BLACK-HOLE ATTRACTORS IN N=1 SUPERGRAVITY.

PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA

RECEIVED: *March 30, 2007*
ACCEPTED: *June 21, 2007*
PUBLISHED: *July 6, 2007*

Black-hole attractors in $N = 1$ supergravity

Laura Andrianopoli,[abc] **Riccardo D'Auria,**[b] **Sergio Ferrara**[cd] **and Mario Trigiante**[b]

[a] *Centro E. Fermi, Compendio Viminale,*
I-00184 Rome, Italy
[b] *Dipartimento di Fisica, Politecnico di Torino,*
Corso Duca degli Abruzzi 24, I-10129, Turin, Italy and
Istituto Nazionale di Fisica Nucleare (INFN) Sezione di Torino, Italy
[c] *CERN PH-TH Division,*
CH 1211 Geneva 23, Switzerland
[d] *Istituto Nazionale di Fisica Nucleare (INFN), Laboratori Nazionali di Frascati and*
Department of Physics and Astronomy, University of California,
Los Angeles, CA, U.S.A.
E-mail: laura.andrianopoli@cern.ch, riccardo.dauria@polito.it,
sergio.ferrara@cern.ch, mario.trigiante@polito.it,

ABSTRACT: We study the attractor mechanism for $N = 1$ supergravity coupled to vector and chiral multiplets and compute the attractor equations of these theories. These equations may have solutions depending on the choice of the holomorphic symmetric matrix $f_{\Lambda\Sigma}$ which appears in the kinetic lagrangian of the vector sector. Models with non trivial electric-magnetic duality group which have or have not attractor behavior are exhibited. For a particular class of models, based on an $N = 1$ reduction of homogeneous special geometries, the attractor equations are related to the theory of pure spinors.

KEYWORDS: Supergravity Models, Black Holes.

Contents

1. Introduction

The Reissner-Nordstrom charged black-hole is a solution of the Maxwell-Einstein system. This solution may have two horizons, one horizon or no horizon whenever $M^2 \gtreqless Q^2$, where M is the mass and Q the charge of the black-hole.

In a supergravity context, such configuration can be either viewed as a particular solution of $N = 2$ pure supergravity [1] or of $N = 1$ supergravity coupled to one vector multiplet [2]. Indeed, these theories have the same number of on-shell bosonic and fermionic degrees of freedom, but with a spin 3/2 gravitino exchanged with a photino. In the context of $N = 2$ supergravity, the solutions with $M^2 \geq Q^2$ can be viewed as BPS or non-BPS [3], while solutions with $M^2 < Q^2$ are forbidden (cosmic censorship) [4, 5].

In the $N = 1$ theory, the bosonic solutions are the same, so $M^2 < Q^2$ is still forbidden in spite of the fact that no supersymmetric black-holes exist in this case. For $M^2 = Q^2$ the horizon geometry is Bertotti-Robinson, with a $AdS_2 \times S_2$ metric [6].

Recent investigation (for recent reviews, see for instance [7−11]) have in fact shown that extremal black-holes with attractor behavior also exist without saturating the BPS bound [12−17]. Many examples in $N = 8$ supergravity [18] as well as in generic $N = 2$ theories have been given [19], so that such configurations may be studied also in theories which do not have BPS black-hole configurations [20, 21].

The aim of this investigation is to consider particular theories of $N = 1$ supergravity coupled to matter multiplets, which may have extremal black-hole solutions with attractor behavior [22 − 24]. We will extend the analysis to $N = 1$ theories with scalar fields, where extremal black-holes are connected to attractor points for the scalars.

Now we have an unspecified number n_V of vector multiplets ($\Lambda = 1, \cdots n_V$) and $n_S \geq 1$ of chiral multiplets. The electric-magnetic duality properties of these lagrangians have been studied in [25], following the general analysis given in [26]. In the general case of a theory coupled to chiral and vector multiplets, to have a consistent solution exhibiting attractor behavior, the crucial element is encoded in a complex symmetric matrix, $f_{\Lambda\Sigma}$, with $\mathrm{Im}f < 0$, which is related to the kinetic term of the gauge fields [27]

$$\sqrt{g}\mathrm{Im}f_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma|\mu\nu} + \frac{1}{2}\mathrm{Re}f_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}. \tag{1.1}$$

The matrix $f_{\Lambda\Sigma}$ must satisfy some particular properties, in particular it has to be a holomorphic function of the scalar fields $\partial_{\bar{\imath}}f_{\Lambda\Sigma} = 0$.

In terms of f the black-hole potential reads [13]

$$V = -\frac{1}{2}(q_\Lambda - f_{\Lambda\Sigma}p^\Sigma)(\mathrm{Im}f^{-1})^{\Lambda\Gamma}(q_\Gamma - \overline{f}_{\Gamma\Delta}p^\Delta) = -\frac{1}{2}Q^T\mathcal{M}(f)Q \tag{1.2}$$

with $Q = (p^\Lambda, q_\Lambda)$ the (constant) charge vector and $\mathcal{M}$ the symmetric, symplectic, negative defined matrix ($\mathcal{M}^T = \mathcal{M}$, $\mathcal{M} \cdot \Omega \cdot \mathcal{M} = \Omega$, where Ω is the $\mathrm{Sp}(2n_V, \mathbb{R})$ invariant metric $\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$) given by

$$\mathcal{M} = \begin{pmatrix} \mathrm{Im}f + \mathrm{Re}f\mathrm{Im}f^{-1}\mathrm{Re}f & -\mathrm{Re}f\mathrm{Im}f^{-1} \\ -\mathrm{Im}f^{-1}\mathrm{Re}f & \mathrm{Im}f^{-1} \end{pmatrix}. \tag{1.3}$$

To have large extremal black-hole solutions we require that the black-hole potential has an extremum $\partial_i V = 0$ at $V|_{\mathrm{extr}} \neq 0$, with Hessian matrix $\partial\partial V$ positive definite. The black-hole entropy is then given by [13]:

$$S_{BH}(p,q) = \pi V|_{\partial_i V = 0}. \tag{1.4}$$

In $N = 1$ theories the vector kinetic matrix $f_{\Lambda\Sigma}$ is not fixed by supersymmetry and it can in principle be a rather arbitrary holomorphic function of the chiral multiplets. However, in theories which originate from higher dimensions, such as the ones coming from superstring compactifications, the matrix $f_{\Lambda\Sigma}$ may have a restricted form due to the symmetries of the theory.

For instance, in section 2 we will consider particular $N = 2$ models whose bosonic sector coincides (without truncations) with $N = 1$ models and which exhibit non-trivial attractor behavior. Examples of $N = 1$ models which have no higher N analogue can be obtained for Grassmannian manifolds $U(n,n)/[U(n) \times U(n)]$, following the results of [28].

As another example, in heterotic string compactifications on Calabi-Yau threefolds the tree-level form of $f_{\Lambda\Sigma}$ is just $f_{\Lambda\Sigma} = S\delta_{\Lambda\Sigma}$ where S is the chiral dilaton-axion multiplet [29]. This is the first example we will encounter in section 3. For Calabi-Yau orientifolds (in type IIA) this matrix is linear in the chiral fields, and a class of examples which share similar properties are the models coming from (orientifolded) homogeneous special geometries.

Their general features will also be discussed in section 3. The $N = 1$ theories obtained from truncation of homogeneous special geometries exhibit the particular feature that the chiral multiplets sector is described by a non linear σ-model of the type $\mathrm{SO}(2,n)/[\mathrm{SO}(2)\times \mathrm{SO}(n)]$ and the vector multiplets are in several copies of the spinor representation of $spin(1, n-1)$ which, combining electric and magnetic field-strengths extends to the electric-magnetic duality group $spin(2, n)$. These theories generally show attractor behavior and their critical points have the nice geometrical interpretation that a certain moduli-dependent spinor, constructed out of the electric and magnetic charges, becomes a pure spinor.[1] At the end of the section, the Hessian matrix of some $N = 1$ models at the critical points is determined, showing the attractor nature of the solution. The paper ends in section 4 with some concluding remarks.

2. Embedding Maxwell-Einstein theory in $N = 1$ supergravity and attractors

We are going to discuss in this section the conditions to have extremal black-hole attractor solutions (for large black-holes) in theories with $N = 1$ supergravity.

The crucial condition on the scalar sector is the request of an holomorphic matrix $f_{\Lambda\Sigma}$. The attractor equation $\partial_i V = 0$, for arbitrary matrix $f_{\Lambda\Sigma}$ (with $\mathrm{Im} f_{\Lambda\Sigma} < 0$) satisfying $\partial_{\bar{\imath}} f_{\Lambda\Sigma} = 0$ may be written as

$$\partial_i V = 0 = \overline{\mathcal{V}}^\Lambda \partial_i f_{\Lambda\Sigma} \overline{\mathcal{V}}^\Sigma \tag{2.1}$$

where:

$$\mathcal{V}_\Lambda \equiv (q_\Lambda - f_{\Lambda\Sigma} p^\Sigma)\,, \qquad \partial_{\bar{\imath}} \mathcal{V}_\Lambda = 0\,; \qquad \mathcal{V}^\Lambda = (\mathrm{Im} f^{-1})^{\Lambda\Sigma} \mathcal{V}_\Sigma, \tag{2.2}$$

and the inverse formula holds:

$$p^\Lambda = -\mathrm{Im}\mathcal{V}^\Lambda\,, \qquad q_\Lambda = \mathrm{Re}\mathcal{V}_\Lambda - \mathrm{Re} f_{\Lambda\Sigma} \mathrm{Im}\mathcal{V}^\Sigma \tag{2.3}$$

Indeed,

$$V = -\frac{1}{2}\overline{\mathcal{V}}_\Lambda (\mathrm{Im} f^{-1})^{\Lambda\Sigma} \mathcal{V}_\Sigma = -\frac{1}{2}\overline{\mathcal{V}}^\Lambda \mathrm{Im} f_{\Lambda\Sigma} \mathcal{V}^\Sigma = -\frac{1}{2}\overline{\mathcal{V}}_\Lambda \mathcal{V}^\Lambda \tag{2.4}$$

and

$$\partial_i \mathcal{V}^\Lambda = -\frac{1}{2\mathrm{i}} (\mathrm{Im} f^{-1})^{\Lambda\Sigma} \partial_i f_{\Sigma\Gamma} \overline{\mathcal{V}}^\Gamma \tag{2.5}$$

so that

$$\partial_i V = -\frac{1}{2}\overline{\mathcal{V}}_\Lambda \partial_i \mathcal{V}^\Lambda = \frac{1}{4\mathrm{i}} \overline{\mathcal{V}}^\Lambda \partial_i f_{\Lambda\Sigma} \overline{\mathcal{V}}^\Sigma \tag{2.6}$$

We consider large black-holes solutions, then we require that at the attractor point eq. (2.1) be satisfied with $\overline{\mathcal{V}}^\Lambda \neq 0$. The interpretation of the black-hole potential in eq. (2.4) is that $\mathcal{V}^\Lambda$ is the shift which appears in the photino supersymmetry variation $\delta_\epsilon \lambda^\Lambda$ in the presence of the charged black-hole background. A bilinear photino-gravitino term [27] in the geodesic action [13] with field strengths $\mathcal{V}^\Lambda$ shows that supersymmetry is spontaneously broken [24, 31].

[1] We call here, with an abuse of language, a pure spinor a spinor ψ for which $\psi\gamma_\mu\psi = 0$ [30].

Examples of non-linear σ-models for chiral multiplets which are compatible with a non-trivial electric-magnetic duality of the Maxwell fields [26] (that is with a scalar-dependent holomorphic matrix $f_{\Lambda\Sigma}$) are

1. $\mathrm{Sp}(2n,\mathbb{R})/U(n)$ coupled to n vector multiplets, with duality group $\mathrm{Sp}(2n,\mathbb{R})$
2. $U(1,n)/U(n)$ coupled to $n+1$ vector multiplets, with duality group $U(1,n) \subset \mathrm{Sp}(2n+2,\mathbb{R})$
3. $\mathrm{SU}(1,1)/U(1)$ coupled to n vector multiplets, with duality group $\mathrm{SL}(2,\mathbb{R}))\times\mathrm{SO}(n) \subset \mathrm{Sp}(2n,\mathbb{R})$
4. $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ coupled to r vector multiplets in the spinor representation of $\mathrm{SO}(1,n-1) \subset \mathrm{SO}(2,n)$, with duality group $spin(2,n) \subset \mathrm{Sp}(2r,\mathbb{R})$, where r is the dimension of the spinor representation of $\mathrm{SO}(1,n-1)$
5. $U(n,n)/U(n)\times U(n)$ coupled to $2n$ vector multiplets

As we will see in the next sections, examples 2,3,4,5 exhibit in general attractor behavior, while example 1 does not. This can be easily understood because in the $\mathrm{Sp}(2n,\mathbb{R})/U(n)$ case the scalar fields $x_{\Lambda\Sigma} = x_{\Sigma\Lambda}$ belong to the symmetric representation of $U(n)$, and we have, for the kinetic matrix of the vector

$$f_{\Lambda\Sigma} = x_{\Lambda\Sigma}. \tag{2.7}$$

Then, from (2.1) we find that the attractor equation for this model is

$$\partial_{\Lambda\Sigma}V = 0 \;\Rightarrow\; \overline{\mathcal{V}}_\Lambda\overline{\mathcal{V}}_\Sigma = 0 \tag{2.8}$$

whose only solution is $\mathcal{V}_\Lambda = 0$, which implies $V|_{\text{extr}} = 0$. This solution may correspond to a small black-hole, while attractor solutions for large black-holes cannot be found for this model.

2.1 $N=1$ theories with special geometry

An attractor behavior is guaranteed in theories where the kinetic matrix $f_{\Lambda\Sigma}$ is defined in a special-Kähler geometry. First of all, to have an $N=1$ theory with special geometry for the scalar sector, it is necessary that the number of Wess-Zumino multiplets and of vector multiplets be related. In particular, if the number of chiral multiplets is $n_S = n$, the number of vector multiplets has to be $n_V = n+1$. Then, the following identity has to hold [32, 24]:

$$V = -\frac{1}{2}Q^T\mathcal{M}Q = |D_iZ|^2 + |Z|^2, \tag{2.9}$$

in terms of a covariantly holomorphic superpotential ($N=2$ central charge)

$$Z(z) = e^{\frac{\mathcal{K}}{2}}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda)\,, \qquad X^\Lambda = (1, z^i)\,, \quad i = 1,\cdots n. \tag{2.10}$$

Using the relations of special geometry the attractor condition is in this case

$$\partial_iV = 0 = 2\overline{Z}D_iZ + \mathrm{i}C_{ijk}\overline{Z}^j\overline{Z}^k\,, \qquad (\overline{Z}^j \equiv g^{j\bar\imath}D_{\bar\imath}\overline{Z}). \tag{2.11}$$

However, for the theory to be $N = 1$ supersymmetric the matrix $f_{\Lambda\Sigma}$ must be holomorphic (for a given choice of coordinates). But for general special-Kähler geometries, the kinetic matrix for the vectors, $\mathcal{N}_{\Lambda\Sigma}$, which is related to the covariantly holomorphic symplectic section $U^M{}_0 = (L^\Lambda, M_\Lambda) = e^{\frac{\mathcal{K}}{2}}(X^\Lambda, F_\Lambda)$ and to its covariant derivative $\overline{U}^M{}_i = D_i U^M{}_0 \equiv (f_i^\Lambda, h_{\Lambda i})$ via

$$\begin{cases} M_\Lambda &= \mathcal{N}_{\Lambda\Sigma} L^\Sigma \\ h_{\Lambda i} &= \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma \end{cases}, \tag{2.12}$$

is in general neither holomorphic nor antiholomorphic. We find indeed, from (2.12)

$$\begin{cases} (\partial_i \mathcal{N}_{\Lambda\Sigma}) L^\Sigma &= -(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f_i^\Sigma \\ (\partial_i \mathcal{N}_{\Lambda\Sigma}) \overline{f}_{\bar{\jmath}}^\Sigma &= 0 \end{cases} \tag{2.13}$$

and

$$\begin{cases} (\partial_{\bar{\imath}} \mathcal{N}_{\Lambda\Sigma}) L^\Sigma &= 0 \\ (\partial_{\bar{\imath}} \mathcal{N}_{\Lambda\Sigma}) \overline{f}_{\bar{\jmath}}^\Sigma &= \mathrm{i} C_{\bar{\imath}\bar{\jmath}\bar{k}} g^{\bar{k}\ell} (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f_\ell^\Sigma \end{cases} \tag{2.14}$$

From (2.14) we find that, for the case $n_V = n_S + 1$, the only way to have a holomorphic kinetic matrix is to make the identification $\mathcal{N}_{\Lambda\Sigma} = f_{\Lambda\Sigma}$ and ask $C_{ijk} = 0$, in which case we have $\partial_{\bar{\imath}} f = 0$. The bosonic sector of the theory found in this way is then an $N = 1$ model which is identical to the one of an $N = 2$ model.[2] The only way to satisfy the above properties is to consider as non-linear σ-model spanned by the scalar sector the series $\frac{U(1,n)}{U(1)\times U(n)}$. For this series of special-Kähler models indeed $C_{ijk} = 0$, and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ is holomorphic. In the basis with prepotential $F(X) = -\frac{\mathrm{i}}{2}\eta_{\Lambda\Sigma} X^\Lambda X^\Sigma$ ($\eta_{\Lambda\Sigma} = (1, -1, \cdots, -1)$) we have

$$\mathcal{N}_{\Lambda\Sigma} = \mathrm{i}\left(\eta_{\Lambda\Sigma} - 2\frac{X_\Lambda X_\Sigma}{X^2}\right) \qquad (X_\Lambda \equiv \eta_{\Lambda\Sigma} X^\Sigma) \tag{2.15}$$

We then find, for the attractor condition

$$\partial_i V = 0 \Rightarrow 2\overline{Z} D_i Z = 0 \tag{2.16}$$

which has two solutions. Either

$$Z \neq 0 \quad D_i Z = 0 \tag{2.17}$$

in which case the black-hole potential at the extremum is

$$V_{\text{extr}} = |Z|^2 = I_2 \tag{2.18}$$

or

$$Z = 0 \quad D_i Z \neq 0 \tag{2.19}$$

giving

$$V_{\text{extr}} = |D_i Z|^2 = -I_2. \tag{2.20}$$

[2] This is the so-called minimal coupling of n vector multiplets to $N = 2$ supergravity [33].

Here I_2 is the quadratic $U(1,n)$ invariant written in terms of the black-hole charge (p_0, q_0, p_i, q_i), $(i = 1, \cdots n)$ as

$$I_2 = q_0^2 + p_0^2 - \sum_i (p_i^2 + q_i^2). \tag{2.21}$$

From the analysis of [34], the solution with $Z \neq 0$ exists for $I_2 > 0$ and is a $N = 2$, BPS critical point which is a genuine attractor, since the Hessian matrix of the black-hole potential is positive definite [24, 13]. For the solution with $Z = 0$, which implies $I_2 < 0$, the critical point is $N = 2$ non-BPS and has $2(n-1)$ flat directions since the Hessian matrix is semidefinite positive with only two non vanishing eigenvalues. Note that for $n = 3$ the model can also be interpreted as the bosonic sector of $N = 3$ supergravity coupled to one vector multiplet [35]. In the latter case, the BPS and non-BPS solutions are exchanged [10] and the four flat directions of the BPS solution in the $N = 3$ model correspond to the hypermultiplet in the $N = 3 \to N = 2$ decomposition.

For more general $N = 2$ σ-models (with $C_{ijk} \neq 0$), to have a (anti-) holomorphic kinetic matrix a truncation in the matter sector is needed to satisfy eq. (2.13) such that $\partial_i \mathcal{N}_{\Lambda\Sigma} = 0$ [36].

2.2 Genuine $N = 1$ examples

Among the class of $N = 1$ supersymmetric theories with a non-trivial electric-magnetic duality group, one can consider a model with n^2 complex scalars coupled to $2n$ vector multiplets. In this case the non-linear σ-model is the Kähler manifold $U(n,n)/U(n) \times U(n)$ and the electric-magnetic duality group is $U(n,n) \subset \mathrm{Sp}(4n, \mathbb{R})$, with the electric and magnetic field-strengths embedded in the $2n + 2\overline{n}$ of $U(n,n)$ [28]. Denoting by $s_{i\bar{\jmath}}$ the holomorphic coordinates on the σ-model, with $F^i_{\alpha\beta} = F^i_{\beta\alpha}$ the self-dual part (in spinor notation) of the field strength of the complex vector A^i and with $F^{\bar{\imath}}_{\alpha\beta}$ the self-dual part of the field-strngth of the complex conjugate vector $A^{\bar{\imath}} = (A^i)^*$, then the vector kinetic term is just

$$\mathcal{L} = \mathrm{Im}\left(s_{i\bar{\jmath}} F^i_{\alpha\beta} F^{\bar{\jmath}\alpha\beta} \right). \tag{2.22}$$

For $n = 1$ this model coincides with the $N = 2$ model previously considered and for $n = 3$ it is the bosonic sector of $N = 3$ supergravity coupled to three vector multiplets, while for other n it does not have a higher N origin. As it was explicitly shown for $n = 1$ and $n = 3$, these models admit in general attractor black-hole solutions [34, 10].

3. $N = 1$ examples as $N = 2$ truncations

The supersymmetry reduction of $N = 2 \to N = 1$ supergravity is obtained by truncating the $N = 1$ spin 3/2 multiplet containing the second gravitino and the graviphoton.

All orientifold models in which the $N = 1$ truncation leaves some vectors and scalars with a non trivial holomorphic matrix $f_{\Lambda\Sigma} = -\overline{\mathcal{N}}_{\Lambda\Sigma}$ (in the subspace which excludes the graviphoton) may be studied to see whether they have attractors or not.

A general analysis of the consistent truncation of $N = 2$ theories to $N = 1$ has been given in [36], to which we refer for all the details. We just quote here the main results for the reduction of the vector multiplet sector. Let us first decompose the coordinates of the $N=2$ special manifold as $z^{\mathcal{I}} \to (z^i, z^\alpha)$, with $i = 1, \cdots n_S$ $N=1$ chiral multiplets while α labels the rest of the coordinates, and the $N=2$ vectors as $A^{\mathbf{\Lambda}} \to (A^\Lambda, A^X)$, with $\Lambda = 1, \cdots n_V$ enumerating the $N=1$ vectors and X the rest of the $N=2$ vectors. A consistent truncation requires, on the $N=1$ theory:

$$A^X = 0\,, \qquad z^\alpha = \text{const.} \tag{3.1}$$

$$L^\Lambda = 0\,, \qquad f_i^\Lambda = 0\,, \qquad f_\alpha^X = 0 \tag{3.2}$$

$$\mathcal{N}_{\Lambda X} = 0\,, \quad C_{\alpha\beta\gamma} = 0\,, \qquad C_{ij\alpha} = 0 \tag{3.3}$$

which in particular imply the truncation of the graviphoton projector:

$$T_\Lambda = (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\mathbf{\Sigma}} L^{\mathbf{\Sigma}} = 0. \tag{3.4}$$

This immediately shows that, whenever an $N=2$ holomorphic prepotential $F(X)$ exists such that $F_\Lambda = \partial F/\partial X^\Lambda$ (and $F_{\Lambda\Sigma} \equiv \partial^2 F/\partial X^\Lambda \partial X^\Sigma$), the $N=1$ vector kinetic matrix is indeed anti-holomorphic, since in that case

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} - 2\mathrm{i}\overline{T}_\Lambda \overline{T}_\Sigma (L^{\mathbf{\Gamma}} \mathrm{Im} F_{\mathbf{\Gamma\Delta}} L^{\mathbf{\Delta}}) \to \overline{F}_{\Lambda\Sigma}\,, \quad \partial_i \overline{F}_{\Lambda\Sigma} = 0 \tag{3.5}$$

so that we can identify $f_{\Lambda\Sigma}$ with $-F_{\Lambda\Sigma}$. However, from the analysis of the previous section, eq. (2.13), it turns out that the matrix $\mathcal{N}$ is always anti-holomorphic in the reduced theory (even in the cases where no prepotentail F exists) since $\mathrm{Im}\mathcal{N}_{\Lambda\mathbf{\Sigma}} f_i^{\mathbf{\Sigma}} \to 0$.

An interesting possibility, considered in [36], is the case of the $N=2$ theory based on the σ-model

$$\frac{\mathrm{SU}(1,1)}{U(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}. \tag{3.6}$$

Let us study in detail the attractor equations for the $N=1$ truncation of this model where only the dilaton chiral multiplet is kept together with n vector multiplets. In this case the kinetic matrix simply becomes

$$f_{\Lambda\Sigma} = S\delta_{\Lambda\Sigma} \tag{3.7}$$

and the duality group reduces to $\mathrm{SL}(2,\mathbb{R})) \times \mathrm{SO}(n) \subset \mathrm{Sp}(2n,\mathbb{R})$, where $\mathrm{SL}(2,\mathbb{R}))$ acts as electric-magnetic duality.

Referring to the discussion in section 2, we have in this case

$$\mathcal{V}_\Lambda = q_\Lambda - Sp_\Lambda \tag{3.8}$$

and, from (2.1)

$$\partial_S V = 0 \Rightarrow \sum_\Lambda \mathcal{V}_\Lambda \mathcal{V}_\Lambda = 0. \tag{3.9}$$

An attractor solution is then found for

$$S = a + \mathrm{i}b\ : \qquad a = -p\cdot q/p^2\,; \quad b = -\sqrt{p^2q^2 - (p\cdot q)^2}/p^2. \tag{3.10}$$

and, substituting in the extremized black-hole potential gives

$$V|_{\text{extr}} = \sqrt{p^2 q^2 - (p \cdot q)^2}, \tag{3.11}$$

with a positive Hessian matrix, since:

$$\frac{\partial^2 V}{\partial a^2}|_{\text{extr}} = \frac{\partial^2 V}{\partial b^2}|_{\text{extr}} = \frac{(p^2)^2}{\sqrt{p^2 q^2 - (p \cdot q)^2}} \, ; \qquad \frac{\partial^2 V}{\partial a \partial b}|_{\text{extr}} = 0. \tag{3.12}$$

Note that the entropy for this model has formally the same expression (with $\mathrm{SO}(6,n)$ replacing $\mathrm{SO}(n)$) as in the general $N = 4$ theory [37−39, 24] since the non trivial electric-magnetic duality $\mathrm{SL}(2,\mathbb{R})$) is the same.

For $n = 6$, the bosonic sector of this model coincides with the bosonic sector of pure $N = 4$ supergravity.

For $n = 2$, its bosonic sector coincides instead with the one of the $N = 2$ theory of the quadratic series, with one ($N = 2$) vector multiplet. Note in fact that the quartic invariant $I_4 = q^2 p^2 - (q \cdot p)^2$ reduces in this case (where we only have q_0, p^0, q_1, p^1) to the square of the quadratic invariant $I_2 = p^0 q_1 - p^1 q_0$, $I_4 = (I_2)^2$.

For $n = 1$, the quartic invariant is zero, since in this case $(q \cdot p)^2 = q^2 p^2$. This case concides with the first example of section 2 (the $\mathrm{Sp}(2n,\mathbb{R})/U(n)$ series) for $n = 1$.

3.1 CY orientifold compactifications and $N = 1$ reduction of homogeneous $N = 2$ models

The model discussed above may be generalized by considering the compactification to four dimensions of Type IIA theory on orientifolds (or of M-theory on a special class of G_2-manifolds), as discussed in [40]. According to [40], by considering a Type IIA orientifold which keeps only the complex Kähler moduli z^A and vectors A^α_μ, with $A = 1, \ldots, h^-_{1,1}$ and $\alpha = 1, \ldots, h^+_{1,1}$, the $N = 1$ kinetic matrix for the bulk vectors has the simple form (which generalizes the expression for the 1-modulus S case)

$$f_{\alpha\beta} = -\overline{\mathcal{N}}_{\alpha\beta} = z^A \, d_{A\alpha\beta} \, . \tag{3.13}$$

Similar expressions exist also for the gauge kinetic matrix of the brane vectors (as a function of the bulk moduli) [40, 41] So one could consider the example of a truncation of the homogeneous (but non symmetric) space $L(0, P, \dot{P})$ in which $z^A = (S, z^2, z^3)$ and $z^\alpha = (z^m, z^{\dot{m}})$ ($m = 1, \ldots, P$ and $\dot{m} = 1, \ldots, \dot{P}$). In this theory the only non vanishing entries for the d tensor are

$$d_{S22} = -d_{S33} = \frac{1}{2} \; ; \;\; d_{2mn} = d_{3mn} = \delta_{mn} \; ; \;\; d_{2\dot{m}\dot{n}} = -d_{3\dot{m}\dot{n}} = \delta_{\dot{m}\dot{n}} \, . \tag{3.14}$$

In the orientifolded theory we would have $z^A = (S, z^2, z^3)$ and $A^\alpha_\mu = (A^m_\mu, A^{\dot{m}}_\mu)$.

Let us now analyze the attractor behavior of the $N = 1$ reduction for more general homogeneous $L(q, P, \dot{P})$ Special Kähler models [42]. These models have $r + q + 3$ complex scalars, with $r = (P + \dot{P})\mathcal{D}_{q+1}$, $\mathcal{D}_{q+1}$ being the irreducible reprsentation of $spin(1, q + 1)$.

JHEP07(2007)019

The truncation to $N=1$ leaves $q+2$ chiral multiplets together with r vector multiplets. In particular, the scalar S corresponding to the dilaton decouples from the rest and after the orientifold projection we are left with the coordinates $z^A = x^A + i\,y^A$, $A = 0,1,\ldots,q+1$ spanning the σ-model $\mathrm{SO}(2,q+2)/[\mathrm{SO}(2)\times\mathrm{SO}(q+2)]$. Let us denote the vector fields by A^α, $\alpha = 1,\ldots,r$.

The holomorphic kinetic matrix is now a particular case of (3.13); written in terms of the γ-matrices of $\mathrm{SO}(1,q+1)$, it is:

$$f_{\alpha\beta} = z^A\,\mathbf{\Gamma}_{A\alpha\beta}\,, \tag{3.15}$$

where

$$\begin{aligned}\mathbf{\Gamma}_{0\alpha\beta} &= -\delta_{\alpha\beta}\ ;\quad \mathbf{\Gamma}_{i\alpha\beta} = (\gamma_i)_{\alpha\beta}\,,\qquad i = 1,\ldots q+1\\ \overline{\mathbf{\Gamma}}_0^{\alpha\beta} &= \delta^{\alpha\beta}\ ;\qquad \overline{\mathbf{\Gamma}}_i^{\alpha\beta} = (\gamma_i)_{\alpha\beta}\,,\\ \mathbf{\Gamma}_{(A}\overline{\mathbf{\Gamma}}_{B)} &= \eta_{AB}\ ;\qquad\quad \eta = \mathrm{diag}(-1,+1,\ldots,+1)\,.\end{aligned} \tag{3.16}$$

are two copies of $\mathrm{SO}(1,q+1)$ γ-matrices. They together compose the $2r\times 2r$ representation of the $\mathrm{SO}(1,q+1)$ gamma matrices, corresponding to the embedding in the electric-magnetic duality group $\mathrm{SO}(2,q+2)$, which reads

$$\Gamma_A = \begin{pmatrix} \mathbf{0} & \mathbf{\Gamma}_A \\ \overline{\mathbf{\Gamma}}_A & \mathbf{0}\end{pmatrix}\,, \tag{3.17}$$

The above equations are in fact written for the case $P=1$, $\dot{P}=0$. An obvious extension is understood for P, $\dot{P}$ generic (when this is the case, in (3.16) $P\to\dot{P}$ requires $\mathbf{\Gamma}_i\to-\mathbf{\Gamma}_i$) and will be used in section 3.1.2.

So $\mathfrak{so}(1,q+1)$ is an electric subalgebra of the electric-magnetic algebra $\mathfrak{so}(2,q+2)$, and the system of electric and magnetic field-strengths $\mathcal{S} = (F^\alpha, G_\alpha)$ compose the spinor representation of $\mathrm{SO}(2,q+2)$. To be more precise, the (real) spinor of electric and magnetic charges is irreducible under $\mathrm{SO}(2,q+2)$ but decomposes as $\mathcal{S} = \mathcal{S}_e^+ + \mathcal{S}_m^-$ for $\mathrm{SO}(2,q+2)\to\mathrm{SO}(1,q+1)\times\mathrm{SO}(1,1)$, where $\mathcal{S}^\pm$ have opposite grading under $\mathrm{SO}(1,1)$ and, for q even, also opposite chirality. The $2r$-dimensional $\mathrm{SO}(2,q+2)$ spinorial representation can be described in terms of the following $2r\times 2r$ matrices $\Gamma_M = \{\Gamma_{-1},\,\Gamma_A,\,\Gamma_{q+2}\}$ and $\overline{\Gamma}_M = \{-\Gamma_{-1},\,\Gamma_A,\,\Gamma_{q+2}\}$ ($M,\,N = -1,\ldots,\,q+2$), where

$$\Gamma_{-1} = \mathbb{1}_r\times\mathbb{1}_2\ ;\ \ \Gamma_{q+2} = \mathbb{1}_r\times\sigma_3\,, \tag{3.18}$$

which satisfy the relations

$$\Gamma_{(M}\overline{\Gamma}_{N)} = \widehat{\eta}_{MN}\ \ ;\ \ \widehat{\eta} = \mathrm{diag}(-1,-1,+1,\ldots,+1)\,. \tag{3.19}$$

The action of the $\mathfrak{so}(2,q+2)$ generators on the $2r$ electric-magnetic charges is defined by the matrices $J_{MN} = \frac{1}{4}\Gamma_{[M}\overline{\Gamma}_{N]}$.

The Kähler potential in a special-coordinate inspired basis is

$$K = -\log Y \tag{3.20}$$

with

$$
\begin{aligned}
Y &= -\frac{1}{4}\left[(z_0-\overline{z}_0)^2-(z_i-\overline{z}_i)^2\right], \quad i=1,\cdots q+1 \\
&= -\frac{1}{4}\eta_{AB}(z^A-\overline{z}^A)(z^B-\overline{z}^B)\equiv ||y||^2
\end{aligned}
\tag{3.21}
$$

and, in terms of $z^A = x^A + \mathrm{i}y^A$, the vector kinetic matrix is

$$
f_{\alpha\beta} = (\mathbf{\Gamma}_A)_{\alpha\beta} z^A\,, \qquad \mathrm{Im} f_{\alpha\beta} < 0. \tag{3.22}
$$

We find also:

$$
\mathrm{Im} f_{\alpha\beta} = y^A\,\mathbf{\Gamma}_{A\alpha\beta}\ ;\quad \mathrm{Im} f^{-1\,\alpha\beta} = \frac{y^A}{||y||^2}\,\overline{\mathbf{\Gamma}}{}^{\alpha\beta}_A\ ;\quad ||y||^2 = y^A\eta_{AB}y^B < 0\,, \tag{3.23}
$$

The black-hole potential reads:[3]

$$
\begin{aligned}
V &= -\frac{1}{2}(q_\alpha - z^A\mathbf{\Gamma}_{A\alpha\gamma}p^\gamma)\mathrm{Im} f^{-1\,\alpha\beta}(q_\beta - \overline{z}^A\mathbf{\Gamma}_{A\beta\delta}p^\delta) = \\
&= -\frac{1}{2||y||^2}\left(y\cdot N - 2\,x^T\,W\,y + (y\cdot M)(||y||^2-||x||^2) + 2\,(x\cdot M)(x\cdot y)\right)\,,
\end{aligned}
\tag{3.24}
$$

where we have introduced the following shorthand notation

$$
\begin{aligned}
N_A &= q_\alpha \overline{\mathbf{\Gamma}}{}^{\alpha\beta}_A q_\beta\ ;\quad M_A = p^\alpha\mathbf{\Gamma}_{A\alpha\beta}p^\beta\ ;\qquad W_{AB} = p^\alpha(\mathbf{\Gamma}_A\overline{\mathbf{\Gamma}}_B)_\alpha{}^\beta q_\beta\,, \\
y\cdot M &\equiv y^A M_A\ ;\quad x\cdot y = x^A\,\eta_{AB}\,y^B\ ;\quad x^T\,W\,y = x^A\,W_{AB}\,y^B\,.
\end{aligned}
\tag{3.25}
$$

The extremization condition may be written in the elegant form

$$
\overline{\mathcal{V}}^\alpha\mathbf{\Gamma}_{A\alpha\beta}\overline{\mathcal{V}}^\beta = 0, \tag{3.26}
$$

in terms of the spinor

$$
\overline{\mathcal{V}}^\alpha = \frac{1}{||y||^2}\left(y^A\,q_\beta(\overline{\mathbf{\Gamma}}_A)^{\beta\alpha} - x^A\,y^B\,p^\beta\,(\mathbf{\Gamma}_A\overline{\mathbf{\Gamma}}_B)_\beta{}^\alpha\right) + \mathrm{i}\,p^\alpha. \tag{3.27}
$$

Equation (3.26) can be written as the following real conditions in the real and imaginary parts of the z^A moduli:

$$
\begin{aligned}
0 &= N_A + M_A\,(||y||^2 - |x||^2) - 2\,y_A\,(y\cdot M) + 2\,W_{[AD]}\,x^D\,, \\
x_A &= \frac{(p\cdot q)}{||M||^2}\,M_A - \frac{1}{(y\cdot M)}\,P_A{}^B\,W_{[BC]}\,y^C\,,
\end{aligned}
\tag{3.28}
$$

where $P_A{}^B$ is the projector in the directions orthogonal to M_A:

$$
P_A{}^B = \delta^B_A - \frac{M_A\,M^B}{||M||^2}\,. \tag{3.29}
$$

[3]For the $L(q,P,\dot{P})$ models with $P\dot{P}\neq 0$ ($q=4m$), since $\mathrm{Im} f_{\alpha\beta}$ is block-diagonal in the $P,\dot{P}$ space, two terms in two separate spinor spaces are understood in eq. (3.24).

Eq.s (2.3), (3.27) and (3.26) allow to write down a general expression for the entropy, given by eq. (1.4):

$$\frac{1}{\pi}S_{BH}(p,q) = V|_{\partial_i V=0} = -\,M_A\,y^A|_{\rm extr}\,. \tag{3.30}$$

Geometrically, the attractor points are the points where $\overline{\mathcal{V}}^\alpha$ becomes a pure spinor.[4] As a remark we observe that eq. (3.26) is identical in form to eq. (4.43) of [45] for the $N=2$ attractors of homogeneous Kähler spaces with vanishing central charge (and vanishing of the $q+2$ matter charges Z_I). On a general ground this is a consequence of the fact that the $N=1$ attractor equations given in eq. (2.1) are similar to the ones for the $N=2$ attractors (eq. (2.11)) with vanishing central change:

$$C_{ijk}\,\overline{Z}^{\,j}\overline{Z}^{\,k} = 0\,, \tag{3.31}$$

if one replaces C_{ijk} by $\partial_i f_{\alpha\beta}$ and $\overline{Z}^{\,i} = g^{i\bar{\jmath}}D_{\bar{\jmath}}\overline{Z}$ by $\overline{\mathcal{V}}^\alpha = {\rm Im}f^{-1\,\alpha\beta}\,\overline{\mathcal{V}}_\beta$.

As the above discussion shows, $L(q,P,\dot{P})$ theories may admit in general attractor extrema, apart from particular cases. We are going to discuss, in the rest of this section, some specific examples.

3.1.1 The $L(q,1)$ cases

This series, for particular values of q: $q=1,2,4,8$, describes $N=2$ symmetric spaces [46]. Let us consider in particular the case $q=8$, which corresponds to the σ-model $E_{7(-25)}/E_6\times$ U(1), when decomposed with respect to SL$(2,\mathbb{R})$) $\times$ SO$(2,10)$ in a truncation where one only keeps the SL$(2,\mathbb{R})$) singlets. Since the representation of the electric and magnetic field-strengths decomposes under SL$(2,\mathbb{R})$) $\times$ SO$(2,10)$ as

$$56 \to (2,12) + (1,32), \tag{3.32}$$

only the 32 electric and magnetic field-strengths belonging to the spinorial representation of SO$(2,10)$ are kept. In the σ-model counterpart

$$E_{7(-25)}/[E_6\times {\rm U}(1)] \to {\rm SO}(2,10)/[{\rm SO}(2)\times{\rm SO}(10)]. \tag{3.33}$$

So, also in this case the final $N=1$ model is based on the SO$(2,10)$/[SO$(2)\times$ SO(10)] σ-model coupled to F,G in the spinorial representation of SO$(2,n+2)$, with electric sub-algebra SO$(1,n+1)$.

For the $L(2,1)$ model, the gamma matrices read

$$\mathbf{\Gamma}_1 = -\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ ;\ \mathbf{\Gamma}_2 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$
$$\mathbf{\Gamma}_3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\ ;\ \mathbf{\Gamma}_4 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{3.34}$$

[4] Here we adopt a definition [43 – 45] which is milder than the mathematical definition when $q>8$ (and $P,\,\dot{P}>1$)

The potential at the extremum has the following expression:

$$V_{\rm extr} = (p^1\, q_2 - p^2\, q_1 + p^4\, q_3 - p^3\, q_4)\,. \tag{3.35}$$

This agrees with the fact that the charge-spinor in this case belongs to the $\mathbf{4} \in \mathrm{SU}(2,2) = spin(4,2)$, which is complex, while the entropy is given in terms of a real bilinear invariant [47].

The $L(1,1)$ case has no attractors, as we will see in the following, as a particular case of the series $L(1,P)$.

Let us now move to analyze various cases of homogeneous spaces $L(q,P)$ and $L(q,P,\dot{P})$.

3.1.2 The $L(0,P,\dot{P})$ cases

For this series ($q=0$), the spinor of charges degenerates, and we have $P+\dot{P}$ spinorial electric and magnetic charges (p^α, q_α) and ($\dot{p}^\alpha$, $\dot{q}_\alpha$). The gamma matrices read

$$\mathbf{\Gamma}_0 = \begin{pmatrix} -\delta_{\alpha\beta} & 0 \\ 0 & -\delta_{\dot\alpha\dot\beta} \end{pmatrix} = -\overline{\mathbf{\Gamma}}_0\,; \qquad \mathbf{\Gamma}_1 = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & -\delta_{\dot\alpha\dot\beta} \end{pmatrix} = \overline{\mathbf{\Gamma}}_1\,, \tag{3.36}$$

so that the scalar potential is

$$V = -\frac{1}{2}\mathcal{V}^\alpha \mathrm{Im} f_{\alpha\beta}\overline{\mathcal{V}}^\beta - \frac{1}{2}\mathcal{V}^{\dot\alpha}\mathrm{Im} f_{\dot\alpha\dot\beta}\overline{\mathcal{V}}^{\dot\beta} \tag{3.37}$$

with

$$\mathrm{Im} f_{\alpha\beta} = -\delta_{\alpha\beta}(y_0 - y_1)\;;\; \mathcal{V}^\alpha = -\frac{1}{y_0-y_1}\left[q^\alpha + (z_0 - z_1)p^\alpha\right]; \tag{3.38}$$

$$\mathrm{Im} f_{\dot\alpha\dot\beta} = -\delta_{\dot\alpha\dot\beta}(y_0 + y_1)\;;\; \mathcal{V}^{\dot\alpha} = -\frac{1}{y_0+y_1}\left[q^{\dot\alpha} + (z_0 + z_1)p^{\dot\alpha}\right]. \tag{3.39}$$

To have $\mathrm{Im} f < 0$ requires $y_0 > y_1 > 0$.

For this series, the potential (3.37) decomposes into the sum of two independent, functionally identical, contributions, each one depending on a different variable:

$$V = V(u) + \dot{V}(\dot{u})\,; \qquad u \equiv z_0 - z_1\,, \quad \dot{u} = z_0 + z_1 \tag{3.40}$$

where:

$$\begin{aligned} V(u) &= -\frac{1}{2}\mathcal{V}^\alpha \mathrm{Im} f_{\alpha\beta}\overline{\mathcal{V}}^\beta(u) = \frac{1}{2\,\mathrm{Im} u}\left(q^2 + 2\,(q\cdot p)\,\mathrm{Re}\,u + p^2\,|u|^2\right) \\ \dot{V}(\dot{u}) &= -\frac{1}{2}\mathcal{V}^{\dot\alpha}\mathrm{Im} f_{\dot\alpha\dot\beta}\overline{\mathcal{V}}^{\dot\beta}(\dot{u}) = \frac{1}{2\,\mathrm{Im}\dot{u}}\left(\dot{q}^2 + 2\,(\dot{q}\cdot\dot{p})\,\mathrm{Re}\,\dot{u} + \dot{p}^2\,|\dot{u}|^2\right) \end{aligned} \tag{3.41}$$

The attractor equations become the equations for two cones, which can be regarded as the pure spinor equations for $\mathrm{SO}(1,1)$:

$$\sum_{\alpha=1,\cdots P} \mathcal{V}^\alpha\mathcal{V}^\alpha = 0 \tag{3.42}$$

$$\sum_{\dot\alpha=1,\cdots \dot{P}} \mathcal{V}^{\dot\alpha}\mathcal{V}^{\dot\alpha} = 0. \tag{3.43}$$

Therefore the attractor points are the ones for which the complex vectors $\mathcal{V}^\alpha$ and $\mathcal{V}^{\dot\alpha}$ have vanishing euclidean norm. The minima of (3.40) are found for

$$u = -\frac{1}{p^2}\left(q\cdot p - \mathrm{i}\sqrt{I_4}\right) \tag{3.44}$$

$$\dot u = -\frac{1}{\dot p^2}\left(\dot q\cdot \dot p - \mathrm{i}\sqrt{\dot I_4}\right) \tag{3.45}$$

where $I_4 \equiv q^2p^2 - (q\cdot p)^2$, $\dot I_4 \equiv \dot q^2 \dot p^2 - (\dot q\cdot \dot p)^2$. The extremum of the black-hole potential is then

$$V|_{\text{extr}} = \sqrt{I_4} + \sqrt{\dot I_4}. \tag{3.46}$$

The Hessian matrix at the extremum, evaluated with respect to the real and imaginary parts of $u, \dot u$, is

$$H(u,\dot u)|_{\text{extr}} = \begin{pmatrix} \frac{p^4}{\sqrt{I_4}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & 0 \\ 0 & \frac{\dot p^4}{\sqrt{\dot I_4}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{pmatrix}, \qquad \det H|_{\text{extr}} > 0. \tag{3.47}$$

showing that for all this class of models the extrema of the potential have indeed an attractor nature.

Note that for all $L(0,P,\dot P)$ the duality group is $\mathrm{SO}(2,2)\times\mathrm{SO}(P)\times\mathrm{SO}(\dot P)$, and the potential at the extremum may be written in terms of the manifest invariant of the duality group

$$I_4 = T_{\alpha\beta}T^{\alpha\beta} = p^2q^2 - (p\cdot q)^2, \tag{3.48}$$

(and similarly for $\dot I_4$), with $T_{\alpha\beta} = -T_{\beta\alpha} \equiv \mathcal{S}_\alpha^T\cdot\Omega\cdot\mathcal{S}_\beta$, $\mathcal{S}_\alpha^T = (p^\alpha, q_\alpha)$ an $\mathrm{SO}(P)$-valued chiral spinor of $\mathrm{SO}(2,2)$ and Ω the invariant metric of $\mathrm{SU}(1,1)\subset\mathrm{SO}(2,2)$. This class of models is particularly interesting because it may correspond to a system of P $D3$ and $\dot P$ $D7$ branes on Calabi-Yau orientifold compactifications [48].

If $P\dot P \neq 0$, both P and $\dot P$ must be bigger than one, otherwise the attractor point does not exist (since then I_4 or $\dot I_4$ vanish, and $\mathrm{Im}\,u$ or $\mathrm{Im}\,\dot u$ would vanish either.).

For $P\dot P = 0$, we have the $L(0,P)$ $(P>1)$ models, in which case one complex modulus (u or $\dot u$) is undetermined on the black-hole solution, the Hessian has two vanishing eigenvalues and the attractor equations have two flat directions.

Let us finally observe that, since the irreducible representation of the spinor of charges in $\mathrm{SO}(2,2)$ is in fact chiral, only a subgroup $\mathrm{SL}(2,\mathbb{R}))\times\mathrm{SO}(P)\times\mathrm{SO}(\dot P)\subset\mathrm{SO}(2,2)\times\mathrm{SO}(P)\times\mathrm{SO}(\dot P)$ of the duality group acts non trivially. The vector-multiplet sector of this theory (in the case $\dot P = 0$) is then identical to the $N=1$ truncation of the $L(-1,P)$ series. In this last case, however, the scalar sector reduces to the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$, so that the attractor condition is one complex equation for one modulus. Then the critical point is a genuine attractor. Note that this truncation gives back the same $\mathrm{SU}(1,1)/\mathrm{U}(1)\times\mathrm{SO}(P)$ model already discussed in section 3, whose entropy has been given in (3.11).

JHEP07(2007)019

3.1.3 The $L(1,2)$ case.

We have four electric and four magnetic charges. The gamma matrices read:

$$\mathbf{\Gamma}_1 = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \mathbf{\Gamma}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ; \mathbf{\Gamma}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.49}$$

The potential at the extremum has the following form:

$$V_{\text{extr}} = (p^2\, q_4 - q_2\, p^4 + p^1\, q_3 - q_1\, p^3)\,. \tag{3.50}$$

From this we see that in the symmetric $L(1,1)$ case, in which $p^3 = q_3 = p^4 = q_4 = 0$, the potential at the extremum is zero. This is in agreement with the fact that the in this case we have a single spinor of charges, belonging to the $\mathbf{4} \in \mathrm{Sp}(4,\mathbb{R}) = spin(3,2)$, which has no antisymmetric bilinear invariant.

4. Concluding remarks

In this investigation we have considered the black-hole potential of charged extremal black-holes in $N = 1$ supergravity coupled to chiral and Maxwell vector multiplets. The attractor equations take the particular simple form (2.1). In a particular class of models, obtained by an orientifold projection of homogeneous special geometries, the attractor equation (3.26) has the geometrical meaning, at least for $q \leq 8$, that the spinor $\mathcal{V}^\alpha$ defined in (3.27) is a pure spinor. Pure spinors have already occurred in the literature in connection to attractor equations for type II compactifications on generalized Calabi-Yau manifolds in [49].

The entropy can be computed and it is given in terms of invariants of the electric-magnetic duality group that, for an $N = 1$ reduction of $L(q, P, \dot{P})$ homogeneous spaces, is in general $spin(2, q+2) \times S_q(P, \dot{P})$, where $S_q(P, \dot{P})$ is the centralizer of the relevant Clifford algebra and it was classified in [42]. For models of the type $L(0, P, \dot{P})$, the underlying special geometry may correspond to D-branes on a CY-orientifold compactification and the attractor points would correspond to extremal black-holes on the branes. From the analysis of section 3, we find that such attractors exist if at least two branes of the same kind are kept. We also find evidence that extremal black-holes with attractor behavior may exist in heterotic string compactifications on Calabi-Yau manifolds, with the dilaton and axion fields fixed in terms of the electric and magnetic charges of the vector bundle. This is the $N = 1$ analogue of the $N = 4$ dilaton-axion black-hole [50 − 52]. On a more general ground, it seems that, whenever the gauge-kinetic matrix is moduli-dependent in $N = 1$ supergravity models coupled to vector multiplets, then charged extremal black-hole solutions with attractor behavior appear as a generic rather than an exceptional feature.

It would be interesting to extend the present analysis by including deviations from the Maxwell-Einstein system, by considering either Born-Infeld contributions to the Maxwell action [53, 54, 28, 55] (as it would be relevant in the case of brane vector fields) and higher curvature terms [56, 7, 14, 8, 9] in the gravitational field.

JHEP07(2007)019

Acknowledgments

We acknowledge enlightening discussions with C. Angelantonj, I. Antoniadis, P. Aschieri and R. Stora. Work supported in part by the European Community's Human Potential Program under contract MRTN-CT-2004-005104 'Constituents, fundamental forces and symmetries of the universe', in which L.A., R.D'A. and M.T. are associated to Torino University. The work of S.F. has been supported in part by European Community's Human Potential Program under contract MRTN-CT-2004-005104 'Constituents, fundamental forces and symmetries of the universe' and the contract MRTN- CT-2004-503369 'The quest for unification: Theory Confronts Experiments', in association with INFN Frascati National Laboratories and by D.O.E. grant DE-FG03-91ER40662, Task C.

References

[1] S. Ferrara and P. van Nieuwenhuizen, *Consistent supergravity with complex spin* 3/2 *gauge fields*, *Phys. Rev. Lett.* **37** (1976) 1669.

[2] S. Ferrara, J. Scherk and P. van Nieuwenhuizen, *Locally supersymmetric Maxwell-Einstein theory*, *Phys. Rev. Lett.* **37** (1976) 1035.

[3] G.W. Gibbons and C.M. Hull, *A bogomolny bound for general relativity and solitons in* $N = 2$ *supergravity*, *Phys. Lett.* **B 109** (1982) 190.

[4] R. Kallosh, A.D. Linde, T. Ortin, A.W. Peet and A. Van Proeyen, *Supersymmetry as a cosmic censor*, *Phys. Rev.* **D 46** (1992) 5278 [hep-th/9205027].

[5] J. Bellorin, P. Meessen and T. Ortin, *Supersymmetry, attractors and cosmic censorship*, *Nucl. Phys.* **B 762** (2007) 229 [hep-th/0606201].

[6] B. Bertotti, *Uniform electromagnetic field in the theory of general relativity*, *Phys. Rev.* **116** (1959) 133;
I. Robinson, *A solution of the Maxwell-Einstein equations*, *Bull. Acad. Polon. Sci.* **7** (19959) 351.

[7] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[8] B. de Wit, *Supersymmetric black holes*, *Fortschr. Phys.* **54** (2006) 183 [hep-th/0511261].

[9] B. Pioline, *Lectures on on black holes, topological strings and quantum attractors*, *Class. and Quant. Grav.* **23** (2006) S981 [hep-th/0607227].

[10] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal black holes in supergravity*, to appear in the book *String theory and fundamental interactions* published in celebration of the 65^{th} birthday of Gabriele Veneziano, Eds. M. Gasperini and J. Maharana (Lecture Notes in Physics, Springer Berlin, Heidelberg, 2007), www.springerlink.com/content/1616-6361, hep-th/0611345.

[11] S. Bellucci, S. Ferrara and A. Marrani, *Attractor horizon geometries of extremal black holes*, to appear in the proceedings of 17^{th} SIGRAV Conference, Turin, Italy, 4-7 Sep. 2006, hep-th/0702019.

[12] R.R. Khuri and T. Ortin, *A non-supersymmetric dyonic extreme reissner-nordstrom black hole*, *Phys. Lett.* **B 373** (1996) 56 [hep-th/9512178];
T. Ortin, *Extremality versus supersymmetry in stringy black holes*, *Phys. Lett.* **B 422** (1998) 93 [hep-th/9612142]; *Non-supersymmetric (but) extreme black holes, scalar hair and other open problems*, hep-th/9705095.

[13] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103].

[14] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177].

[15] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Non-supersymmetric attractors*, *Phys. Rev.* **D 72** (2005) 124021 [hep-th/0507096].

[16] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022 [hep-th/0510024].

[17] R. Kallosh, N. Sivanandam and M. Soroush, *The non-BPS black hole attractor equation*, *JHEP* **03** (2006) 060 [hep-th/0602005].

[18] S. Ferrara and R. Kallosh, *On $N = 8$ attractors*, *Phys. Rev.* **D 73** (2006) 125005 [hep-th/0603247].

[19] P.K. Tripathy and S.P. Trivedi, *Non-supersymmetric attractors in string theory*, *JHEP* **03** (2006) 022 [hep-th/0511117].

[20] D.Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, *Phys. Rev.* **D 69** (2004) 104027 [hep-th/0312055].

[21] A. Ceresole and G. Dall'Agata, *Flow equations for non-BPS extremal black holes*, *JHEP* **03** (2007) 110 [hep-th/0702088].

[22] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[23] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[24] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136]; *Universality of supersymmetric attractors*, *Phys. Rev.* **D 54** (1996) 1525 [hep-th/9603090].

[25] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[26] M.K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, *Nucl. Phys.* **B 193** (1981) 221.

[27] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Yang-Mills theories with local supersymmetry: lagrangian, transformation laws and superhiggs effect*, *Nucl. Phys.* **B 212** (1983) 413.

[28] P. Aschieri, D. Brace, B. Morariu and B. Zumino, *Nonlinear self-duality in even dimensions*, *Nucl. Phys.* **B 574** (2000) 551 [hep-th/9909021].

[29] E. Witten, *Dimensional reduction of superstring models*, *Phys. Lett.* **B 155** (1985) 151.

[30] E. Cartan, *Leçons sur la theorie des spineurs*, Paris, Hermann (1937);
C. Chevalley, *The algebraic theory of spinors and Clifford Algebras*, Collected Works, Springer Verlag (1996).

[31] S. Bellucci, S. Ferrara and A. Marrani, *On some properties of the attractor equations*, *Phys. Lett.* **B 635** (2006) 172 [hep-th/0602161].

[32] A. Ceresole, R. D'Auria and S. Ferrara, *The symplectic structure of $N = 2$ supergravity and its central extension*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 67 [hep-th/9509160].

[33] J.F. Luciani, *Coupling of $O(2)$ supergravity with several vector multiplets*, *Nucl. Phys.* **B 132** (1978) 325.

[34] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of symmetric special geometries and attractors*, *Int. J. Mod. Phys.* **A 21** (2006) 5043 [hep-th/0606209].

[35] L. Castellani et al., *The complete $N = 3$ matter coupled supergravity*, *Nucl. Phys.* **B 268** (1986) 317.

[36] L. Andrianopoli, R. D'Auria and S. Ferrara, *Supersymmetry reduction of N-extended supergravities in four dimensions*, *JHEP* **03** (2002) 025 [hep-th/0110277]; *Consistent reduction of $N = 2 \to N = 1$ four dimensional supergravity coupled to matter*, *Nucl. Phys.* **B 628** (2002) 387 [hep-th/0112192].

[37] M. Cvetič and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, *Phys. Rev.* **D 53** (1996) 584 [hep-th/9507090].

[38] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys.* **B 459** (1996) 125 [hep-th/9508094].

[39] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [hep-th/0508042].

[40] T.W. Grimm and J. Louis, *The effective action of $N = 1$ Calabi-Yau orientifolds*, *Nucl. Phys.* **B 699** (2004) 387 [hep-th/0403067]; *The effective action of type IIA Calabi-Yau orientifolds*, *Nucl. Phys.* **B 718** (2005) 153 [hep-th/0412277].

[41] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, hep-th/0610327.

[42] D.V. Alekseevsky, *Classification of quaternionic spaces with a transitive solvable group of motions*, *Math. USSR Izvestija* **9** (1975) 297;
B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, *Nucl. Phys.* **B 400** (1993) 463 [hep-th/9210068];
V. Cortés, *Alekseevskian spaces*, *Diff. Geom. Appl.* **6** (1996) 129;
P. Fre et al., *Tits-Satake projections of homogeneous special geometries*, *Class. and Quant. Grav.* **24** (2007) 27 [hep-th/0606173].

[43] N. Berkovits and N. Nekrasov, *The character of pure spinors*, *Lett. Math. Phys.* **74** (2005) 75 [hep-th/0503075];

[44] P.A. Grassi and N. Wyllard, *Lower-dimensional pure-spinor superstrings*, *JHEP* **12** (2005) 007 [hep-th/0509140].

[45] R. D'Auria, S. Ferrara and M. Trigiante, *Critical points of the black-hole potential for homogeneous special geometries*, *JHEP* **03** (2007) 097 [hep-th/0701090].

[46] M. Günaydin, G. Sierra and P.K. Townsend, *Exceptional supergravity theories and the magic square*, *Phys. Lett.* **B 133** (1983) 72;
M. Günaydin, G. Sierra and P.K. Townsend, *The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras*, *Nucl. Phys.* **B 242** (1984) 244.

[47] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, U.S.A. (1978).

[48] C. Angelantonj, R. D'Auria, S. Ferrara and M. Trigiante, $D^3 \times T^2/\mathbb{Z}_2$ *orientifolds with fluxes, open string moduli and critical points*, *Phys. Lett.* **B 583** (2004) 331 [hep-th/0312019].

[49] J.P. Hsu, A. Maloney and A. Tomasiello, *Black hole attractors and pure spinors*, *JHEP* **09** (2006) 048 [hep-th/0602142].

[50] R. Kallosh and A.W. Peet, *Dilaton black holes near the horizon*, *Phys. Rev.* **D 46** (1992) 5223 [hep-th/9209116].

[51] R. Kallosh, T. Ortin and A.W. Peet, *Entropy and action of dilaton black holes*, *Phys. Rev.* **D 47** (1993) 5400 [hep-th/9211015].

[52] E. Bergshoeff, R. Kallosh and T. Ortin, *Stationary axion/dilaton solutions and supersymmetry*, *Nucl. Phys.* **B 478** (1996) 156 [hep-th/9605059].

[53] G.W. Gibbons and D.A. Rasheed, $\mathrm{SL}(2,r)$ *invariance of non-linear electrodynamics coupled to an axion and a dilaton*, *Phys. Lett.* **B 365** (1996) 46 [hep-th/9509141].

[54] J. Castelino, S.-M. Lee and I.W. Taylor, *Longitudinal 5-branes as 4-spheres in matrix theory*, *Nucl. Phys.* **B 526** (1998) 334 [hep-th/9712105].

[55] B. Chandrasekhar, H. Yavartanoo and S. Yun, *Non-supersymmetric attractors in BI black holes*, hep-th/0611240.

[56] R.M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038];
V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [gr-qc/9403028];