A note on bisexual Galton-Watson branchingprocesses in random environments

A NOTE ON BISEXUAL GALTON-WATSON BRANCHING PROCESSES IN RANDOM ENVIRONMENTS

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Abstract. A bisexual Galton-Watson branching process is a two–type branching model, in which matings in one generation give rise to random numbers of both males and females in the next. The mating function describes how many mating units are formed from given numbers of males and females. In this paper we consider the case that the distributions of the random numbers of males and females produced by the mating units depend on some fertility parameters evolving randomly in time. By means of a main stochastic comparison result, we show that the total population increases, in some stochastic sense, as the positive dependence between the fertility indexes increases. Simple examples of applications of this result are provided, together with other similar results for a different model of population growth.

1 Introduction Galton-Watson branching processes constitute an appropriate mathematical model for the description of populations' growth where individuals produce offsprings according to some stochastic law (see, e.g., Kimmel and Axelrod [9]).

Recently Fernández–Ponce et al. [6] considered branching processes defined in random environments, and studied the influence on the population sizes of the autocorrelation in the environmental process. In their Corollary 4.2 they showed that the population size, at any fixed generation, increases in increasing convex order (whose definition is recalled next) as the autocorrelation of the environmental process increases.

Here we show that such property can be actually extended to bisexual Galton-Watson branching processes, i.e., to the modification of the standard branching process introduced by Daley [3] in order to allow for sexual reproduction. In his model, matings in one generation give rise to random numbers of both males and females in the next, and the number of mating units is described by a mating function of the number of males and females in the previous generation. See Molina et al. [12] or Mota et al. [13] and references therein for recent results and applications of this model.

For a formal description of bisexual Galton-Watson branching processes defined on a random environment consider a sequence of couples of fertility indexes $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \{(\theta_n, \lambda_n) \in \mathbb{T} \times \mathbb{L}, n \in \mathbb{N}\}$ defined on an appropriate space $\mathbb{T} \times \mathbb{L} \subseteq \mathbb{R}^2$, and denote with $(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)$ the finite sequence of couples $\{(\theta_k, \lambda_k) \in \mathbb{T} \times \mathbb{L}, k = 1, \ldots, n\}$, for every $n \in \mathbb{N}$. The bisexual Galton-Watson branching process we consider here is the bivariate process $(\mathbf{F}, \mathbf{M})_{(\boldsymbol{\theta}, \boldsymbol{\lambda})} = \{(F_n, M_n)_{(\boldsymbol{\theta}_{n-1}, \boldsymbol{\lambda}_{n-1})}, n \in \mathbb{N}^+\}$ defined recursively by

$$Z_1(\theta_0, \lambda_0) = N_{(\theta_0, \lambda_0)}$$

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and

$$\begin{aligned}
(F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)} &= \sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})} (f_{n,i}(\theta_n), m_{n,i}(\lambda_n)) \\
&= \left(\sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})} f_{n,i}(\theta_n), \sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})} m_{n,i}(\lambda_n) \right), \quad n \in \mathbb{N}, \\
Z_{n+1}(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n) &= L((F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)})
\end{aligned}$$

where $N_{(\theta_0,\lambda_0)}$ is a positive random integer, the mating function L is assumed to be non-decreasing in each argument and the empty sum is considered to be (0,0). The sequence $(\mathbf{f}, \mathbf{m})_{(\boldsymbol{\theta},\boldsymbol{\lambda})} = \{(f_{n,i}(\theta_n), m_{n,i}(\lambda_n)), n \in \mathbb{N}\}$ represents the numbers of females and males produced by the i-th mating unit in the n-th generation. All the couples $(f_{n,i}(\theta_n), m_{n,i}(\lambda_n)), \forall i, \forall n$, are assumed to be independent once the sequence $(\boldsymbol{\theta}, \boldsymbol{\lambda})$ of the fertility parameters is fixed.

In many practical contexts, one can assume that the offsprings distributions depend on environmental conditions (see, e.g., Athreya and Karlin [1]). Thus, in order to describe dependence of the process on random evolutions in time of the environment, we can assume the parameters $(\boldsymbol{\theta}, \boldsymbol{\lambda})$ to be the realization of an environmental bivariate process $(\Theta, \Lambda) = \{(\Theta_n, \Lambda_n), n \in \mathbb{N}\}$ having state space $\mathbb{T} \times \mathbb{L}$, that describes such evolutions.

Here we study the influence of the autocorrelation of the environmental bivariate process (Θ, Λ) on the population sizes $\{(F_n, M_n), n \in \mathbb{N}^+\}$. We show that the population size, for any fixed generation n, increases in increasing convex sense as the positive dependence between the random fertility indexes increases. As described in details in the next section, the increasing convex order is a stochastic comparison that jointly consider the "size" and the "variability" of random variables.

The paper proceeds as follows. In Section 2 we provide notation and tools on stochastic comparisons and multivariate stochastic convexity that will be used along the paper. In Section 3 we state and prove the main result mentioned above, and we also provide some simple examples of application. In the last section, we provide a similar complementary result for a different compound immigration model.

Some conventions and notations that are used throughout the paper are given previously. Let $\leq$ denote the coordinatewise ordering (that is, for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $i = 1, 2, ..., n$) and $[\mathbf{x}, \mathbf{y}] \leq \mathbf{z}$, as a shorthand for $\mathbf{x} \leq \mathbf{z}$ and $\mathbf{y} \leq \mathbf{z}$. The operators $+$, $\vee$ and $\wedge$ denote, respectively, the componentwise sum, maximum and minimum. The notation $=_{st}$ stands for equality in law, and a.s. as a shorthand for almost surely. For any family $\{X_\theta \,|\theta \in \mathbb{T}\}$ of parameterized random variables, with $\mathbb{T} \subseteq \mathbb{R}$, we denote by $X(\Theta)$ the mixture of $\{X_\theta \,|\theta \in \mathbb{T}\}$ with a mixing variable Θ. For any random variable (or vector) X and an event A, $[X \,|A]$ denotes a random variable whose distribution is the conditional distribution of X given A. Also, throughout this paper we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing".

2 Utility notions and preliminary results In this section we provide notations and mathematical tools for the statement of the results presented along the paper. In particular, we will recall the definitions of some stochastic orders as well as multivariate notions of stochastic convexity for a family of parameterized random variables. For that, we will consider different notions of convexity in the multivariate setting.

Let us recall first the definition of two of the most well-known univariate stochastic orders, which are considered in this paper.

Definition 2.1 *Let X and Y be two non-negative random variables, then X is said to be smaller than Y in the stochastic [increasing convex] order (denoted by $X \leq_{st} [\leq_{icx}] Y$) if*

$$\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)],$$

for all increasing [increasing convex] functions ϕ for which the expectations exist.

It should be mentioned that, for non-negative random variables, $X \leq_{\mathrm{st}} Y$ if and only if for all $t \geq 0$ it holds $F_X(t) \geq F_Y(t)$.

While the *st* order compares the probabilities of random variables to assume high values, the *icx* order is essentially a comparison of variables based both on their "magnitude" and on their variability, in the sense that $X \leq_{icx} Y$ means that X is both "smaller" and "less variable" than Y in some stochastic sense (see Shaked and Shanthikumar [16] for details). In particular observe that both $X \leq_{st} Y$ and $X \leq_{icx} Y$ imply $\mathrm{E}[X] \leq \mathrm{E}[Y]$, and that $X \leq_{icx} Y$ implies $\mathrm{Var}[X] \leq \mathrm{Var}[Y]$ whenever $\mathrm{E}[X] = \mathrm{E}[Y]$.

A characterization of the stochastic ordering that will play a crucial role in this paper is recalled now (see Theorem 1.A.1 in Shaked and Shanthikumar [16]). Given two random variables X and Y, then $X \leq_{st} Y$ if and only if there exist two random variables $\widehat{X}$ and $\widehat{Y}$, defined on the same probability space, such that $X =_{st} \widehat{X}$, $Y =_{st} \widehat{Y}$ and $\widehat{X} \leq \widehat{Y}$, a.s.

These two orders can be generalized in different ways in the multivariate settings. We will consider here two multivariate definitions of the increasing convex order, that will be used in the next sections. For it, recall that a real-valued function ϕ defined on $\mathbb{R}^n$ is said to be *convex* (denoted here by $\phi \in cx$) if

$$\phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\phi(\mathbf{x}) + (1-\alpha)\phi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in [0,1]$, and is said to be *directionally convex* (denoted by $\phi \in dcx$) if for any $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1,2,3,4$, such that $\mathbf{x}_1 \leq [\mathbf{x}_2, \mathbf{x}_3] \leq \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$, then

$$\phi(\mathbf{x}_1) + \phi(\mathbf{x}_4) \geq \phi(\mathbf{x}_2) + \phi(\mathbf{x}_3).$$

If in addition, ϕ is increasing, that is, for all $\mathbf{x} \leq \mathbf{y}$ then $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$, then we say that ϕ is, respectively, *increasing and convex* (denoted by $\phi \in icx$) or *increasing and directionally convex* (denoted by $\phi \in idcx$).
Moreover, a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $\phi(\mathbf{x}) = (\phi_1(\mathbf{x}), ..., \phi_m(\mathbf{x}))$ is said to be directionally convex if each of the coordinate functions ϕ_i, $i = 1, 2, ..., m$ is directionally convex.

For a complete discussion on convex and directionally convex functions see Meester and Shanthikumar [11]. In particular we note here that directional convexity neither implies nor is implied by usual convexity, and the composition of functions preserves increasing directional convexity. Also, the composition of an *icx* function with an *idcx* function is an *idcx* function.

As already stated, the stochastic order and the increasing convex order can be extended to the multivariate case in several ways. Here we consider three of them.

Definition 2.2 *Let $\mathbf{X} = (X_1, ..., X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two n-dimensional random vectors, then $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the stochastic [increasing convex, increasing directionally convex] order (denoted by $\mathbf{X} \leq_{st} [\leq_{icx}, \leq_{idcx}] \mathbf{Y}$) if*

$$\mathrm{E}\left[\phi(\mathbf{X})\right] \leq \mathrm{E}\left[\phi(\mathbf{Y})\right]$$

for all increasing [increasing convex, increasing directionally convex] real-valued functions ϕ defined on $\mathbb{R}^n$ for which the expectations exist.

For a survey on these stochastic orderings and their properties we refer the reader to Shaked and Shanthikumar [16]. Here we just point out that multivariate *st* and *icx* order have the same intuitive interpretation as in the univariate case, while, on the other hand, the *idcx* order can be used to compare positive dependence properties of random vectors having the same marginal distributions. In this case, in fact, $\mathbf{X} \leq_{idcx} \mathbf{Y}$ means that the components of $\mathbf{X}$ are less positively dependent than the components of $\mathbf{Y}$ (see again Shaked and Shanthikumar [16] for details). For example, it can be shown that if $\mathbf{X}$ and $\mathbf{Y}$ are multivariate random vectors with the same mean vector $\mathbf{M}$ and variance–covariance matrices Σ and $\Sigma + \mathbf{D}$, respectively, then $\mathbf{X} \leq_{idcx} \mathbf{Y}$ when $\mathbf{D}$ is a matrix that has zero diagonal elements and all other entries non-negative. Also, it can be shown that, given a vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ of *associated* random variables (see Denuit et al. [5] for definition), if the X_i are marginally identically distributed then

$$\mathbf{X}^{\perp} \leq_{idcx} \mathbf{X} \leq_{idcx} \mathbf{X^0}$$

where $\mathbf{X}^{\perp} = (X_1^{\perp}, X_2^{\perp}, \ldots, X_n^{\perp})$ is the corresponding vector of independent variables, while $\mathbf{X^0} = (X_1^0, X_2^0, \ldots, X_n^0)$ is such that $X_1^0 = X_2^0 = \ldots = X_n^0 = X_1$ a.s. Further examples and sufficient conditions for *idcx* comparison of random vectors may be found in Meester and Shanthikumar [10] (see in particular Lemma 2.10), and in Rüschendorf [14] (Corollary 3.2 and Corollary 4.1 therein).

At this point, we recall some notions of multivariate stochastic convexity for a family of parameterized random variables, introduced in Shaked and Shanthikumar [15] and further studied also in Meester and Shanthikumar [10] and [11] . For it, consider a family of multivariate random variables $\mathbf{X}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathbb{T}$, where $\mathbb{T}$ is a sublattice of either $\mathbb{R}^n$ or $\mathbb{N}^n$.

Definition 2.3 *A family $\{\mathbf{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{T}\}$ of multivariate random variables is said to be:*
i) stochastically increasing (denoted by $\{\mathbf{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{T}\} \in SI$) if for any $\boldsymbol{\theta}_i \in \mathbb{T}$, $i = 1, 2$, $\boldsymbol{\theta}_1 \leq \boldsymbol{\theta}_2$, then $\mathbf{X}(\boldsymbol{\theta}_1) \leq_{st} \mathbf{X}(\boldsymbol{\theta}_2)$;
ii) stochastically increasing and directionally convex in the sample path sense (denoted by $\{\mathbf{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{T}\} \in SI{-}DCX(sp)$) if for any four $\boldsymbol{\theta}_i \in \mathbb{T}$, $i = 1, .., 4$, such that $\boldsymbol{\theta}_1 \leq [\boldsymbol{\theta}_2, \boldsymbol{\theta}_3] \leq \boldsymbol{\theta}_4$ and $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_4 = \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3$, there exist random variables $\widehat{\mathbf{X}}_i$, $i = 1, ..., 4$, defined on a common probability space, such that $\widehat{\mathbf{X}}_i =_{st} \mathbf{X}(\boldsymbol{\theta}_i)$, $i = 1, ..., 4$ and

$$\left[\widehat{\mathbf{X}}_2, \widehat{\mathbf{X}}_3\right] \leq \widehat{\mathbf{X}}_4, \; a.s. \tag{1}$$

and

$$\widehat{\mathbf{X}}_1 + \widehat{\mathbf{X}}_4 \geq \widehat{\mathbf{X}}_2 + \widehat{\mathbf{X}}_3, \; a.s. \tag{2}$$

In case both the parameter and the random variables are univariate, then we will use the notation $SI - CX(sp)$.

Stochastic increasing directional convexity in sample path sense is closed by composition with *idcx* functions (see for example, Lemma 2.15 in Meester and Shanthikumar [10]). Also, this notion of stochastic convexity is closed by conjunction of independent random variables (see Lemma 2.16 in Meester and Shanthikumar [10] or Theorem 3.3 and Theorem 4.4 in Meester and Shanthikumar [11]).

Some examples of stochastic directional convexity for parameterized families of random variables can be found in the literature: see Shaked and Shanthikumar [15], Chang et al. [2] or Meester and Shanthikumar [11]. For example, the Bernoulli distribution and the Poisson distribution are $SI-CX(sp)$ in their parameters, and the Multinomial distribution and the Gamma distribution are $SI-DCX(sp)$. Other examples can be obtained by using the preservation properties above. For example, it is easy to verify that when the families $\{X(\theta),\theta\in\mathbb{T}\}$ and $\{Y(\lambda),\lambda\in\mathbb{L}\}$ are independent and SI-CX(sp), then $\{(X(\theta),Y(\lambda)),(\theta,\lambda)\in\mathbb{T}\times\mathbb{L}\}$ is SI-DCX(sp). Also, under appropriate conditions, some applied stochastic models have stochastic directional convexity properties (see references above).

The following two properties will be used in the next sections. The proof of Lemma 2.2 may be found in Shaked and Shanthikumar [15].

Lemma 2.1 *Let* $(X_1,\dots,X_n)\le_{idcx}(Y_1,\dots,Y_n)$ *or* $(X_1,\dots,X_n)\le_{icx}(Y_1,\dots,Y_n)$. *Then* $\sum_{i=1}^n X_i\le_{icx}\sum_{i=1}^n Y_i$.

Proof. Let ϕ be any *icx* real function. Then it is easy to verify that the function $h(u_1,\dots,u_n)=\phi(u_1+\dots+u_n)$ is both *idcx* and *icx* (see Corollary 2.5 in Meester and Shanthikumar [10] for details). Thus

$$\mathrm{E}[\phi(\sum_{i=1}^n X_i)]=\mathrm{E}[h(X_1,\dots,X_n)]\le\mathrm{E}[h(Y_1,\dots,Y_n)]=\mathrm{E}[\phi(\sum_{i=1}^n Y_i)]$$

■

Lemma 2.2 *Let the family* $\{\mathbf{X}(\boldsymbol{\theta}),\boldsymbol{\theta}\in\mathbb{T}\}$ *be SI-DCX(sp). Then* $\mathrm{E}[\phi(\mathbf{X}(\boldsymbol{\theta}))]$ *is increasing and directionally convex in* $\boldsymbol{\theta}$ *for any idcx function* ϕ.

3 Main result

The following result describes conditions for SI-DCX(sp) property of the generations sequence of males and females.

Theorem 3.1 *Consider the sum*

$$\begin{aligned}(F_{n+1},M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)} &= \sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})}(f_{n,i}(\theta_n),m_{n,i}(\lambda_n))\\ &= \left(\sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})} f_{n,i}(\theta_n),\ \sum_{i=1}^{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})} m_{n,i}(\lambda_n)\right)\end{aligned}$$

If
i) all the variables $f_{n,i}(\theta_n)$, $m_{n,i}(\lambda_n)$ *and* $Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})$ *are independent for fixed values of* $(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)$;
ii) the families $\{f_{n,i}(\theta),\theta\in\mathbb{T}\}$ *and* $\{m_{n,i}(\lambda),\lambda\in\mathbb{L}\}$ *are* $SI-CX(sp)$ *for every* n *and* i;
iii) $\{Z_n(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1}),(\boldsymbol{\theta}_{n-1},\boldsymbol{\lambda}_{n-1})\in\mathbb{T}^{n-1}\times\mathbb{L}^{n-1}\}$ *is* $SI-DCX(sp)$;
iv) all the variables $f_{n,i}(\theta)$ *are identically distributed for every fixed* $\theta\in\mathbb{T}$ *and, similarly, all the variables* $m_{n,i}$ *are identically distributed for every fixed* $\lambda\in\mathbb{L}$;
then

$$\{(F_{n+1},M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)},(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)\in\mathbb{T}^n\times\mathbb{L}^n\}$$

is $SI-DCX(sp)$.

Proof. Let $\mathbf{u}_k=(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)^{(k)}$, $k=1,\dots,4$ be such that $\mathbf{u}_1\le[\mathbf{u}_2,\mathbf{u}_3]\le\mathbf{u}_4$ and $\mathbf{u}_1+\mathbf{u}_4=\mathbf{u}_2+\mathbf{u}_3$. By the assumptions, we can build on the same probability space the random variables $\widehat{f}_{n,i}^{(k)} =_{st} f_{n,i}(\theta_n^{(k)})$, $\widehat{m}_{n,i}^{(k)} =_{st} m_{n,i}(\lambda_n^{(k)})$, $i \in \mathbb{N}$, and $\widehat{Z}^{(k)} =_{st} Z(\boldsymbol{\theta}_{n-1}, \boldsymbol{\lambda}_{n-1})^{(k)}$, for $k = 1, ..., 4$, such that, almost surely,

$$\widehat{f}_{n,i}^{(1)} + \widehat{f}_{n,i}^{(4)} \geq \widehat{f}_{n,i}^{(2)} + \widehat{f}_{n,i}^{(3)}, \quad \widehat{f}_{n,i}^{(4)} \geq [\widehat{f}_{n,i}^{(2)}, \widehat{f}_{n,i}^{(3)}],$$

$$\widehat{m}_{n,i}^{(1)} + \widehat{m}_{n,i}^{(4)} \geq \widehat{m}_{n,i}^{(2)} + \widehat{m}_{n,i}^{(3)}, \quad \widehat{m}_{n,i}^{(4)} \geq [\widehat{m}_{n,i}^{(2)}, \widehat{m}_{n,i}^{(3)}],$$

and

$$\widehat{Z}^{(1)} + \widehat{Z}^{(4)} \geq \widehat{Z}^{(2)} - \widehat{Z}^{(3)}, \quad \widehat{Z}^{(4)} \geq [\widehat{Z}^{(2)}, \widehat{Z}^{(3)}].$$

Let

$$(\widehat{F}^{(k)}, \widehat{M}^{(k)}) = \sum_{i=1}^{\widehat{Z}^{(k)}} (\widehat{f}_{n,i}^{(k)}, \widehat{m}_{n,i}^{(k)}), \quad k = 1, \ldots, 4.$$

and observe that $(\widehat{F}^{(k)}, \widehat{M}^{(k)}) =_{st} (F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)^{(k)}}$.

Consider now the partition of Ω in the sets

$$\begin{aligned}
\Omega_{123} &= \{\omega \in \Omega : \widehat{Z}^{(1)} \geq \widehat{Z}^{(2)} \geq \widehat{Z}^{(3)}\} \\
\Omega_{132} &= \{\omega \in \Omega : \widehat{Z}^{(1)} \geq \widehat{Z}^{(3)} > \widehat{Z}^{(2)}\} \\
\Omega_{213} &= \{\omega \in \Omega : \widehat{Z}^{(2)} > \widehat{Z}^{(1)} \geq \widehat{Z}^{(3)}\} \\
\Omega_{312} &= \{\omega \in \Omega : \widehat{Z}^{(3)} > \widehat{Z}^{(1)} \geq \widehat{Z}^{(2)}\} \\
\Omega_{321} &= \{\omega \in \Omega : \widehat{Z}^{(3)} > \widehat{Z}^{(2)} > \widehat{Z}^{(1)}\} \\
\Omega_{231} &= \{\omega \in \Omega : \widehat{Z}^{(2)} > \widehat{Z}^{(3)} > \widehat{Z}^{(1)}\}
\end{aligned}$$

For almost all $\omega \in \Omega_{123}$ we have:

$$\begin{aligned}
(\widehat{F}^{(1)}, \widehat{M}^{(1)}) + (\widehat{F}^{(4)}, \widehat{M}^{(4)}) &= \sum_{j=1}^{\widehat{Z}^{(1)}} (\widehat{f}_{n,i}^{(1)}, \widehat{m}_{n,i}^{(1)}) + \sum_{j=1}^{\widehat{Z}^{(4)}} (\widehat{f}_{n,i}^{(4)}, \widehat{m}_{n,i}^{(4)}) \\
&\geq \sum_{j=1}^{\widehat{Z}^{(2)}} \Big((\widehat{f}_{n,i}^{(1)}, \widehat{m}_{n,i}^{(1)}) + (\widehat{f}_{n,i}^{(4)}, \widehat{m}_{n,i}^{(4)}) \Big) \\
&\geq \sum_{j=1}^{\widehat{Z}^{(2)}} \Big((\widehat{f}_{n,i}^{(2)}, \widehat{m}_{n,i}^{(2)}) + (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) \Big) \\
&\geq \sum_{j=1}^{\widehat{Z}^{(2)}} (\widehat{f}_{n,i}^{(2)}, \widehat{m}_{n,i}^{(2)}) + \sum_{j=1}^{\widehat{Z}^{(3)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) \\
&= (\widehat{F}^{(2)}, \widehat{M}^{(2)}) + (\widehat{F}^{(3)}, \widehat{M}^{(3)}).
\end{aligned}$$

Thus, for almost all $\omega \in \Omega_{123}$ it holds

$$(\widehat{F}^{(1)}, \widehat{M}^{(1)}) + (\widehat{F}^{(4)}, \widehat{M}^{(4)}) \geq (\widehat{F}^{(2)}, \widehat{M}^{(2)}) + (\widehat{F}^{(3)}, \widehat{M}^{(3)}) \tag{3}$$

It is easy to verify that (3) holds even for almost all $\omega \in \Omega_{132}$.

For almost all $\omega \in \Omega_{213}$ we have:

$$
\begin{aligned}
(\widehat{F}^{(1)}, \widehat{M}^{(1)}) + (\widehat{F}^{(4)}, \widehat{M}^{(4)}) &= \sum_{j=1}^{\widehat{Z}^{(1)}} (\widehat{f}^{(1)}_{n,i}, \widehat{m}^{(1)}_{n,i}) + \sum_{j=1}^{\widehat{Z}^{(4)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
= \; & \sum_{j=1}^{\widehat{Z}^{(3)}} \Big((\widehat{f}^{(1)}_{n,i}, \widehat{m}^{(1)}_{n,i}) + (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \Big) + \sum_{j=\widehat{Z}^{(3)}+1}^{\widehat{Z}^{(1)}} (\widehat{f}^{(1)}_{n,i}, \widehat{m}^{(1)}_{n,i}) + \sum_{j=\widehat{Z}^{(3)}+1}^{\widehat{Z}^{(4)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
\geq \; & \sum_{j=1}^{\widehat{Z}^{(3)}} \Big((\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) \Big) + \sum_{j=\widehat{Z}^{(3)}+1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
\geq \; & \sum_{j=1}^{\widehat{Z}^{(3)}} (\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + \sum_{j=1}^{\widehat{Z}^{(3)}} (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) + \sum_{j=\widehat{Z}^{(3)}+1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) \\
= \; & \sum_{j=1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + \sum_{j=1}^{\widehat{Z}^{(3)}} (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) = (\widehat{F}^{(2)}, \widehat{M}^{(2)}) + (\widehat{F}^{(3)}, \widehat{M}^{(3)}).
\end{aligned}
$$

Thus, even in this case (3) holds.

Using the same arguments as in the case Ω_{213}, one can see that (3) holds even for almost all $\omega \in \Omega_{312}$.

Consider now the subset Ω_{321}. For almost all $\omega \in \Omega_{321}$ it holds

$$
\begin{aligned}
(\widehat{F}^{(1)}, \widehat{M}^{(1)}) + (\widehat{F}^{(4)}, \widehat{M}^{(4)}) &= \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}^{(1)}_{n,i}, \widehat{m}^{(1)}_{n,i}) + \sum_{i=1}^{\widehat{Z}^{(4)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
= \; & \sum_{i=1}^{\widehat{Z}^{(1)}} \Big((\widehat{f}^{(1)}_{n,i}, \widehat{m}^{(1)}_{n,i}) + (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \Big) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(4)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
\geq \; & \sum_{i=1}^{\widehat{Z}^{(1)}} \Big((\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) \Big) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) + \sum_{i=\widehat{Z}^{(2)}+1}^{\widehat{Z}^{(4)}} (\widehat{f}^{(4)}_{n,i}, \widehat{m}^{(4)}_{n,i}) \\
\geq \; & \sum_{i=1}^{\widehat{Z}^{(1)}} \Big((\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) \Big) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) \\
& \qquad + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(4)}-\widehat{Z}^{(2)}+\widehat{Z}^{(1)}} (\widehat{f}^{(4)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}, \widehat{m}^{(4)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}) \\
\geq \; & \sum_{i=1}^{\widehat{Z}^{(2)}} (\widehat{f}^{(2)}_{n,i}, \widehat{m}^{(2)}_{n,i}) + \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i}) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(3)}} (\widehat{f}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}, \widehat{m}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}})
\end{aligned}
$$

the last inequality being $\widehat{Z}^{(4)} - \widehat{Z}^{(2)} + \widehat{Z}^{(1)} \geq \widehat{Z}^{(3)}$, $\widehat{f}^{(4)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}} \geq \widehat{f}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}$ and $\widehat{m}^{(4)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}} \geq \widehat{m}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}$.

Let now $(\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i})' = (\widehat{f}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}, \widehat{m}^{(3)}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}})$. Note that by assumption (iv) it holds $(\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i})' =_{st} (\widehat{f}^{(3)}_{n,i}, \widehat{m}^{(3)}_{n,i})$ for all $i = \widehat{Z}^{(1)} + 1, \dots, \widehat{Z}^{(3)}$. Moreover, the couples

$(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})'$, with $i=\widehat{Z}^{(1)}+1,\ldots,\widehat{Z}^{(3)}$, are independent from the couples $(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})$, with $i=1,\ldots,\widehat{Z}^{(1)}$. Thus, by inequalities above, we have

$$(\widehat{F}^{(1)},\widehat{M}^{(1)})+(\widehat{F}^{(4)},\widehat{M}^{(4)})\geq\sum_{i=1}^{\widehat{Z}^{(2)}}(\widehat{f}^{(2)}_{n,i},\widehat{m}^{(2)}_{n,i})+\sum_{i=1}^{\widehat{Z}^{(1)}}(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})+\sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(3)}}(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})',$$

i.e., for almost all $\omega\in\Omega_{321}$ it holds

$$(\widehat{F}^{(1)},\widehat{M}^{(1)})+(\widehat{F}^{(4)},\widehat{M}^{(4)})\geq(\widehat{F}^{(2)},\widehat{M}^{(2)})+(\widehat{F}^{(3)},\widehat{M}^{(3)})',\tag{4}$$

where

$$(\widehat{F}^{(3)},\widehat{M}^{(3)})'=\sum_{i=1}^{\widehat{Z}^{(1)}}(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})+\sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(3)}}(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})'.$$

In a similar way one can see that for almost all $\omega\in\Omega_{231}$ it holds

$$(\widehat{F}^{(1)},\widehat{M}^{(1)})+(\widehat{F}^{(4)},\widehat{M}^{(4)})\geq(\widehat{F}^{(2)},\widehat{M}^{(2)})'+(\widehat{F}^{(3)},\widehat{M}^{(3)}),\tag{5}$$

where

$$(\widehat{F}^{(2)},\widehat{M}^{(2)})'=\sum_{i=1}^{\widehat{Z}^{(1)}}(\widehat{f}^{(2)}_{n,i},\widehat{m}^{(2)}_{n,i})+\sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(2)}}(\widehat{f}^{(2)}_{n,i},\widehat{m}^{(2)}_{n,i})',$$

and where the $(\widehat{f}^{(2)}_{n,i},\widehat{m}^{(2)}_{n,i})'$ are defined as the $(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})'$ above.

Combining (3), (4) and (5) one has

$$(\widehat{F}^{(1)},\widehat{M}^{(1)})+(\widehat{F}^{(4)},\widehat{M}^{(4)})\geq(\widetilde{F}^{(2)},\widetilde{M}^{(2)})+(\widetilde{F}^{(3)},\widetilde{M}^{(3)}),\tag{6}$$

where

$$(\widetilde{F}^{(2)},\widetilde{M}^{(2)})=\begin{cases}(\widehat{F}^{(2)},\widehat{M}^{(2)}) & \text{for } \omega\notin\Omega_{231}\\ (\widehat{F}^{(2)},\widehat{M}^{(2)})' & \text{for } \omega\in\Omega_{231}\end{cases}$$

and

$$(\widetilde{F}^{(3)},\widetilde{M}^{(3)})=\begin{cases}(\widehat{F}^{(3)},\widehat{M}^{(3)}) & \text{for } \omega\notin\Omega_{321}\\ (\widehat{F}^{(3)},\widehat{M}^{(3)})' & \text{for } \omega\in\Omega_{321}\end{cases}.$$

It is easy to verify that

$$(\widetilde{F}^{(2)},\widetilde{M}^{(2)})=_{st}(F_{n+1},M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)^{(2)}}$$

and

$$(\widetilde{F}^{(3)},\widetilde{M}^{(3)})=_{st}(F_{n+1},M_{n+1})_{(\boldsymbol{\theta}_n,\boldsymbol{\lambda}_n)^{(3)}}.$$

Moreover, it is easy to verify that

$$(\widehat{F}^{(4)},\widehat{M}^{(4)})\geq[(\widetilde{F}^{(2)},\widetilde{M}^{(2)}),(\widetilde{F}^{(3)},\widetilde{M}^{(3)})]\tag{7}$$

In fact, for example, we have

$$(\widetilde{F}^{(3)},\widetilde{M}^{(3)})=(\widehat{F}^{(3)},\widehat{M}^{(3)})=\sum_{i=1}^{\widehat{Z}^{(3)}}(\widehat{f}^{(3)}_{n,i},\widehat{m}^{(3)}_{n,i})\geq\sum_{i=1}^{\widehat{Z}^{(4)}}(\widehat{f}^{(4)}_{n,i},\widehat{m}^{(4)}_{n,i})=(\widehat{F}^{(4)},\widehat{M}^{(4)})$$

when $\omega \notin \Omega_{321}$, and

$$\begin{aligned}
(\widetilde{F}^{(3)}, \widetilde{M}^{(3)}) &= \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(3)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)})' \\
&= \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(3)}} (\widehat{f}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}^{(3)}, \widehat{m}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}^{(3)}) \\
&\leq \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) + \sum_{i=\widehat{Z}^{(1)}+1}^{\widehat{Z}^{(4)}-\widehat{Z}^{(2)}+\widehat{Z}^{(1)}} (\widehat{f}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}^{(3)}, \widehat{m}_{n,i+\widehat{Z}^{(2)}-\widehat{Z}^{(1)}}^{(3)}) \\
&= \sum_{i=1}^{\widehat{Z}^{(1)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) + \sum_{i=\widehat{Z}^{(2)}+1}^{\widehat{Z}^{(4)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) \\
&\leq \sum_{i=1}^{\widehat{Z}^{(4)}} (\widehat{f}_{n,i}^{(3)}, \widehat{m}_{n,i}^{(3)}) \leq \sum_{i=1}^{\widehat{Z}^{(4)}} (\widehat{f}_{n,i}^{(4)}, \widehat{m}_{n,i}^{(4)}) = (\widehat{F}^{(4)}, \widehat{M}^{(4)})
\end{aligned}$$

when $\omega \in \Omega_{321}$.

Thus, by inequalities (6), (7) and recalling that $(\widehat{F}^{(i)}, \widehat{M}^{(i)}) =_{st} (F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)^{(i)}}$ when $i \in \{1, 4\}$ and $(\widetilde{F}^{(i)}, \widetilde{M}^{(i)}) =_{st} (F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)^{(i)}}$ when $i \in \{2, 3\}$, one get the assertion. ■

By the closure property of *idcx* functions with respect to composition, mentioned in Section 2, it immediately follows the next statement (see also Lemma 2.15 in Meester and Shanthikumar [10]).

Corollary 3.1 *Let the mating function L be an idcx function. Then, under the assumptions of Theorem 3.1, $\{Z_{n+1}((\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)), (\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n) \in \mathbb{T}^n \times \mathbb{L}^n\}$ is $SI - DCX(sp)$.*

Note that the *idcx* property of the mating function L is not too restrictive; for example the functions $L(x, y) = \min\{x, y\}, L(x, y) = x + y$ and $L(x, y) = xy$ satisfy this property. Moreover, the *idcx* property implies superadditivity of L, that is a quite common assumption for mating functions (see, e.g., Daley et al. [4]).

Now, by induction one can easily prove the following corollary.

Corollary 3.2 *If assumptions i) and ii) of of Theorem 3.1 are satisfied, if the mating function L is an idcx function, and if $\{Z_1(\theta_0, \lambda_0), (\theta_0, \lambda_0) \in \mathbb{T} \times \mathbb{L}\}$ is $SI - DCX(sp)$, then $\{(F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)}, (\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n) \in \mathbb{T}^n \times \mathbb{L}^n\}$ is $SI - DCX(sp)$ for every $n \in \mathbb{N}$.*

We can now state the main result of this section. For it, let us denote with $(\boldsymbol{\Theta}_n, \Lambda_n) = \{(\Theta_i, \Lambda_i), i = 1, \ldots, n\}$ the first n pairs of the random sequence (Θ, Λ) (and similarly for (Θ'_n, Λ'_n))

Corollary 3.3 *Let the assumptions i) and ii) of of Theorem 3.1 be satisfied, and let the mating function L be an idcx function. If $\{Z_1(\theta_0, \lambda_0), (\theta_0, \lambda_0) \in \mathbb{T} \times \mathbb{L}\}$ is $SI - DCX(sp)$, then for every $n \geq 1$ the inequality*

$$(\Theta_n, \Lambda_n) \leq_{idcx} (\Theta'_n, \Lambda'_n)$$

implies

$$(F_{n+1}, M_{n+1})_{(\Theta_n, \Lambda_n)} \leq_{idcx} (F_{n+1}, M_{n+1})_{(\Theta'_n, \Lambda'_n)},$$

$$[F_{n+1} + M_{n+1}|(\Theta_n, \Lambda_n)] \leq_{icx} [F_{n+1} + M_{n+1}|(\Theta'_n, \Lambda'_n)]$$

and

$$[Z_{n+1}|(\Theta_n, \Lambda_n)] \leq_{icx} [Z_{n+1}|(\Theta'_n, \Lambda'_n)].$$

Proof. Let u be any increasing and directionally convex function. By Corollary 3.2 and Lemma 2.2 it follows that the function $h(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n) = \mathrm{E}[u((F_{n+1}, M_{n+1})_{(\boldsymbol{\theta}_n, \boldsymbol{\lambda}_n)})]$ is increasing and directionally convex. Now, the first assertion follows from Lemma 2.11 in Meester and Shanthikumar [10], the second from the first assertion and Lemma 2.1, and the last one from the first assertion and Lemma 2.2. ■

In words, Corollary 3.3 essentially says that, under appropriate assumptions, the total population at any fixed generation n increases in *icx* order as the environmental process increases in *idcx* order, i.e., as the positive dependence between the fertility indexes increases.

Remark 3.1 *Note that the results above can be actually further generalized considering mating functions $L_n(F_n, M_n, \delta_n)$ depending also on one more environmental parameter δ_n, or mating functions $L_n(F_n + IF_n, M_n + IM_n)$ depending also on an immigration process $\boldsymbol{I} = \{(IF_n, IM_n), n \in \mathbb{N}\}$ (see González et al. [7]).*

Remark 3.2 *As a particular case one can assume that $\Theta_n = \Lambda_n$ a.s. for every fixed n. In this case it can be obtained a simple generalization to bisexual branching processes of Corollary 4.2 in Fernández–Ponce et al. [6], that deals with standard branching processes. This is a simple interesting case, that shows that population increases in* icx *order as the positive autocorrelation in time between the random fertility indexes increases.*

Assume for example that the random evolutions of the environment are described by a stationary discrete–time homogeneous Markov process $\Theta = \{\Theta_n : n \in \mathbb{N}\}$ that is stochastically monotone (i.e., such that $[\Theta_2|\Theta_1 = \theta]$ is stochastically increasing in θ). Using the criteria described above one can define stochastic bounds for the total population at any generation. In fact, let $\Theta'_n = \{\Theta'_k : k = 1, \dots, n\}$ be a finite sequence of variables such that $\Theta'_k = \Theta'_1$ a.s. for all k where Θ'_1 has the same distribution of Θ_1 (i.e., the stationary marginal distribution of Θ). Then it is well-known that $(\Theta_1, \Theta_2, \dots, \Theta_n) \leq_{idcx} (\Theta'_1, \Theta'_2, \dots, \Theta'_n)$ for every $n \in \mathbb{N}$. Let now $\Theta''_n = \{\Theta''_k : k = 1, \dots, n\}$ be a finite sequence of independent and identically distributed variables such that $\Theta''_k =_{st} \Theta_1$ (i.e., having as distribution the stationary marginal distribution of Θ). It has been shown (see, e.g., Hu and Pan [8]) that in this case it holds $(\Theta''_1, \Theta''_2, \dots, \Theta''_n) \leq_{idcx} (\Theta_0, \Theta_1, \dots, \Theta_n)$ for every $n \in \mathbb{N}$.

Therefore, for a bisexual Galton–Watson branching process defined on a random environment as described before, and subjected to an underlying stationary discrete–time homogeneous Markov process Θ, it holds

$$[F_{n+1} + M_{n+1}|\Theta''_n] \leq_{icx} [F_{n+1} + M_{n+1}|\Theta_n] \leq_{icx} [F_{n+1} + M_{n+1}|\Theta'_n]$$

and, in particular

$$\mathrm{E}[F_{n+1} + M_{n+1}|\Theta''_n] \leq \mathrm{E}[F_{n+1} + M_{n+1}|\Theta_n] \leq \mathrm{E}[F_{n+1} + M_{n+1}|\Theta'_n]$$

for all $n \in \mathbb{N}^+$.

Remark 3.3 *Otherwise, one can assume that $\Theta_n = \Theta$ and $\Lambda_n = \Lambda$ a.s. for all n (i.e., the fertility indexes do not change in time). This is another simple interesting case, that shows that population increases in* icx *order as the positive dependence between the two random* *fertility indexes increases. For example, let* (Θ, Λ) *and* (Θ', Λ') *be two bivariate normally distributed random pairs, having the same mean vector* $\boldsymbol{M}$ *and variance-covariances matrices*

$$\Sigma = \begin{pmatrix} \sigma_\Theta^2 & \rho \\ \rho & \sigma_\Lambda^2 \end{pmatrix} \quad \text{and} \quad \Sigma' = \begin{pmatrix} \sigma_\Theta^2 & \rho' \\ \rho' & \sigma_\Lambda^2 \end{pmatrix},$$

respectively. Then using the results above we have that for all $n \in \mathbb{N}^+$ *it holds*

$$[F_{n+1} + M_{n+1} | (\Theta, \Lambda)] \leq_{icx} [F_{n+1} + M_{n+1} | (\Theta', \Lambda')],$$

and in particular

$$\mathrm{E}[F_{n+1} + M_{n+1} | (\Theta, \Lambda)] \leq \mathrm{E}[F_{n+1} + M_{n+1} | (\Theta', \Lambda')],$$

whenever $\rho \leq \rho'$.

4 A complementary result In this section we consider a different population growth model, and we show that it satisfies properties similar to those provided in the previous section.

Consider a population whose total number of individuals is given by the sum of immigrates coming from m different fonts of immigration. From each font i, immigrates arrive at random times $T_{j,i}$ according to a counting process $N_{\delta_i}^i$ (parametrized by $\delta_i \in \mathbb{R}^+$), and at each time $T_{j,i}$ total population increases of a random amount $X_{j,i}(\lambda)$ of individuals whose distribution depends on some common environmental parameter $\lambda \in \mathbb{R}^+$. Thus, the total number of immigrates up to time t due by font i of immigration is

$$\sum_{j=1}^{N_{\delta_i}^i(t)} X_{j,i}(\lambda),$$

and the total number of immigrates up to time t is

$$W(t) = W_{(\delta_1, \dots, \delta_m, \lambda)}(t) = \sum_{i=1}^{m} \sum_{j=1}^{N_{\delta_i}^i(t)} X_{j,i}(\lambda). \tag{8}$$

One can assume that all variables $N_{\delta_i}^i$ and $X_{j,i}(\lambda)$ are independent when the parameters are fixed, and that they are actually given by a realization of a random vector $(\Delta_1, \dots, \Delta_m, \Lambda)$, defined on some appropriate space contained in $\mathbb{R}^{m+1}$, describing random environmental conditions.

For the main result on this model, we firstly recall a preliminary result that has been proved in Fernández–Ponce et al. [6], whose proof is similar to the proof of Theorem 3.1.

Theorem 4.1 *Consider the family of random sums* $\{Z(\boldsymbol{\theta}, \boldsymbol{\lambda}), (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathbb{T} \times \mathbb{L}\}$ *defined by*

$$Z(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \sum_{j=1}^{N(\boldsymbol{\theta})} X_j(\boldsymbol{\lambda})$$

If
i) all the families $\{X_j(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathbb{L}\}$, $j \in \mathbb{N}$, *and* $\{N(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{T}\}$ *are independent;*
ii) $\{X_j(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathbb{L}\} \in SI - DCX(sp)$ *for every* $j \in \mathbb{N}$;
iii) $\{N(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{T}\} \in SI - DCX(sp)$;
iv) $\{X_j(\boldsymbol{\lambda}), j \in \mathbb{N}\} \in SI$ *for every* $\boldsymbol{\lambda} \in \mathbb{L}$;
then $\{Z(\boldsymbol{\theta}, \boldsymbol{\lambda}), (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathbb{T} \times \mathbb{L}\} \in SI - DCX(sp)$.

Using the above Theorem 4.1 it is easy to prove the following.

Theorem 4.2 *Let*

$$W(t) = W_{(\delta_1,\dots,\delta_m,\lambda)}(t) = \sum_{i=1}^{m} \sum_{j=1}^{N^i_{\delta_i}(t)} X_{j,i}(\lambda).$$

i) If the families $\{X_{j,i}(\lambda), \lambda \in \mathbb{R}\}$ and the the families of processes $\{N^i_{\delta_i}, \delta_i \in \mathbb{R}\}$ are all independent;
ii) if the families $\{X_{j,i}(\lambda), \lambda \in \mathbb{R}\} \in SI - -CX(sp)$ for all fixed i and j;
iii) if the families $\{N^i_{\delta_i}(t), \delta_i \in \mathbb{R}\} \in SI - -CX(sp)$ for all fixed i and all t,
iv) if the families $\{X_{j,i}(\lambda), j \in \mathbb{N}\} \in SI$ for all i and $\lambda \in \mathbb{R}$;
then the family $\{W_{(\delta_1,\dots,\delta_m,\lambda)}(t), (\delta_1, \dots, \delta_m, \lambda) \in \mathbb{R}^{m+1}\}$ is SI–DCX(sp).

Proof. From Theorem 4.1 we have that

$$\left\{ \sum_{j=1}^{N^i_{\delta_i}(t)} X_{j,i}(\lambda), (\delta_i, \lambda) \in \mathbb{R}^{m+1} \right\},$$

is SI–DCX(sp) for all $t \geq 0$ and $i = 1, 2, ..., m$. Now, by the independence assumptions and by Lemma 2.16 in Meester and Shanthikumar [10], we have that

$$\left\{ \left(\sum_{j=1}^{N^1_{\delta_1}(t)} X_{j,1}(\lambda), \dots, \sum_{j=1}^{N^m_{\delta_m}(t)} X_{j,m}(\lambda) \right), (\delta_1, ..., \delta_m, \lambda) \in \mathbb{R}^{m+1} \right\}$$

is SI–DCX(sp). Finally, the assertion follows by Lemma 2.15 in Meester and Shanthikumar [10] since the function $f(\mathbf{x}) = \sum_{i=1}^{m} x_i$ is *idcx*. ■

Note that Lemma 2.16 can be applied here since this result holds when all or some of the parameters are the same, and it also holds if SI–CX(sp) is replaced by SI-DCX(sp).

Now we can state the main result of this section.

Corollary 4.1 *Let*

$$W(t) = W_{(\delta_1,\dots,\delta_m,\lambda)}(t) = \sum_{i=1}^{m} \sum_{j=1}^{N^i_{\delta_i}(t)} X_{j,i}(\lambda).$$

If the assumptions (i)–(iv) in the previous theorem hold, then

$$(\Delta_1, \dots, \Delta_m, \Lambda) \leq_{idcx} (\Delta'_1, \dots, \Delta'_m, \Lambda')$$

implies

$$W_{(\Delta_1,\dots,\Delta_m,\Lambda)}(t) \leq_{icx} W_{(\Delta'_1,\dots,\Delta'_m,\Lambda')}(t)$$

for all $t \geq 0$,

Proof. Let u be any increasing and directionally convex function. Then by Theorem 4.2 and Lemma 2.2 it follows that the function $h_t(\delta_1, \dots, \delta_m, \lambda) = \mathrm{E}[u(W_{(\delta_1,\dots,\delta_m,\lambda)}(t))]$ is increasing and directionally convex. Thus, the assertion follows from Corollary 2.12 in Meester and Shanthikumar [10]. ■

As for the main result presented in Section 3, the statement above essentially says that under appropriate assumptions the total population at every fixed time t increases as the positive dependence between the environmental parameters increases.

For an example of application, consider two populations W and W' defined as in (4.1), but subjected to two different environments $(\Delta_1, \dots, \Delta_m, \Lambda)$ and $(\Delta'_1, \dots, \Delta'_m, \Lambda')$, respectively. For fixed values of the parameters δ_i and λ, let the counting processes $N^i_{\delta_i}$ be homogeneous Poisson processes having rates $\delta_i, i = 1, \dots, m$, and let the variables $X_{j,i}(\lambda)$ be Bernoulli distributed with mean λ. Also, let $(\Delta_1, \dots, \Delta_m, \Lambda)$ be a vector of independent normally distributed variables, having means $d_1, \dots, d_m, l$, and variances $v_1, \dots, v_m, u$, with $d_i >> v_i, i = 1, \dots, m$ and $l >> u$ (here $l >> u$ means that l is much bigger than u, so that probability of having negative parameters is negligible). Let now $(\Delta'_1, \dots, \Delta'_m, \Lambda')$ be a multivariate normally distributed vector whose components have the same means and variances than $(\Delta_1, \dots, \Delta_m, \Lambda)$ and non–negative covariances. Then making use of Corollary 4.1 it immediately follows

$$\mathrm{E}[W'(t)] \geq \mathrm{E}[W(t)] = \sum_{i=1}^{m} t\mathrm{E}[\Delta_i]\mathrm{E}[\Lambda] = lt \sum_{i=1}^{m} d_i$$

for every $t \geq 0$.

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