

Existence of nontrivial solutions for semilinear problems with strictly differentiable nonlinearity

Original

Existence of nontrivial solutions for semilinear problems with strictly differentiable nonlinearity / Lancelotti, Sergio. - In: ABSTRACT AND APPLIED ANALYSIS. - ISSN 1085-3375. - 2006:(2006). [10.1155/AAA/2006/62458]

Availability:

This version is available at: 11583/1447673 since:

Publisher:

Hindawi Publishing Corporation

Published

DOI:10.1155/AAA/2006/62458

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SEMILINEAR PROBLEMS WITH STRICTLY DIFFERENTIABLE NONLINEARITY

SERGIO LANCELOTTI

Received 8 March 2005; Accepted 13 July 2005

The existence of a nontrivial solution for semilinear elliptic problems with strictly differentiable nonlinearity is proved. A result of homological linking under nonstandard geometrical assumption is also shown. Techniques of Morse theory are employed.

Copyright © 2006 Sergio Lancelotti. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Since the paper of Amann and Zehnder [1], the existence of nontrivial solutions u for semilinear elliptic problems of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

with $g(0) = 0$, has been the object of several studies, in which topological and variational methods are successfully applied. We refer the reader to [2, 3, 8, 10]. In particular, since the combination of linking theorems and Morse theory has turned out to be very fruitful, it is customary to impose conditions on g that guarantee that the associated functional $f : H_0^1(\Omega) \rightarrow \mathbb{R}$, given by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(u) dx, \quad G(s) = \int_0^s g(t) dt, \quad (1.2)$$

is of class C^2 .

In a recent paper [12], Perera and Schechter have proved a result of Amann-Zehnder type under assumptions that imply f to be only of class C^1 . More precisely, about the regularity of g , they assume that g is continuous, there exist in \mathbb{R} the limits

$$\lim_{s \rightarrow -\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow +\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow 0} \frac{g(s)}{s} \quad (1.3)$$

2 Please provide a short running title

and that

$$\frac{g(s)}{s} \text{ is Lipschitz continuous in a neighbourhood of } 0. \quad (1.4)$$

One could observe that hypothesis (1.4) allows f not to be of class C^2 , but it does not include every g satisfying the usual assumption that g is of class C^1 and g' is bounded. In particular, condition (1.4) is not stable if we add to g a term of the form

$$\frac{|s|^{3/2}}{1+s^2}. \quad (1.5)$$

The first purpose of this paper is to extend the result of [12] in such a way that also the classical smooth case is included. Our result is the following.

THEOREM 1.1. *Let Ω be a bounded open subset of \mathbb{R}^n and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $g(0) = 0$ and*

(a) *there exists $C \geq 0$ such that*

$$|g(s)| \leq C(1 + |s|); \quad (1.6)$$

(b) *there exists $\alpha \in \mathbb{R}$ such that*

$$\lim_{s \rightarrow \pm\infty} \frac{g(s)}{s} = \alpha. \quad (1.7)$$

If we denote by (λ_m) the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition, let us assume that $\alpha \neq \lambda_m$ for any $m \in \mathbb{N}$. Moreover, let us suppose that g is strictly differentiable at 0 (see Definition 3.1 below) and that there exists $m \in \mathbb{N}$ with either $g'(0) < \lambda_m < \alpha$ or $g'(0) > \lambda_m > \alpha$.

Then (1.1) admits a nontrivial solution.

Theorem 1.1 is in fact a particular case of a more general result, which will be presented in Section 2.

Remark 1.2. If, as in [12], we have $g(s) = \gamma s(s)$, with γ Lipschitz continuous in a neighbourhood of 0, then it is easy to see that g is strictly differentiable at 0.

A second purpose of the paper is to improve the saddle theorem proved in [11, Theorem 1.4], also mentioned in [12], in which the functional is of class C^2 , but nonstandard geometrical assumptions are considered. We will prove the following.

THEOREM 1.3. *Let H be a Hilbert space such that $H = H_- \oplus H_+$ with $\dim H_- < \infty$ and H_+ closed in H . Let $f : H \rightarrow \mathbb{R}$ be a functional of class C^2 and assume that*

$$c_0 = \inf_{H_+} f > -\infty, \quad c_1 = \sup_{H_-} f < +\infty, \quad (1.8)$$

f satisfies (PS)_c for every $c \in [c_0, c_1]$, $f''(u)$ is a Fredholm operator at every critical point u in $f^{-1}([c_0, c_1])$.

Then there exists a critical point u of f with $c_0 \leq f(u) \leq c_1$ and $m(f, u) \leq \dim H_- \leq m^(f, u)$.*

Please provide a short running title (a short representation of the main title appearing at the top of every even page) that should not exceed 60 characters (including spaces).

In [11] it is only shown that there exist critical points \underline{u} , \bar{u} with $c_0 \leq f(\bar{u}) \leq f(\underline{u}) \leq c_1$ and $m(f, \underline{u}) \leq \dim H_- \leq m^*(f, \bar{u})$, but one cannot say if there exists a critical point $u = \underline{u} = \bar{u}$, as in the case with standard geometrical assumptions (see [8]), or not. Our improvement is related to the fact that, according to [Proposition 4.3](#) below, also under the nonstandard geometrical assumptions of [Theorem 1.3](#), it is possible to recognize a homological linking structure.

The paper is organized as follows: in [Section 2](#) we state the result of existence of nontrivial solutions; [Sections 3](#) and [4](#) are devoted to prove some auxiliary results, while in [Section 5](#) we prove the main theorems.

2. Existence of a nontrivial solution

Let Ω be a bounded open subset of \mathbb{R}^n and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

- (g₀) $g(x, 0) = 0$ for a.e. $x \in \Omega$;
- (g₁) there exists $C \geq 0$ such that $|g(x, s)| \leq C(1 + |s|)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;
- (g₂) for a.e. $x \in \Omega$, the function $\{s \mapsto g(x, s)\}$ is strictly differentiable at 0 (see [Definition 3.1](#) below) with $D_s g(\cdot, 0) \in L^\infty(\Omega)$;
- (g₃) there exist $\hat{C} \geq 0$ and $\delta > 0$ such that, for a.e. $x \in \Omega$, we have

$$\forall s, t \in] - \delta, \delta[: |g(x, s) - g(x, t)| \leq \hat{C}|s - t|. \quad (2.1)$$

If we set $G(x, s) = \int_0^s g(x, t) dt$, it is well known that the functional $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx \quad (2.2)$$

is of class C^1 .

We denote by $m(f, 0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, 0) u^2 dx \quad (2.3)$$

is negative definite, and by $m^*(f, 0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where Q is negative semidefinite. We call $m(f, 0)$ (resp., $m^*(f, 0)$) the strict (resp., large) Morse index of f at 0.

THEOREM 2.1. *Assume that $H_0^1(\Omega) = X_- \oplus X_+$ with $\dim X_- < \infty$ and X_+ closed in $H_0^1(\Omega)$. Suppose also that*

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty, \quad (2.4)$$

and that f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$,

If it is $\dim X_- \notin [m(f, 0), m^*(f, 0)]$, then the problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.5)$$

We changed “(g0)” to “(g₀).” Please check similar cases throughout.

4 Please provide a short running title

admits a nontrivial solution u .

Remark 2.2. Under the assumption of **Theorem 1.1**, it is well known that f satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and the geometrical assumptions of **Theorem 2.1**. Since it is clear that also (g_0) – (g_3) are satisfied, **Theorem 1.1** is a consequence of **Theorem 2.1**.

3. Computations of critical groups

Definition 3.1. Let Φ be a map from an open subset U of a normed space X to a normed space Y and let $u \in U$. We say that Φ is *strictly differentiable* at u (*strongly differentiable* in the sense of [6]), if there exists a continuous linear map $L : X \rightarrow Y$ such that

$$\lim_{\substack{(w_1, w_2) \rightarrow (u, u) \\ w_1 \neq w_2}} \frac{\Phi(w_1) - \Phi(w_2) - L(w_1 - w_2)}{\|w_1 - w_2\|} = 0. \quad (3.1)$$

Of course, in such a case Φ is Fréchet differentiable at u and $L = \Phi'(u)$.

Definition 3.2. Let \mathbb{K} be a field, X be a metric space and $f : X \rightarrow \mathbb{R}$ be a continuous function. For $u \in X$ and $c = f(u)$, let us set

$$\forall q \in \mathbb{Z} : C_q(f, u) = H_q(f^c, f^c \setminus \{u\}), \quad (3.2)$$

where $f^c = \{v \in X : f(v) \leq c\}$ and $H_q(A, B)$ denotes the q th singular homology group of the pair (A, B) , with coefficients in \mathbb{K} (see, e.g., [14]). The vector space $C_q(f, u)$ is called *the q th critical group* of f at u . Because of the excision property, we may replace f by $f|_U$ for any neighborhood U of u in X .

Definition 3.3. Let X be a Banach space, U an open subset of X and $f : U \rightarrow \mathbb{R}$ be a function of class C^1 . Let C be a closed subset of X with $C \subseteq U$. We say that f satisfies *the Palais-Smale condition* ((PS) , for short) *on C* , if every sequence (u_h) in C with $f(u_h)$ bounded and $f'(u_h) \rightarrow 0$ admits a convergent subsequence. In the case $C = A = X$, we simply say that f satisfies (PS) .

Let $c \in \mathbb{R}$. We say that f satisfies *the Palais-Smale condition at level c* ($(PS)_c$, for short), if every sequence (u_h) in U with $f(u_h) \rightarrow c$ and $f'(u_h) \rightarrow 0$ admits a convergent subsequence.

Let Ω be a bounded open subset of \mathbb{R}^n ($n \geq 3$), $1 \leq p < (n+2)/(n-2)$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

(g'_1) there exists $C \geq 0$ such that $|g(x, s)| \leq C(1 + |s|^p)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.
Let $u_0 \in H_0^1(\Omega)$ be an isolated weak solution of the semilinear problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.3)$$

By regularity theory, we automatically have $u_0 \in L^\infty(\Omega)$. Moreover, let us assume that:

(g'_2) for a.e. $x \in \Omega$, the function $\{s \mapsto g(x, s)\}$ is strictly differentiable at $u_0(x)$ and $D_s g(\cdot, u_0) \in L^\infty(\Omega)$;

(g'_3) there exist $\widehat{C} \geq 0$ and $\delta > 0$ such that for a.e. $x \in \Omega$

$$\forall s, t \in]-\delta, \delta[: |g(x, u_0(x) + s) - g(x, u_0(x) + t)| \leq \widehat{C}|s - t|. \quad (3.4)$$

Let $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the functional

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx, \quad (3.5)$$

where $G(x, s) = \int_0^s g(x, t) dt$, and let $Q : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, u_0) u^2 dx. \quad (3.6)$$

Finally, let $m(f, u_0)$ and $m^*(f, u_0)$ be defined as in [Section 2](#).

THEOREM 3.4. *We have that $C_q(f, u_0) = \{0\}$ for every $q \leq m(f, u_0) - 1$ and every $q \geq m^*(f, u_0) + 1$.*

The proof will be given at the end of the section.

As a first step, we approximate the functional f with suitable functionals f_{λ} of class C^1 with f'_{λ} strictly differentiable at u_0 and such that the critical groups of f_{λ} at u_0 are independent of λ .

Let us denote by $\|\cdot\|_q$ the norm of $L^q(\Omega)$ and by $\|\cdot\|_{1,2}$ the norm of $H_0^1(\Omega)$.

Remark 3.5. Up to substitute g with $\tilde{g} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\tilde{g}(x, s) = g(x, u_0(x) + s) - g(x, u_0(x)), \quad (3.7)$$

we may assume that $u_0 = 0$ and that $g(x, 0) = 0$.

LEMMA 3.6. *There exists a constant $\overline{C} > 0$ such that, for a.e. $x \in \Omega$ and for any $s \in \mathbb{R}$, we have*

$$|g(x, s)| \leq \overline{C}(1 + |s|^{p-1})|s|. \quad (3.8)$$

Proof. If $0 < |s| < \delta$, then by (g'_3) it is

$$\left| \frac{g(x, s)}{s} \right| \leq \widehat{C}. \quad (3.9)$$

Otherwise, if $|s| \geq \delta$, then it is

$$\left| \frac{g(x, s)}{s} \right| \leq \frac{C(1 + |s|^p)}{|s|} \leq \frac{C}{\delta} + C|s|^{p-1}. \quad (3.10)$$

Hence the assertion follows. \square

6 Please provide a short running title

Now let $\delta > 0$ be as in (g'_3) and $\vartheta \in C_c^\infty(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\text{supt}(\vartheta) \subseteq]-\delta, \delta[$ and

$$\begin{aligned} \vartheta(s) &= 1 \quad \text{if } s \in \left[-\frac{\delta}{4}, \frac{\delta}{4}\right], \\ 0 \leq \vartheta \leq \frac{1}{2} \quad &\text{if } s \in [-\delta, \delta] \setminus \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]. \end{aligned} \quad (3.11)$$

For every $\lambda \in [0, 1]$ let us define $g_\lambda(x, s) = g(x, \vartheta(\lambda s)s)$ and let $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the functional

$$f_\lambda(u) = \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega G_\lambda(x, u) dx, \quad (3.12)$$

where $G_\lambda(x, s) = \int_0^s g_\lambda(x, t) dt$. It is clear that:

- (a) for every $\lambda > 0$ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x, s)\}$ is Lipschitz continuous uniformly with respect to x ;
- (b) for every λ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x, s)\}$ is strictly differentiable at 0 with $D_s g_\lambda(x, 0) = D_s g(x, 0)$;
- (c) for a.e. $x \in \Omega$, the functions $\{(\lambda, s) \mapsto g_\lambda(x, s)\}$ and $\{(\lambda, s) \mapsto G_\lambda(x, s)\}$ are continuous;
- (d) there exists $\bar{C} \geq 0$ such that $|g_\lambda(x, s)| \leq \bar{C}(1 + |s|^p)$, $|G_\lambda(x, s)| \leq \bar{C}(1 + |s|^{p+1})$.

THEOREM 3.7. *The following facts hold:*

- (i) for every $\lambda \in [0, 1]$, the functional f_λ is of class C^1 ;
- (ii) there exists an open bounded neighbourhood U of 0 in $H_0^1(\Omega)$ such that, for every $\lambda \in [0, 1]$, 0 is the only critical point of f_λ in \bar{U} ;
- (iii) for every $\lambda \in]0, 1]$, f'_λ is strictly differentiable at 0 with $\langle f''_\lambda(0)v, v \rangle = Q(v)$.

Proof. It is readily seen that assertion (i) holds.

Let us consider assertion (ii). By contradiction, let us assume that there exist (λ_h) in $[0, 1]$ and (u_h) in $H_0^1(\Omega)$ with $u_h \neq 0$ and $u_h \rightarrow 0$ strongly in $H_0^1(\Omega)$ such that $f'_{\lambda_h}(u_h) = 0$. Up to a subsequence, $\lambda_h \rightarrow \lambda$ in $[0, 1]$. Since u_h is a critical point of f_{λ_h} , we have that u_h is a weak solution of

$$-\Delta u = g_{\lambda_h}(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.13)$$

Let

$$a_h = \begin{cases} \frac{g_{\lambda_h}(x, u_h)}{u_h} & \text{where } u_h \neq 0, \\ 0 & \text{where } u_h = 0. \end{cases} \quad (3.14)$$

By **Lemma 3.6** it is

$$|a_h| \leq \left| \frac{g_{\lambda_h}(x, u_h)}{u_h} \right| = \left| \frac{g(x, \vartheta(\lambda_h u_h) u_h)}{u_h} \right| \leq \bar{C}(1 + |\vartheta(\lambda_h u_h) u_h|^{p-1}) \leq \bar{C}(1 + |u_h|^{p-1}). \quad (3.15)$$

Since u_h is bounded in $L^{2n/(n-2)}(\Omega)$, then a_h belongs to $L^q(\Omega)$ with $q > n/2$ and

$$\|a_h\|_q \leq C' \left(1 + \|u_h\|_{2n/(n-2)}^{p-1}\right) \leq M. \quad (3.16)$$

Hence u_h is a weak solution of the linear problem

$$-\Delta u = a_h u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.17)$$

By [7, Theorem 3.13.1] $u_h \in L^\infty(\Omega)$ and there exists $C > 0$ such that $\|u_h\|_\infty \leq C \|Du_h\|_2$. Hence $u_h \rightarrow 0$ in $L^\infty(\Omega)$. Since $\vartheta = 1$ on $[-\delta/4, \delta/4]$, for h sufficiently large we have that u_h is a weak solution of (3.3). It follows that 0 is not an isolated solution of (3.3): a contradiction.

Finally, let us consider assertion (iii). Let $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ be the continuous linear operator such that

$$\langle Lv, w \rangle = \langle Lw, v \rangle, \quad \langle Lv, v \rangle = Q(v). \quad (3.18)$$

Let $(u_h), (v_h), (w_h)$ in $H_0^1(\Omega)$ be such that $u_h \rightarrow 0, w_h \rightarrow 0$ in $H_0^1(\Omega)$ and $\|v_h\|_{1,2} \leq 1$. Up to a subsequence, $w_h \rightarrow 0$ and $u_h \rightarrow 0$ a.e. in Ω . We have that

$$\begin{aligned} & \left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right| \\ &= \left| \int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left[\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right] (w_h - u_h) v_h dx \right| \\ &\leq C \left(\int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \\ &\quad \times \|w_h - u_h\|_{1,2} \|v_h\|_{1,2}. \end{aligned} \quad (3.19)$$

Then it is

$$\begin{aligned} & \frac{\left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right|}{\|w_h - u_h\|_{1,2}} \\ &\leq C \left(\int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \|v_h\|_{1,2} \\ &\leq C \left(\int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} dx \right)^{2/n}. \end{aligned} \quad (3.20)$$

By (a) and (b) we can apply Lebesgue's theorem, obtaining

$$\left(\int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} dx \right)^{2/n} \rightarrow 0. \quad (3.21)$$

Therefore

$$\lim_{h \rightarrow +\infty} \frac{\langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle}{\|w_h - u_h\|_{1,2}} = 0 \quad (3.22)$$

8 Please provide a short running title

and assertion (iii) follows. \square

THEOREM 3.8. *The critical groups $C_q(f_\lambda, 0)$ are independent of λ . In particular*

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.23)$$

Proof. Let U be an open bounded neighbourhood of 0 in $H_0^1(\Omega)$ as in assertion (ii) of **Theorem 3.7**. We claim that if $\lambda_h \rightarrow \lambda$ in $[0, 1]$, then $\|f_{\lambda_h}|_{\bar{U}} - f_\lambda|_{\bar{U}}\|_{1, \infty} \rightarrow 0$. Let (u_h) be a sequence in \bar{U} . Up to a subsequence, $u_h \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_h \rightarrow u$ a.e. in Ω . It is

$$\begin{aligned} f_{\lambda_h}(u_h) - f_\lambda(u_h) &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u_h)] dx \\ &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx + \int_{\Omega} [G_\lambda(x, u) - G_\lambda(x, u_h)] dx. \end{aligned} \quad (3.24)$$

By (c), (d) and Lebesgue's theorem we deduce that

$$\int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx \rightarrow 0. \quad (3.25)$$

Therefore $f_{\lambda_h} \rightarrow f_\lambda$ uniformly on \bar{U} .

Now, let $v_h \in H_0^1(\Omega)$ with $\|v_h\|_{1,2} \leq 1$. Up to a subsequence $v_h \rightharpoonup v$ in $H_0^1(\Omega)$, $v_h \rightarrow v$ in $L^{2n/(n-2)}(\Omega)$ and $v_h \rightarrow v$ a.e. in Ω . It is

$$\begin{aligned} &|\langle f'_{\lambda_h}(u_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle| \\ &= \left| \int_{\Omega} [g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)] v_h dx \right| \\ &= \left| \int_{\Omega} [g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)] v_h dx \right| \\ &\leq C \left(\int_{\Omega} |g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)|^{2n/(n+2)} dx \right)^{(n+2)/2n} \|v_h\|_{1,2}. \end{aligned} \quad (3.26)$$

As before we have that

$$\int_{\Omega} |g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)|^{2n/(n+2)} dx \rightarrow 0. \quad (3.27)$$

It follows that $f'_{\lambda_h} \rightarrow f'_\lambda$ uniformly on \bar{U} . Finally, since U is bounded and g has subcritical growth, we have that for every $\lambda \in [0, 1]$ f_λ satisfies (PS) in \bar{U} . By [5, Theorem 5.2] the assertion follows. \square

In the second part of this section we deduce from [6] a generalization of the classical Shifting theorem (see [3, Theorem I.5.4], [10, Theorem 8.4]).

Let H be a Hilbert space, U be an open subset of H , $u_0 \in U$ and $f : U \rightarrow \mathbb{R}$ be a function of class C^1 such that f' is strictly differentiable at u_0 and $f''(u_0)$ is a Fredholm operator. In particular, f' is Lipschitz continuous in a neighbourhood of u_0 . Let $L : H \rightarrow H$

be the linear operator defined by

$$\forall v, w \in H : \langle Lv, w \rangle = \langle f''(u_0)v, w \rangle, \quad (3.28)$$

let $V_0 = \ker L$ and let P_{V_0} be the orthogonal projection on V_0 . We also denote by $m(f, u_0)$ (resp., $m^*(f, u_0)$) the strict (resp., large) Morse index of f at u_0 .

THEOREM 3.9. *Let u_0 be an isolated critical point of f . Then there exist a neighbourhood \hat{U} of $P_{V_0}u_0$ in V_0 and a function $\hat{f} : \hat{U} \rightarrow \mathbb{R}$ of class C^1 with locally Lipschitz gradient such that $P_{V_0}u_0$ is an isolated critical point of \hat{f} and*

$$\forall q \in \mathbb{Z} : C_q(f, u_0) \approx \begin{cases} C_{q-m(f, u_0)}(\hat{f}, P_{V_0}u_0) & \text{if } m(f, u_0) < \infty, \\ \{0\} & \text{if } m(f, u_0) = \infty, \end{cases} \quad (3.29)$$

$$\forall q \leq m(f, u_0) - 1 : C_q(f, u_0) = \{0\}, \quad (3.30)$$

$$\forall q \geq m^*(f, u_0) + 1 : C_q(f, u_0) = \{0\}.$$

Proof. Without loss of generality, we may assume that $u_0 = 0$. From [6, Theorem 1.2] we also see that the generalized Morse lemma holds also in this setting. Arguing as in the proof of [10, Theorem 8.4], we find that (3.29) holds. Actually, in our case f is of class C^{2-0} instead of C^2 , but the proof of [10, Theorem 8.4] remains valid also in this case.

On the other hand, also the proof of [10, Theorem 8.5] can be easily adapted from the C^2 to the C^{2-0} case. Therefore we have that $C_q(\hat{f}, P_{V_0}u_0) = \{0\}$ if $q \geq \dim V_0 + 1$. Since $m^*(f, u_0) = m(f, u_0) + \dim V_0$, the other assertions follow from (3.29). \square

Finally, let us prove **Theorem 3.4**.

Proof. By **Remark 3.5** we may assume that $u_0 = 0$. Let $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ be as in (3.12). By **Theorem 3.7** we have that f_1 is of class C^1 with f_1' strictly differentiable at 0 and 0 is an isolated critical point of f_1 . Moreover, $f_1''(0)$ is a Fredholm operator. By **Theorem 3.8** it is

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.31)$$

On the other hand, since $Q(u) = \langle f_1''(0)u, u \rangle$, we have that $m(f, 0) = m(f_1, 0)$ and $m^*(f, 0) = m^*(f_1, 0)$. From **Theorem 3.9** the assertion follows. \square

4. Homological linking

Throughout this section, X will denote a Banach space, $B_r(u)$ the open ball of center $u \in X$ and radius r and $f : X \rightarrow \mathbb{R}$ a function of class C^1 . We set $K = \{u \in X : f'(u) = 0\}$ and, for every $c \in \mathbb{R}$,

$$K_c = \{u \in X : f'(u) = 0, f(u) = c\}. \quad (4.1)$$

We also denote by H_* singular homology.

First of all, let us recall from [4] an extension of the homological linking of [3].

10 Please provide a short running title

Definition 4.1. Let D, S, A be three subsets of X , $m \in \mathbb{N}$ and \mathbb{K} a field. We say that (D, S) links A homologically in dimension m (over \mathbb{K}), if $S \subseteq D$, $S \cap A = \emptyset$ and there exists $z \in H_m(X, S; \mathbb{K})$ belonging to the image of $i_* : H_m(D, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$ but not of $j_* : H_m(X \setminus A, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$, where $i : (D, S) \rightarrow (X, S)$ and $j : (X \setminus A, S) \rightarrow (X, S)$ are the inclusion maps.

It is clear that, if (D, S) links A homologically, then $D \cap A \neq \emptyset$.

THEOREM 4.2. *Let D, S, A be three subsets of X such that (D, S) links A homologically in dimension m and let $z \in H_m(X, S; \mathbb{K})$ be as in [Definition 4.1](#). Assume that*

$$\inf_A f > -\infty, \quad \sup_D f < +\infty, \quad \forall u \in S: f(u) < \inf_A f \quad (4.2)$$

and define

$$c = \inf \{b \in \mathbb{R} : S \subseteq f^b \text{ and } z \text{ belongs to the image of the homomorphism induced by inclusion } H_m(f^b, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})\}. \quad (4.3)$$

Suppose that f satisfies (PS) and that each element of K_c is isolated in K .

Then $\inf_A f \leq c \leq \sup_D f$ and there exists $u \in K_c$ with $C_m(f, u) \neq \{0\}$.

To prove our main results we need the following.

PROPOSITION 4.3. *Let $X = X_- \oplus X_+$, with $\dim X_- < \infty$ and X_+ closed in X . Assume that*

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty \quad (4.4)$$

and that f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$.

Then there exists a compact pair (D, S) in X such that

$$\max_D f \leq c_1, \quad \forall u \in S: f(u) < c_0 \quad (4.5)$$

and such that (D, S) links X_+ homologically in dimension $\dim X_-$ over all \mathbb{K} .

Proof. Since f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$, there exists $r > 0$ such that $K \cap f^{-1}([c_0, c_1]) \subseteq (B_r(0) \cap X_-) \oplus X_+$. Moreover, there exist $\delta, \sigma > 0$ such that

$$\|P_{X_-} u\| \geq r, \quad c_0 - \delta \leq f(u) \leq c_1 + \delta \implies \|f'(u)\| > \sigma, \quad (4.6)$$

where P_{X_-} denotes the projection on X_- induced by the decomposition $X = X_- \oplus X_+$. Let $c > 0$ be such that $\|P_{X_-} u\| \leq c\|u\|$ for any $u \in X$ and let

$$R = c \frac{c_1 - c_0 + \delta}{\sigma} + r + \delta, \quad \rho_1 = 1, \quad \rho_2 = R - r - \delta, \quad (4.7)$$

$$C = X \setminus [(B_{r+\rho_1+\rho_2}(0) \cap X_-) \oplus X_+].$$

By [5, Theorem 2.1] applied to the function $f|_{\{u \in X: f(u) \geq c_0 - \delta\}}$, there exist a continuous function

$$\tau : \overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \longrightarrow [0, +\infty) \quad (4.8)$$

and a continuous map

$$\eta : \left(\overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \right) \times [0, 1] \longrightarrow \{u \in X : f(u) \geq c_0 - \delta\} \quad (4.9)$$

such that

- (a) $\tau(u) = 0 \Leftrightarrow f(u) = c_0 - \delta$;
- (b) $\|\eta(u, t) - u\| \leq \tau(u)t$;
- (c) $f(\eta(u, t)) \leq f(u) - \sigma\tau(u)t$;
- (d) $f(\eta(u, 1)) = c_0 - \delta$.

Let $\vartheta_1 : \mathbb{R} \rightarrow [0, 1]$ be a continuous function such that

$$\vartheta_1(s) = 1 \quad \text{if } s \leq c_1, \quad \vartheta_1(s) = 0 \quad \text{if } s \geq c_1 + \delta/2, \quad (4.10)$$

and let $\vartheta_2 : X \rightarrow [0, 1]$ be a continuous function such that

$$\vartheta_2(u) = 1 \quad \text{if } \|u\| \geq R, \quad \vartheta_2(u) = 0 \quad \text{if } \|u\| \leq R - \delta. \quad (4.11)$$

Let $\mathcal{H} : X \times [0, 1] \rightarrow X$ be the deformation defined by

$$\mathcal{H}(u, t) = \begin{cases} \eta(u, \vartheta_1(f(u))\vartheta_2(P_{X_-}u)t) & \text{if } u \in \overline{B_{\rho_1}(C)}, c_0 - \delta \leq f(u) \leq c_1 + \delta, \\ u & \text{if } f(u) \leq c_0 - \delta, \\ u & \text{if } f(u) \geq c_1 + \delta/2, \\ u & \text{if } \|P_{X_-}u\| \leq R - \delta. \end{cases} \quad (4.12)$$

If $u \in X_-$, we have that

$$\|P_{X_-}\mathcal{H}(u, t) - u\| \leq c\|\mathcal{H}(u, t) - u\| \leq c \frac{f(u) - f(\mathcal{H}(u, t))}{\sigma} \leq c \frac{c_1 - c_0 + \delta}{\sigma} < R - r. \quad (4.13)$$

It follows

$$\begin{aligned} \|P_{X_-}u\| \leq r &\implies \mathcal{H}(u, t) = u, \\ u \in X_-, &\implies f(\mathcal{H}(u, 1)) < c_0, \\ \|u\| \geq R &\implies \|P_{X_-}(\mathcal{H}(u, t))\| \geq r, \quad \forall t \in [0, 1]. \end{aligned} \quad (4.14)$$

It is clear that $(X, (X_- \setminus B_r(0)) \oplus X_+)$ links X_+ homologically in dimension $\dim X_-$ and that the inclusion map

$$i : \left(\overline{B_R(0)} \cap X_-, \partial B_R(0) \cap X_- \right) \longrightarrow (X, (X_- \setminus B_r(0)) \oplus X_+) \quad (4.15)$$

12 Please provide a short running title

induces an isomorphism in homology. Let $m = \dim X_-$ and

$$B = \overline{B_R(0)} \cap X_-, \quad E = \partial B_R(0) \cap X_-, \quad F = (X_- \setminus B_r(0)) \oplus X_+. \quad (4.16)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & \longleftarrow & H_m(X \setminus X_+, E) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.17)$$

where horizontal rows are induced by the inclusions and the vertical rows are isomorphisms. We have that there exists $z \in H_m(X, E)$ belonging to the image of $H_m(B, E) \rightarrow H_m(X, E)$ such that $i_*(z) \in H_m(X, F)$, but not to the image of $H_m(X \setminus X_+, F) \rightarrow H_m(X, F)$. Let us consider the compact sets $D = \mathcal{H}(B, 1)$ and $S = \mathcal{H}(E, 1)$. We have that

$$\max_D f \leq c_1, \quad \max_S f < c_0, \quad S \subseteq F. \quad (4.18)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & & \\ \mathcal{H}_*(\cdot, 1) \downarrow & & \mathcal{H}_*(\cdot, 1) \downarrow & & \\ H_m(D, S) & \longrightarrow & H_m(X, S) & \longleftarrow & H_m(X \setminus X_+, S) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.19)$$

Since $\mathcal{H}(\cdot, 1) : (X, E) \rightarrow (X, F)$ is homotopically equivalent to the identity map, then (D, S) links X_+ homologically in dimension $m = \dim X_-$ and the assertions follows. \square

5. Proof of the main results

proof of Theorem 2.1. By contradiction, let us assume that 0 is the unique solution of (2.5). Since $m = \dim X_- \notin [m(f, 0), m^*(f, 0)]$, by Theorem 3.4 it is $C_m(f, 0) = \{0\}$. By Proposition 4.3 there exists a compact pair (D, S) in $H_0^1(\Omega)$ such that

$$\forall u \in S: f(u) < \inf_{X_+} f \quad (5.1)$$

and (D, S) links X_+ homologically in dimension m over all \mathbb{K} . By Theorem 4.2 there exists a critical point $u \in H_0^1(\Omega)$ of f such that $C_m(f, u) \neq \{0\}$. Hence $u \neq 0$ and u is a weak solution of (2.5): a contradiction. \square

proof of Theorem 1.3. Let (D, S) be as in Proposition 4.3. By [13, Proposition 3.9 and Remark] there exists $\delta > 0$ such that f satisfies $(PS)_c$ for every $c \in [c_0 - \delta, c_1 + \delta]$ and

$f''(u)$ is a Fredholm operator at every critical point u in $f^{-1}([c_0 - \delta, c_1 + \delta])$. Let us argue by contradiction and set

$$\begin{aligned} K_1 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m^*(f, u) < \dim H_-\}, \\ K_2 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m(f, u) > \dim H_-\}. \end{aligned} \quad (5.2)$$

Then K_1, K_2 are two disjoint compact sets whose union is the critical set of f in $f^{-1}([c_0 - \delta, c_1 + \delta])$. By Marino-Prodi perturbation lemma [9, Teorema 2.2], there exists a functional $\hat{f} : H \rightarrow \mathbb{R}$ of class C^2 such that

$$\inf_{H_+} \hat{f} > c_0 - \delta/2, \quad \sup_{H_-} \hat{f} < c_1 + \delta/2, \quad \max_S \hat{f} < \inf_{H_+} \hat{f}, \quad (5.3)$$

\hat{f} satisfies $(PS)_c$ for every $c \in [c_0 - \delta/2, c_1 + \delta/2]$, \hat{f} has only non-degenerate critical points u in $\hat{f}^{-1}([c_0 - \delta/2, c_1 + \delta/2])$, with either $m(\hat{f}, u) < \dim H_-$ or $m^*(\hat{f}, u) > \dim H_-$. If we apply **Theorem 4.2** to \hat{f} , we find a critical point u of \hat{f} with $c_0 - \delta/2 \leq \hat{f}(u) \leq c_1 + \delta/2$ and $C_m(\hat{f}, u) \neq \{0\}$, where $m = \dim H_-$. By the Morse lemma, we have $m(\hat{f}, u) = m$ and a contradiction follows. \square

Acknowledgment

The author wishes to thank Prof. Marco Degiovanni for helpful discussions and valuable hints.

References

- [1] H. Amann and E. Zehnder, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1980), no. 4, 539–603.
- [2] K. C. Chang, *Solutions of asymptotically linear operator equations via Morse theory*, Comm. Pure Appl. Math. 34 (1981), no. 5, 693–712.
- [3] ———, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 6, Birkhäuser, Massachusetts, 1993.
- [4] S. Cingolani and M. Degiovanni, *Nontrivial solutions for p -Laplace equations with right-hand side having p -linear growth at infinity*, Comm. Partial Differential Equations 30 (2005), no. 8, 1191–1203.
- [5] J.-N. Corvellec and A. Hantoute, *Homotopical stability of isolated critical points of continuous functionals*, Set-Valued Anal. 10 (2002), no. 2-3, 143–164.
- [6] A. A. de Moura and F. M. de Souza, *A Morse lemma for degenerate critical points with low differentiability*, Abstr. Appl. Anal. 5 (2000), no. 2, 113–118.
- [7] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Nauka Press, Moscow, 1964, Academic Press, New York, 1968.
- [8] A. C. Lazer and S. Solimini, *Nontrivial solutions of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal. 12 (1988), no. 8, 761–775.
- [9] A. Marino and G. Prodi, *Metodi perturbativi nella teoria di Morse*, Boll. Un. Mat. Ital. (4) 11 (1975), no. 3, suppl., 1–32.
- [10] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer, New York, 1989.

14 Please provide a short running title

- [11] K. Perera and M. Schechter, *Morse index estimates in saddle point theorems without a finite dimensional closed loop*, Indiana Univ. Math. J. **47** (1998), no. 3, 1083–1095.
- [12] ———, *Applications of Morse theory to the solution of semilinear problems depending on C^1 functionals*, Nonlinear Anal. Ser. A: Theory Methods **45** (2001), no. 1, 1–9.
- [13] S. Solimini, *Morse index estimates in min-max theorems*, Manuscripta Math. **63** (1989), no. 4, 421–453.
- [14] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

Sergio Lancelotti: Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, I 10129 Torino, Italy

E-mail address: sergio.lancelotti@polito.it

1. Comment on ref. [4]: We updated the information of this reference. Please check.