracted minimal non totally geodesic orbits in complex projective spaces.

An immediate consequence of the above results is the following general result about Lorentzian holonomy, whose second part is due to Marcel Berger [B1, B2] (see also [BI]).

Corollary 1.5 *Let M^N be a locally indecomposable Lorentzian manifold. Then its restricted holonomy group either acts transitively on hyperbolic space or transitively on a horosphere. Moreover, if the restricted holonomy group acts irreducibly it coincides with $SO_0(N,1)$.*

2 Preliminaries and basic facts

Let $\mathbb{R}^{N,1} = (\mathbb{R}^{N+1}, \langle , \rangle)$ be the usual Lorentzian space and let $SO(N,1)$ be its special group of isometries. Let G be a Lie subgroup of $SO(N,1)$. The action of G is said irreducible if G does not leave invariant any proper subspace of $\mathbb{R}^{N,1}$ and weakly irreducible if any $G-$invariant subspace has a degenerate induced metric.

A Lorentzian manifold is said locally indecomposable if it is not a local product or, equivalently, if its restricted holonomy group acts weakly irreducibly.

We will always regard hyperbolic space as $H^N = \{v \in \mathbb{R}^{N,1} : \langle v, v\rangle = -1\}$. In this way $SO_0(N,1)$ is identified with the connected component of isometry group of H^N.

A Lie subgroup G of $SO(N,1)$ is said to act almost effectively on H^N if $\{g \in G : g.p = p \; \forall p \in H^N\}$ is a discrete subgroup of G.

To any X in the Lie algebra of SO(N,1) we associate, as usual, the Killing field on H^N defined by $\bar{X}(p) := X.p$ where $X.p := \frac{d}{ds}\phi_s^X.p \mid_{s=0}$ (ϕ_s^X is the one-parameter group of isometries defined by X, i.e. $\phi_s^X = Exp(sX)$). Then $\langle \nabla_v \bar{X}, v\rangle = 0$, for all $v \in TH^N$ (Killing equation).

We recall that the infinity $H^N(\infty)$ of hyperbolic space H^N, i.e. the classes of equivalence of asymptotic geodesics, can be identified with the unit tangent sphere at a point (see [BGS]). In this way, each point $z \in H^N(\infty)$ determines a unique unit tangent field ξ on H^N. Recall that a horosphere can be viewed as the intersection of H^N with a degenerate hyperplane. Any horosphere is completely determined by specifying a point in H^N and another one in $H^N(\infty)$. It is a well known fact that horospheres are umbilical flat submanifolds of codimension 1 which are isometric to a Euclidean space (see [Sp]). Observe that ξ is also a (umbilical) parallel normal field to any submanifold of H^N which is contained in the horosphere Q.

Let M be a Riemannian submanifold of hyperbolic space H^N and let $\nu(M)$ be its normal bundle. The endpoint map $exp^\nu : \nu(M) \to H^N$ (see [PT, pp.67]) is defined by $exp^\nu := exp_{|_{\nu(M)}}$, where exp is the usual exponential map and $\nu(M)$ is regarded as a subset of TH^N. If ξ is a parallel normal field and $t \in R$, then one can construct a new set M_t, *parallel* to M, by the image of the composition of the endpoint map by the parallel normal field $t\xi$.

Let now Q be a horosphere centered at $z \in H^N(\infty)$ and let ξ be the unit (parallel) normal field to Q associated with z. Then, the foliation by horospheres centered at z coincides with the foliation $Q_t, t \in \mathbb{R}$. It is a well known fact that the foliation by horospheres centered at $z \in H^N(\infty)$ coincides with the foliation given by the intersection of H^N with a family of parallel degenerate hyperplanes of $\mathbb{R}^{N,1}$.

If $M = G.p$, where G acts by isometries on H^N and ξ is an equivariant normal parallel field, we have that $M_t = G.exp(t\xi(p))$ and therefore it is also a $G-$orbit (see [PT, pp.87]). If $exp^\nu \circ (t\xi)$ is an immersion, then M_t is called a parallel manifold to M.

The following lemma is standard.

Lemma 2.1 *Let V be a connected totally geodesic submanifold of a horosphere Q centered at $z \in H^N(\infty)$. Then*

$$T = \{exp(t\xi(q)) : q \in V, t \in \mathbb{R}\} = \bigcup_t V_t$$

is a connected totally geodesic submanifold of H^N, where ξ is the unit normal vector field to Q associated with z. Moreover, for all t, V_t is a connected totally geodesic submanifold of the horosphere Q_t centered at $z \in H^N(\infty)$.

Remark 2.2 *Let G be a connected Lie subgroup of the isometries of hyperbolic space H^N which fixes a point $z \in H^N(\infty)$. Let L be the subgroup of G which leaves invariant any horosphere centered at z. If V_Q is the union of the totally geodesic orbits of the action of L on the horosphere Q then $(V_Q)_t = V_{(Q_t)}$. The proof follows from the above Lemma 2.1 and the below Remark 2.4.*

We will need the following characterization of homogeneous submanifolds of Euclidean space, obtained by the first author [D] (cf. [O1, appendix]).

Theorem 2.3 *Let $M = G.v$ be a homogeneous irreducible submanifold of $\mathbb{R}^N$, where G is a Lie subgroup of the isometry group $I(\mathbb{R}^N)$ of $\mathbb{R}^N$. Then, the universal cover $\tilde{G}$ of G splits as $K \times \mathbb{R}^k$, where K is a compact simply connected Lie group. Moreover, the representation ρ of $K \times \mathbb{R}^k$ into $I(\mathbb{R}^N)$ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is a representation of $K \times \mathbb{R}^k$ into $SO(\mathbb{R}^d)$*

and ρ_2 is a linear map of $\mathbb{R}^k$ into $\mathbb{R}^e$, (N=d+e), regarding $\mathbb{R}^e$ as its group of translations.

Remark 2.4 *From the above theorem (and its proof) one has the following fact: If G is a connected Lie subgroup of isometries of $\mathbb{R}^n$, then the union V of all totally geodesic $G-$orbits is an affine subspace. Moreover, the orbits of G on V define a (totally geodesic) parallel foliation.*

Remark 2.5 *If G acts by isometries on hyperbolic space and one orbit is contained in a horosphere, let us say Q, then any other orbit is contained in a horosphere. In fact, G must preserve any parallel manifold to Q, which is again a horosphere centered at the same point at infinity.*

3 Proof of the theorems

We prove our theorems via a sequence of lemmas. The following uniqueness result will play a central role in the proof of Theorem 1.3.

Lemma 3.1 *Let G be a connected Lie subgroup of the isometries of hyperbolic space H^N which has a totally geodesic orbit (may be a fixed point). Then no other orbit of positive dimension is minimal.*

Proof. Let $G.p$ be the totally geodesic orbit and let $G.q \neq \{q\}$ be another orbit. Let γ be a geodesic in H^N that minimizes the distance between q and $G.p$. Without loss of generality we may assume that $\gamma(0) = p$ and $\gamma(1) = q$ (eventually by changing the base point p by another in the orbit). It is standard to show that $\dot\gamma(0)$ is perpendicular to $T_p(G.p)$, or equivalently $\langle X.p, \dot\gamma(0)\rangle = 0$ for all X in the Lie algebra of G. Observe that this implies that $\langle X.\gamma(t), \dot\gamma(t)\rangle = 0$ for all t, because of $\frac{d}{dt}\langle X.\gamma(t), \dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma(t)} X.\gamma(t), \dot\gamma(t)\rangle = 0$, by the Killing equation. So, $\dot\gamma(t)$ is perpendicular to $T_{\gamma(t)}(G.\gamma(t))$, for all t. Let X be a Killing field in the Lie algebra of G such that $X.q \neq 0$ and let ϕ^X_s be the one-parameter group of isometries generated by X. Define $h : I \times \mathbb{R} \to H^N$ by $h_s(t) := \phi^X_s.\gamma(t)$. Note that $X.h_s(t) = \frac{\partial h}{\partial s}$ and that, for a fixed s, $h_s(t)$ is a geodesic.

Let $A_{\dot\gamma(t)}$ be the shape operator at the point $\gamma(t)$ of the orbit $G.\gamma(t)$. Define $f(t) := -\langle A_{\dot\gamma(t)}(X.\gamma(t)), X.\gamma(t)\rangle = \langle \frac{D}{\partial s}\frac{\partial h}{\partial t}, X.h_s(t)\rangle\mid_{s=0}$. Let us compute

$$\frac{d}{dt} f(t) = \langle \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial h}{\partial t}, X.h_s(t)\rangle\mid_{s=0} + \langle \frac{D}{\partial s}\frac{\partial h}{\partial t}, \frac{D}{\partial t} X.h_s(t)\rangle\mid_{s=0}$$

$$= \langle R(\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s})\frac{\partial h}{\partial t}, X.h_s(t)\rangle \mid_{s=0} + \langle \frac{D}{\partial t}\frac{\partial h}{\partial s}, \frac{D}{\partial t}X.h_s(t)\rangle \mid_{s=0}$$
$$= \langle R(\dot\gamma(t), X.\gamma(t))\dot\gamma(t), X.\gamma(t)\rangle + \|\nabla_{\dot\gamma(t)}(X.\gamma(t))\|^2$$
$$= \|\dot\gamma(t)\|^2 \|X.\gamma(t)\|^2 + \|\nabla_{\dot\gamma(t)}(X.\gamma(t))\|^2.$$

Then $\frac{d}{dt}f(t) \geq 0$ and $\frac{d}{dt}f(1) > 0$ (because of $X.q \neq 0$). Since $f(0) = 0$, due to the fact that $G.p$ is totally geodesic, we obtain that $f(1) = -\langle A_{\dot\gamma(1)}(X.q), X.q\rangle > 0$. Hence $A_{\dot\gamma(1)}$ is negative definite and so $G.q$ can not be minimal.

Remark 3.2 *From the proof of the above lemma one can prove the following: let G be a subgroup of the isometries of a space H of nonpositive curvature. If $G.p$ is a totally geodesic orbit and there exists another orbit which is minimal, then it must be also totally geodesic. Moreover, both orbits are contained in a flat totally geodesic $G-$invariant submanifold of one dimension higher, where they are parallel. In particular, if H is a symmetric space then $rank(H) \geq \dim(G.p) + 1$.*

Remark 3.3 *If a normal subgroup F of an isometry subgroup G of hyperbolic space has a totally geodesic orbit (different from a point), then this orbit is also an orbit of G. Namely, since G permutes $F-$orbits, this is a consequence of the above uniqueness result of Lemma 3.1.*

Let X be a Killing field in the Lie algebra of $SO(N, 1)$ and let ϕ_s^X be its one-parameter group of isometries. It is a standard fact that either ϕ_s^X has fixed points in H^N, or it translate a unique geodesic or it has a unique fixed point $z \in H^N(\infty)$. If ϕ_s^X fixes a point $z \in H^N(\infty)$ and does not translate a geodesic then any horosphere at z is invariant by ϕ_s^X. The following lemma is a generalization of this fact (cf. [BGS, pp. 86]).

Lemma 3.4 *Let A be a connected abelian Lie subgroup of SO(N,1). Then one of the following properties holds:*

i) A translates a unique geodesic of H^N.

ii) A has a fixed point in H^N.

iii) A has a unique fixed point in $H^N(\infty)$.

Proof. Let $X \neq 0$ be a Killing field in the Lie algebra of A and let ϕ_s^X be its one-parameter group of isometries. If ϕ_s^X translates a geodesic γ then Remark 3.3 implies that A also translates γ.

If ϕ_s^X has a unique fixed point $z \in H^N(\infty)$ then z is also fixed by A, since ϕ_s^X is a normal subgroup of A.

Assume that ϕ_s^X has fixed points in H^N. The set $(H^N)^X$ of fixed points of ϕ_s^X is a proper $A-$invariant connected totally geodesic submanifold of H^N. Then A acts on $(H^N)^X$, an hyperbolic space of lower dimension, and we can repeat the argument there.

Remark 3.5 *Let X be a Killing field on H^N whose associated one-parameter group of isometries fixes some $z \in H^N(\infty)$. If $X.q$ is tangent to the horosphere foliation $\mathcal{Q}(z)$ at some point q, then it must be always tangent to this foliation. If $X.q$ is not tangent to a horosphere Q then there exists a unique point $p \in Q$ such that $X.p$ is perpendicular to Q. This is equivalent to the fact that the orbits of a one-dimensional Lie subgroup of isometries of H^N, which has a fixed point at infinity, must be contained in horospheres or this group translates a unique geodesic (the uniqueness follows, for instance, from Lemma 3.1). This can be proved by observing that the function $q \to \langle X(q), X(q)\rangle$ of Q, grows quadratically and so it must have a critical point which has the desired property (see also [BGS, pp. 86]).*

Lemma 3.6 *Let G be a connected Lie subgroup of the isometries of hyperbolic space H^N which fix a point $z \in H^N(\infty)$. Then the same conclusion of Theorem 1.3 holds.*

Proof. We prove the statement by induction on the dimension of hyperbolic space. If $N = 2$ then either the action is transitive or any orbit has dimension one. In the last case by Remark 3.5 we are done. Let us then assume $N > 2$ and that some orbit of G (and hence any, by Remark 2.5) is not contained in a horosphere. Let L be the subgroup of G which leaves invariant some (and hence any) horosphere centered at $z \in H^N(\infty)$. Since G preserves the foliation by horospheres centered at $z \in H^N(\infty)$, it is standard to show that L is a connected normal Lie subgroup of G of codimension one. Let X be a Killing field in the Lie algebra of G which does not belong to the Lie algebra of L. Let Q be a horosphere centered at $z \in H^N(\infty)$. We identify Q with a Euclidean space. We have, by Remark 2.4, that the union V_Q of the totally geodesic orbits of L is a (connected) totally geodesic submanifold of Q. It is clear that if L acts transitively on the horosphere then G must act transitively on H^N, and we are done. Assume first that $V_Q \neq Q$. By making use of Lemma 2.1 we construct a proper connected totally geodesic submanifold T of H^N which contains V_Q. We claim that T is invariant by G. If $g \in G$ then $V_{g.Q} = g.V_Q$, since L is normal subgroup of G. Then $g.T = T$ since $T = \bigcup_t (V_Q)_t$ and $(V_Q)_t = V_{(Q_t)}$ by Lemma 2.1 (see Remark 2.2). So, we are done by induction. If $V_Q = Q$ then there exist a unique point $q \in Q$ such that $X(q)$ is perpendicular to Q (see Remark 3.5). We claim that the proper connected totally geodesic submanifold $\tilde{T}$ determined, as in Lemma 2.1 by $L.q$ is invariant by G (moreover $G.q = \tilde{T}$). Let $\mathcal{F}$ be the distribution of H^N defined by $\mathcal{F}_p := \nu_p(Q(p)) \oplus T_p(L.p)$, where $Q(p)$ is the horosphere through p centered at $z \in H^N(\infty)$. By Lemma 2.1 and Remark 2.2 we have that $\mathcal{F}$ is integrable with totally geodesic leaves. Moreover, $\mathcal{F}$ is $G-$invariant, since L is a normal subgroup of G (note that G permutes $L-$orbits). Observe that $\tilde{T}$ is the leaf of $\mathcal{F}$ through q. Let ϕ_t^X be the one-parameter group of isometries generated by X. Then ϕ_t^X translates the geodesic perpendicular to Q at q, which is contained in $\tilde{T}$. Therefore, $\phi_t^X(\tilde{T}) = \tilde{T}$. Since $\tilde{T}$ is $L-$invariant and G is generated by ϕ_t^X and L we obtain that $G.q = \tilde{T}$.

Remark 3.7 *A Lie subgroup of $SO(N,1)$ which has a fixed point at $H^N(\infty)$ does not act irreducibly. In fact, it preserves a parallel foliation by horospheres and so, it must leave invariant the degenerate linear hyperplane associated with this foliation.*

Lemma 3.8 *Let G be a non semisimple connected Lie subgroup of $SO(N,1)$. Then either G leaves invariant some proper connected totally geodesic submanifold of hyperbolic space or the same conclusion of Theorem 1.3 holds.*

Proof. Let A be a normal abelian subgroup of G. By Lemma 3.4 there are three cases for A. In the first case, using Remark 3.2 we are done. In the second case, the fixed set of A is a proper $G-$invariant connected totally geodesic submanifold. In third case, G must fix the same point at infinity as A. So, by Lemma 3.6 we are done.

Lemma 3.9 *Let G be a simple simply connected Lie group of the noncompact type which acts almost effectively on H^N. Then G has a minimal orbit which is also the orbit of a connected proper Lie subgroup of G.*

For the proof we will need the following auxiliary result, since we do not assume G to be closed in $SO(N,1)$.

Sublemma 3.10 *If K is a maximal compact connected subgroup of G and $p \in H^N$ is fixed by K (which must exist because H^N is negatively curved), then $K = (G_p)_0$, the connected component of the isotropy of G at p.*

Proof. The group $(G_p)_0$ is isomorphic, via the differential at p, to a Lie subgroup of $SO(T_pH^N)$ and hence it admits a biinvariant (positive definite) Riemannian metric. So, the universal cover of $(G_p)_0$ splits as $\mathbb{R}^k \times C$, where C is a compact Lie group. Consider the obvious (almost effectively) action of this last group on the simple symmetric space G/K of the noncompact type and regard the orbit $(\mathbb{R}^k \times C).[e]$, where $[e]$ is the base point of G/K. Since K is contained in $(G_p)_0$, it must preserve this orbit and therefore $V = T_{[e]}((\mathbb{R}^k \times C).[e])$ is a $K-$invariant subspace of $T_{[e]}(G/K)$. Since K acts irreducibly on this last space we obtain that either $V = \{0\}$ or $V = T_{[e]}(G/K)$. In the first case $(G_p)_0$ must be contained in the isotropy of G/K and we are done. In the second case we obtain that $\mathbb{R}^k \times C$ acts transitively on G/K. Since G/K is nonpositively curved there must exist $[g] \in G/K$ which is fixed by C (see [KN, pp. 111]). Then $G/K = (\mathbb{R}^k \times C).[g] = \mathbb{R}^k.[g]$ which is impossible, otherwise G/K would be flat.

Proof of Lemma 3.9. Since G/K has a unique, up to a constant mutiple, $G-$invariant metric we may fix it so that the orbit $G.p$ is locally isometric to G/K (see Sublemma 3.10). Let $\nu_p^K(G.p)$ be the subspace of the normal space $\nu_p(G.p)$ to the orbit $G.p$, which consists of the fixed vectors of $K = (G_p)_0$. Observe that any $\eta \in \nu_p^K(G.p)$ defines locally a $G-$invariant normal vector field $\tilde{\eta}$ of $G.p$. Let $\xi \in \nu_p^K(G.p)$ and let $q = exp_p(\xi) = \gamma_\xi(1)$. Then $K.q = \{q\}$ and, by Sublemma 3.10, $K = (G_q)_0$. So, any $G-$orbit through a point of $T := exp_p(\nu_p^K(G.p))$ is locally homothetic to G/K. Let now $G = NAK$ be an Iwasawa decomposition of G. Then $G.q = NAK.q = S.q$, for all $q \in T$, where $S = NA$. In order to finish the proof it suffices to show, since $\dim(S) < \dim(G)$, that there exists $q \in T$ such that $G.q$ is a minimal submanifold of H^N. Observe that the mean curvature vector of $G.p$ at p lies in $\nu_p^K(G.p)$, since it must be fixed by the isotropy. So, we may assume that $\nu_p^K(G.p) \neq \{0\}$, or equivalently $T \neq \{p\}$ (otherwise $G.p$ would be minimal). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition associated to G/K and let $X \in \mathfrak{p}$ with $\|X\| = 1$. Then $\|X.q\|$ is the factor of the local homothecy from G/K to the orbit $G.q$, $q \in T$ (observe that this does not depend on $X \in \mathfrak{p}$ of unit length). Let dV_q be the volume element of $G.q$ at q. Then $dV_q = \|X.q\|^n dV_p$, where $n = \dim(G/K)$. Let $\xi \in \nu_p^K(G.p)$ with $\|\xi\| = 1$ and consider the Jacobi field $Y_\xi(t) := X.\gamma_\xi(t)$ along the geodesic $\gamma_\xi(t) = exp^\nu(t\xi)$, where $X \in \mathfrak{p}$ has unit length. The Jacobi field $Y_\xi(t)$ is determined by $Y_\xi(0) = X.p$ and $\frac{D}{dt}Y_\xi(t)|_{t=0} = -A_\xi(X.p) + \nabla^\perp_{X.p}\tilde{\xi}$, where $\tilde{\xi}$ is the normal $G-$invariant field defined by ξ (see above). By solving explicitly the Jacobi equation (using parallel fields along $\gamma_\xi(t)$) one gets that

$$\|Y_\xi(t)\|^2 = \|\cosh(t)X.p - \sinh(t)A_\xi(X.p)\|^2 + \|\sinh(t)\nabla^\perp_{X.p}\tilde{\xi}\|^2$$

$$= \cosh^2(t).\|X.p\|^2 - 2\cosh(t)\sinh(t)\langle X.p, A_\xi(X.p)\rangle +$$
$$+ \sinh^2(t)\|A_\xi(X.p)\|^2 + \sinh^2(t)\|\nabla^\perp_{X.p}\tilde{\xi}\|^2.$$

Therefore $\frac{d}{dt}|_{t=0}\|Y_\xi(t)\|^2 = -2\langle X.p, A_\xi(X.p)\rangle$. Since $\|Y_\xi(t)\|$ does not depend on $X \in \mathfrak{p}$, $\|X\| = 1$, we obtain that $\langle A_\xi v, v\rangle$ does not depend on $v \in T_p(G.p), \|v\| = 1$. Hence $A_\xi = \lambda(\xi)Id$, where λ is linear on $\nu_p^K(G.p)$. This also implies that $\|\nabla^\perp_v\tilde{\xi}\|^2$ does not depend on v if $\|v\| = 1$.

We claim that, either $\|\nabla^\perp_v\tilde{\xi}\| \neq 0$ or $\lambda(\xi) \neq 1$ for all $\xi \in \nu_p^K(G.p)$ with $\|\xi\| = 1$. In fact, if for some ξ we have that $\|\nabla^\perp_v\tilde{\xi}\| = 0$ and $\lambda(\xi) = 1$ then the field $Z(q) = \tilde{\xi}(q) + q$ is constant on $G.p$, regarding $G.p \subset \mathbb{R}^{N,1}$. Hence, $G.p$ is contained in a horosphere, let us say Q, defined by some hyperplane parallel to the degenerate hyperplane $\{x \in \mathbb{R}^{N,1} : \langle Z, x\rangle = 0\}$. The affine span E of $G.p$ is contained in this hyperplane and is $G-$invariant. The intersection $E \bigcap H^N = E \bigcap Q$ is $G-$invariant and totally umbilical in Q (i.e. a sphere or a Euclidean subspace of Q). This is a contradiction. In fact, in the first case G would act on a sphere which is impossible since G is of non-compact type. In the second case, it contradicts [V] (see also [O1, appendix]).

Thus, by a standard argument involving compactness, there exists $\epsilon > 0$ such that $(1 - \lambda(\xi))^2 + \|\nabla^\perp_v\tilde{\xi}\|^2 \geq \epsilon$ for all $\xi \in \nu_p^K(G.p)$ with $\|\xi\| = 1$. Then

$$\|Y_\xi(t)\|^2 = \sinh^2(t)(\|\coth(t)X.p - \lambda(\xi)X.p)\|^2 + \|\nabla^\perp_{X.p}\tilde{\xi}\|^2)$$

$$= \sinh^2(t)((\coth(t) - \lambda(\xi))^2 + \|\nabla^\perp_{X.p}\tilde{\xi}\|^2) \geq \sinh^2(t)\,\frac{\epsilon}{2}$$

for large t, independent of ξ with $\|\xi\| = 1$.

Since this lower bound is independent of ξ with $\|\xi\| = 1$ we obtain that the length of the Killing field defined by $X \in \mathfrak{p}$ achieves its minimum at some point $q \in T$. This implies that the volume element dV_q is a minimum. Hence by [HL, Th.1] $G.q$ is a minimal orbit and the result follows.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. We proceed by induction on $r = \dim(G) + N$, where $N = \dim(H^N)$. If $r = 3$ it is a well known fact. Assume $r > 3$, if G is not semisimple then by Lemma 3.8 we are done. If G semisimple we may assume that G has a simple noncompact factor F (otherwise by the well known Theorem of Cartan ([KN, pp. 111]) we are done). By Lemma 3.9 F has a minimal orbit which is also a minimal orbit of a proper subgroup. By induction this proper subgroup has a totally geodesic orbit, because horospheres can not contain minimal submanifolds of H^N (horospheres are umbilical). Then by the uniqueness result of Lemma 3.1 we have that this minimal orbit is totally geodesic. Since F is normal in G this orbit is also $G-$orbit (see Remark 3.3) and we are done.

We are now able to prove our main results.

Proof of Theorem 1.2 . By Theorem 1.3 either G acts transitively on H^N or all orbits are included in horospheres centered at the same point z at the infinity. In the last case, if the action on the horospheres at z is not transitive, we can construct, as in Lemma 3.6 a connected totally geodesic $G-$invariant proper submanifold of H^N. This implies that the action is not weakly irreducible, and we are done.

Proof of Theorem 1.1 If the action is irreducible then we claim that G is semisimple. In fact, this follows from Lemma 3.8 and Remark 3.7. Let F be a simple factor of G. Then by Theorem 1.3 F also acts transitively on hyperbolic space. In fact, F must have a minimal orbit by Lemma 3.9 and by Corollary 1.4 this orbit must be totally geodesic. But this implies that this $F-$orbit is also a $G-$orbit (see Remark 3.3). Since $H^N = F/F_p$ is simply connected, we obtain that F_p is connected and by Sublemma 3.10 a maximal compact subgroup of F. Hence $F = SO_0(N, 1)$, since (F, F_p) is a symmetric pair.

4 Final Remarks

1- In [OW] it was proved that there are no non-trivial irreducible homogeneous submanifolds of hyperbolic space with non-vanishing parallel mean curvature. Corollary 1.4 extends this result also for the minimal case.

2- If the (restricted) holonomy group of a Lorentzian locally symmetric space is not $SO_0(N,1)$ then it must act transitively on a horosphere. This follows from Corollary 1.5 and the classification of reducible indecomposable locally symmetric Lorentzian spaces given by Wallach and Cahen [CW]. In fact, these spaces have abelian holonomy which cannot act transitively on hyperbolic space. A direct proof of the non-transitivity, in the reducible case, follows from [BBG]. In fact, if the holonomy were transitive on hyperbolic space the Lorentzian symmetric space would be Osserman and hence of constant curvature.

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Fa.M.A.F., Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

e-mail address: discala@mate.uncor.edu

olmos@mate.uncor.edu