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# Analysis of Random Mobility Models with Partial Differential Equations

Michele Garetto, *Member, IEEE*, and Emilio Leonardi, *Member, IEEE*

**Abstract**—In this paper, we revisit two classes of mobility models which are widely used to represent users' mobility in wireless networks: Random Waypoint (RWP) and Random Direction (RD). For both models, we obtain systems of partial differential equations which describe the evolution of the users' distribution. For the RD model, we show how the equations can be solved analytically both in the stationary and transient regime, adopting standard mathematical techniques. Our main contributions are 1) simple expressions which relate the transient duration to the model parameters and 2) the definition of a generalized random direction model whose stationary distribution of mobiles in the physical space corresponds to an assigned distribution.

**Index Terms**—Mobility models, partial differential equations.

## 1 INTRODUCTION

MOBILITY models play a fundamental role in the analysis and design of wireless systems [1], [2]. In the past several years, researchers have proposed a number of mobility models for the purpose of simulating the movement of users in a wireless network. Two widely used models are the Random Waypoint model (RWP) [3] and the Random Direction model (RD) [4]. In both models, users independently follow a sequence of linear segments and traverse each segment at a constant speed. The two models differ in how a user chooses the next segment to traverse: Under the RWP model, a user selects a random destination point within the space; instead, under the RD model, a user chooses a direction to travel in and a duration for the travel. In both cases, the speed on a segment is taken from some given distribution. Moreover, before starting to travel on the new segment, users can stop for a random time, thus alternating phases in which they move with phases in which they keep still.

Despite their wide use in simulation studies, properties of the above mobility models have only recently been established and fully understood. In [5], Le Boudec and Vojnovic used Palm calculus to study the stationary regime of a large class of mobility models (including RWP and RD), explaining a number of previously observed phenomena such as speed decay [12] and nonuniform distribution of nodes [9], [10]. Their analysis generalizes findings in [7], [6], [11] about the existence and uniqueness of a stationary regime and provides the correct methodology to start a simulation in steady state so as to avoid transient effects (perfect simulation). Moreover, the proposed perfect sampling technique applies to quite general area shapes (e.g.,

the Swiss Cross) without requiring the computation of complex geometric integrals (like, for example, in [8]).

It turns out that the RWP mobility pattern is more difficult to analyze and control in terms of the stationary distributions of location and speed of mobiles and does not usually lead to a uniform density of nodes in the space. On the contrary, the RD mobility pattern has the nice property that users always tend to be uniformly distributed in the space, irrespective of the boundary conditions imposed (wrap around or reflection). Moreover, the distribution of location and speed at a random time instant are the same as at a transition instant [5], which greatly simplifies the analysis.

Two important issues in the analysis of mobility models still need to be solved. The first is the study of the convergence rate to the stationary regime from arbitrary initial conditions. That is, how long does it take to approach the stationary distribution if the simulation starts away from the equilibrium? So far, in the literature, the transient behavior of mobility models has mostly been considered a nuisance and many efforts have been devoted precisely to eliminate transient effects from simulations. However, capacity planning, network resilience and reliability, etc., usually require testing applications and protocols in time-varying, critical conditions, not in the steady state. Take, for example, the case of a large number of mobile nodes forming an ad hoc network initially confined in a small area (such as a conference room, a football stadium, or the like), who at some point start dispersing away. One would like to simulate such a scenario to see how the network behaves while nodes get more and more far apart till connectivity is lost. This paper will show that theoretical mobility models permit doing this in a controlled and predictable fashion, i.e., it is possible to choose parameters of the mobility model to obtain a desired nodes' dispersion rate and duration of the transient. Our dynamical viewpoint thus brings what might be regarded as ideal, unrealistic mobility models much closer to practical applications.

The second issue is the reverse of the problem considered so far in analytical studies that have appeared in the literature: Is it possible to devise a mobility pattern that achieves a desired stationary distribution of nodes in space? So far, theoretical studies have just predicted the

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stationary distribution generated by a given mobility model. However, the ability to design a mobility model that produces an assigned distribution of nodes in the area would be of much greater interest in real problems, where node densities are almost always nonuniform. As an example, one could be interested in simulating scenarios in which nodes are more densely concentrated in some portions of the area, like in an urban context.

In this paper, we propose an analysis of the RWP and RD models that allows us to address both issues above, filling the existing gap in the analysis of mobility models. We use partial differential equations (PDEs) to describe how the mobiles' state distribution evolves over time. Our novel formulation provides the analytical basis for solving both transient and nonuniform cases. In particular, it permits studying the transient dynamics of a system starting from an arbitrary initial condition for both RWP and RD models. For the RD model, we show how the partial differential equations can be solved *analytically* in the transient regime, adopting standard mathematical techniques. Moreover, we rederive known results about the stationary distribution of the RD model in a more straightforward manner than previous approaches based on Palm Calculus. Our methodology allows, for the first time to the best of our knowledge, 1) to derive simple expressions relating the transient duration to the model parameters and 2) to generalize the RD model so as to obtain a desired stationary distribution of nodes in the space.

The remainder of the paper is organized as follows: In Sections 2 and 3, we present the equations describing the behavior of a single user moving according to the RD and RWP model, respectively. In Section 4, we statistically reinterpret the previously obtained equations for a single user, showing that they can be used as well to describe the dynamics of a large population of mobile users. The steady state analysis of the standard RD model is provided in Section 5, whereas the extension of this model to achieve arbitrary distributions of nodes in the area is described in Section 6. The transient analysis of the RD model is presented in Section 7. In Section 8, we validate our analysis by simulation on a few examples and present possible applications of our methodology. Finally, Section 9 concludes the paper.

## 2 EQUATIONS OF THE RD MODEL

We start considering a single user moving according to the random direction model over a unidimensional domain, further assuming that move and pause times are exponentially distributed. Then, we generalize our approach to the case in which move and pause times have a general distribution. Finally, we extend our equations to the multidimensional case.

### 2.1 Unidimensional Case with Exponential Phases

We assume that the domain in which the mobile can move is the interval  $[x_l, x_u]$ . Move and pause times are taken from an exponential distribution of parameters  $\mu$  and  $\lambda$ , respectively. When the mobile starts traveling on a new segment, it selects a speed from the generic distribution  $f_V(v)$ . We further assume that the absolute speed value is upper bounded by a constant  $V_{\max}$ , i.e., support of  $f_V(v)$  is in the interval  $[-V_{\max}, V_{\max}]$ . This is a reasonable assumption for all cases of practical interest.

The dynamics of the mobile can be described in terms of a Markov Process over a general space state [13], in which the instantaneous mobile state  $K(t)$  is characterized by 1) the phase  $P(t) \in \mathcal{P} = \{move, pause\}$ , 2) the instantaneous position  $X(t) \in [x_l, x_u]$ , and 3) the current speed  $V(t)$  (in case  $P(t) = move$ ).

Let  $N(x, v, t)$  be the cumulative probability that, at time  $t$ , the mobile is in the *move* phase at a position  $X(t) \in [x_l, x]$  with a speed  $V(t) \in [-V_{\max}, v]$ :

$$N(x, v, t) \triangleq \Pr\{P(t) = move, X(t) \in [x_l, x], V(t) \in [-V_{\max}, v]\}.$$

Let  $S(x, t)$  be the cumulative probability that, at time  $t$ , the mobile is in the *pause* phase at a position  $x \in [x_l, x]$ :

$$S(x, t) \triangleq \Pr\{P(t) = pause, X(t) \in [x_l, x]\}.$$

Consider a small interval  $T = [t, t + \Delta t)$ . Conditionally over the fact that no phase transition occurs in  $T$ , according to the RD model at time  $t + \Delta t$ , state  $K(t) = (move, x, v)$  is deterministically transformed into state  $K(t + \Delta t) = (move, x + v\Delta t, v)$ , whereas state  $K(t) = (pause, x)$  is deterministically transformed into state  $K(t + \Delta t) = (pause, x)$ ; thus, conditionally over the fact that no phase transition occurs in  $T$ , we have (with abuse of notation):

$$\begin{aligned} N(x + v\Delta t, v, t + \Delta t) &= N(x, v, t) && \text{(no phase transition),} \\ S(x, t + \Delta t) &= S(x, t) && \text{(no phase transition).} \end{aligned}$$

If a transition occurs at  $t + \tau \in T$ , state  $K(t) = (pause, x - v(\Delta t - \tau))$  is deterministically transformed into state  $K(t + \Delta t) = (move, x, v)$ , whereas state  $K(t) = (move, x - v\tau, v)$  is transformed into state  $K(t + \Delta t) = (pause, x)$ . Thus, conditionally over the fact that a phase transition occurs in  $T$ , it results (again, with abuse of notation):

$$N(x, v, t + \Delta t) = S(x, t) \int_{-V_{\max}}^v f_V(v) dv + O(\Delta t)$$

(with phase transition),

$$S(x, t + \Delta t) = \int_{-V_{\max}}^{V_{\max}} N(x, v, t) dv + O(\Delta t)$$

(with phase transition).

Due to the exponential distribution of the move and pause times, the probability that a phase transition occurs in  $T$  from *pause* to *move* is  $\lambda\Delta t + o(\Delta t)$ ; the probability that a phase transition occurs during  $T$  from *move* to *pause* is  $\mu\Delta t + o(\Delta t)$ ; the probability that more than one phase transition occurs in  $T$  is instead  $o(\Delta t)$ . Therefore, we can write:

$$\begin{aligned} N(x + v\Delta t, v, t + \Delta t) &= \\ N(x, v, t)(1 - \mu\Delta t) + \lambda\Delta t S(x + v\Delta t, t) &\int_{-V_{\max}}^v f_V(v) dv + o(\Delta t), \end{aligned} \tag{1}$$

$$S(x, t + \Delta t) = S(x, t)(1 - \lambda\Delta t) + \mu\Delta t \int_{-V_{\max}}^{V_{\max}} N(x, v, t) dv + o(\Delta t). \tag{2}$$

Subtracting  $N(x + v\Delta t, v, t)$  from both members of (1) (and, similarly, subtracting  $S(x, t)$  from both members of (2)), dividing by  $\Delta t$  and letting  $\Delta t \rightarrow 0$ , under the assumption that  $f_V(v)$  is a continuous and derivable function, we obtain the following coupled differential equations, which can be regarded as the Chapman-Kolmogorov equations of the Markov process:

$$\frac{\partial n(x, v, t)}{\partial t} = -v \frac{\partial n(x, v, t)}{\partial x} + \lambda f_V(v) s(x) - \mu n(x, v, t), \quad (3)$$

$$\frac{\partial s(x, t)}{\partial t} = -\lambda s(x, t) + \mu \int_v n(x, v, t) dv, \quad (4)$$

where

$$n(x, v, t) = \frac{\partial^2 N(x, v, t)}{\partial x \partial v}; \quad s(x, t) = \frac{\partial S(x, t)}{\partial x}.$$

**Boundary Conditions.** A problem that arises in the RD model is what to do when the mobile hits a boundary. Several strategies have been proposed; among them, the most popular are *wrap around* and *reflection*. In the *wrap around* model, the mobile hitting a boundary with speed  $v$  instantaneously reappears at the opposite side maintaining the same speed. Thus, the boundary conditions of the *wrap around* model for (3) and (4) are

$$n(x_l, v, t) = n(x_u, v, t); \quad \lim_{x \rightarrow x_l^+} \frac{\partial n(x, v, t)}{\partial x} = \lim_{x \rightarrow x_u^-} \frac{\partial n(x, v, t)}{\partial x} \quad \forall v, t,$$

$$s(x_l, t) = s(x_u, t); \quad \lim_{x \rightarrow x_l^+} \frac{\partial s(x, t)}{\partial x} = \lim_{x \rightarrow x_u^-} \frac{\partial s(x, t)}{\partial x} \quad \forall t.$$

In the *reflection* model, the mobile is instead bounced back, reversing its speed. The boundary conditions of the *reflection* model for (3) are

$$n(x_l, v, t) = n(x_l, -v, t); \quad n(x_u, v, t) = n(x_u, -v, t) \quad \forall v, t.$$

In both cases, the initial condition is assumed to be given:

$$n(x, v, 0) = n_o(x, v); \quad s(x, 0) = s_o(x).$$

We remark that the initial condition must satisfy the constraints related to its physical interpretation as a probability density function (pdf) of the mobile position, speed, and phase at time  $t = 0$ . In particular,  $n_o(x, v) \geq 0$ ,  $s_o(x) \geq 0$ , and

$$\iint n_o(x, v) dx dv + \int s_o(x) dx = 1.$$

**Uniqueness of Solution.** In the appendix, we prove that the mathematical problem defined by (3) and (4) subject to the boundary and initial conditions defined above either for the *wrap-around* or *reflection* models admits no more than one solution. We emphasize that this is a fundamental step of our analysis; indeed, under uniqueness assumptions, if we find a solution of the equations with assigned initial and boundary conditions, we can conclude that it corresponds to the actual system trajectory.

## 2.2 Extension to General Phase Times Distributions

Let  $g(y)$  be the pdf of *move* time and  $\mu(y)$  be the associated hazard function, defined as  $\mu(y) = g(y)/(1 - G(y))$ , where  $G(y)$  is the cumulative distribution function (cdf) of *move*

time. Similarly, let  $h(z)$  be the pdf of *pause* times and  $\lambda(z)$  be the associated hazard function  $\lambda(z) = h(z)/(1 - H(z))$ , where  $H(z)$  is the cdf of *pause* time. The systems dynamics can still be described as a Markov process over a general space state. However, in this case, the state space becomes more complex since phase durations are not memoryless: The mobile state  $K(t)$  is now characterized by the time  $W(t)$  elapsed since the last phase transition in addition to its current phase  $P(t)$ , instantaneous position  $X(t)$ , and current speed  $V(t)$  (in case  $P(t) = \text{move}$ ). The mobile dynamics satisfy the following system of differential equations:

$$\frac{\partial n(x, v, y, t)}{\partial t} = -v \frac{\partial n(x, v, y, t)}{\partial x} - \frac{\partial n(x, v, y, t)}{\partial y} + \delta(y) \int_z \lambda(z) f_V(v) s(x, z, t) dz - \mu(y) n(x, v, y, t), \quad (5)$$

$$\frac{\partial s(x, z, t)}{\partial t} = -\frac{\partial s(x, z, t)}{\partial z} - \lambda(z) s(x, z, t) + \delta(z) \iint \mu(y) n(x, v, y, t) dv dy, \quad (6)$$

where  $\delta()$  represents the Dirac function.

## 2.3 The Multidimensional Case

The extension to a  $k$ -dimensional domain  $\in \mathbb{R}^k$  is rather straightforward. Let  $\mathbf{x} = (x_1, x_2, \dots, x_k)$  be the position of the mobile and  $\mathbf{v} = (v_1, v_2, \dots, v_k)$  be the current speed vector (each component represents the mobile's position/speed along the corresponding dimension). In case of exponential move and pause times, the Chapman-Kolmogorov equations of the system are

$$\frac{\partial n(\mathbf{x}, \mathbf{v}, t)}{\partial t} = -\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t) + \lambda f_V(\mathbf{v}) s(\mathbf{x}) - \mu n(\mathbf{x}, \mathbf{v}, t), \quad (7)$$

$$\frac{\partial s(\mathbf{x}, t)}{\partial t} = -\lambda s(\mathbf{x}, t) + \mu \int n(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (8)$$

where  $\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t)$  is the inner product between  $\mathbf{v}$  and  $\nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t)$ .

In the case of general distributions of phase duration, we have

$$\frac{\partial n(\mathbf{x}, \mathbf{v}, y, t)}{\partial t} = -\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, y, t) - \frac{\partial n(\mathbf{x}, \mathbf{v}, y, t)}{\partial y} + \delta(y) \int_z \lambda(z) f_V(\mathbf{v}) s(\mathbf{x}, z) dz - \mu(y) n(\mathbf{x}, \mathbf{v}, y, t), \quad (9)$$

$$\frac{\partial s(\mathbf{x}, z, t)}{\partial t} = -\frac{\partial s(\mathbf{x}, z, t)}{\partial z} - \lambda(z) s(\mathbf{x}, z, t) + \delta(z) \iint \mu(y) n(\mathbf{x}, \mathbf{v}, y, t) d\mathbf{v} dy. \quad (10)$$

## 3 EQUATIONS OF THE RWP MODEL

Similarly to what we have done for the RD model, we start considering a mobile moving along a unidimensional domain, assuming that pause times are exponentially distributed. Notice that, in the RWP model, users do not choose a duration for the move phase, which instead depends on the selected destination point and speed. Next,

we generalize our approach to the case in which pause times are generally distributed and, finally, to the multi-dimensional case.

### 3.1 The Unidimensional Case with Exponential Pauses

Let  $[x_l, x_u]$  be the domain in which the mobile can move and  $\lambda$  be the parameter of the exponentially distributed pause time. The mobile in  $x$ , when choosing the next segment to travel in, first selects a destination point  $d$  according to the distribution  $r(d)$ , then selects a speed according to the distribution  $f_V(v|d, x)$ . We notice that, if  $d > x$ , it must be  $f_V(v|d, x) = 0$  for  $v < 0$ , while if  $d < x$ , it must be  $f_V(v|d, x) = 0$  for  $v > 0$ . We again assume that the absolute speed value is upper bounded by a constant  $V_{max}$ ; i.e., support of  $f_V(v|d, x)$  falls in the interval  $[-V_{max}, V_{max}]$ ,  $\forall d, x$ .

The dynamics of the mobile can be described in terms of a Markov Process over a general space state in which the instantaneous state  $K(t)$  is characterized by

1. the phase  $P(t) \in \mathcal{P} = \{move, pause\}$ ,
2. the instantaneous position  $X(t) \in [x_l, x_u]$ ,
3. the current destination  $D(t) \in [x_l, x_u]$ , and
4. the current speed  $V(t) \in [-V_{max}, V_{max}]$  (in case  $P(t) = move$ ).

Let  $N(x, v, d, t)$  be the cumulative probability that, at time  $t$ , the mobile is in the *move* phase at a position  $X(t) \in [x_l, x]$  with a destination  $D(t) \in [x_l, d]$  and a speed  $V(t) \in [-V_{max}, v]$ :

$$N(x, v, d, t) \triangleq \Pr\{P(t) = move, X(t) \in [x_l, x], D(t) \in [x_l, d], V(t) \in [-V_{max}, v]\}.$$

Let  $S(x, t)$  be the cumulative probability that, at time  $t$ , the mobile is in the *pause* phase at a position  $X(t) \in [x_l, x]$ :

$$S(x, t) \triangleq \Pr\{P(t) = pause, X(t) \in [x_l, x]\}.$$

Introducing the densities

$$n(x, v, d, t) = \frac{\partial^3 N(x, v, d, t)}{\partial x \partial v \partial d}; \quad s(x, t) = \frac{\partial S(x, t)}{\partial x},$$

we obtain the following pair of equations in a way similar to what has been done for the RD model:

$$\frac{\partial n(x, v, d, t)}{\partial t} = -v \frac{\partial n(x, v, d, t)}{\partial x} + \lambda f_V(v|d) r(d) s(x, t), \quad (11)$$

$$\frac{\partial s(x, t)}{\partial t} = -\lambda s(x, t) + \int v n(x, v, x, t) dv, \quad (12)$$

where (11) is defined for  $d \geq x$  and  $v > 0$ , or  $d \leq x$  and  $v < 0$ .

**Boundary Conditions.** In the RWP model, the boundary conditions express the fact that the probability for the mobile to hit the boundaries is null:

$$\begin{aligned} n(x_l, v, d, t) = 0; \quad n(x_u, v, d, t) = 0 \quad \forall v, d, t, \\ s(x_l, t) = 0; \quad s(x_u, t) = 0 \quad \forall t. \end{aligned}$$

In addition, we impose the initial conditions

$$n(x, v, d, 0) = n_o(x, v, d); \quad s(x, 0) = s_o(x),$$

which must be a proper pdf for the mobile's initial position, speed, and destination.

### 3.2 Extension to General Pause Time Distribution

Let  $h(z)$  be the pdf of pause time and  $\lambda(z)$  be the associated hazard function. The system dynamics can still be described by a Markov Process over a general state space; we only need to add to the state associated to the pause phase the time  $z$  elapsed since the mobile entered the pause phase. The model equations become

$$\begin{aligned} \frac{\partial n(x, v, d, t)}{\partial t} = -v \frac{\partial n(x, v, d, t)}{\partial x} \\ + f_V(v|x, d) r(d) \int \lambda(z) s(x, z, t) dz \end{aligned} \quad (13)$$

defined for  $d \geq x$  and  $v > 0$  or  $d \leq x$  and  $v < 0$ , and

$$\begin{aligned} \frac{\partial s(x, z, t)}{\partial t} = -\frac{\partial s(x, z, t)}{\partial z} - \lambda(z) s(x, z, t) \\ + \delta(z) \int v n(x, v, x, t) dv. \end{aligned} \quad (14)$$

### 3.3 Multidimensional Case

Let  $\mathbf{x} = (x_1, x_2, \dots, x_k)$  be the position of the mobile,  $\mathbf{d} = (d_1, d_2, \dots, d_k)$  be the current destination, and be  $v$  the current speed.

Considering the case in which the pause time is generally distributed, with hazard function  $\lambda(z)$ , we obtain

$$\begin{aligned} \frac{\partial n(\mathbf{x}, \mathbf{v}, \mathbf{d}, t)}{\partial t} = -\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, \mathbf{d}, t) \\ + f_V(\mathbf{v}|\mathbf{d}, \mathbf{x}) r(\mathbf{d}) \int_z \lambda(z) s(\mathbf{x}, z, t) dz, \\ \frac{\partial s(\mathbf{x}, z, t)}{\partial t} = -\frac{\partial s(\mathbf{x}, z, t)}{\partial z} - \lambda(z) s(\mathbf{x}, z, t) \\ + \delta(z) \int \|\mathbf{v}\| n(\mathbf{x}, \mathbf{v}, \mathbf{x}, t) dv. \end{aligned}$$

## 4 STATISTICAL INTERPRETATION OF PREVIOUS EQUATIONS

In this section, we provide a statistical interpretation of the equations derived in Sections 2 and 3, valid when the population of mobile users becomes large. We restrict ourselves to the unidimensional random direction model under general phase distributions, however, the same interpretation holds in all other cases.

Consider a population of  $N$  mobiles moving independently of each other. The complete state for mobile  $i$  at time  $t$  is denoted by  $K_i(t) = (P(t), X(t), V(t), W(t))$ . Let  $\mathcal{M}$  be the set of all states in which the mobile is in the *move* phase and  $\mathcal{S}$  be the set of all states in which the mobile is in the *pause* phase. Let  $A$  be any (Lebesgue measurable) set of states and define  $A_{\mathcal{M}} = A \cap \mathcal{M}$  and  $A_{\mathcal{S}} = A \cap \mathcal{S}$ . Let  $\mathbb{1}_{K_i(t) \in A}$  be an indicator function which returns 1 if mobile  $i$  at time  $t$  is in a state belonging to  $A$ , i.e.,  $K_i(t) \in A$ , and 0 otherwise.

By the strong law of large numbers, it results that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{K_i(t) \in A} = E[\mathbb{1}_{K_1(t) \in A}] = \Pr\{K_1(t) \in A\} \\ = \int_{A_{\mathcal{M}}} n(x, v, y, t) dA_{\mathcal{M}} + \int_{A_{\mathcal{S}}} s(x, z, t) dA_{\mathcal{S}}. \end{aligned}$$

Now, we observe that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{K_i(t) \in A}$$

has an immediate physical interpretation as the fraction of mobiles whose instantaneous state at time  $t$  belongs to  $A$ ; as a consequence, (5) and (6) describe the statistical density evolution of a large population of users moving according to the considered mobility model.

## 5 STEADY STATE ANALYSIS

In this section, we compute the steady-state solutions (i.e., solutions which are invariant with respect to time) of the RD model. We start considering the unidimensional case with exponential phase times. Next, we generalize our solution to the case in which phases are generally distributed and, finally, to the multidimensional case.

### 5.1 The Exponential Case

The system dynamics are described by a Markov process over an uncountable compact<sup>1</sup> space state whose properties have recently been studied, proving that the steady state distribution exists uniquely [5], [6].<sup>2</sup> In particular, [5] shows the existence and uniqueness of the time-stationary distribution for a broad class of random trip models that includes both random direction and random waypoint.

Moreover, regardless of the initial condition,  $n(x, v, t)$  and  $s(x, t)$  tend to the steady state distribution for  $t \rightarrow \infty$ , as proven in [5] for general random trip models.

By setting the derivative with respect to time equal to zero in both (3) and (4), we obtain that steady state solutions  $n(x, v)$ ,  $s(x)$  must satisfy the following equations:

$$v \frac{\partial n(x, v)}{\partial x} = \lambda f_V(v) s(x) - \mu n(x, v), \quad (15)$$

$$\lambda s(x) = \mu \int n(x, v) dv \quad (16)$$

with the boundary conditions defined in Section 2.

Considering product-form candidate solutions for  $n(x, v)$ , i.e.,  $n(x, v) = \alpha(x)\beta(v)$ , we obtain the following solution of steady-state equations:

$$n(x, v) = \frac{\lambda f_V(v)}{(\lambda + \mu)|x_u - x_l|}; \quad s(x) = \frac{\mu}{(\lambda + \mu)|x_u - x_l|},$$

which satisfies the boundary conditions for both *wrap around* and *reflection*, in the latter case under the mild assumption that the speed distribution is symmetric, i.e.,  $f_V(v) = f_V(-v)$ . Notice that we have basically reobtained the known result that the steady state distribution of nodes is uniform in space, while the speed distribution is the same as that used to select a new speed at the transition points.

### 5.2 General Phase Times Distributions

When phase times have a general distribution, the steady state solution of the RD model still uniquely exists under

the single condition that average phase durations are finite [5], [6]. In addition, regardless of the initial condition,  $n(x, v, t)$  and  $s(x, t)$  tend to the steady state distribution for  $t \rightarrow \infty$ .

Now, we show how the steady state analysis of RD models with generally distributed phases can be reconducted to the analysis of RD models with exponential phases. Setting the derivative with respect to time equal to zero in both (5) and (6), we have

$$-v \frac{\partial n(x, v, y)}{\partial x} = -\frac{\partial n(x, v, y)}{\partial y} + \delta(y) \int_z \lambda(z) f_V(v) s(x, z) dz - \mu(y) n(x, v, y), \quad (17)$$

$$\frac{\partial s(x, z)}{\partial z} = -\lambda(z) s(x, z) + \delta(z) \iint \mu(y) n(x, v, y) dv dy. \quad (18)$$

Considering product-form candidate solutions of the type  $n(x, v, y) = m(x, v)k(y)$  and  $s(x, z) = p(x)h(z)$  with  $\int_{0-}^{\infty} h(z) dz = \int_{0-}^{\infty} k(y) dy = 1$  and defining

$$\lambda_{\text{eff}} = \int_{0-}^{\infty} \lambda(z) h(z) dz = \frac{1}{E[T_{\text{pause}}]};$$

$$\mu_{\text{eff}} = \int_{0-}^{\infty} \mu(y) k(y) dy = \frac{1}{E[T_{\text{move}}]},$$

it results that  $m(x, v)$ ,  $p(x)$ ,  $k(y)$ , and  $h(z)$  must satisfy

$$v \frac{\partial m(x, v)}{\partial x} = \lambda_{\text{eff}} f_V(v) p(x) - \mu_{\text{eff}} m(x, v), \quad (19)$$

$$\lambda_{\text{eff}} p(x) = \mu_{\text{eff}} \int m(x, v) dv, \quad (20)$$

$$\frac{\partial k(y)}{\partial y} p_{\text{move}} = \lambda_{\text{eff}} \delta(y) p_{\text{pause}} - \mu(y) k(y) p_{\text{move}}, \quad (21)$$

$$\frac{\partial h(z)}{\partial y} p_{\text{pause}} = \mu_{\text{eff}} \delta(z) p_{\text{move}} - \lambda(z) h(z) p_{\text{pause}}, \quad (22)$$

where  $p_{\text{pause}}$  and  $p_{\text{move}}$  are, respectively, the probability for the mobile of being in pause and move phase at steady state

$$p_{\text{pause}} = \frac{E[T_{\text{pause}}]}{E[T_{\text{move}}] + E[T_{\text{pause}}]}; \quad p_{\text{move}} = \frac{E[T_{\text{move}}]}{E[T_{\text{move}}] + E[T_{\text{pause}}]}.$$

We observe that (19) and (20) are structurally identical to (15) and (16); thus, they admit the same solution (with proper parameter substitutions). Instead, (21) and (22) admit the following solutions:

$$k(y) = \frac{e^{-\int_0^y \mu(\alpha) d\alpha}}{\int_{0-}^{\infty} e^{-\int_0^y \mu(\alpha) d\alpha} dy} = \frac{1 - G(y)}{E[T_{\text{move}}]},$$

$$h(z) = \frac{e^{-\int_0^z \lambda(\alpha) d\alpha}}{\int_{0-}^{\infty} e^{-\int_0^z \lambda(\alpha) d\alpha} dz} = \frac{1 - H(z)}{E[T_{\text{pause}}]},$$

which correspond, as expected, to the residual time spent in the move or in the pause phase when sampling the system at a random point in time. Also, in this case, we have found the unique steady state solution for both *wrap around* and *reflection* (in the latter case, under the assumption that  $f_V(v) = f_V(-v)$ ).

1. Any closed bounded subset of  $\mathbb{R}^n$ ,  $\forall n$ , is compact.

2. In [5] and [6], the properties of RD models have been analyzed by considering the embedded discrete time Markov process which is obtained by sampling the system dynamics at instants in which the mobile changes phase. An exhaustive analysis of Markov processes over uncountable space states can be found in [13] for the discrete time case.

### 5.3 The Multidimensional Case

Previous results can be immediately generalized to a multidimensional rectangular domain since, in this case, steady-state equations admit product form solutions

$$\begin{aligned} n(\mathbf{x}, \mathbf{v}, y) &= n_1(x_1, v_1)n_2(x_1, v_1) \cdots n_k(x_k, v_k) k(y), \\ s(\mathbf{x}, z) &= s_1(x_1)s_2(x_1) \cdots s_k(x_k) h(z), \end{aligned}$$

and, thus, can be decoupled into unidimensional equations which are structurally identical to those presented in the previous section.

We observe that the RD model with *wrap around* is meaningful only on rectangular domains. Instead, the RD model with *reflection* can be defined on a general domain  $\mathcal{D}$ , provided that it is compact, strictly connected, and with a regular frontier.<sup>3</sup> Let  $\partial\mathcal{D}$  denote the frontier of  $\mathcal{D}$ . The boundary conditions can be expressed for RD with *reflection* as follows: For any  $\mathbf{x} \in \partial\mathcal{D}$  and for any pair  $(\mathbf{v}_i, \mathbf{v}_r)$ , where  $\mathbf{v}_r$  is the billiard-like reflected velocity of a mobile hitting the border at  $\mathbf{x}$  with velocity  $\mathbf{v}_i$ , we have that  $n(\mathbf{x}, \mathbf{v}_i, t) = n(\mathbf{x}, \mathbf{v}_r, t)$ .

Under the assumption that  $f_V(\mathbf{v})$  has circular symmetry, the unique steady state solution of the RD model with reflection on domain  $\mathcal{D}$  is

$$n(x, v, t) = \frac{\lambda f_V(\mathbf{v}) \mathbb{1}_{\mathcal{D}}(\mathbf{x})}{(\lambda + \mu) \int \mathbb{1}_{\mathcal{D}}(\mathbf{x}) d\mathbf{x}}; \quad s(x) = \frac{\mu \mathbb{1}_{\mathcal{D}}(\mathbf{x})}{(\lambda + \mu) \int \mathbb{1}_{\mathcal{D}}(\mathbf{x}) d\mathbf{x}},$$

where  $\mathbb{1}_{\mathcal{D}}(x)$  is an indicator function which returns 1 if  $\mathbf{x} \in \mathcal{D}$  and 0 otherwise. Indeed, it is straightforward to verify that the above solution satisfies the steady-state RD equations with the *reflection* boundary condition.

We remark that the assumption that the speed vector  $f_V(\mathbf{v})$  has completely symmetric density has also been used in [19] to show the stability of RD model with reflection over multidimensional domains.

### 5.4 Discussion

As a final remark of our steady-state analysis, we emphasize that our approach based on differential equations allows us to obtain the steady-state distribution of RD models with *wrap-around* or *reflection* (in the latter case, under the condition that  $f_V(\mathbf{v})$  is a rotationally symmetric function) in a way different from previous approaches based on Palm Calculus [6], [5]. Indeed, our analysis provides an interesting connection between the theory of differential equations and the stochastic techniques adopted in [6], [5].

## 6 GENERALIZED RD MODEL WITH NONUNIFORM STATIONARY SOLUTION

The standard random direction model brings to a steady state in which nodes are uniformly distributed in space. However, in many practical cases, one would like to have an anisotropic node density in the area. For this reason, we now generalize the RD model in such a way that the stationary distributions of nodes in the move and/or pause phases are not necessarily uniform in space, but follow a

desired (assigned) distribution. In particular, we consider a random direction model in which 1) the pause time may depend on the position  $x$  where the mobile stops and 2) the speed of mobiles during the move phase can vary with the instantaneous position  $x$ .

When a mobile starts traveling on a new segment, we assume it chooses a “base speed”  $\zeta$  from a generic distribution  $f_V(\zeta)$ . The actual speed  $v$  is a deterministic function of the position  $x$  and the base speed  $\zeta$ . For simplicity, we assume that the actual speed is simply proportional to the base speed  $\zeta$  through a factor  $\psi(x)$  that depends only on the position, i.e.,  $v(x, \zeta) = \psi(x)\zeta$ .

The equations of the generalized RD model are

$$\begin{aligned} \frac{\partial n(x, \zeta, y)}{\partial t} &= -\frac{\partial v(x, \zeta)n(x, \zeta, y)}{\partial x} - \frac{\partial n(x, \zeta, y)}{\partial y} \\ &+ \delta(y)f_V(v) \int_z \lambda(x, z)s(x, z)dz - \mu(y)n(x, \zeta, y), \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial s(x, z)}{\partial t} &= -\frac{\partial s(x, z)}{\partial z} - \lambda(x, z)s(x, z) \\ &+ \delta(z) \iint \mu(y)n(x, v, y)dv dy, \end{aligned} \quad (24)$$

where we assume that  $\lambda(x, z)$  has the product form  $\lambda(x, z) = \lambda_1(x)\lambda_2(z)$ , i.e., the average pause time depends on  $x$ , whereas the shape of the distribution of pause time does not depend on  $x$ .

Profiles  $\tilde{n}(x)$  and  $\tilde{s}(x)$  of the mobiles’ density in the move and pause phases, respectively, which are not vanishing at any point  $x$ , can be obtained (as shown in the appendix) by setting

$$\psi(x) = \frac{c}{\tilde{n}(x)}, \quad \lambda(x) = \frac{\mu \tilde{n}(x)}{\tilde{s}(x)}, \quad (25)$$

where  $c$  is any strictly positive constant. For example, consider a segment  $[x_l = -1, x_u = 1]$  and desired spatial profiles  $\tilde{n}(x) = |x + 1|$  and  $\tilde{s}(x) = x^2 + 1$ . According to our methodology, such desired densities can be obtained by setting  $\psi(x) = \frac{1}{|x+1|}$  and  $\lambda(x) = \mu \frac{|x+1|}{x^2+1}$ , having arbitrarily chosen  $c = 1$ .

Note that, in the case in which we are not interested in obtaining specific profiles in the move and pause phases, but just an overall density  $\tilde{\rho}(x) = \tilde{n}(x) + \tilde{s}(x)$ , we can avoid making the mobile’s speed vary with the instantaneous position  $x$ . Indeed, in this case, we can proceed simply as follows: We set  $\psi(x) = 1$ , thus using the original speed distribution  $f_V(v)$  when selecting a new speed value at the transition from pause to move, which is kept constant during the move phase. From (25) (left equation), we observe that, with this choice, we obtain a constant density in the move phase over the whole area. We can actually choose an arbitrary value  $\tilde{n} < \min_x \rho(x)$  to be the resulting probability of being in the move phase at any point and then use (25) (right equation) to compute the transition rate  $\lambda(x)$  that produces the desired overall local density

$$\lambda(x) = \frac{\mu \tilde{n}}{\rho(x) - \tilde{n}}.$$

3. We say that the frontier is regular if there exists a tangent line at every point of it except at most finite numbers.

This technique allows us to achieve an arbitrary node density distribution in a very simple manner, however, no control on individual profiles  $\tilde{n}(x)$  and  $\tilde{s}(x)$  is possible.

## 7 TRANSIENT ANALYSIS

In this section, we present an analytical solution for the transient regime of the RD model. We start considering the case of *wrap around* boundary conditions. As we will see at the end of this section, the transient analysis of the RD model with *reflection* comes for free once we know how to solve the RD model with *wrap around*. As usual, we first consider the unidimensional case with exponential phase times. Then, we extend the analysis to the case of general phase times and, finally, to the multidimensional case.

### 7.1 Unidimensional Case with Exponential Phase Times

We apply the methodology of separation of variables to find solutions for the system of (3) and (4) in case of *wrap around* boundary conditions. Consider product-form candidate solutions:  $n(x, v, t) = \tau(t)\alpha(x)\beta(v)$  and  $s(x, t) = \tau(t)\alpha(x)$ . It turns out (see the appendix for details) that the following elementary functions are solution of the RD equations satisfying the *wrap-around* boundary conditions:

$$n_k(x, v, t) = \frac{\lambda\mu f_V(v) e^{j2\pi f_x kx} e^{\gamma t}}{(\lambda + \gamma)(\mu + \gamma + j2\pi k f_x v)}, \quad (26)$$

$$s_k(x, t) = \frac{\mu}{\lambda + \gamma} e^{j2\pi f_x kx} e^{\gamma t}, \quad (27)$$

where  $f_x = 1/(x_u - x_l)$ ,  $k \in \mathbb{Z}$  and  $\gamma$  is a solution of the following equation:

$$\frac{\lambda\mu}{\lambda + \gamma} \int_v \frac{f_V(v)}{\mu + \gamma + j2\pi k f_x v} dv = 1. \quad (28)$$

When  $k = 0$ , (28) admits the solution  $\gamma_1 = 0$  corresponding to the steady-state distribution of the system already found in Section 5. There is also the solution  $\gamma_2 = -(\lambda + \mu)$ , which has a different physical interpretation: It is the rate at which the system converges to the steady state distribution from the condition in which the probability of being in the move or pause phases are uniform over space but not in equilibrium.

For  $k \neq 0$ , the existence of real solutions for  $\gamma$  can be guaranteed when the probability density function of nodes' speed is symmetric, i.e.,  $f_V(v) = f_V(-v)$ . In this case, the imaginary component of  $\int_v \frac{f_V(v)}{(\mu + \gamma + j2\pi kv)} dv$  is null.

For example, in case  $f_V(v)$  is uniform in the interval  $[-V, V]$ , (28) reduces to

$$\frac{\lambda\mu}{(\lambda + \gamma)2\pi k f_x V} \arctan\left(\frac{2\pi k f_x V}{\mu + \gamma}\right) = 1,$$

from which  $\gamma$  can be easily obtained numerically. In general, it can be shown that  $\gamma$  has two negative solutions,  $\gamma_1$  and  $\gamma_2$ , for every  $k$  ( $\gamma_2 < \gamma_1 < 0$ ).

Moreover, letting  $\hat{n}_k(x, t) = \int n_k(x, v, t) dv$ , we obtain the elementary solution vector

$$\begin{pmatrix} \hat{n}_k(x, t) \\ s_k(x, t) \end{pmatrix} = \begin{pmatrix} \lambda + \gamma \\ \mu \end{pmatrix} e^{j2\pi f_x kx} e^{\gamma t}. \quad (29)$$

Recalling that, for every value of  $k$ , there exist two solutions of  $\gamma$ , it turns out that any solution of (3) and (4) in which the initial distribution profiles in the move and pause phases are continuous with respect to the space coordinate (i.e.,  $\in C^0([x_u, x_l])$ ) can be expanded in series of the above elementary vectors.<sup>4</sup>

The procedure to compute the system state at an arbitrary time instant  $t$  can be summarized into the following steps:

1. Compute the values of  $\gamma_1(k)$  and  $\gamma_2(k)$  associated to every elementary vector.
2. Compute the Fourier series expansion of the initial distribution of mobiles in terms of elementary vectors evaluated at time  $t = 0$ .
3. Multiply each coefficient of the (possibly truncated) series expansion by the exponential decay factor of the corresponding solution vector (either  $e^{-\gamma_1(k)t}$  or  $e^{-\gamma_2(k)t}$ ).
4. Reconstruct the distribution of mobiles using the new values of coefficients at time  $t$ .

Of course, Steps 1 and 2 have to be performed only once, not for any  $t$ . As expected, as time tends to infinity, all "propagation modes"  $\alpha_k(x)$ , with  $k \neq 0$ , tend to vanish exponentially, leaving only the uniform distribution associated to  $k = 0$  ( $\gamma_1 = 0$ ). We remark that the duration of the transient is essentially determined by the periodic component with the minimum absolute value of  $\gamma_i$ .

### 7.2 General Phase Times Distributions

We have not tried to solve exactly the transient analysis of the system in case the move and/or pause times have general (nonexponential) distributions. However, an approximate analysis can be performed using a stage decomposition approach. This means that a separate differential equation has to be written for each stage of the decomposition. For example, consider the case in which the move time is described by a hyper-exponential distribution of the second order:

$$H_2(t) = p_1\mu_1 e^{-\mu_1 t} + p_2\mu_2 e^{-\mu_2 t}.$$

This simple 2-stage approximation allows us to match the first two moments of any distribution having a coefficient of variation larger than one. Let  $n_1(x, v, t)$  and  $n_2(x, v, t)$  be the pdf over  $(x, v)$  of the mobile in move stages 1 and 2, respectively, at time  $t$ . The transient behavior is then described by the following system of differential equations:

$$\begin{cases} \frac{\partial n_1}{\partial t} = -v \frac{\partial n_1}{\partial t} + \lambda p_1 f_V s - \mu_1 n_1 \\ \frac{\partial n_2}{\partial t} = -v \frac{\partial n_2}{\partial t} + \lambda p_2 f_V s - \mu_2 n_2 \\ \frac{\partial s}{\partial t} = -\lambda s + \mu_1 \int n_1 dv + \mu_2 \int n_2 dv, \end{cases} \quad (30)$$

which can be solved in a way analogous to the exponential case. In this case, the equation relating  $\gamma$  to  $\eta$  is

$$p_1\mu_1 \int \frac{f_V(v)}{\mu_1 + \gamma + \eta v} dv + p_2\mu_2 \int \frac{f_V(v)}{\mu_2 + \gamma + \eta v} dv = \frac{\lambda + \gamma}{\lambda}.$$

4. This is due to the fact that the Fourier system represents a complete orthogonal system in the class of square summable functions which comprise functions in  $C^0([x_u, x_l])$ .

Similarly, we can analyze the case in which the pause time is described by a hyper-exponential distribution of the second order, obtaining the set of equations

$$\begin{cases} \frac{\partial n}{\partial t} = -v \frac{\partial n}{\partial t} + \lambda_1 f_V s_1 + \lambda_2 f_V s_2 - \mu n \\ \frac{\partial s_1}{\partial t} = -\lambda_1 s_1 + p_1 \mu \int n dv \\ \frac{\partial s_2}{\partial t} = -\lambda_2 s_2 + p_2 \mu \int n dv. \end{cases} \quad (31)$$

In this case, the equation relating  $\gamma$  to  $\eta$  becomes

$$\left( \frac{p_1 \lambda_1}{\lambda_1 + \gamma} + \frac{p_2 \lambda_2}{\lambda_2 + \gamma} \right) \mu \int_v \frac{f_V(v)}{\mu + \gamma + \eta v} dv = 1.$$

Instead, in the case in which the move time is described by an Erlang distribution of the second order (the sum of two exponential distributions with rate  $\mu_1$  and  $\mu_2$ ), we obtain:

$$\begin{cases} \frac{\partial n_1}{\partial t} = -v \frac{\partial n_1}{\partial t} + \lambda p_1 f_V s - \mu_1 n_1 \\ \frac{\partial n_2}{\partial t} = -v \frac{\partial n_2}{\partial t} + \mu n_1 - \mu_2 n_2 \\ \frac{\partial s}{\partial t} = -\lambda s + \mu_2 \int n_2 dv. \end{cases} \quad (32)$$

In this case, the equation relating  $\gamma$  to  $\eta$  is

$$\mu_1 \mu_2 \int \frac{f_V(v)}{(\mu_1 + \gamma + \eta v)(\mu_2 + \gamma + \eta v)} dv = \frac{\lambda + \gamma}{\lambda}.$$

From the examples above, it is clear that one can write a set of equations for a generic stage decomposition of the move and/or pause time and obtain an implicit equation of  $\gamma$  for each feasible value of  $\eta$ . We remark that, to analyze the transient, it is not necessary to solve the entire set of differential equations, just to derive (numerically) the proper value of  $\gamma$  for each value of  $\eta$  and apply the same procedure described at the end of Section 7.1. Thus, using a stage decomposition approach, the analysis of the transient behavior of the system in the general case has the same computational complexity as the exponential case.

### 7.3 The Multidimensional Case

The transient analysis can be extended also to the multidimensional case. Since the most interesting applications of mobility models arise in the bidimensional space, here, we describe in more detail the transient analysis in the 2D case, restricting ourselves to the case of exponential sojourn times in the move and pause states. We assume that mobiles are free to move in the rectangular area  $[x_u, x_l] \times [y_u, y_l]$  and choose the speed components along  $x$  and  $y$  from an arbitrary bidimensional distribution  $f_V$ . Similar to the monodimensional case, we apply the methodology of separation of variables, looking for solutions for (7) and (8) in the form

$$\begin{aligned} n(x, y, v_x, v_y, t) &= \tau(t) \alpha_X(x) \alpha_Y(y) \beta_X(v_x, v_y), \\ s(x, y, t) &= \tau(t) r_X(x) r_Y(y) \end{aligned}$$

with  $\alpha_X(x) = e^{j2\pi k_x x}$  and  $\alpha_Y(y) = e^{j2\pi k_y y}$ , with  $(k_x, k_y) \in \mathbb{R}^2$ .

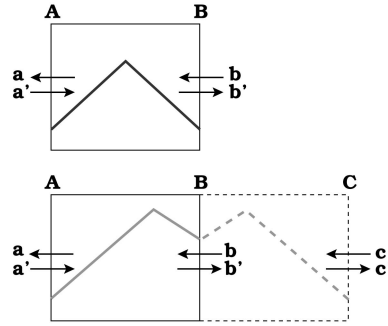


Fig. 1. Reduction of RD model with *reflection* to RD model with *wrap around*.

Plugging these expressions into (7) and (8), we have  $\frac{d\tau(t)}{dt} = \gamma\tau(t)$ , while boundary conditions of the wrap around model imply that the only feasible solutions for  $\alpha_X(x)$  and  $\alpha_Y(y)$  are obtained for  $(k_x, k_y) \in \mathbb{Z}^2$ . Now, we have that exponents  $\gamma$  associated to each pair  $(k_x, k_y) \in \mathbb{Z}^2$  have to satisfy the equation

$$\frac{\lambda \mu}{\lambda + \gamma} \iint \frac{f_V(v_x, v_y)}{\mu + \gamma + j2\pi(k_x f_x v_x + k_y f_y v_y)} dv_x dv_y = 1. \quad (33)$$

The procedure to evaluate the system state at a generic time instant  $t$  follows the same steps described in Section 7.1, except that now we need to perform a bidimensional (Fourier) expansion of the initial distribution of mobiles in the move and pause phases over the rectangular region.

### 7.4 Extension to Reflection Boundary Conditions

The transient analysis of the RD model with *reflection* can be easily reconducted to the analysis of the RD model with *wrap around*. Here, we provide an intuitive explanation of how this can be done. A formal proof is reported in the appendix. Consider first the simple case in which the initial distribution of mobiles in the *move* and *pause* phases are symmetric (top of Fig. 1). In this case, the solution of the RD model with *reflection* is exactly the same as that of the RD model with *wrap around*. Indeed, the flows of mobiles hitting the boundaries are the same,  $\mathbf{a} = \mathbf{b}'$ , and since, in the *wrap around*  $\mathbf{b}' = \mathbf{a}'$ , we have  $\mathbf{a} = \mathbf{a}'$  (similarly,  $\mathbf{b} = \mathbf{b}'$ ), which means that the dynamics are the same as in the *reflection* model. If the initial conditions are not symmetric, we double the area, adding a specular “image” of the initial domain to the right (or to the left), as shown in the bottom part of Fig. 1. Doing so, we obtain a scenario in which the initial conditions are symmetric, thus,  $\mathbf{a} = \mathbf{c}' = \mathbf{a}'$ . Moreover, by construction, we have  $\mathbf{b} = \mathbf{b}'$ . Therefore, the dynamics of the *wrap around* model in the extended area “contain” those of the *reflection* model in the restricted area.

## 8 VALIDATION AND APPLICATIONS

In this section, we validate our analysis of the RD model comparing analytical prediction with simulation results obtained from an event-driven simulator. At the same time, we offer examples of possible applications of our methodology.

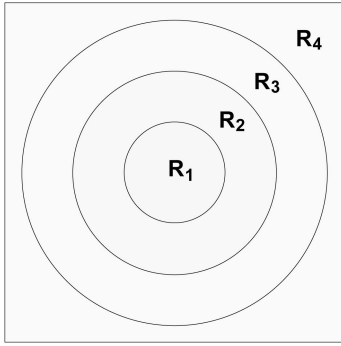


Fig. 2. Regions of the metropolitan area.

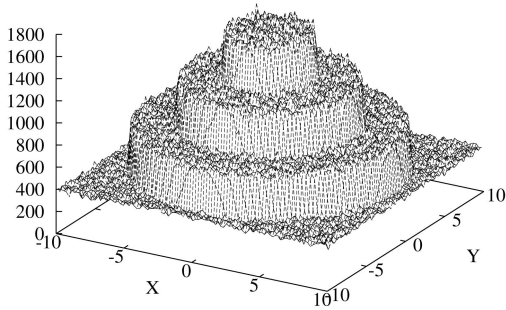


Fig. 3. Distribution of mobiles measured on simulation after 5 hours, resulting from the generalized RD model.

### 8.1 Generalized Mobility Model

Suppose that we want to achieve a given nonuniform stationary distribution of mobiles on the 2D plane. In particular, consider a metropolitan area divided into three concentric rings,  $R_1$ ,  $R_2$ , and  $R_3$ , in a square area of edge 20 kilometers, as depicted in Fig. 2.

The outer region is denoted by  $R_4$ . Let us assume the population density is maximum in  $R_1$ , equal to  $\rho_1$ , whereas the density in region  $R_i$ , ( $i > 1$ ), is  $\rho_i = \rho_1/i$ . We can design a generalized RD model whose stationary distribution of the mobiles' location follows exactly this distribution. Assuming an equal fraction of users in the move and pause states ( $\lambda = \mu$ ), we have to set the scaling factor of speed velocity  $\psi(x, y) = i$ ,  $\forall (x, y) \in R_i$ .

Fig. 3 contains the results of a simulation in which 8 million mobiles move according to the generalized mobility model specified above. Irrespective of their initial position, distributions of move/pause times, and distribution of speed, they tend to the desired nonuniform density. In particular, the plot in Fig. 3 reports the total number of users measured in simulation in each square of edge 10 m after 5 hours of simulated time, assuming a base speed uniformly distributed at  $[-10 \text{ km/h}, 10 \text{ km/h}]$  in each direction.

### 8.2 Transient Analysis in 2D

We now present an example of transient analysis on the 2D plane. We assume that mobiles are initially uniformly distributed within a circle of radius 2 in the middle of square area of edge 20. Move and pause times are exponentially distributed with mean 1. The speed distribution  $f_V(\mathbf{v})$  has a circular symmetry; the speed modulus is uniformly distributed in  $[0, 1]$ .

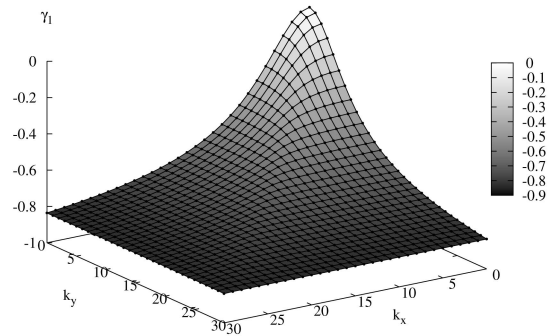


Fig. 4. Parameter  $\gamma_1$  for all combinations of  $(k_x, k_y)$ ,  $1 \leq k_x, k_y \leq 30$ .

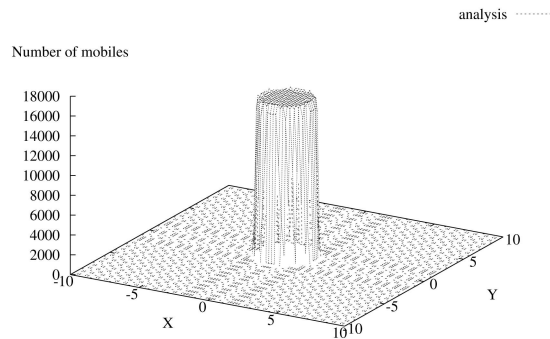


Fig. 5. Representation of the initial distribution of the mobiles' location, limiting the Fourier series expansion to  $128 \times 128$  terms.

This scenario could represent how the center of a city empties at the end of a working day.

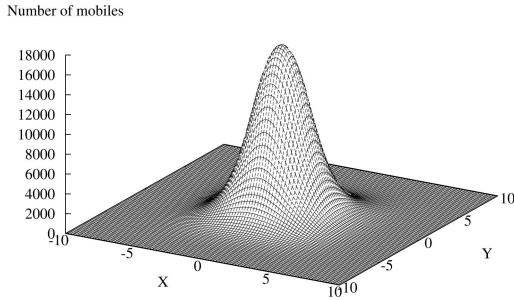
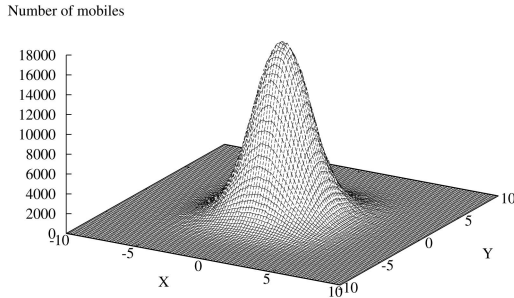
We follow the steps to perform the transient analysis of the system as described in Section 7. First, we compute parameters  $\gamma_1(k_x, k_y)$  and  $\gamma_2(k_x, k_y)$  associated to each elementary vector (see Section 7.3). Fig. 4 depicts parameter  $\gamma_1$  for all combinations of the first 30 positive values of  $k_x$  and  $k_y$  as a continuous surface (for a better representation). Note that

$$\gamma_1(k_x, k_y) = \gamma_1(-k_x, k_y) = \gamma_1(k_x, -k_y) = \gamma_1(-k_x, -k_y),$$

thus, positive values of  $k_x$  and  $k_y$  provide all information. We observe that  $\gamma_1$ , which is a negative number, decreases rapidly for increasing  $k_x$  or  $k_y$ . In practice, the duration of the transient is determined by the smallest absolute values of  $\gamma_1$ . This suggests that there is no need to keep too many terms of the Fourier series expansion of the initial distributions: Solutions corresponding to large  $k_x$  or  $k_y$  decay very fast over time and therefore do not provide a significant contribution to the overall solution except in the very beginning of the transient.

Next, we compute the elementary vector expansion of the initial distribution of mobiles' location (a bidimensional Fourier series expansion), truncating the series to  $128 \times 128$  coefficients  $k_x$  and  $k_y$ . This is enough to produce a satisfactory representation of the initial distribution since the very beginning (i.e.,  $t = 0$ ), as illustrated in Fig. 5.

Now, suppose that we want to compute the distribution of mobiles at an arbitrary time instant  $t > 0$ . It is sufficient to reconstruct the distribution from the elementary vectors series expansion, having multiplied each term of the series by the corresponding factor (either  $e^{-\gamma_1(k_x, k_y)t}$  or  $e^{-\gamma_2(k_x, k_y)t}$ ).

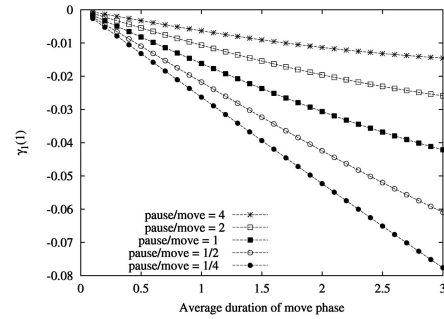
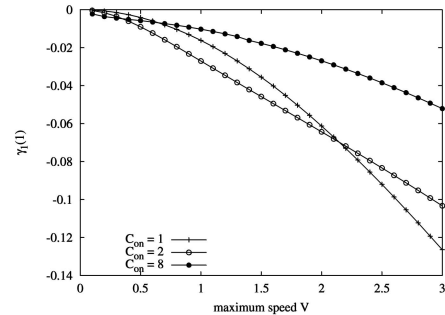
Fig. 6. Distribution of mobiles according to analysis at time  $t = 10$ .Fig. 7. Distribution of mobiles according to simulation at time  $t = 10$ .

For example, Figs. 6 and 7 report the distribution of mobiles at time  $t = 10$  according to analysis and simulation, respectively. The mobiles' distribution has been obtained in simulation, considering 10 million nodes and counting how many of them are present at  $t = 10$  in each square of a  $100 \times 100$  grid. Note that we have used such a large number of nodes to obtain a clean distribution on the chosen grid after a single simulation run. We could have considered a smaller number of nodes (or even a single node), but, in this case, it would have been necessary to average the results of many independent simulation runs.

Recall that we can regard the analytical prediction as the probability of finding a single mobile at a given point of the plane after time  $t$ , starting from an initial pdf of its location at  $t = 0$ . Actually, this point of view opens a wide range of possible applications of our analysis. For example, one could study the persistence of a wireless connection between a mobile and a base station and use our probabilistic analysis to design better hands-off strategies. Another application is the computation of meeting or hitting times among nodes starting from given locations in the space, which is instrumental in the design of mobility-assisted routing schemes [15]. Another interesting application is the analysis of link duration and availability in mobile ad hoc networks, a problem that has been receiving increasing attention [14], [16], [17], [18].

### 8.3 Impact of Parameters on the Transient Duration

We now turn to the 1D case and study the impact of various parameters of the random direction model on the duration of the transient. As already observed, the time constant of the system is essentially given by the smallest absolute value of  $\gamma_1$ , i.e., the one associated with the fundamental mode  $k = 1$ . Thus, we look now at how  $\gamma_1(1)$  depends on the system parameters. We fix the region where mobiles move to the interval  $[-10, 10]$ . First, we consider the impact of move/pause dynamics, while keeping the speed

Fig. 8. The dependence of  $\gamma_1(1)$  on move/pause dynamics.Fig. 9. The dependence of  $\gamma_1(1)$  on maximum speed  $V$  for different variation coefficients of move time.

uniformly distributed in  $[-1, 1]$ . Fig. 8 reports the value of  $\gamma_1(1)$  as a function of the average duration of move time for different ratios between pause and move times. Both of them are assumed to be exponentially distributed.

We observe that, for a given pause/move ratio,  $\gamma_1(1)$  becomes more negative (which implies a shorter transient) for an increasing duration of the move time because mobiles spread faster if they keep the same direction and speed for a prolonged period of time. For a given value of the average move time, the absolute value of  $\gamma_1(1)$  decreases (which implies a longer transient) for increasing persistence in the pause state.

Next, we fix the average duration of the move and pause times equal to 1 and vary the maximum speed  $V$  of mobiles, which is assumed to be uniformly distributed in  $[-V, V]$ . Fig. 9 reports the value of  $\gamma_1(1)$  as a function of  $V$  for different values of the variation coefficient  $C_{on}$  of move time, whose distribution is assumed to be a hyper-exponential of the second order. Pause times are instead exponentially distributed, i.e.,  $C_{off} = 1$ .

While the effect of speed distribution is more intuitive, i.e.,  $\gamma_1(1)$  becomes more negative for increasing  $V$  (shorter transient), the dependency on the variation coefficient  $C_{on}$  is quite intriguing with multiple intersections among curves corresponding to  $C_{on} = 1, 2, 8$ . Therefore, we decided to check on simulating this peculiar behavior. In particular, we consider the case of  $V = 1$ . According to Fig. 9, the fastest transient should be for  $C_{on} = 2$ , whereas  $C_{on} = 8$  should produce the slower transient. We take a Gaussian distribution with a variance of 1 as the initial distribution of the mobiles' position, centered in the origin. Figs. 10, 11, and 12, report, respectively, the distribution mobiles for  $C_{on} = 1, 2, 8$  sampled every 10 time units.

Analytical predictions match perfectly with simulation results in all cases and confirm the impact of the variation

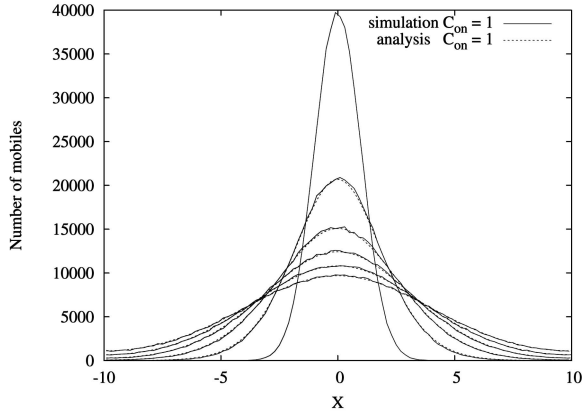


Fig. 10. Distribution of mobiles for  $C_{on} = 1$ . Comparison between analysis and simulation at different time instants.

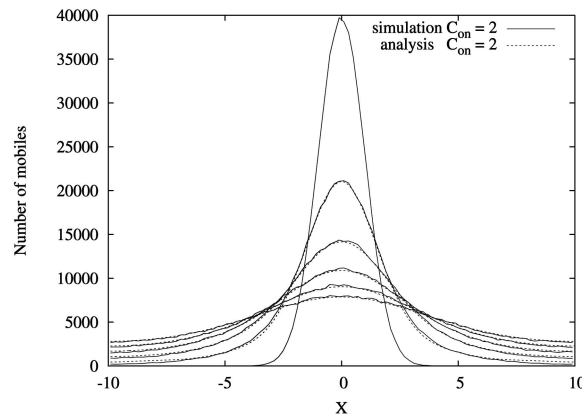


Fig. 11. Distribution of mobiles for  $C_{on} = 2$ . Comparison between analysis and simulation at different time instants.

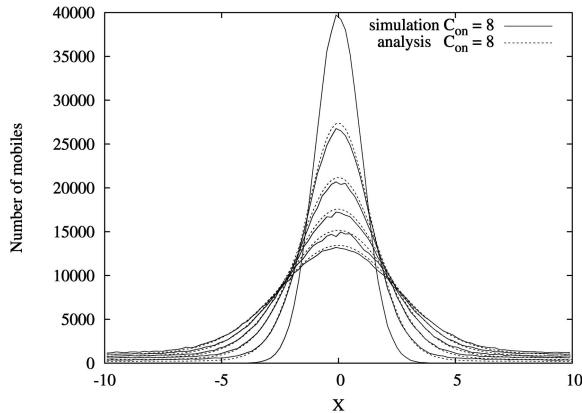


Fig. 12. Distribution of mobiles for  $C_{on} = 8$ . Comparison between analysis and simulation at different time instants.

coefficient: The curves referring to  $C_{on} = 2$  flatten more rapidly than the curves for  $C_{on} = 1$ , which in turn flatten more rapidly than the curves for  $C_{on} = 8$ .

Finally, Fig. 13 reports the values of  $\gamma_1(k)$  up to  $k = 1,000$  for the three considered values of  $C_{on}$ . The most significant values of  $\gamma_1(k)$  in determining the duration of the transient are shown in the inset of Fig. 13.

#### 8.4 Transient Analysis with Reflection

For the RD model with reflection boundary conditions, we present an example of transient analysis carried out

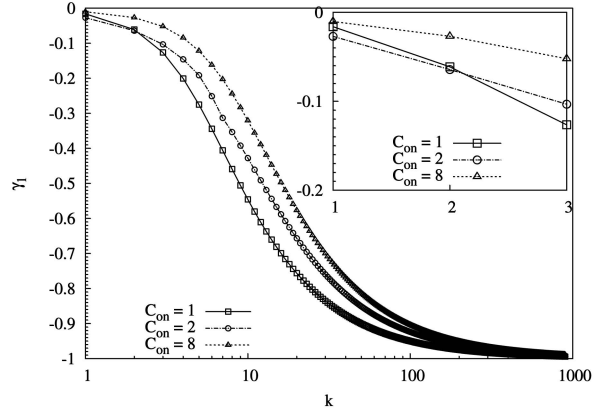


Fig. 13. Values of  $\gamma_1(k)$  for different variation coefficients.

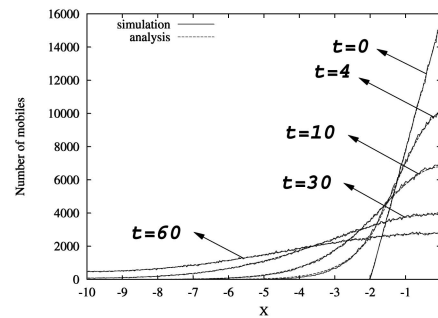


Fig. 14. Distributions of mobiles according to analysis and simulation in a scenario with reflection boundary condition at various time instants.

adopting the procedure described in Section 7.4. We consider an asymmetric initial condition in which the node density has the shape of a triangle, put against the right boundary of the segment  $[-10, 0]$ . In Fig. 14, we compare the node density distribution obtained in simulation and analysis, sampled at  $t = 0, 4, 10, 30, 60$ . The duration of move and pause times are assumed to be exponentially distributed with mean 1 and the speed is chosen uniformly in  $[-1, 1]$ . As expected, a perfect match between analytical prediction and simulation results can be observed.

## 9 CONCLUSIONS

So far in the literature, the theoretical investigation of random direction and random waypoint mobility models has mainly focused on the analysis of the steady state distributions. The approach proposed in this paper permits us to extend the analysis to the transient regime. We have started from the observation that Chapman-Kolmogorov equations describing the dynamics of a single mobile can be used to describe the dynamics of a large population of users. We have obtained Chapman-Kolmogorov equations of a mobile moving according to either the RD or RWP model. Then, we have applied standard mathematical techniques to *analytically* solve the equations for RD models in both the steady state and transient regimes, either with *wrap around* or *reflection* boundary conditions. We have derived simple expressions relating the transient duration to the model parameters; moreover, we have proposed generalized RD models to achieve a desired stationary distribution of mobiles in the space, a problem that has

received little attention so far. So far, we have not found an analogous analytical solution for the equations of the RWP model. We suspect that more sophisticated mathematical techniques are needed in this case to solve the Chapman-Kolmogorov equations. Nevertheless, our formulation of the RWP allows one to also compute the transient behavior for this model using numerical methods (e.g., finite elements). Our dynamical viewpoint indeed opens many new directions in the theory and practice of random mobility models.

## APPENDIX

Here, we prove that the problem defined by (3) and (4) with boundary and initial conditions specified in Section 2.1 admits at most one solution in both the *wrap-around* and *reflection* cases. First, we introduce the following two lemmas which will be used to prove our main result:

**Lemma 1.** *Suppose for simplicity that  $f_V(v)$  is uniformly distributed between  $[-1/2, 1/2]$ . Then, the functional*

$$L(t) = \frac{\mu}{2} \iint n^2(x, v, t) dx dv + \frac{\lambda}{2} \int s(x, t)^2 dx$$

*is a nonincreasing function of time for any pair of functions  $n(x, v, t)$ ,  $s(x, t)$  which are solutions of (3) and (4), respectively.*

**Proof.** First, we define

$$\|s(x, t)\|^2 = \int s^2(x, t) dx = \iint s^2(x, t) \mathbb{1}(v) dv dx,$$

$$\|f_V(v)\|^2 = \int f^2(v) dv = 1,$$

$$\begin{aligned} \|s(x, t) f_V(v)\|^2 &= \iint s^2(x, t) f^2(v) dv dx \\ &= \|s(x, t)\|^2 \|f_V(v)\|^2 = \|s(x, t)\|^2. \end{aligned}$$

Now, let us evaluate

$$\begin{aligned} \frac{dL(t)}{dt} &= \frac{d}{dt} \left[ \frac{\mu}{2} \iint n(x, v, t)^2 dx dv + \frac{\lambda}{2} \int s(x, t)^2 dx \right] \\ &= \mu \iint n(x, v, t) \frac{\partial n(x, v, t)}{\partial t} dx dv + \lambda \int s(x, t) \frac{\partial s(x, t)}{\partial t} dx \\ &= \iint n(x, v, t) \left[ -v \frac{\partial n(x, v, t)}{\partial x} + \lambda f_V(v) s(x) - \mu n(x, v, t) \right] dx dv \\ &\quad + \lambda \iint s(x, t) \left[ -\lambda s(x, t) + \mu \int n(x, v, t) dv \right] dx \\ &= -\mu \iint v \frac{\partial n^2(x, v, t)}{\partial x} dx dv \\ &\quad + \lambda \mu \iint f_V(v) s(x) n(x, v, t) dv dx - \mu^2 \iint n^2(x, v, t) dv \\ &\quad + \mu \lambda \iint s(x) n(x, v, t) dv dx - \lambda^2 \int s^2(x, t) dx \\ &\leq -\mu^2 \|n^2(x, v, t)\| + 2\lambda \mu \|s(x)\| \|n(x, v, t)\| - \lambda^2 \|s(x)\|^2 \leq 0. \end{aligned}$$

In case  $f_V(v)$  is not uniform, the previous result can be extended, redefining the function  $L(t)$ .  $\square$

**Lemma 2.** *Let  $f_V(v)$  be a regular pdf whose associated cdf is  $F_V(v)$ . Having defined  $m(x, v, t) = n(x, v, t)/f_V(v)$  for any  $\epsilon > 0$ , the functional*

$$L(t) = \frac{\mu}{2} \iint_{v: f_V(v) > \epsilon} m(x, v, t)^2 dx dF_V(v) + \frac{\lambda}{2} \int s(x, t)^2 dx$$

*is not increasing.*

**Proof.** This statement can be proved, repeating the passages of the previous proof.  $\square$

From the monotonicity of functional  $L(t)$ , we can easily show that solutions of (3) and (4) are unique.

**Proof.** By contradiction, suppose that two different pairs of functions  $n_1(x, v, t)$ ,  $s_1(x, t)$  and  $n_2(x, v, t)$ ,  $s_2(x, t)$  are solutions of (3) and (4) with the same initial and boundary conditions; then, by linearity of (3) and (4),  $n_1(x, v, t) - n_2(x, v, t)$  and  $s_1(x, t) - s_2(x, t)$  are solutions of (3) and (4) with null initial conditions; i.e.,  $n_1(x, v, 0) - n_2(x, v, 0) = 0$  and  $s_1(x, 0) - s_2(x, 0) = 0$ . As a consequence, since  $L(t) \geq 0$  (by definition),  $L(0) = 0$ , and  $L(t)$  is not increasing, it results that  $L(t) = 0$ ,  $\forall t$ . Therefore, it must be  $n_1(x, v, t) - n_2(x, v, t) = 0$  and  $s_1(x, t) - s_2(x, t) = 0$ ; i.e.,  $n_1(x, v, t) = n_2(x, v, t)$  and  $s_1(x, t) = s_2(x, t)$ .  $\square$

The steady-state equations for the generalized RD model are

$$\begin{aligned} \frac{\partial v(x, \zeta) n(x, \zeta, y)}{\partial x} &= - \frac{\partial n(x, \zeta, y)}{\partial y} \\ &\quad + \delta(y) f_V(v) \lambda_1(x) \int_z \lambda_2(z) s(x, z) dz - \mu(y) n(x, \zeta, y), \end{aligned} \quad (34)$$

$$\frac{\partial s(x, z)}{\partial z} = -\lambda_1(x) \lambda_2(z) s(x, z) + \delta(z) \iint \mu(y) n(x, \zeta, y) dv dy. \quad (35)$$

Considering product-form candidate solutions of the type  $n(x, \zeta, y) = m(x, \zeta) k(y)$  and  $s(x, z) = p(x) h(z)$  with  $\int_{0-}^{\infty} h(z) dz = \int_{0-}^{\infty} k(y) dy = 1$ , and defining

$$\begin{aligned} \lambda_{\text{eff}}(x) &= \lambda_1(x) \int_{0-}^{\infty} \lambda_2(z) h(z) dz = \frac{1}{E[T_{\text{pause}}(x)]}, \\ \mu_{\text{eff}} &= \int_{0-}^{\infty} \mu(y) k(y) dy = \frac{1}{E[T_{\text{move}}]}, \end{aligned}$$

it results that  $m(x, \zeta)$ ,  $p(x)$  must satisfy

$$\frac{\partial v(x, \zeta) m(x, \zeta)}{\partial x} = \lambda_{\text{eff}}(x) f_V(v) p(x) - \mu_{\text{eff}} m(x, \zeta), \quad (36)$$

$$\lambda_{\text{eff}}(x) p(x) = \mu_{\text{eff}} \int m(x, \zeta) dv, \quad (37)$$

while  $k(y)$  and  $h(z)$  satisfy equations similar to (21) and (22), respectively.

Substituting the expression of  $p(x)$  obtained from (37) into (36) yields

$$\frac{\partial [v(x, \zeta) m(x, \zeta)]}{\partial x} = f_V(\zeta) \mu \int m(x, \zeta) d\zeta - \mu m(x, \zeta). \quad (38)$$

Now, considering product-form candidate solutions, i.e., solutions of the form  $m(x, \zeta) = l(x)\beta(\zeta)$ , with  $\int \beta(\zeta)d\zeta = 1$ , it results that

$$\zeta\beta(\zeta)\frac{\partial[\psi(x)l(x)]}{\partial x} = \mu l(x)[f_V(\zeta) - \beta(\zeta)] \quad (39)$$

and we can decouple the previous equation into two ordinary differential equations:

$$\frac{d[\psi(x)l(x)]}{dx} = C\mu l(x), \quad C\zeta = \frac{f_V(\zeta)}{\beta(\zeta)} - 1,$$

from which it results that  $\beta(\zeta) = \frac{f_V(\zeta)}{[C\zeta+1]}$ . Since  $\beta(\zeta) \geq 0$  and  $\int \beta(\zeta)d\zeta = 1$ , it results that  $C = 0$ ; hence,  $\beta(\zeta) = f_V(\zeta)$  and  $l(x) = \frac{a}{\psi(x)}$  for some  $a$  such that  $\int l(x)dx = 1$ .

In conclusion, we can obtain any assigned profiles  $\tilde{n}(x)$  and  $\tilde{s}(x)$  of the mobiles' density in the move and pause phases, respectively, by setting

$$\psi(x) = \frac{c}{\tilde{n}(x)}, \quad \lambda(x) = \frac{\mu \tilde{n}(x)}{\tilde{s}(x)}.$$

Consider product-form candidate solutions:  $n(x, v, t) = \tau(t)m(x, v)$  and  $s(x, t) = \tau(t)r(x)$ ; substituting into (3) and (4), we obtain

$$\begin{aligned} \frac{d\tau(t)}{dt}m(x, v) &= -v\tau(t)\frac{\partial m(x, v)}{\partial x} + \lambda f_V(v)r(x)\tau(t) \\ &\quad - \mu m(x, v)\tau(t), \\ \frac{d\tau(t)}{dt}r(x) &= -\lambda r(x)\tau(t) + \mu\tau(t) \int m(x, v)dv, \end{aligned}$$

from which we can separate the dependency on time from the dependency on space and speed, yielding

$$\frac{d\tau(t)}{dt} = \gamma\tau(t), \quad (40)$$

$$v\frac{\partial m(x, v)}{\partial x} = \lambda f_V(v)r(x) - (\mu + \gamma)m(x, v), \quad (41)$$

$$r(x) = \frac{\mu}{\lambda + \gamma} \int m(x, v)dv. \quad (42)$$

Now, substituting the expression of  $r(x)$  provided by (42) into (41), we have

$$v\frac{\partial m(x, v)}{\partial x} = f_V(v)\frac{\lambda\mu}{\lambda + \gamma} \int m(x, v)dv - (\mu + \gamma)m(x, v).$$

For  $m(x, v)$ , we consider again product-form candidate solutions  $m(x, v) = \alpha(x)\beta(v)$ , obtaining

$$v\beta(v)\frac{d\alpha(x)}{dx} = f_V(v)\frac{\lambda\mu}{\lambda + \gamma}\alpha(x)\int \beta(v)dv - (\mu + \gamma)\alpha(x)\beta(v),$$

in which we can separate the functions which depend on  $x$  from the functions which depend on  $v$ :

$$\frac{d\alpha(x)}{dx} = \eta\alpha(x), \quad (43)$$

$$\frac{\beta(v)}{\int \beta(w)dw} = f_V(v)\frac{\lambda\mu}{(\lambda + \gamma)(\mu + \gamma + \eta v)}. \quad (44)$$

Functions  $\alpha(x) = e^{\eta x}$ , where  $\eta$  is any complex number, are solutions of (43). Instead from (44), since  $\int_v \frac{\beta(v)}{\int \beta(w)dw} dv = 1$ , we obtain a fundamental relation between  $\gamma$  and  $\eta$ :

$$\frac{\lambda\mu}{\lambda + \gamma} \int_v \frac{f_V(v)}{\mu + \gamma + \eta v} dv = 1. \quad (45)$$

Wrap around boundary conditions require that  $\alpha(x_l) = \alpha(x_u)$ ,  $\lim_{x \rightarrow x_l^+} \frac{\partial \alpha(x)}{\partial x} = \lim_{x \rightarrow x_u^-} \frac{\partial \alpha(x)}{\partial x}$ . This constraint is satisfied when  $\alpha(x)$  is periodic with period  $1/f_x = x_u - x_l$ . It follows that wrap around boundary conditions are satisfied when  $\eta = j2\pi f_x k$ , with  $k \in \mathbb{Z}$ . Notice that solutions  $\alpha_k(x) = e^{j2\pi f_x k x}$  correspond to the standard Fourier basis for the interval  $[x_u, x_l]$ , which is dense in  $C^0([x_u, x_l])$ , the class of continuous functions defined over  $[x_u, x_l]$ .

For any given  $k \in \mathbb{Z}$ , (45) provides an implicit equation that defines exponent  $\gamma(k)$ :

$$\frac{\lambda\mu}{\lambda + \gamma} \int_v \frac{f_V(v)}{\mu + \gamma + j2\pi k f_x v} dv = 1. \quad (46)$$

Here, we prove that the transient analysis of the RD model with *reflection* can be reduced to the analysis of the RD model with *wrap around*, as explained in Section 7.4. We assume that the speed distribution is symmetric, i.e.,  $f_V(v) = f_V(-v)$ . The proof is articulated in four steps.

**Step 1.** Consider the unidimensional RD model with *wrap around*. Without loss of generality, let the domain be the interval  $[x_l = -1, x_u = 1]$ . If  $n(x, v, t)$  and  $s(x, t)$  are the solutions of (3) and (4) corresponding to the initial conditions  $n_o(x, v)$  and  $s_o(x)$ , then  $n(-x, -v, t)$  and  $s(-x, t)$  are the solutions of (3) and (4) corresponding to the initial conditions  $n_o(-x, -v)$  and  $s_o(-x)$ . This property can be easily checked directly on (3) and (4) through the change of variables  $(x, v) \rightarrow (-x, -v)$ .

**Step 2.** As a consequence of the previous step, the following property follows: If the initial condition is symmetrical, i.e.,  $n_o(x, v) = n_o(-x, -v)$ ,  $s_o(x) = s_o(-x)$ , then the solution  $n(x, v, t)$ ,  $s(x, t)$  is symmetrical for all  $t$ , i.e.,  $n(x, v, t) = n(-x, -v, t)$  and  $s(x, t) = s(-x, t)$ .

**Step 3.** For any symmetrical initial condition  $n_o(x, v) = n_o(-x, -v)$  and  $s_o(x) = s_o(-x)$ , the RD models with *wrap around* and *reflection* admit the same solution. Indeed, the *wrap around* solution must satisfy the boundary conditions  $n(1, v, t) = n(-1, v, t)$ . This, combined with the invariance under the transformation  $(x, v) \rightarrow (-x, -v)$ , implies that  $n(1, v, t) = n(1, -v, t)$  and, similarly,  $n(-1, v, t) = n(-1, -v, t)$ ; thus, the *wrap around* solution also satisfies the *reflection* boundary conditions and, therefore, provides a solution for the reflection model. Finally, from the uniqueness of the solution of the RD model, no other solution for the *reflection* model exists.

**Step 4.** Now, without loss of generality, consider a *reflection* model over the domain  $[0, 1]$ , under an arbitrary initial condition  $n_o(x, v)$  and  $s_o(x, t)$ . We compare the solution of this model with that of a *wrap around* model over the extended domain  $[-1, 1]$ , under the initial condition  $n_o(x, v) + n_o(-x, -v)$  and  $s_o(x) + s_o(-x)$ : We claim that the restriction of the latter model (with wrap around) over the domain  $[0, 1]$  provides the solution of the

reflection model over the same domain. Indeed, consider  $n(x, v, t)$  and  $s(x, t)$ , the solution of the wrap around model over  $[-1, 1]$ . Observe that, by construction, the initial conditions of this wrap around model are symmetric. Since  $n(x, v, t)$  and  $s(x, t)$  are invariant under the transformation  $(x, v) \rightarrow (-x, -v)$ , we have  $n(1, v, t) = n(-1, v, t) = n(1, -v, t)$ ; therefore, the solution satisfies the reflection condition at boundary  $x = 1$ . Moreover, by construction,  $n(0, v, t) = n(0, -v, t)$ ; thus, the reflection boundary conditions are also satisfied at  $x = 0$ . Since  $n(x, v, 0) = n_o(x, v)$  and  $s(x, 0) = s_o(x)$  over domain  $[0, 1]$ , the restriction of  $n(x, v, t)$ ,  $s(x, t)$  over  $[0, 1]$  provide the unique solution of the reflection model over the same domain under initial conditions  $n_o(x, v)$  and  $s_o(x)$ .

## REFERENCES

- [1] T. Camp, J. Boleng, and V. Davies, "A Survey of Mobility Models for Ad Hoc Network Research," *Wireless Comm. and Mobile Computing*, special issue on mobile ad hoc networking: research, trends, and applications, vol. 2, no. 5, pp. 483-502, 2002.
- [2] D.B. Johnson and D.A. Maltz, "Dynamic Source Routing in Ad Hoc Wireless Networks," *Mobile Computing*, chapter 5, pp. 153-181. Kluwer Academic, 1996.
- [3] J. Broch, D.A. Maltz, D.B. Johnson, Y.-C. Hu, and J. Jetcheva, "A Performance Comparison of Multi-Hop Wireless Ad Hoc Network Routing Protocols," *Proc. ACM MobiCom*, pp. 85-97, Oct. 1998.
- [4] C. Bettstetter, "Mobility Modeling in Wireless Networks: Categorization, Smooth Movement, and Border Effects," *ACM Mobile Computing and Comm. Rev.*, vol. 5, no. 6, pp. 55-66, July 2001.
- [5] J.-Y. Le Boudec and M. Vojnovic, "Perfect Simulation and Stationarity of a Class of Mobility Models," *Proc. IEEE INFOCOM*, Mar. 2005.
- [6] P. Nain, D. Towsley, B. Liu, and Z. Liu, "Properties of Random Direction Models," *Proc. IEEE INFOCOM*, Mar. 2005.
- [7] J. Yoon, M. Liu, and B. Noble, "Sound Mobility Models," *Proc. ACM MobiCom*, pp. 205-216, Sept. 2003.
- [8] W. Navidi and T. Camp, "Stationary Distributions for the Random Waypoint Model," *IEEE Trans. Mobile Computing*, vol. 3, no. 1, 2004.
- [9] C. Bettstetter, G. Resta, and P. Santi, "The Node Distribution of the Random Waypoint Mobility Model for Wireless Ad Hoc Networks," *IEEE Trans. Mobile Computing*, vol. 2, no. 3, pp. 257-269, July-Sept. 2003.
- [10] E. Hyttia, P. Lassila, and J. Virtamo, "Spatial Node Distribution of the Random Waypoint Mobility Model with Applications," *IEEE Trans. Mobile Computing*, vol. 5, no. 6, pp. 680-694, June 2006.
- [11] M. McGuire, "Stationary Distributions of Random Walk Mobility Models for Wireless Ad Hoc Networks," *Proc. ACM MobiHoc*, May 2005.
- [12] J. Yoon, M. Liu, and B. Noble, "Random Waypoint Considered Harmful," *Proc. IEEE INFOCOM*, Apr. 2003.
- [13] S. Meyn and R. Tweedie, *Markov Chains and Stochastic Stability*. Springer-Verlag, 1994.
- [14] G. Carofoglio, C.-F. Chiasserini, M. Garetto, and E. Leonardi, "Analysis of Route Stability under the Random Direction Mobility Model," *Proc. Workshop Math. Performance Modeling and Analysis (MAMA '06)*, June 2006.
- [15] T. Spyropoulos, K. Psounis, and C.S. Raghavendra, "Performance Analysis of Mobility-Assisted Routing," *Proc. ACM MobiHoc*, pp. 49-60, May 2006.
- [16] S. Jiang, "An Enhanced Prediction-Based Link Availability Estimation for MANETs," *IEEE Trans. Comm.*, vol. 52, no. 2, pp. 183-186, Feb. 2004.
- [17] F. Bai, N. Sadagopan, B. Krishnamachari, and A. Helmy, "Modeling Path Duration Distributions in MANETs and Their Impact on Reactive Routing Protocols," *IEEE J. Selected Areas in Comm.*, vol. 22, no. 7, pp. 1357-1373, Sept. 2004.
- [18] Y. Han, R.J. La, and H. Zhang, "Path Selection in Mobile Ad-Hoc Networks and Distribution of Path Duration," *Proc. IEEE INFOCOM*, Apr. 2006.
- [19] J.-Y. Le Boudec and M. Vojnovic, "The Random Trip Model: Stability, Stationary Regime, and Perfect Simulation," *IEEE/ACM Trans. Networking*, Dec. 2006.



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