Orbit dynamics and kinematics with full quaternions

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# ORBIT DYNAMICS AND KINEMATICS WITH FULL QUATERNIONS

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Abstract: Full quaternions constitute a compact notation for describing the motion of a body in the space. An important result about full quaternions is that they can be partitioned into a unit quaternion (which describes the orientation with respect to a suitable reference), and a modulus (which represents the translational motion along the direction indicated by the unit quaternion). Since vectors and scalars are also full quaternions, the equations of body motion can be rewritten in quaternion form. In this paper the orbit dynamics and kinematics of a point mass moving in the space are transformed in quaternion form. Simple application examples are presented. 

Keywords: Aerospace Trajectories, System Models, Quaternions, Satellite Control Applications.

## 1. INTRODUCTION

When dealing with satellite attitude and orbit control, one of the first design issue is the formulation of spacecraft dynamics. According to classical approach, rigid body motion can be decomposed into:

1. *orbital motion*, depending on position and velocity of the satellite Centre of Mass (COM) ;
2. *attitude kinematics and dynamics*, described by Euler parameters (i.e.: unit quaternions) or Euler angles.

This methodology is very well known, has been widely treated in literature (Wertz, J.R., 1978 and Kaplan, M.H., 1976), and is commonly used in applications: for example it has been employed in the design of a *drag-free* controller for the European satellite GOCE (Canuto, E. *et al*., 2002). In this case, satellite attitude corresponds to the orientation of a body-fixed reference frame w.r.t. a *local orbital* frame, univocally defined by orbit position and velocity. Assuming that the orientation of the body frame w.r.t. an inertial frame is known, it becomes necessary to parameterize the orientation of the orbital frame w.r.t. the inertial reference. The problem, apparently straightforward, which suggested the present work, is transforming the inertial coordinates of the three unit vectors constituting the orbital frame into a set of four Euler parameters. Two alternative solutions have been considered:

1. to build the rotation matrix and then exploit the well known conversion rules allowing to pass to quaternion parameterization;
2. to associate a *full quaternion* notation (i.e.: non-unitary quaternion) to orbital frame.

The former solution has been employed in attitude determination of the GOCE satellite. The latter one, which is credited to be original, has been developed with the aim of finding a direct way to express the motion of the local orbital frame entrained by the COM motion.

A full quaternion can describe the modulus and the orientation of a vector w.r.t. a given reference frame. This implies, considering the satellite orbit, that *position and velocity can be alternatively denoted with a vector or with the associated full quaternion*. Since that, orbital dynamics and kinematics can be rewritten substituting vector notation with full quaternions. This results in harmonization of motion equations: both orbital dynamics/kinematics and attitude dynamics/kinematics can be rewritten in quaternion form. Then the orientation of the orbital frame can be directly extracted from the related full quaternion at any time.

This paper is devoted to lay down the foundations of this technique with the help of simple applications. First of all, definition and elementary algebra of full quaternions will be introduced in Section 2. Next, how full quaternions can represent vector magnifications and finite rotations will be shown. First and second derivatives of full quaternions are then derived in order to rewrite orbital motion equations in quaternion form. This will be explained in Section 3, where quaternion kinematics and dynamics will be derived. In Section 4, quaternion kinematic and dynamic equations will be applied to a pair of typical orbital references: the *Local Orbital Reference Frame* and the *Local Vertical – Local Horizontal* frame. In both cases the associated full quaternion will be defined, as well as orbital kinematics and dynamics. A simple case of uniform circular motion will enlighten the similarities between classical vector form and quaternion expression of orbital motion. Finally, in Section 5 some simulation results will be presented.

## 2. FULL QUATERNIONS

*2.1 Definition*

A quaternion $\mathcal{A}$ is defined as a complex number:

$$\mathcal{A} \triangleq a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = a_0 + \boldsymbol{a}\,. \tag{1}$$

Quaternions can be also expressed in column vector form w.r.t. the basis (1, ***i***, ***j*** ,***k***):

$$\mathcal{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T = \begin{bmatrix} a_0 & \boldsymbol{a}^T \end{bmatrix}^T. \tag{2}$$

*Remark*. To alleviate notation, the script $\mathcal{A}$ will denote at the same time:

1. quaternions in complex number form (1),
2. and quaternions in column vector form (2).

A vector quaternion is a three-dimensional vector ***b*** represented in quaternion notations, i.e.:

$$\mathcal{B} = \begin{bmatrix} 0 & \boldsymbol{b}^T \end{bmatrix}^T. \tag{3}$$

In this case the notations $\mathcal{B}$ and ***b*** will have the same meaning.

*2.2 Algebra*

A brief summary of the full quaternion algebra is provided, leaving the details to the Appendix and Chou, J.C.K (1992).

The norm of a quaternion $\mathcal{A}$, denoted by $|\mathcal{A}|$, is a scalar quaternion and is defined as follows:

$$|\mathcal{A}|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 = \left\| \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T \right\|_2^2. \tag{4}$$

If $|\mathcal{A}|=1$, $\mathcal{A}$ is called a *unit quaternion*, and deserves its own notation $\underline{\mathcal{A}}$. If $\mathcal{A}$ has non-unitary norm, it is called a *full quaternion*.

*Remark.* Since scalars and vectors are quaternions, scalar and vector algebra applies.

Let $\mathcal{A} = a_0 + \boldsymbol{a}$, $\mathcal{B} = b_0 + \boldsymbol{b}$ and $\mathcal{C} = c_0 + \boldsymbol{c}$ be three quaternions.

2.2.1 <u>Multiplication</u>

According with Chou, J.C.K. (1992), quaternion multiplication is defined as:

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B} = a_0 b_0 + a_0\boldsymbol{b} + b_0\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{b}\,, \tag{5}$$

where the symbols $\cdot$ and $\times$ stand for dot product and cross product. An alternative expression of the norm in (4) can be obtained through quaternion multiplication, namely:

$$|\mathcal{A}|^2 = \mathcal{A} \otimes \mathcal{A}^*, \tag{6}$$

where $\mathcal{A}^* = a_0 - \boldsymbol{a}$ denotes quaternion conjugate. The product in (5) can be also expressed in matrix form as shown below:

$$\begin{bmatrix} c_0 \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0 I + C(\boldsymbol{a}) \end{bmatrix} \begin{bmatrix} b_0 \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} b_0 & -\boldsymbol{b}^T \\ \boldsymbol{b} & b_0 I - C(\boldsymbol{b}) \end{bmatrix} \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix}, \tag{7}$$

where matrix expressions in (8) for dot and cross product have been employed.

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}^T\boldsymbol{b}, \quad \boldsymbol{a}\times\boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \boldsymbol{b} = C(\boldsymbol{a})\boldsymbol{b}\,. \tag{8}$$

2.2.2 <u>Commutative property</u>

Although commutative law does not hold in general, the matrix expression (7) shows $\mathcal{A}$ and $\mathcal{B}$ to commute through sign change. Therefore, the following matrix representations of quaternions can be introduced:

$$\mathcal{A}^+ = \begin{bmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0 I + C(\boldsymbol{a}) \end{bmatrix}, \mathcal{B}^- = \begin{bmatrix} b_0 & -\boldsymbol{b}^T \\ \boldsymbol{b} & b_0 I - C(\boldsymbol{b}) \end{bmatrix}, \tag{9}$$

where superscripts + and – denote the sign of the cross product matrix $C(\cdot)$ and $I$ denotes the identity matrix. Using notations defined in (9), the commutative property, which is hidden in (7), can be expressed in the following compact form:

$$\mathcal{C} = \mathcal{A}^+\mathcal{B} = \mathcal{B}^-\mathcal{A}\,, \tag{10}$$

where one must pay attention that $\mathcal{A}$ and $\mathcal{B}$ are meant to be in column vector form.

2.2.3 <u>Inverse</u>

Each nonzero quaternion $\mathcal{A}$ admits an inverse $\mathcal{A}^{-1}$ such that $\mathcal{A} \otimes \mathcal{A}^{-1} = 1$. The *inverse quaternion* $\mathcal{A}^{-1}$ is related to $\mathcal{A}$ by:

$$\mathcal{A}^{-1} = \mathcal{A}^* \big/ |\mathcal{A}|^2\,. \tag{11}$$

*2.3 Magnification and finite rotations*

As it will be shown below, full quaternions allow to describe at the same time vector rotation as unit quaternions and vector magnification. To this end, consider a full quaternion $\mathcal{R}$, a quaternion $\mathcal{B}$ and the following transformation of $\mathcal{B}$ into $\mathcal{B}'$:

$$\mathcal{B}' = \mathcal{R} \otimes \mathcal{B} \otimes \mathcal{R}^*. \tag{12}$$

Any full quaternion $\mathcal{R}$ can be factorized into the product of the norm and of the unit quaternion:

$$\mathcal{R} = |\mathcal{R}| \otimes \underline{\mathcal{R}}\,. \tag{13}$$

Then, using factorization (13), one can rewrite (12) by separating it into two terms, as shown below

$$\mathcal{B}' = |\mathcal{R}|^2 \otimes \left( \underline{\mathcal{R}} \otimes \mathcal{B} \otimes \underline{\mathcal{R}}^* \right). \tag{14}$$

Every unit quaternion admits the *Euler parameters* representation. Therefore, it is possible to express $\underline{\mathcal{R}}$ in terms of a rotation angle $\theta$ around the instantaneous axis of rotation ***u***:

$$\underline{\mathcal{R}} = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 \end{bmatrix}^T = \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2)\boldsymbol{u}^T \end{bmatrix}^T \tag{15}$$

By applying (10) and (A.2), it follows:

$$\mathcal{B}' = \mathcal{R}^+ \left(\mathcal{R}^-\right)^T \mathcal{B} = |\mathcal{R}|^2 \left[ \underline{\mathcal{R}}^+ \left(\underline{\mathcal{R}}^-\right)^T \right] \mathcal{B}\,. \tag{16}$$

Further, employing matrices $E^+$ and $E^-$ defined in (A.3) yields:

$$\begin{aligned}\mathcal{B}' &= |\mathcal{R}|^2 \begin{bmatrix} 1 & 0 \\ 0 & E^-\left(\underline{\mathcal{R}}^*\right)\left(E^+\left(\underline{\mathcal{R}}^*\right)\right)^T \end{bmatrix} \mathcal{B} = |\mathcal{R}|^2 \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \mathcal{B} = \\ &= |\mathcal{R}|^2\, \boldsymbol{R}\mathcal{B}\end{aligned} \tag{17}$$

The matrix ***R*** is a 4×4 quaternion transformation in a four-dimension space. As remarked by Chou, J.C.K., (1992), ***R*** is a linear operator with the property of leaving invariant quaternion norms. From expression (17) it is possible to separate a 3×3 rotation matrix:

$$R = E^-\left(\underline{\mathcal{R}}^*\right)\left(E^+\left(\underline{\mathcal{R}}^*\right)\right)^T = \left(r_0^2 - \boldsymbol{r}^T\boldsymbol{r}\right)I + 2\left(\boldsymbol{r}\boldsymbol{r}^T + r_0 C(\boldsymbol{r})\right) \tag{18}$$

Note that the above definition of $R$ is consistent[1] with the definition of direction cosine matrix given by Wertz, J.R. (1978).

It is clear from previous equations, that the product (12) applies two different transformations:

[1] Actually, the 3×3 matrix $R$ in (18) is the transpose of the direction cosine matrix in Wertz, J.R., (1978), because the opposite rotation direction has been used.

1. a magnification, by $|\mathcal{R}|^2$, of the $\mathcal{B}$ norm,
2. and a rotation of $\mathcal{B}$ by an angle $\theta$ around the axis $\boldsymbol{u}$ (as stated by Euler Theorem).

In the case $\mathcal{B}$ is a vector quaternion, the factorization (17) reduces to:

$$\begin{bmatrix} 0 \\ \boldsymbol{b}' \end{bmatrix} = |\mathcal{R}|^2 \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{b} \end{bmatrix} \Rightarrow \boldsymbol{b}' = \rho R \boldsymbol{b},\ \rho = |\mathcal{R}|^2 . \quad (19)$$

The use of full quaternions allows to generalize the description of the motion of an object in the three-dimensional space: not only rotations but also translations can be parameterized.

## 3. QUATERNION KINEMATICS AND DYNAMICS

As stated in Section 1, the goal of this paper is to rewrite orbital dynamic and kinematic equations using full quaternions. To this end, first and second derivatives of a quaternion will be determined.

Let $\boldsymbol{r}_i$ and be nonzero vectors which, according to Section 2.1, can be considered as vector quaternions. Then, as in (12), it is possible to define a full quaternion $\mathcal{P}$ relating the vector $\boldsymbol{r}_o$ to the reference vector $\boldsymbol{r}_i$ through a rotation and a magnification:

$$\boldsymbol{r}_o = \mathcal{P} \otimes \boldsymbol{r}_i \otimes \mathcal{P}^* \Leftrightarrow \boldsymbol{r}_i = (\mathcal{P})^{-1} \otimes \boldsymbol{r}_o \otimes (\mathcal{P}^*)^{-1} . \quad (20)$$

### 3.1 Kinematics

Differentiating (20) yields:

$$\begin{aligned} \dot{\boldsymbol{r}}_o &= \dot{\mathcal{P}} \otimes (\mathcal{P})^{-1} \otimes \boldsymbol{r}_o + \boldsymbol{r}_o \otimes (\mathcal{P}^*)^{-1} \otimes \dot{\mathcal{P}}^* + \boldsymbol{q}_i , \\ \boldsymbol{q}_i &= \mathcal{P} \otimes \dot{\boldsymbol{r}}_i \otimes \mathcal{P}^* \end{aligned} \quad (21)$$

Then, by deriving the product $\mathcal{P} \otimes (\mathcal{P})^{-1} = 1$, one can define the quaternion $\mathcal{W}$ as shown below:

$$\dot{\mathcal{P}} \otimes (\mathcal{P})^{-1} = -\mathcal{P} \otimes (\dot{\mathcal{P}}^{-1}) = \mathcal{W} . \quad (22)$$

From the above definition the quaternion kinematic equation follows:

$$\dot{\mathcal{P}} = \mathcal{W} \otimes \mathcal{P} . \quad (23)$$

After factorization of $\mathcal{P}$ as in (13), it is possible to rewrite $\mathcal{W}$ as (Chou, J.C.K, 1992):

$$\mathcal{W} = |\dot{\mathcal{P}}| \big/ |\mathcal{P}| + \dot{\underline{\mathcal{P}}} \otimes \underline{\mathcal{P}}^* = w_0 + \boldsymbol{w} = w_0 + \boldsymbol{w}_\perp + \boldsymbol{w}_\parallel , \quad (24)$$

where the decomposition of $\boldsymbol{w}$ into normal and parallel components $\boldsymbol{w}_\perp$ and $\boldsymbol{w}_\parallel$ w.r.t. $\boldsymbol{r}_o$ has been exploited. Substituting (24) into (21) enlightens that the derivative of $\boldsymbol{r}_o$ is unaffected by the parallel component $\boldsymbol{w}_\parallel$:

$$\begin{aligned} \dot{\boldsymbol{r}}_o &= \mathcal{W} \otimes \boldsymbol{r}_o + \boldsymbol{r}_o \otimes \mathcal{W}^* + \boldsymbol{q}_i = 2 w_0 \boldsymbol{r}_o + 2\boldsymbol{w} \times \boldsymbol{r}_o + \boldsymbol{q}_i = \\ &= 2\mathcal{W}_\perp \otimes \boldsymbol{r}_o + \boldsymbol{q}_i ,\ \ \mathcal{W}_\perp = w_0 + \boldsymbol{w}_\perp \end{aligned} . \quad (25)$$

Since $\boldsymbol{r}_o$ is a vector quaternion, equation (19) applies:

$$\begin{bmatrix} 0 \\ \boldsymbol{r}_o \end{bmatrix} = |\mathcal{P}|^2 \begin{bmatrix} 1 & 0 \\ 0 & R_P \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{r}_i \end{bmatrix} \Rightarrow \boldsymbol{r}_o = \rho_P R_P \boldsymbol{r}_i ,\ \rho_P = |\mathcal{P}|^2 , \quad (26)$$

where $R_P$ is a rotation matrix. Comparing (25) with the first derivative of (26):

$$\dot{\boldsymbol{r}}_o = \left\{ I \dot{\rho}_P / \rho_P + \left( \dot{R}_P R_P^T \right) \right\} \boldsymbol{r}_o + \rho_P R_P^T \dot{\boldsymbol{r}}_i = \Omega \boldsymbol{r}_o + \rho_P R_P^T \dot{\boldsymbol{r}}_i , \quad (27)$$

yields the following equalities:

$$\begin{aligned} 2 w_0 &= \dot{\rho}_P / \rho_P \\ 2\boldsymbol{w} \times &= C(2\boldsymbol{w}) = \left( \dot{R}_P R_P^T \right) \Rightarrow \dot{R}_P = C(2\boldsymbol{w}) R_P \end{aligned} . \quad (28)$$

One can recognize that $2\boldsymbol{w}$ represents the angular velocity in the three dimensional space (Wertz J.R., 1978) and $2w_o$ represents the translation velocity along the $\boldsymbol{r}_o$ direction, normalized by $\rho_P$. The ensemble $(w_o+\boldsymbol{w})=\mathcal{W}$ forms a full quaternion referred to as *generalized angular velocity*. This term has been chosen because it is the composition of a normalized linear velocity and an angular velocity, both expressed in $s^{-1}$ units. Rewriting (23) and (25) in matrix notations yields:

$$\begin{aligned} &\dot{\mathcal{P}} = \mathcal{W}^+ \mathcal{P} \qquad \dot{\boldsymbol{r}}_o = 2\mathcal{W}_\perp^+ \boldsymbol{r}_o + \mathcal{P}^+ \left( \mathcal{P}^- \right)^T \dot{\boldsymbol{r}}_i \\ &\mathcal{W}^+ = \begin{bmatrix} w_0 & -\boldsymbol{w} \\ \boldsymbol{w} & \Omega/2 \end{bmatrix}\ \mathcal{W}_\perp^+ = \begin{bmatrix} w_0 & -\boldsymbol{w}_\perp^T \\ \boldsymbol{w}_\perp & \Omega_\perp/2 \end{bmatrix} \end{aligned} , \quad (29)$$

where $\Omega_\perp$ equals $\Omega$ under the constraint $\boldsymbol{w} = \boldsymbol{w}_\perp$.

*Remark*. As shown in (13), $\mathcal{P}$ is composed by its norm and unit quaternion. The latter can represent not only a simple vector rotation in three dimensional space, but a more general rotation of an entire reference frame, in agreement with the Euler Theorem. Therefore, the quaternion kinematics in (23) has four degrees of freedom (d.o.f.), i.e.:

1. one d.o.f. related to the translation velocity $w_0$ (the derivative of $\rho_P$ defined in (26));
2. three d.o.f. related to the angular rate $\boldsymbol{w}$ of a reference frame whose one unit vector is parallel to $\boldsymbol{r}_o$ (the derivative of the rotation matrix $R_P$ defined in (26)).

Since the vector kinematics in (25) describes only the motion of $\boldsymbol{r}_o$, the information about the rotation of the other two axes of the frame defined above becomes unnecessary. This is confirmed by the vanishing of the parallel component $\boldsymbol{w}_\parallel$. Then, in agreement with classical mechanics, d.o.f. of vector kinematics reduce to three because of such an orthogonality constraint.

Equations (25) and (26) can be viewed as output equations of the state equation (23).

### 3.2 Dynamics

First define the *generalized angular acceleration* $\mathcal{A}$ as the derivative of the generalized angular rate $\mathcal{W}$:

$$\mathcal{A} = \dot{\mathcal{W}} \Rightarrow a_0 + \boldsymbol{a} = a_0 + \boldsymbol{a}_\perp + \boldsymbol{a}_\parallel = \dot{w}_0 + \dot{\boldsymbol{w}} . \quad (30)$$

In (30), the decomposition of $\boldsymbol{a}$ into normal and parallel components w.r.t. $\boldsymbol{r}_o$ has been exploited. Be aware that $\dot{\boldsymbol{w}}_\perp \neq \boldsymbol{a}_\perp$ and $\dot{\boldsymbol{w}}_\parallel \neq \boldsymbol{a}_\parallel$.

Quaternion dynamics follows by taking the derivative of quaternion kinematics (23):

$$\ddot{\mathcal{P}} = \dot{\mathcal{W}} \otimes \mathcal{P} + \mathcal{W} \otimes \dot{\mathcal{P}} = [\mathcal{A} + \mathcal{W} \otimes \mathcal{W}] \otimes \mathcal{P} = \mathcal{D} \otimes \mathcal{P} , \quad (31)$$

where the quaternion $\mathcal{D}$ gathers the effect of angular rate and acceleration. Scalar and vector parts of $\mathcal{D}$ are related to the components of $\mathcal{W}$ and $\mathcal{A}$ through:

$$\mathcal{D} = d_0 + \boldsymbol{d} = \left( a_0 + w_0^2 - |\boldsymbol{w}|^2 \right) + \left( \boldsymbol{a} + 2 w_0 \boldsymbol{w} \right) . \quad (32)$$

The second derivative of $\boldsymbol{r}_o$ can be obtained by exploiting (25) and (31):

$$\begin{aligned} &\ddot{\boldsymbol{r}}_o = \mathcal{D} \otimes \boldsymbol{r}_o + \boldsymbol{r}_o \otimes \mathcal{D}^* + 2\mathcal{W} \otimes \boldsymbol{r}_o \otimes \mathcal{W}^* + \boldsymbol{q} = \\ &= (2a_0) \boldsymbol{r}_o + (2w_0)^2 \boldsymbol{r}_o + 2\left[ (2w_0)(2\boldsymbol{w}) \times \boldsymbol{r}_o \right] + (2\boldsymbol{a}) \times \boldsymbol{r}_o + \\ &+ (2\boldsymbol{w}) \times \left[ (2\boldsymbol{w}) \times \boldsymbol{r}_o \right] + \boldsymbol{q}, \quad \boldsymbol{q} = (2w_0) \boldsymbol{q}_i + (2\boldsymbol{w}) \times \boldsymbol{q}_i + \dot{\boldsymbol{q}}_i \end{aligned} . \quad (33)$$

This expression has a clear similarity with the ordinary equation of the relative motion (see Kaplan, M.H., 1976 or Greenwood, D.T. *et al.*, 1965). Therefore, a physical meaning can be assigned to each term in (33):

1. $\boldsymbol{q}$ represents the acceleration of the vector $\boldsymbol{r}_i$;
2. $(2a_0)\boldsymbol{r}_o + (2\boldsymbol{a})\times\boldsymbol{r}_o$ is the apparent acceleration of $\boldsymbol{r}_o$ w.r.t. $\boldsymbol{r}_i$. In particular:
   2.1. $(2a_0)\boldsymbol{r}_o$ is the apparent acceleration along $\boldsymbol{r}_o$;
   2.2. $(2\boldsymbol{a})\times\boldsymbol{r}_o$ is the apparent acceleration along a normal direction to $\boldsymbol{r}_o$;
3. $2\left[(2\boldsymbol{w})\times(2w_0)\boldsymbol{r}_o\right]$ is the Coriolis acceleration;
4. $(2\boldsymbol{w})\times\left[(2\boldsymbol{w})\times\boldsymbol{r}_o\right]+(2w_0)^2\boldsymbol{r}_o$ is the centrifugal term.

Further developing (33) shows that the acceleration of $\boldsymbol{r}_o$ does not depend on the parallel component $\boldsymbol{a}_{\parallel}$:

$$\begin{array}{l}\ddot{\boldsymbol{r}}_o = 2\mathcal{D}_\perp \otimes \boldsymbol{r}_o + 2\mathcal{W}\otimes\boldsymbol{r}_o\otimes\mathcal{W}^* + \boldsymbol{q} \\ \mathcal{D}_\perp = \mathcal{A}_\perp + \mathcal{W}\otimes\mathcal{W},\ \ \mathcal{A}_\perp = a_0 + \boldsymbol{a}_\perp\end{array}. \qquad (34)$$

*Remark*. As observed for kinematics, quaternion dynamics (31) is more general than vector dynamics (34). Since the angular acceleration $\boldsymbol{a}$ is related to the rotation of a frame aligned with $\boldsymbol{r}_o$, it is unconstrained and then the former equation has four d.o.f.. In (34), the parallel component $\boldsymbol{a}_{\parallel}$ vanishes, showing an orthogonality constraint on $\boldsymbol{a}$, which corresponds to a d.o.f. reduction. Then, (34) has only three d.o.f., in agreement with classical mechanics.

## 4. APPLICATIONS

Once obtained the general kinematic and dynamic equations of full quaternions, the last step to be done is applying them to orbital motion. To this end, consider a point *P* with mass *m* moving in the space, subject to a force $\boldsymbol{F}$. Six different local reference frames can be attached to the point mass. They can be defined by taking all possible combinations of two elements from the set composed by position, velocity and acceleration vectors. Two of the six available frames are well known in Astronautics and will be considered for the following application examples:

1. the Local Orbital Reference Frame (LORF);
2. the Local Vertical – Local Horizontal frame (LVLH).

Both frames and their orientation w.r.t. an inertial reference are shown in Figure 1. The inertial frame $\mathrm{R}=\{O,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is a Cartesian reference with origin in *O* and unit vectors corresponding to $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ already introduced in (1). For each of the two orbital frames, the following problems will be solved:

1. complete definition of the frame axes;
2. assignment of a full quaternion to the frame;
3. formulation of the differential equation of the full quaternion.

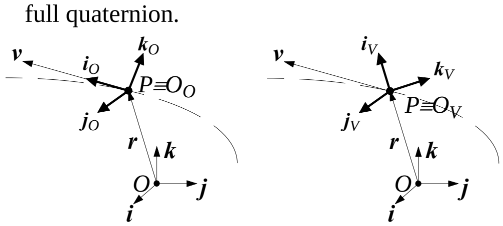


Figure 1 - LORF and LVLH w.r.t. the inertial frame

### 4.1 LORF Reference Frame

The LORF $\mathrm{R}_O=\{O_O,\boldsymbol{i}_O,\boldsymbol{j}_O,\boldsymbol{k}_O\}$ is a Cartesian reference frame defined by velocity $\boldsymbol{v}$ and position $\boldsymbol{r}$ as follows:

1. the origin $O_O$ coincides with *P*;
2. $\boldsymbol{i}_O$ lies along the velocity direction;
3. $\boldsymbol{j}_O$ is normal to the instantaneous orbit plane (defined by position and velocity);
4. $\boldsymbol{k}_O$ completes the frame.

$$\boldsymbol{i}_O = \boldsymbol{v}/|\boldsymbol{v}|,\ \boldsymbol{j}_O = (\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}\times\boldsymbol{v}|,\ \boldsymbol{k}_O = \boldsymbol{i}_O\times\boldsymbol{j}_O\,. \quad (35)$$

The velocity vector and the orientation of the LORF triple can be expressed through the *LORF quaternion* $\mathcal{R}_O$. The definition of $\mathcal{R}_O$ is arbitrary: for example the axis $\boldsymbol{i}$ rotates into $\boldsymbol{i}_O$ and the axis $\boldsymbol{j}$ rotates into $\boldsymbol{j}_O$:

$$\boldsymbol{v} \triangleq \mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*,\quad \boldsymbol{j}_O \triangleq \underline{\mathcal{R}}_O\otimes\boldsymbol{j}\otimes\underline{\mathcal{R}}_O^*\,. \quad (36)$$

Because there exist an infinite number of rotations satisfying the left equation in (36), a further constraint must be introduced: the right equation specifies that the $\boldsymbol{j}$-axis of the inertial frame must be rotated into the orbital plane normal direction.

Factorizing the left equation in (36) as in (14) enlightens the norm $|\mathcal{R}_O|$ of the LORF quaternion to be equal to the square root of the velocity modulus, and the unitary part $\underline{\mathcal{R}}_O$ to represent the orientation of the velocity unit vector w.r.t. the inertial frame:

$$\begin{array}{l}\boldsymbol{v}=v\boldsymbol{n}_v=|\mathcal{R}_O|^2\left(\underline{\mathcal{R}}_O\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_O^*\right),\quad \|\boldsymbol{n}_v\|=1 \ \Rightarrow \\ \Rightarrow|\mathcal{R}_O|=\sqrt{v},\quad \underline{\mathcal{R}}_O\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_O^*=\boldsymbol{n}_v\end{array}. \quad (37)$$

Now, one can apply formula (25) of quaternion kinematics to compute the acceleration $\dot{\boldsymbol{v}}$:

$$\dot{\boldsymbol{v}}=2w_{O,0}\boldsymbol{v}+2\boldsymbol{w}_O\times\boldsymbol{v}=2\mathcal{W}_{O\perp}\otimes\boldsymbol{v},\ \ \mathcal{W}_{O\perp}=w_{O,0}+\boldsymbol{w}_{O\perp}\,, \quad (38)$$

where the derivative of $\boldsymbol{i}$, being zero by definition, disappears and $\boldsymbol{w}_O$ has been decomposed into the normal and parallel components $\boldsymbol{w}_{O\perp}$ and $\boldsymbol{w}_{O\parallel}$ w.r.t. $\boldsymbol{v}$. The acceleration of the point mass can be related to the force $\boldsymbol{F}$ through Newton's Law and remembering that $\boldsymbol{w}_{O\perp}\cdot\boldsymbol{v}=0$:

$$\boldsymbol{F}/m=\dot{\boldsymbol{v}}\Rightarrow w_{O,0}=\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{2m|\boldsymbol{v}|^2},\ \ \boldsymbol{w}_{O\perp}=\frac{\boldsymbol{v}\times\boldsymbol{F}}{2m|\boldsymbol{v}|^2}\,. \quad (39)$$

Since $\boldsymbol{v}$ and $\boldsymbol{F}$ are vector quaternions, a more compact expression for the LORF angular rate can be used:

$$\mathcal{W}_O=\mathcal{W}_{O\perp}+\boldsymbol{w}_{O\parallel}=-(1/2m)\left(\boldsymbol{v}\otimes\boldsymbol{v}^*\right)^{-1}\boldsymbol{F}\otimes\boldsymbol{v}+\boldsymbol{w}_{O\parallel}\,. \quad (40)$$

Therefore, the orbital equations for LORF quaternion can be written in quaternion form:

$$\begin{cases}\dot{\mathcal{R}}_O=\mathcal{W}_O\otimes\mathcal{R}_O, & \mathcal{R}_O(0)=\mathcal{R}_{O0} \\ \dot{\boldsymbol{r}}=\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*, & \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}, \quad (41)$$

or matrix form, exploiting (10) and (A.2), as follows:

$$\begin{cases}\dot{\mathcal{R}}_O=\mathcal{R}_O^-\mathcal{W}_O=\mathcal{R}_O^-\left[-\dfrac{\boldsymbol{v}^-\boldsymbol{F}}{2m|\boldsymbol{v}|^2}+\boldsymbol{w}_{O\parallel}\right], & \mathcal{R}_O(0)=\mathcal{R}_{O0} \\ \dot{\boldsymbol{r}}=\mathcal{R}_O^+\left(\mathcal{R}_O^-\right)^T\boldsymbol{i}, & \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}. \quad (42)$$

*Remark*. As stated in Section 3.1, the parallel component $\boldsymbol{w}_{O\parallel}$ does not give contribution in (38). This confirms the existence of an orthogonality constraint to $\boldsymbol{w}_O$, meaning that the vector kinematics (38) has three d.o.f. The application of Newton's

Law shows $\boldsymbol{w}_{O\parallel}$ to be completely independent on $\boldsymbol{F}$. The angular rate $\boldsymbol{w}_{O\parallel}$ affects only (41), and represents the angular rate of the unit vectors $\boldsymbol{j}_O$ and $\boldsymbol{k}_O$ around the axis $\boldsymbol{i}_O$. But if such vectors underwent a rotation, the LORF frame would be lost. Therefore the kinematic quaternion constraint $\boldsymbol{w}_{O\parallel}=0$ follows. Consequently, the orbital equation (41) of the LORF quaternion is subject to a kinematic constraint ($\boldsymbol{w}_{O\parallel}=0$) and a static constraint ($\boldsymbol{v}=\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*$) that forces $\boldsymbol{v}$ to be a vector quaternion. Then the number of d.o.f. reduces from the eight possible to only six, in agreement with classical mechanics.

*4.2 LVLH reference frame*

The LVLH frame $\mathrm{R}_V=\{O_V,\boldsymbol{i}_V,\boldsymbol{j}_V,\boldsymbol{k}_V\}$ is a Cartesian reference defined by position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ as follows:

1. the origin $O_V$ coincides with $P$;
2. $\boldsymbol{i}_V$ lies along the position direction;
3. $\boldsymbol{j}_V$ is normal to the instantaneous orbit plane;
4. $\boldsymbol{k}_V$ completes the frame.

$$\boldsymbol{i}_V=\boldsymbol{r}/|\boldsymbol{r}|,\quad \boldsymbol{j}_V=(\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}\times\boldsymbol{v}|,\quad \boldsymbol{k}_V=\boldsymbol{i}_V\times\boldsymbol{j}_V\,. \quad (43)$$

The position vector and the orientation of the LVLH triple can be expressed through the *LVLH quaternion* $\mathcal{R}_V$. In accordance with (36) it can be defined as:

$$\boldsymbol{r}\triangleq\mathcal{R}_V\otimes\boldsymbol{i}\otimes\mathcal{R}_V^*,\quad \boldsymbol{j}_V\triangleq\underline{\mathcal{R}}_V\otimes\boldsymbol{j}\otimes\underline{\mathcal{R}}_V^*\,. \quad (44)$$

Factorizing the left equation in (44) enlightens the norm $|\mathcal{R}_V|$ of $\mathcal{R}_V$ to be equal to the square root of the position modulus, and the unitary part $\underline{\mathcal{R}}_V$ to be the orientation of $\boldsymbol{r}$ w.r.t. the inertial frame:

$$\begin{aligned}&\boldsymbol{r}=r\boldsymbol{n}_r=|\mathcal{R}_V|^2\left(\underline{\mathcal{R}}_V\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_V^*\right),\quad \|\boldsymbol{n}_r\|=1\quad\Rightarrow\\&\Rightarrow|\mathcal{R}_V|=\sqrt{r},\quad \underline{\mathcal{R}}_V\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_V^*=\boldsymbol{n}_r\end{aligned}\,. \quad (45)$$

Now, one can apply the formula (25) of quaternion kinematics to compute the velocity $\boldsymbol{v}$:

$$\dot{\boldsymbol{r}}=\boldsymbol{v}=2w_{V,0}\boldsymbol{r}+2\boldsymbol{w}_V\times\boldsymbol{r}=2\mathcal{W}_{V\perp}\otimes\boldsymbol{r}\,, \quad (46)$$

where the decomposition of $\boldsymbol{w}_V$ into normal and parallel components $\boldsymbol{w}_{V\perp}$ and $\boldsymbol{w}_{V\parallel}$ w.r.t. $\boldsymbol{r}$ has been exploited. The quaternion kinematics of the LVLH follows by (23):

$$\dot{\mathcal{R}}_V=\mathcal{W}_V\otimes\mathcal{R}_V\,, \quad (47)$$

and the LVLH dynamics follows from (31):

$$\ddot{\mathcal{R}}_V=\left[\mathcal{A}_V+\mathcal{W}_V\otimes\mathcal{W}_V\right]\otimes\mathcal{R}_V=\mathcal{D}_V\otimes\mathcal{R}_V\,. \quad (48)$$

Then, recalling (33) and (34), the acceleration can be determined as:

$$\begin{aligned}\ddot{\boldsymbol{r}}=\dot{\boldsymbol{v}}&=(2a_{V,0})\boldsymbol{r}+(2w_{V,0})^2\boldsymbol{r}+2\left[(2w_{V,0})(2\boldsymbol{w}_V)\times\boldsymbol{r}\right]+\\&+(2\boldsymbol{a}_V)\times\boldsymbol{r}+(2\boldsymbol{w}_V)\times\left[(2\boldsymbol{w}_V)\times\boldsymbol{r}\right]=\\&=2\mathcal{D}_{V\perp}\otimes\boldsymbol{r}+2\mathcal{W}_V\otimes\boldsymbol{r}\otimes\mathcal{W}_V^*,\ \mathcal{D}_{V\perp}=\mathcal{A}_{V\perp}+\mathcal{W}_V\otimes\mathcal{W}_V\end{aligned}\,. \quad (49)$$

As done for LORF kinematics, one can relate acceleration expression to force $\boldsymbol{F}$ through Newton's Law. Taking the dot product between position and force yields:

$$a_{V,0}=\left\{(\boldsymbol{r}\cdot\boldsymbol{F})/m|\boldsymbol{r}|^2-\left[(2w_{V,0})^2-|2\boldsymbol{w}_{V\perp}|^2\right]\right\}/2\,. \quad (50)$$

Cross product between position and force brings to:

$$\boldsymbol{a}_{V\perp}=\frac{1}{2}\left\{\frac{\boldsymbol{r}\times\boldsymbol{F}}{m|\boldsymbol{r}|^2}-2(2w_{V,0})(2\boldsymbol{w}_{V\perp})-2\boldsymbol{w}_{V\parallel}\times2\boldsymbol{w}_{V\perp}\right\}. \quad (51)$$

Expressions (50) and (51) can be compacted into:

$$\begin{aligned}&\mathcal{A}_V=\mathcal{A}_{V\perp}+\boldsymbol{a}_{V\parallel}=a_{V,0}+\boldsymbol{a}_{V\perp}+\boldsymbol{a}_{V\parallel}=-\frac{1}{2|\boldsymbol{r}|^2}\mathcal{Y}(\boldsymbol{F})+\boldsymbol{a}_{V\parallel}\\&\mathcal{Y}(\boldsymbol{F})=\frac{\boldsymbol{F}}{m}\otimes\boldsymbol{r}+2|\boldsymbol{r}|^2\mathcal{W}_V\otimes\mathcal{W}_V+2(\mathcal{W}_V\otimes\boldsymbol{r})\otimes(\boldsymbol{r}\otimes\mathcal{W}_V)^*\end{aligned} \quad (52)$$

Finally, the orbital equations for LVLH quaternion can be written in quaternion form:

$$\begin{cases}\dot{\mathcal{R}}_V=\mathcal{W}_V\otimes\mathcal{R}_V\,, & \mathcal{R}_V(0)=\mathcal{R}_{V0}\\ \dot{\mathcal{W}}_V=-\left(1/2|\boldsymbol{r}|^2\right)\mathcal{Y}(\boldsymbol{F})+\boldsymbol{a}_{V\parallel}\,, & \mathcal{W}_V(0)=\mathcal{W}_{V0}\end{cases}\,. \quad (53)$$

*Remark*. As stated in Section 3.2, the parallel component $\boldsymbol{a}_{V\parallel}$ does not give contribution in (49). This confirms the existence of an orthogonality constraint to $\boldsymbol{a}_V$, meaning that the vector dynamics (49) has three d.o.f. The application of the Newton's Law, shows that $\boldsymbol{a}_{V\parallel}$ is unforced by $\boldsymbol{F}$. The angular acceleration $\boldsymbol{a}_{V\parallel}$ affects only (53), and represents an angular acceleration of the unit vectors $\boldsymbol{j}_V$ and $\boldsymbol{k}_V$ around the axis $\boldsymbol{i}_V$. But if such vectors underwent a rotation, the LVLH frame would be lost. Therefore the dynamic and kinematic quaternion constraints $\boldsymbol{a}_{V\parallel}=0$ and $\boldsymbol{w}_{V\parallel}=0$ follow.

Finally, the orbital equation (53) of the LVLH quaternion is subject to a kinematic constraint ($\boldsymbol{w}_{V\parallel}=0$) and a dynamic constraint ($\boldsymbol{a}_{V\parallel}=0$). Therefore the number of d.o.f. reduces from the eight possible to only six, in agreement with classical mechanics.

*4.3 Uniform Circular Motion*

This section ends with a simple example: the uniform circular motion of $P$ around $O$, sketched in Figure 2.

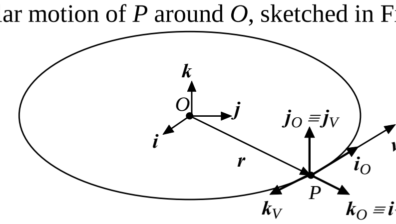


Figure 2 - Uniform circular motion around *O*

LVLH and LORF quaternion definitions are the same as in (44) and (36). First, quaternion kinematics is applied, starting from LVLH case. The generalized angular velocity of the LVLH quaternion is:

$$\mathcal{W}_V=\boldsymbol{w}_{V\perp},\ \boldsymbol{w}_{V\parallel}=0\,. \quad (54)$$

The generalized angular velocity is coincident with the angular rate of $P$ around $O$, denoted with $\boldsymbol{\omega}=(\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}|^2$. This leads to the next result showing quaternion kinematic equation to be similar to classical vector form:

$$\begin{cases}\dot{\boldsymbol{r}}=\boldsymbol{\omega}\times\boldsymbol{r}\\ \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}\quad\begin{cases}\dot{\mathcal{R}}_V=\boldsymbol{w}_V\otimes\mathcal{R}_V\\ \mathcal{R}_V(0)=\mathcal{R}_{V0}\end{cases}, \quad (55)$$

where $\boldsymbol{w}_V=\boldsymbol{\omega}/2$. By applying (31), quaternion dynamic equation can be obtained. Quaternion dynamics, as seen for kinematics, appears similar to classical vector form:

$$\begin{cases}\dot{\boldsymbol{v}}=-|\boldsymbol{\omega}|^2\boldsymbol{r}\\ \boldsymbol{v}(0)=\boldsymbol{v}_0\end{cases}\quad\begin{cases}\ddot{\mathcal{R}}_V=\left[\boldsymbol{a}_V+\boldsymbol{w}_V\otimes\boldsymbol{w}_V\right]\otimes\mathcal{R}_V=-|\boldsymbol{w}_V|^2\mathcal{R}_V\\ \mathcal{R}_V(0)=\mathcal{R}_{V0}\end{cases}, \quad (56)$$

where $\boldsymbol{a}_V=0$ by definition of uniform motion.

By using LORF instead of LVLH, dynamics is represented by quaternion kinematics, because the quaternion describes point velocity, instead of position. LORF dynamics is:

$$\begin{cases}\dot{\boldsymbol{v}}=\boldsymbol{\omega}\times\boldsymbol{v}\\ \boldsymbol{v}(0)=\boldsymbol{v}_0\end{cases}\quad\begin{cases}\dot{\mathcal{R}}_O=\boldsymbol{w}_O\otimes\mathcal{R}_O\\ \mathcal{R}_O(0)=\mathcal{R}_{O0}\end{cases}. \tag{57}$$

Kinematics follows from definition (36):

$$\begin{cases}\dot{\boldsymbol{r}}=\boldsymbol{v}\\ \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}\quad\begin{cases}\dot{\boldsymbol{r}}=\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*\\ \boldsymbol{r}(0)=\mathcal{R}_O(0)\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*(0)\end{cases}. \tag{58}$$

The last four equations show that the angular velocities of the LORF and LVLH quaternions are the same, namely $\boldsymbol{\omega}/2=\boldsymbol{w}_O=\boldsymbol{w}_V$. This follows from the fact that, for uniform circular motion, position and velocity are always orthogonal, then rotating with the same angular rate.

## 5. SIMULATION RESULTS

The LORF and LVLH orbital equations in (41) and (53) have been implemented and tested in a MATLAB – Simulink orbit simulator, with a gravity model accounting for central field and $J_2$ contributions. The GOCE orbit has been taken as a reference (see Canuto, E. *et al.*, 2002). The latter is a near-circular ($\varepsilon$ < 0.005) sun-synchronous, quasi-polar ($i \cong 96°$) orbit at a mean altitude $h$ = 250Km. Figure 3 shows the time series of the four components of the LORF quaternion during a time horizon of four orbits. The quaternion has a clear periodicity of two orbits that is equivalent to half the orbital frequency $f_O \cong 0.19$mHz. Since the nonlinear definition (36) of the LORF quaternion applies, the velocity $\boldsymbol{v}$ is periodic with orbital frequency $f_O$. The same considerations can also be applicable on the LVLH quaternion, that is illustrated in Figure 4. Figure 5 shows the GOCE orbit around the Earth w.r.t. inertial coordinates. One can note from the picture that the orbit is near-circular and quasi-polar.

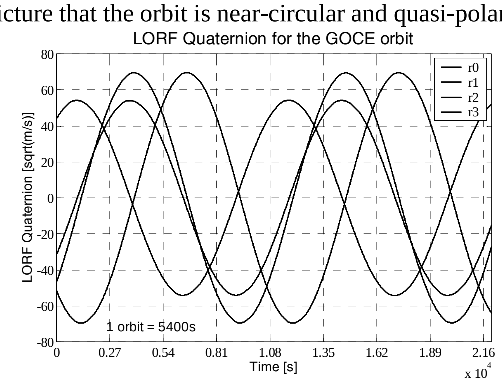


Figure 3 - Time series of the LORF quaternion components

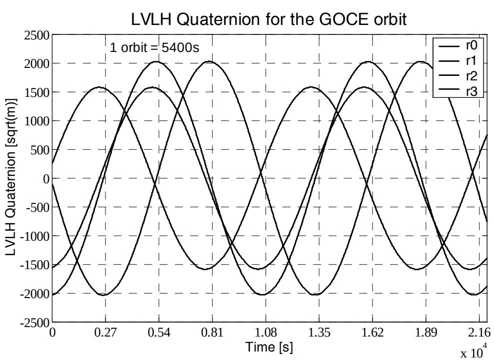


Figure 4 - Time series of the LVLH quaternion components

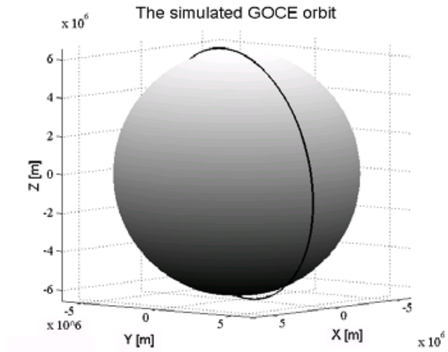


Figure 5 - The GOCE orbit generated by LORF/LVLH simulator

## 6. CONCLUSIONS AND FUTURE DEVELOPMENTS

The orbit dynamics and kinematics for the point mass motion has been transformed from classical vector notation into a new quaternion form. Among future developments, the design of quaternion observer and control will cover the most important role.

## 7. REFERENCES

[1] Wertz, J.R. (1978). *Spacecraft Attitude Determination and Control*. D.Reidel Publishing Company.

[2] Kaplan, M.H. (1976). *Modern Spacecraft Dynamics & Control.* John Wiley & Sons.

[3] Canuto, E., B.Bona, G.Calafiore and M.Indri (2002). Drag Free Control for the European Satellite GOCE. Part I: Modelling. In: *Proc. of 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada USA

[4] Chou, J.C.K. (1992). Quaternion Kinematic and Dynamic Differential Equations. In: *IEEE Trans. on Robotics and Automation*, Vol.8, N.1

[5] Greenwood, D.T. and W.M. Greenfield (1965). *Principles of Dynamics.* Prentice Hall

## 8. APPENDIX

*Algebra - Conjugate Multiplication properties*

When quaternion multiplication involves conjugates, commutative property (10) still hold. Then the product $C=\mathcal{A}^*\otimes\mathcal{B}^*$ can be written in matrix notation through:

$$C=(\mathcal{A}^*)^+\mathcal{B}^*=(\mathcal{B}^*)^-\mathcal{A}^*=(\mathcal{A}^+)^T\mathcal{B}^*=(\mathcal{B}^-)^T\mathcal{A}^*. \tag{A.1}$$

From previous equation it follows:

$$(\mathcal{A}^*)^+=(\mathcal{A}^+)^T \quad\text{and}\quad (\mathcal{B}^*)^-=(\mathcal{B}^-)^T. \tag{A.2}$$

*Algebra - Some Interesting Matrices*

It is useful to introduce the following matrix notations:

$$\begin{aligned}E^+(\mathcal{X})&=\begin{bmatrix}-\boldsymbol{x} & (x_0 I+C(\boldsymbol{x}))\end{bmatrix}\\ E^-(\mathcal{X})&=\begin{bmatrix}-\boldsymbol{x} & (x_0 I-C(\boldsymbol{x}))\end{bmatrix}\end{aligned}. \tag{A.3}$$

By exploiting the new notation, the matrix expression of quaternions introduced in (9) can be rewritten as:

$$\begin{aligned}\mathcal{A}^+&=\begin{bmatrix}\mathcal{A}^* & (E^-(\mathcal{A}^*))^T\end{bmatrix}^T=\begin{bmatrix}\mathcal{A} & (E^-(\mathcal{A}))^T\end{bmatrix}\\ \mathcal{B}^-&=\begin{bmatrix}\mathcal{B}^* & (E^+(\mathcal{B}^*))^T\end{bmatrix}^T=\begin{bmatrix}\mathcal{B} & (E^+(\mathcal{B}))^T\end{bmatrix}\end{aligned}. \tag{A.4}$$