Primes between consecutive powers

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PRIMES BETWEEN CONSECUTIVE POWERS

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ABSTRACT. A well-known conjecture about the distribution of primes asserts that between two consecutive squares there is always at least one prime number. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. This paper is concerned with the distribution of prime numbers between two consecutive powers of integers, as a natural generalization of the aforementioned conjecture. The results follow from the proprieties of the exceptional set for the distribution of prime in short intervals.

1. Introduction. A well-known conjecture about the distribution of primes asserts that all intervals of type $[n^2,(n+1)^2]$ contain at least one prime.

The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann hypothesis. Goldston proved the conjecture assuming a strong form of the Montgomery conjecture, see [**3**]. The author improved this result proving that all intervals of type $[n^2,(n+1)^2]$ contain the expected number of primes, for $n \to \infty$, assuming a weaker hypothesis about the behavior of Selberg's integral in short intervals, see Bazzanella [**1**].

This paper is concerned with the distribution of prime numbers between two consecutive powers of integers, as a natural generalization of the above conjecture.

The well-known result of Huxley about the distribution of primes in short intervals, see [**5**], implies that the intervals $[n^\alpha,(n+1)^\alpha]$ contain the expected number of primes for $\alpha > (12/5)$ and $n \to \infty$.

As was observed, we are unable to prove a similar result for $\alpha = 2$, even under the assumption of the Riemann hypothesis. However, we can obtain some results under the assumption of the Lindelöf

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hypothesis, which states that the Riemann zeta-function satisfies

$$\zeta(\sigma+it) \ll t^{\eta}, \quad \sigma \geq \frac{1}{2}, \quad t \geq 2,$$

for any $\eta > 0$.

Theorem 1. *Assume the Lindelöf hypothesis and let $\alpha > 2$. The intervals $[n^\alpha, (n+1)^\alpha]$ contain the expected number of primes for $n \to \infty$.*

Moreover, we can relax our request and investigate if "almost all" intervals of type $[n^\alpha, (n+1)^\alpha]$ contain the expected number of primes. By this we mean that the number of integers $X \leq n \leq 2X$ for which the interval $[n^\alpha, (n+1)^\alpha]$ does not contain the expected number of primes is $o(X)$. Huxley's zero density estimate [**5**], in conjunction with the method of Selberg [**8**], shows that almost all intervals $[n^\alpha, (n+1)^\alpha]$ contain the expected number of primes for $\alpha > (6/5)$. Obviously we cannot put $\alpha = 1$, but we can handle α close to 1 assuming the Lindelöf hypothesis again.

Theorem 2. *Assume the Lindelöf hypothesis and let $\alpha > 1$. Almost all intervals $[n^\alpha, (n+1)^\alpha]$ contain the expected number of primes.*

The two theorems can be proved following the line of Ingham [**6**] and Selberg [**8**], with the use of the best known zero density estimates. In this note we present a different proof of Theorems 1 and 2 that follows essentially from a result of Bazzanella and Perelli [**2**] about the structure of the exceptional set for the distribution of primes in short intervals.

The peculiarity of our proofs is that we obtain the asymptotic behavior of $\psi((n+1)^\alpha) - \psi(n^\alpha)$ from well-known mean value estimates with the use of information about the structure of the exceptional set for the distribution of primes in short intervals.

2. The basic lemmas. The first lemma is a result about the structure of the exceptional set for the asymptotic formula

$$\psi(x+h(x)) - \psi(x) \sim h(x) \quad \text{as} \quad x \to \infty. \tag{1}$$

Let X be a large positive number, $\delta > 0$, and let $|\ |$ denote the modulus of a complex number or the Lebesgue measure of a set. Let $h(x)$ be an increasing function such that $x^{\varepsilon} \le h(x) \le x$ for some $\varepsilon > 0$ and

$$E_\delta(X, h) = \{X \le x \le 2X : |\psi(x + h(x)) - \psi(x) - h(x)| \ge \delta h(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists an $X_0(\delta)$ such that $E_\delta(X, h) = \varnothing$ for $X \ge X_0(\delta)$. Hence, for small $\delta > 0$, X tending to ∞ and $h(x)$ suitably small with respect to x, the set $E_\delta(X, h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, h) \subset E_{\delta'}(X, h) \quad \text{if} \quad 0 < \delta' < \delta.$$

We will consider increasing functions $h(x)$ of the form $h(x) = x^{\theta + \varepsilon(x)}$, with some $0 < \theta < 1$ and a function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing,

$$\varepsilon(x) = o(1) \quad \text{and} \quad \varepsilon(x + y) = \varepsilon(x) + O\left(\frac{|y|}{x \log x}\right).$$

A function satisfying these requirements will be called of *type* θ.

The first lemma provides the basic structure of the exceptional set $E_\delta(X, h)$.

Lemma 1. *Let $0 < \theta < 1$, and let $h(x)$ be of type θ, X sufficiently large depending on the function $h(x)$ and $0 < \delta' < \delta$ with $\delta - \delta' \ge \exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, h)$, then $E_{\delta'}(X, h)$ contains the interval $[x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_\delta(X, h) \ne \varnothing$, then*

$$|E_{\delta'}(X, h)| \gg_\theta (\delta - \delta')h(X).$$

Lemma 1 essentially says that if we have a single exception in $E_\delta(X, h)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X, h)$, with δ' a little smaller than δ. The interesting consequence of this lemma is that we can use an average estimate to prove the nonexistence of the exceptions.

The second lemma concerns an estimate for the fourth-power integral linked to the distribution of primes in short intervals.

Lemma 2. *Assume the Lindelöf hypothesis, and let $\varepsilon > 0$. Then there exists a function $\Delta(y,T)$ such that*

$$\int_X^{2X} \left|\psi\left(y+\frac{y}{T}\right)-\psi(y)-\frac{y}{T}+\Delta(y,T)\right|^4 dy \ll X^{4+\varepsilon}T^{-3} \tag{2}$$

and

$$\Delta(y,T) \ll \frac{y}{T\ln y} \tag{3}$$

uniformly for $X \geq 2$, $1 \leq T \leq X$ and $X \leq y \leq 2X$.

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [**2**], and Lemma 2 is Lemma B of Yu, see [**9**].

3. Proof of Theorem 1. Theorem 1 asserts that

$$\psi((n+1)^\alpha)-\psi(n^\alpha) \sim (n+1)^\alpha - n^\alpha \quad \text{as} \quad n \to \infty. \tag{4}$$

In order to prove the theorem we assume that (4) does not hold. Then there exist $\delta_0 > 0$ and a sequence $n_j \to \infty$ such that

$$\left|\psi((n_j+1)^\alpha)-\psi(n_j^\alpha)-[(n_j+1)^\alpha-n_j^\alpha)]\right| \geq \delta_0[(n_j+1)^\alpha-n_j^\alpha)].$$

In the remainder of the proof we will always assume that n_j is sufficiently large as prescribed by the various statements. Putting $X_j = n_j^\alpha$ and $H = [(X_j^{1/\alpha}+1)^\alpha - X_j]$, we then have

$$|\psi(X_j+H)-\psi(X_j)-H| \geq \delta_0 H.$$

We now substitute H with X_j/T, where $T = X_j^{1/\alpha}/\alpha$, and obtain

$$\left|\psi(X_j+\frac{X_j}{T})-\psi(X_j)-\frac{X_j}{T}\right| \geq \frac{\delta_0}{2}\frac{X_j}{T},$$

since

$$H = \frac{X_j}{T} + O(X_j^{1-2/\alpha})$$

and

$$\psi(X_j + H) = \psi(X_j + \frac{X_j}{T}) + O(X_j^{1-2/\alpha}).$$

Now we set $h(y) = y/T$. By our choice we see that

$$|\psi(X_j + h(X_j)) - \psi(X_j) - h(X_j)| \geq \frac{\delta_0}{2} h(X_j),$$

and hence we can conclude that $X_j \in E_{\delta_0/2}(X_j, h)$. The use of Lemma 1 with $\delta' = \delta_0/4$ leads to

(5) $$|E_{\delta'}(X_j, h)| \gg h(X_j) \gg X_j^{1-1/\alpha}.$$

On the other hand we can bound $|E_{\delta'}(X_j, h)|$ using Lemma 2. For any $y \in E_{\delta'}(X_j, h)$, with $h(y) = y/T$ and $T = X_j^{1/\alpha}/\alpha$, we can write

$$\left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T}\right| \geq \delta' \frac{y}{T},$$

and therefore

(6) $$\left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T)\right| \geq \frac{\delta'}{2} \frac{y}{T},$$

for every function

$$\Delta(y, T) \ll \frac{y}{T \ln T}.$$

Integrating on the exceptional set $E_{\delta'}(X_j, h)$ the fourth-power of (6) we can deduce

$$\begin{aligned}
|E_{\delta'}(X_j, h)| \left(\frac{X_j}{T}\right)^4 & \\
&\ll \int_{E_{\delta'}(X_j,h)} \left|\frac{y}{T}\right|^4 dy \\
&\ll \int_{E_{\delta'}(X_j,h)} \left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T)\right|^4 dy \\
&\leq \int_{X_j}^{2X_j} \left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T)\right|^4 dy.
\end{aligned}$$

Lemma 2 implies that, for every $\varepsilon > 0$,

$$\int_{X_j}^{2X_j} \left|\psi\left(y+\frac{y}{T}\right)-\psi(y)-\frac{y}{T}+\Delta(y,T)\right|^4 dy \ll X_j^{4+\varepsilon}T^{-3}, \tag{7}$$

and hence

$$|E_{\delta'}(X_j,h)| \ll X_j^{\varepsilon}T \ll X_j^{1/\alpha+\varepsilon}. \tag{8}$$

For $\alpha > 2$ and X_j sufficiently large, we have a contradiction between (5) and (8), and this completes the proof of Theorem 1. □

4. Proof of Theorem 2. We define $H = (n+1)^\alpha - n^\alpha$ and

$$A_\delta(N,\alpha) = \Big\{N^{1/\alpha} \le n \le (2N)^{1/\alpha} : \\ |\psi((n+1)^\alpha) - \psi(n^\alpha) - H| \ge \delta H\Big\}.$$

This set contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of the type $[n^\alpha,(n+1)^\alpha]$ in $[N,2N]$.

To prove Theorem 2, it suffices to show $|A_\delta(N,\alpha)| = o(N^{1/\alpha})$, for every $\delta > 0$ and $\alpha > 1$. The main step of the proof is to connect the exceptional set $A_\delta(N,\alpha)$ with the exceptional set for the distribution of primes in short intervals and show that

$$|A_\delta(N,\alpha)| \ll \frac{|E_{\delta/2}(N,h)|}{N^{1-1/\alpha}} + 1 \tag{9}$$

for every $\delta > 0$, $\alpha > 1$ and $h(x) = (x^{1/\alpha}+1)^\alpha - x$.

In order to prove (9) we let $n \in A_\delta(N,\alpha)$ and $x = n^\alpha \in [N,2N]$. From the definition of the set $A_\delta(N,\alpha)$, we get

$$|\psi((n+1)^\alpha) - \psi(n^\alpha) - H| \ge \delta H,$$

and then

$$|\psi(x+h(x)) - \psi(x) - h(x)| \ge \delta h(x),$$

which implies $x \in E_\delta(N,h)$. Using Lemma 1, with $\delta' = \delta/2$, we obtain that there exists an effective constant c such that

$$[x, x+ch(x)] \cap [N, 2N] \subset E_{\delta/2}(N,h).$$

Let $m \in A_\delta(N,\alpha)$, $m > n$. As before, we can define $y = m^\alpha \in [N, 2N]$ such that

$$[y, y+ch(y)] \cap [N, 2N] \subset E_{\delta/2}(N,h).$$

Choosing $c < 1$, we find

$$y - x = m^\alpha - n^\alpha \geq (n+1)^\alpha - n^\alpha > ch(x)$$

and then

$$[x, x+ch(x)] \cap [y, y+ch(y)] = \varnothing.$$

Hence (9) is proved, since for every $n \in A_\delta(N,\alpha)$ and $x = n^\alpha$, with at most one exception, we have

$$[x, x+ch(x)] \subset [N, 2N].$$

Now we bound the measure of the exceptional set $E_{\delta/2}(N,h)$ using Lemma 2. To perform this, given any $\varepsilon > 0$, we subdivide $[N, 2N]$ into $\ll N^\varepsilon$ intervals of type $I_j = [N_j, N_j + Y]$ with $N \leq N_j < 2N$ and $Y \ll N^{1-\varepsilon}$. For every $y \in E_{\delta/2}(N,h)$, we have

$$|\psi(y+h(y)) - \psi(y) - h(y)| \gg N^{1-1/\alpha}$$

and then

$$\begin{aligned} |E_{\delta/2}(N,h)| N^{4-4/\alpha} &\ll \int_{E_{\delta/2}(N,h)} |\psi(y+h(y)) - \psi(y) - h(y)|^4 \, dy \\ &\ll \sum_j \int_{E^j_{\delta/2}(N,h)} |\psi(y+h(y)) - \psi(y) - h(y)|^4 \, dy \end{aligned}$$

where $E^j_{\delta/2}(N,h) = E_{\delta/2}(N,h) \cap [N_j, N_j + Y]$.

Lemma 2 yields that for every $T_j = N_j^{1/\alpha}/\alpha$ there exists a function $\Delta_j(y, T_j)$ which satisfies the conditions (2) and (3). Applying the Brunn-Titchmarsh inequality we can deduce

$$\Big(\psi(y+h(y)) - \psi(y) - h(y)\Big) - \left(\psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j)\right) \ll \frac{y}{T_j \log X},$$

for every j and $y \in E^j_{\delta/2}(N,h)$, and then we obtain

$$\begin{aligned}|E_{\delta/2}(N,h)|&N^{4-4/\alpha}\\ &\ll \sum_j \int_{E^j_{\delta/2}(N,h)} \left|\psi\left(y+\frac{y}{T_j}\right)-\psi(y)-\frac{y}{T_j}+\Delta_j(y,T_j)\right|^4 dy\\ &\le \sum_j \int_N^{2N} \left|\psi\left(y+\frac{y}{T_j}\right)-\psi(y)-\frac{y}{T_j}+\Delta_j(y,T_j)\right|^4 dy.\end{aligned}$$

The condition (2) of Lemma 2 also implies

$$\begin{aligned}(10)\quad |E_{\delta/2}(N,h)|&\\ \ll N^{-4+4/\alpha}&\sum_j \int_N^{2N} \left|\psi\left(y+\frac{y}{T_j}\right)-\psi(y)-\frac{y}{T_j}+\Delta_j(y,T_j)\right|^4 dy\\ \ll N^{-4+4/\alpha}&\sum_j N^{4+\varepsilon}T_j^{-3} \ll N^{1/\alpha+2\varepsilon},\end{aligned}$$

for every $\varepsilon > 0$. From (9) and (10) it follows that

$$|A_\delta(N,\alpha)| \ll \frac{|E_{\delta/2}(N,h)|}{N^{1-1/\alpha}}+1 \ll \frac{N^{1/\alpha+2\varepsilon}}{N^{1-1/\alpha}}+1 \ll N^{2/\alpha-1+2\varepsilon}+1 = o(N^{1/\alpha}),$$

for every $\alpha > 1$, $\delta > 0$ and ε suitable small. This completes the proof of Theorem 2. □

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