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LAGRANGIAN SYSTEMS WITH LIPSCHITZ OBSTACLE ON MANIFOLDS

SERGIO LANCELOTTI — MARCO MARZOCCHI

ABSTRACT. Lagrangian systems constrained on the closure of an open subset with Lipschitz boundary in a manifold are considered. Under suitable assumptions, the existence of infinitely many periodic solutions is proved.

1. Introduction

The study of Lagrangian functionals of the form

$$(1.1) \quad f(\gamma) = \int_0^1 L(s, \gamma(s), \gamma'(s)) ds$$

on a manifold M , where $L(s, (q, v)): \mathbb{R} \times TM \rightarrow \mathbb{R}$, constitutes a well studied topic in Mechanics and Global analysis. In particular, about the existence and multiplicity of periodic solutions γ of the associated Euler equation, we refer the reader to [1], where the case in which M is a compact manifold without boundary is considered. Starting from [1], some extensions have been considered in the literature, when M is embedded in an Euclidean space. In [3] the case where M is a compact submanifold with boundary in \mathbb{R}^n has been considered.

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In such a case, the associated Euler equation has the form

$$(1.2) \quad \frac{d}{ds}(D_v L(s, \gamma, \gamma')) - D_q L(s, \gamma, \gamma') \in N_{\gamma(s)} M,$$

where $N_q M$ is the outer normal cone to M at q . The main feature is that the natural domain of the functional (1.1) is

$$(1.3) \quad X = \{\gamma \in W^{1,2}(0, 1; \mathbb{R}^n) : \gamma(0) = \gamma(1), \gamma(s) \in M \text{ for all } s\}$$

which is naturally a metric space, but not a smooth manifold (even with boundary). Moreover, solutions γ of (1.2) are not of class C^2 , but only $W^{2,\infty}$ and satisfy (1.2) almost everywhere. In the same direction, the case in which M is a compact p -convex subset of \mathbb{R}^n has been considered in [4]. The class of p -convex subsets [8] includes in particular subsets with corners of convex type and concave parts of class C^2 . This direction of research was started by [18], where the case of an n -dimensional submanifold with boundary of class C^2 in \mathbb{R}^n had been considered.

Another development has been started more recently in [11], [19], considering the case in which M is the closure of a bounded open subset of \mathbb{R}^n with Lipschitz boundary. Also in this case the set X is naturally only a metric space. Moreover, since in this case we cannot expect the solution γ of (1.2) to be of class C^1 , the Euler equation itself requires a reformulation.

The purpose of this paper is to consider the intrinsic case in which M is the closure of a bounded open subset of a differentiable manifold N , instead of \mathbb{R}^n , and also to relax the convexity condition on L , which was in [19] of uniform quadratic type, to the mere convexity with coercivity of order $p > 1$.

Our approach follows the lines of [19], but it is completely intrinsic. Of course the lack of strict convexity in L causes new technical difficulties.

The paper is organized as follows: in Section 2 we state our main results, while Section 3 is devoted to some recalls of nonsmooth analysis. Finally, in Section 4 we prove the main results.

2. Statement of the main results

Let N be a differentiable manifold without boundary of class C^2 and $M \subseteq N$. In the sequel, each $\gamma \in W^{1,p}(a, b; N)$ will be identified with its continuous representative $\tilde{\gamma}: [a, b] \rightarrow N$. We set

$$W^{1,p}(a, b; M) := \{\gamma \in W^{1,p}(a, b; N) : \gamma(s) \in M \text{ for each } s \in [a, b]\}.$$

REMARK 2.1. Let g and \tilde{g} be two Riemannian structures on N and let d and \tilde{d} be the induced distances on N . Then there exists a continuous function $c: N \rightarrow]0, \infty[$ such that, for all $q \in N$ and all $v \in T_q N$,

$$g(q)(v, v) \leq c(q)\tilde{g}(q)(v, v), \quad \tilde{g}(q)(v, v) \leq c(q)g(q)(v, v).$$

In particular, for every compact subset $K \subseteq N$ there exists $C > 0$ such that, for all $q_1, q_2 \in K$,

$$d(q_1, q_2) \leq C\tilde{d}(q_1, q_2), \quad \tilde{d}(q_1, q_2) \leq Cd(q_1, q_2).$$

Let $1 < p < \infty$ and $L: \mathbb{R} \times TN \rightarrow \mathbb{R}$ be a function of class C^1 such that there exist two continuous functions $c, k: M \rightarrow]0, \infty[$ and $d \in \mathbb{R}$ such that for every $s \in \mathbb{R}$ and $q \in M$ one has

$$(2.1) \quad k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_q N,$$

$$(2.2) \quad |D_{(q,v)}L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_q N,$$

$$(2.3) \quad L(s, q, \cdot) \text{ is convex} \quad \text{on } T_q N,$$

where $|v| = \sqrt{g(q)(v, v)}$.

In (2.1), (2.2) we mean that N is provisionally endowed with a Riemannian structure. By Remark 2.1 the above conditions do not depend on the Riemannian structure chosen on N .

In charts, (2.1), (2.2) mean that for every $s \in \mathbb{R}$ and $q \in M$ it is

$$k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_q N,$$

$$|D_q L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_q N,$$

$$|D_v L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_q N.$$

Let us remark that (2.1), (2.3) imply that for every $s \in \mathbb{R}$, $q \in M$ and any $v, w \in T_q N$ we have

$$|D_v L(s, q, v)w| \leq \hat{c}(q)(1 + |v|^{p-1})|w|$$

namely, in charts,

$$|D_v L(s, q, v)| \leq \hat{c}(q)(1 + |v|^{p-1}),$$

where $\hat{c}: M \rightarrow]0, \infty[$ is continuous.

Define a continuous functional $f_{a,b}: W^{1,p}(a, b; M) \rightarrow \mathbb{R}$ by

$$f_{a,b}(\gamma) = \int_a^b L(s, \gamma(s), \gamma'(s)) ds.$$

Given a Riemannian structure on N , for every $\gamma, \eta \in W^{1,p}(a, b; M)$ we set

$$d_1(\gamma, \eta) = \int_a^b d(\gamma(s), \eta(s)) ds,$$

$$d_\infty(\gamma, \eta) = \max\{d(\gamma(s), \eta(s)) : a \leq s \leq b\},$$

where d is the distance on N associated with the Riemannian structure.

DEFINITION 2.2. We say that $\gamma \in W^{1,p}(a, b; M)$ is *L-stationary*, if it is not possible to find $r, c, \sigma > 0$ and a map

$$\mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \\ \rightarrow W^{1,p}(a, b; M)$$

such that:

- (a) \mathcal{H} is continuous from the product of the topology of the uniform convergence and that of \mathbb{R} to that of the uniform convergence;
- (b) for every $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $t \in [0, r]$ we have

$$\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b), \\ d_1(\mathcal{H}(\eta, t), \eta) \leq ct, \quad f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t.$$

Again we mean that the assertion holds after introducing a Riemannian structure on N . By Remark 2.1 this definition does not depend on the choice of the Riemannian structure itself.

PROPOSITION 2.3. Let $\gamma \in W^{1,p}(a, b; M)$ be *L-stationary*. Then for every $[\alpha, \beta] \subseteq [a, b]$ the restriction $\gamma|_{[\alpha, \beta]}$ is *L-stationary*.

PROOF. Set $\hat{\gamma} = \gamma|_{[\alpha, \beta]}$. By contradiction, assume that there exist $r, c, \sigma > 0$ and

$$\mathcal{H}: \{\eta \in W^{1,p}(\alpha, \beta; M) : d_\infty(\eta, \hat{\gamma}) < r, f_{\alpha, \beta}(\eta) < f_{\alpha, \beta}(\hat{\gamma}) + r\} \times [0, r] \\ \rightarrow W^{1,p}(\alpha, \beta; M)$$

according to Definition 2.2.

We claim that there exists $r' \in]0, r[$ such that if $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r'$ and $f_{a,b}(\eta) < f_{a,b}(\gamma) + r'$, then $f_{\alpha, \beta}(\hat{\eta}) < f_{\alpha, \beta}(\hat{\gamma}) + r$, where $\hat{\eta} = \eta|_{[\alpha, \beta]}$.

Again by contradiction, let $(\eta_h) \subseteq W^{1,p}(a, b; M)$ with η_h convergent to γ with respect to the uniform convergence and $\limsup_h f_{a,b}(\eta_h) \leq f_{a,b}(\gamma)$ such that $f_{\alpha, \beta}(\hat{\eta}_h) \geq f_{\alpha, \beta}(\hat{\gamma}) + r$. By (2.1) and (2.3) we have

$$\limsup_h f_{\alpha, \beta}(\hat{\eta}_h) \leq \limsup_h f_{a,b}(\eta_h) - \liminf_h \int_{]a, b[\setminus]\alpha, \beta[} L(s, \eta_h, \eta'_h) ds \\ \leq f_{a,b}(\gamma) - \int_{]a, b[\setminus]\alpha, \beta[} L(s, \gamma, \gamma') ds = f_{\alpha, \beta}(\hat{\gamma}),$$

whence a contradiction. Then, for any $\eta \in W^{1,p}(a, b; M)$ define

$$\mathcal{K}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r', f_{a,b}(\eta) < f_{a,b}(\gamma) + r'\} \times [0, r'] \\ \rightarrow W^{1,p}(a, b; M)$$

by

$$\mathcal{K}(\eta, t)(s) = \begin{cases} \mathcal{H}(\widehat{\eta}, t)(s) & \text{if } s \in [\alpha, \beta], \\ \eta(s) & \text{if } s \notin [\alpha, \beta]. \end{cases}$$

It is readily seen that \mathcal{K} has all the properties required in Definition 2.2. It follows that γ is not L -stationary, which is absurd. \square

DEFINITION 2.4. Let I be an interval in \mathbb{R} with $\text{int}(I) \neq \emptyset$. A continuous map $\gamma: I \rightarrow M$ is said to be a *generalized solution* of the Lagrangian system associated to L on M , if every $s \in \text{int}(I)$ admits a neighbourhood $[a, b]$ in I such that $\gamma|_{[a, b]}$ belongs to $W^{1,p}(a, b; M)$ and is L -stationary.

DEFINITION 2.5. Given $T > 0$, a T -periodic *generalized solution* of the Lagrangian system associated to L on M is a generalized solution $\gamma: \mathbb{R} \rightarrow M$ which is periodic of period T .

We now state our main existence result.

THEOREM 2.6. *Assume that M is the closure of an open subset of N with locally Lipschitz boundary. Suppose also that M is compact, 1-connected and non-contractible in itself and that*

$$(2.4) \quad L(s+1, q, v) = L(s, q, v) \quad \text{for all } s \in \mathbb{R} \text{ and all } (q, v) \in TN.$$

Then there exists a sequence (γ_h) of 1-periodic generalized solutions of the Lagrangian system associated to L on M with

$$\lim_h \int_0^1 L(s, \gamma_h(s), \gamma'_h(s)) ds = +\infty.$$

The notion of generalized solution we have introduced follows the approach of [11, Definition 3.3] and [19, Definition 2.6] and has the advantage to be intrinsically connected to M , although quite indirect. However, at least in the particular case $p = 2$, it is possible to deduce further informations on the generalized solutions.

For every $q \in M$, denote by $N_q M$ the normal cone to M at q (see e.g. Definition 3.2 below).

THEOREM 2.7. *Let $p = 2$ and assume that there exists a continuous function $\omega: N \rightarrow]0, \infty[$ such that for every $s \in \mathbb{R}$, $q \in M$ it is*

$$D_v L(s, q, v)(v - w) - D_v L(s, q, w)(v - w) \geq \omega(q)|v - w|^2 \quad \text{for all } v, w \in T_q N.$$

Let $\gamma \in W^{1,2}(a, b; M)$ be L -stationary. Then $\gamma \in W^{1,\infty}(a, b; M)$, $D_{(q,v)} L(s, \gamma, \gamma') \in L^\infty(a, b; T^(TN))$ and there exist a finite Borel measure μ on $]a, b[$ and a bounded Borel function $\nu:]a, b[\rightarrow T^*N$ such that $\nu(s) \in N_{\gamma(s)} M$ for μ -a.e. $s \in]a, b[$ and*

$$\int_a^b D_{(q,v)} L(s, \gamma, \gamma')(\delta, \delta') ds = - \int_a^b \nu(\delta) d\mu$$

for any $\delta \in W_0^{1,1}(a, b; TN)$ with $\delta(s) \in T_{\gamma(s)}N$ for every $s \in [a, b]$.

Also in this assertion we mean that N is provisionally endowed with a Riemannian structure. Since γ is continuous, by Remark 2.1 the assertion is independent of the choice of the structure.

PROOF OF THEOREM 2.7. By Proposition 2.3, we may assume that $\gamma([a, b])$ is contained in a coordinated neighbourhood. Then the assertion follows from [19, Theorem 2.10]. \square

3. Some relevant results of nonsmooth analysis

In the first part of this section let N be a differentiable manifold of class C^2 and M be the closure of an open set in N with locally Lipschitz boundary.

If X is a Banach space, $E \subseteq X$ and $x \in E$, we denote by $T_x E$ the tangent cone to E at x , according to [6]. We also denote by $B_r(x)$ the open ball of center x and radius r .

DEFINITION 3.1. Let $x \in E$ and $v \in X$. We say that v is *hypertangent to E at x* if there exists $\delta > 0$ such that $B_\delta(x) + [0, \delta]B_\delta(v) \subseteq E$. Let us denote by $\text{Hyp}_x E$ the set of the v 's hypertangent to E at x .

DEFINITION 3.2. Let $q \in M$ and $v \in T_q N$. We say that v is *tangent to M at q* if there exists a chart (U, φ) at q such that $d\varphi(q)v \in T_{\varphi(q)}\varphi(U \cap M)$. The set of the v 's tangent to M at q is denoted by $T_q M$ and is called the *tangent cone to M at q* .

We say that v is *hypertangent to M at q* if there exists a chart (U, φ) at q such that $d\varphi(q)v$ is hypertangent to $\varphi(U \cap M)$ at $\varphi(q)$. The set of the v 's hypertangent to M at q is denoted by $\text{Hyp}_q M$ and is called the *hypertangent cone to M at q* . Finally, we set $N_q M = \{\varphi \in T_q^* N : \varphi(v) \leq 0 \text{ for all } v \in T_q M\}$. $N_q M$ is called the *normal cone to M at q* .

REMARK 3.3. For every $q \in M$ it is $\text{Hyp}_q M \neq \emptyset$ (see [6]) and $\text{Hyp}_q M \subseteq T_q M$.

THEOREM 3.4. *There exists a section $\nu: N \rightarrow TN$ of class C^1 such that*

$$\nu(q) \in \text{Hyp}_q M \quad \text{for all } q \in M.$$

PROOF. For all $q \in N$, let

$$\Psi(q) = \begin{cases} \text{Hyp}_q M & \text{if } q \in M, \\ T_q N & \text{if } q \in N \setminus M. \end{cases}$$

Then for every $q \in N$, $\Psi(q)$ is convex in $T_q N$ and for every $q \in N$ there exists a chart (U, φ) at q such that

$$\bigcap_{\xi \in U} (d\varphi(\xi)(\Psi(\xi))) \neq \emptyset.$$

It follows that there exists $\nu: N \rightarrow TN$ of class C^1 with $\nu(q) \in \Psi(q)$ for every $q \in N$, hence the assertion. \square

LEMMA 3.5. *Let \tilde{N} be a submanifold of class C^2 of \mathbb{R}^n , \tilde{M} be the closure of an open subset of \tilde{N} with locally Lipschitz boundary, A be an open subset of \mathbb{R}^n with $\tilde{N} \subseteq A$ and $\pi: A \rightarrow \tilde{N}$ be a retraction of class C^2 such that π is Lipschitz continuous of constant 2. Then there exists a map $\nu: \tilde{N} \rightarrow \mathbb{R}^n$ of class C^1 such that the following facts hold:*

- (a) for any $q \in \tilde{N}$ we have $\nu(q) \in T_q \tilde{N}$;
 (b) for any $q \in \tilde{M}$ there exists $\delta > 0$ such that

$$\text{if } \begin{cases} \xi \in B_\delta(q), \\ \pi(\xi) \in \tilde{M}, \\ 0 < t \leq \delta, \\ v \in B_\delta(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + tv) \in \text{int}(\tilde{M});$$

- (c) for every compact subset $K \subseteq \tilde{M}$ there exist $\hat{r}, \hat{c} > 0$ satisfying

$$\pi((1-t)q + t\pi(\xi + \rho\nu(\xi))) \in \tilde{M}$$

whenever $q \in \tilde{M}$, $\xi \in K$, $\hat{c}|q - \xi| \leq \rho \leq \hat{r}$ and $t \in [0, 1]$.

PROOF. By Theorem 3.4 there exists a map $\nu: \tilde{N} \rightarrow \mathbb{R}^n$ of class C^1 such that for any $q \in \tilde{N}$ it is $\nu(q) \in T_q \tilde{N}$.

To prove (b), assume by contradiction that $q \in \tilde{M}$, $\xi_h \rightarrow q$, $t_h \rightarrow 0^+$ and $v_h \rightarrow \nu(q)$ with $\pi(\xi_h) \in \tilde{M}$ and $\pi(\xi_h + t_h v_h) \notin \text{int}(\tilde{M})$.

Let (U, φ) be the chart at q such that $\varphi: U \rightarrow T_q \tilde{N}$, $\varphi(q) = 0$ and $\pi(q + \varphi(\xi)) = \xi$ for any $\xi \in U$; in particular, $\nu(q) \in \text{Hyp}_0 \varphi(U \cap \tilde{M})$.

Then we have

$$\varphi(\pi(\xi_h + t_h v_h)) \notin \text{int}(\varphi(U \cap \tilde{M})).$$

Since

$$\varphi(\pi(\xi_h + t_h v_h)) = \varphi(\pi(\xi_h)) + t_h(d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h)$$

with $\varepsilon_h \rightarrow 0$ in $T_q \tilde{N}$, it follows that $d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h \in T_q \tilde{N}$ and

$$\varphi(\pi(\xi_h + t_h v_h)) \in \text{int}(\varphi(U \cap \tilde{M}))$$

for large h , which is absurd.

Now let us prove (c). By contradiction, let (q_h) in \tilde{M} , (ξ_h) in K , (t_h) in $[0, 1]$, $\rho_h \rightarrow 0$ with $h|q_h - \xi_h| \leq \rho_h \leq 1/h$ and

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) \notin \tilde{M}.$$

Up to a subsequence $\xi_h \rightarrow \xi$ in K , $q_h \rightarrow \xi$ in \tilde{M} and $t_h \rightarrow t$ in $[0, 1]$. It is

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) = \pi\left(q_h + t_h\rho_h\left(\frac{\pi(\xi_h + \rho_h\nu(\xi_h)) - q_h}{\rho_h}\right)\right).$$

On the other hand,

$$\begin{aligned} \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} - \nu(\xi) &= \frac{\pi(\xi_h + \rho_h \nu(\xi)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} \\ &\quad + \frac{\xi_h - q_h}{\rho_h} + \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h}. \end{aligned}$$

By [11, Theorem 4.4], it is

$$\lim_h \frac{\pi(\xi_h + \rho_h \nu(\xi)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} = 0.$$

Moreover, by the lipschitzianity of π it is also

$$\left| \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h} \right| \leq 2|\nu(\xi_h) - \nu(\xi)|.$$

It follows that

$$\lim_h \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} = \nu(\xi),$$

hence by (a) it is

$$\pi\left(q_h + t_h \rho_h \left(\frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h}\right)\right) \in \widetilde{M}$$

for large h , which is a contradiction. \square

DEFINITION 3.6. A subset E of N is said to be a LNR in N if there exists an open neighbourhood U of E in N and a locally Lipschitzian retraction $r: U \rightarrow E$.

THEOREM 3.7. *The set M is a LNR in N .*

PROOF. By [14, §2, Theorems 2.10 and 2.14], we may assume that N is a smooth submanifold of \mathbb{R}^n . By [14, §4, Theorem 5.1], there exist an open subset A of \mathbb{R}^n with $N \subseteq A$ and a retraction $\pi: A \rightarrow N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Let $\nu: N \rightarrow \mathbb{R}^n$ be as in Lemma 3.5. By (b) of Lemma 3.5, for every $q \in M$ there exists $\delta_q > 0$ such that

$$\text{if } \begin{cases} \xi \in B_{\delta_q}(q), \\ \pi(\xi) \in M, \\ 0 < t \leq \delta_q, \\ v \in B_{\delta_q}(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + tv) \in \text{int}(M).$$

Let $\delta'_q \in]0, \delta_q]$ be such that

$$\text{if } \begin{cases} \xi \in B_{\delta'_q}(q), \\ 0 \leq t \leq \delta'_q, \end{cases} \quad \text{then } \begin{cases} \xi + t\nu(\xi) \in B_{\delta_q}(q), \\ \nu(\xi) \in B_{\delta_q/2}(\nu(q)), \\ |\xi - q| + \delta_q |\nu(\xi) - \nu(q)| \leq \delta_q^2/4. \end{cases}$$

For every $q \in M$, define

$$U_q = \{\xi \in B_{\delta'_q}(q) : \pi(\xi + \delta'_q \nu(\xi)) \in \text{int}(M)\}, \quad U = \bigcup_{q \in M} U_q.$$

For every $\xi \in U$, let $T(\xi) = \min\{t \geq 0 : \pi(\xi + t\nu(\xi)) \in M\}$. It is easy to see that, if $q \in M$ and $\xi \in U_q$, then

$$T(\xi) < \delta'_q, \quad \xi + T(\xi)\nu(\xi) \in B_{\delta'_q}(q), \quad \pi(\xi + T(\xi)\nu(\xi)) \in M$$

and

$$(3.1) \quad \text{if } \begin{cases} 0 \leq t \leq \delta_q, \\ v \in B_{\delta_q}(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + T(\xi)\nu(\xi) + tv) \in M.$$

Let now $q \in M$ and $\xi_1, \xi_2 \in U_q$ with $\xi_1 \neq \xi_2$. We set

$$s = \frac{2}{\delta_q} (|\xi_1 - \xi_2| + T(\xi_1)|\nu(\xi_1) - \nu(\xi_2)|)$$

and

$$v = \nu(\xi_2) - \frac{1}{s}(\xi_1 - \xi_2 + T(\xi_1)(\nu(\xi_1) - \nu(\xi_2))).$$

We have $s \in]0, \delta_q]$ and $v \in B_{\delta_q}(\nu(q))$. If we consider $t = T(\xi_1) + s$, an easy calculation shows that

$$\xi_2 + t\nu(\xi_2) = \xi_1 + T(\xi_1)\nu(\xi_1) + sv.$$

By (3.1) it follows that $\pi(\xi_2 + t\nu(\xi_2)) \in M$, hence $T(\xi_2) \leq t$. Therefore we get

$$T(\xi_2) \leq T(\xi_1) + s \leq T(\xi_1) + \frac{2}{\delta_q} (|\xi_1 - \xi_2| + \delta_q |\nu(\xi_1) - \nu(\xi_2)|);$$

exchanging the role of ξ_1 and ξ_2 we have

$$|T(\xi_1) - T(\xi_2)| \leq \frac{2}{\delta_q} (|\xi_1 - \xi_2| + \delta_q |\nu(\xi_1) - \nu(\xi_2)|),$$

hence T is locally Lipschitzian. It follows that the map $r: U \rightarrow M$ defined by $r(\xi) = \pi(\xi + T(\xi)\nu(\xi))$ is a locally Lipschitzian retraction. Therefore M is an LNR in \mathbb{R}^n , in particular in N . \square

In the second part of this section, we recall some abstract notions and results of nonsmooth analysis.

Let Y be a metric space endowed with the metric d and let $f: Y \rightarrow \overline{\mathbb{R}}$ be a function. We set

$$\text{epi}(f) = \{(u, \lambda) \in Y \times \mathbb{R} : f(u) \leq \lambda\}.$$

In the following, $Y \times \mathbb{R}$ will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = (d(u, v)^2 + (\lambda - \mu)^2)^{1/2}$$

and $\text{epi}(f)$ with the induced metric.

DEFINITION 3.8. For every $u \in Y$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ 's in $[0, \infty[$ such that there exist $r > 0$ and a continuous map

$$\mathcal{H}: (B_r(u, f(u)) \cap \text{epi}(f)) \times [0, r] \rightarrow Y$$

satisfying

$$d(\mathcal{H}((v, \mu), t), v) \leq t, \quad f(\mathcal{H}((v, \mu), t)) \leq \mu - \sigma t,$$

whenever $(v, \mu) \in B_r(u, f(u)) \cap \text{epi}(f)$ and $t \in [0, r]$.

The extended real number $|df|(u)$ is called *the weak slope* of f at u .

The above notion has been introduced in [9], following an equivalent approach. When f is continuous, it has been independently introduced also in [17], while a variant appears in [15], [16]. The version we have recalled here is taken from [2].

PROPOSITION 3.9. *Let $u \in Y$ with $f(u) \in \mathbb{R}$. Assume there exist $r, c, \sigma > 0$ and a continuous map*

$$\mathcal{H}: \{v \in B_r(u) : f(v) < f(u) + r\} \times [0, r] \rightarrow Y$$

such that for any $v \in B_r(u)$ with $f(v) < f(u) + r$ and any $t \in [0, r]$ it is

$$d(\mathcal{H}(v, t), v) \leq ct, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

Then we have $|df|(u) \geq \sigma/c$.

PROOF. See [11, Proposition 2.3]. □

Now, according to [8], we define a function $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$ by $\mathcal{G}_f(u, \lambda) = \lambda$. Of course, \mathcal{G}_f is Lipschitzian of constant 1.

PROPOSITION 3.10. *For every $u \in Y$ with $f(u) \in \mathbb{R}$, we have $f(u) = \mathcal{G}_f(u, f(u))$ and*

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ \infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

PROOF. See [2, Proposition 2.3]. □

The previous proposition allows us to reduce, at some extent, the study of the general function f to that of the continuous function \mathcal{G}_f . For this purpose, the next result will be useful.

PROPOSITION 3.11. *Let $(u, \lambda) \in \text{epi}(f)$ with $f(u) < \lambda$. Assume that for every $\varepsilon > 0$ there exist $r > 0$ and a continuous map*

$$\mathcal{H}: \{v \in B_r(u) : f(v) < \lambda + r\} \times [0, r] \rightarrow Y$$

such that for any $v \in B_r(u)$ with $f(v) < \lambda + r$ and any $t \in [0, r]$ it is

$$\begin{aligned} d(\mathcal{H}(v, t), v) &\leq \varepsilon t, \\ f(\mathcal{H}(v, t)) &\leq (1 - t)f(v) + t(f(u) + \varepsilon). \end{aligned}$$

Then we have $|d\mathcal{G}_f|(u, \lambda) = 1$.

PROOF. See [10, Corollary 2.11]. \square

Definition 3.8 may be simplified, when f is continuous.

PROPOSITION 3.12. *Let $f: Y \rightarrow \mathbb{R}$ be continuous. Then $|df|(u)$ is the supremum of the σ 's in $[0, +\infty[$ such that there exist $r > 0$ and a continuous map*

$$\mathcal{H}: B_r(u) \times [0, r] \rightarrow Y$$

satisfying

$$(3.2) \quad d(\mathcal{H}(v, t), v) \leq t, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,$$

whenever $v \in B_r(u)$ and $t \in [0, r]$.

PROOF. See [2, Proposition 2.2]. \square

By means of the weak slope, we can now introduce the two main notions of critical point theory.

DEFINITION 3.13. We say that $u \in Y$ is a (lower) critical point of f , if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f , if there exists a (lower) critical point $u \in Y$ of f with $f(u) = c$.

REMARK 3.14. Let \tilde{d} be another metric on Y and let $u \in Y$. Assume that there exist a neighbourhood U of u and $c > 0$ such that, for all $v, w \in U$,

$$d(v, w) \leq c\tilde{d}(v, w), \quad \tilde{d}(v, w) \leq cd(v, w).$$

Then one has $|df|(u) = 0$ if and only if $|\tilde{d}f|(u) = 0$, where $|\tilde{d}f|(u)$ is the weak slope with respect to \tilde{d} .

DEFINITION 3.15. Let $c \in \mathbb{R}$. A sequence (u_h) in Y is said to be a Palais–Smale sequence at level c ($(PS)_c$ -sequence, for short) for f , if $f(u_h) \rightarrow c$ and $|df|(u_h) \rightarrow 0$.

We say that f satisfies the Palais–Smale condition at level c ($(PS)_c$, for short), if every $(PS)_c$ -sequence (u_h) for f admits a convergent subsequence (u_{h_k}) in Y .

DEFINITION 3.16. A topological space Z is said to be weakly locally contractible, if every $u \in Z$ admits a neighbourhood U which is contractible in Z .

THEOREM 3.17. *Let Y be weakly locally contractible with $\text{cat}Y = \infty$, let $f: Y \rightarrow \mathbb{R}$ be continuous and bounded from below and assume that $\{u \in Y : f(u) \leq c\}$ is complete and $(\text{PS})_c$ hold for every $c \in \mathbb{R}$. Then there exists a sequence (u_h) of critical points of f with $f(u_h) \rightarrow \infty$.*

PROOF. See [7, Theorem 3.6] and [5, Theorem 1.4.13]. □

COROLLARY 3.18. *Let Z be a metrizable topological space and $f: Z \rightarrow \mathbb{R}$ a continuous function. Assume that*

- (a) Z is weakly locally contractible and $\text{cat}Z = \infty$;
- (b) for every $c \in \mathbb{R}$, the set $\{u \in Z : f(u) \leq c\}$ is compact.

Then, for every compatible metric on Z , there exists a sequence (u_h) of critical points of f with $f(u_h) \rightarrow \infty$.

4. Proof of the main results

In the first part of this section, let N be a differentiable manifold of class C^2 and M be a LNR in N . Let us consider

$$\Lambda(M) = \{\gamma \in C([0, 1]; M) : \gamma(0) = \gamma(1)\}$$

endowed with the uniform topology ($\Lambda(M)$ is called the *free loop space* of M) and

$$X = \{\gamma \in W^{1,p}(0, 1; M) : \gamma(0) = \gamma(1)\}.$$

Let $L: \mathbb{R} \times TN \rightarrow \mathbb{R}$ be a function of class C^1 satisfying (2.1)–(2.4) and define a lower semicontinuous functional $f: \Lambda(M) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$f(\gamma) = \begin{cases} \int_0^1 L(s, \gamma(s), \gamma'(s)) ds & \text{if } \gamma \in X, \\ \infty & \text{if } \gamma \in \Lambda(M) \setminus X. \end{cases}$$

In the following, we will consider the metrizable topological space $\text{epi}(f)$, endowed with the topology induced by $\Lambda(M) \times \mathbb{R}$, and the continuous function $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$.

Given a Riemannian structure on N , for every $\gamma, \eta \in W^{1,p}(0, 1; M)$, we set as before

$$d_1(\gamma, \eta) = \int_0^1 d(\gamma(s), \eta(s)) ds,$$

$$d_\infty(\gamma, \eta) = \max\{d(\gamma(s), \eta(s)) : 0 \leq s \leq 1\},$$

where d is the distance on N associated with the Riemannian structure.

LEMMA 4.1. *Consider a Riemannian structure on N . Let (γ_h) be a sequence in $W^{1,p}(0,1;M)$ convergent to $\gamma \in W^{1,p}(0,1;M)$ with respect to the topology induced by d_1 and such that $(f(\gamma_h))$ is bounded. Then (γ_h) is convergent to γ with respect to the uniform convergence.*

PROOF. Let U be an open subset of M with \bar{U} compact such that $\gamma([0,1]) \subseteq U$. First of all we claim that $\gamma_h([0,1]) \subseteq U$ for h large enough. By contradiction, let $h_k \rightarrow \infty$ and $(s_k) \subseteq [0,1]$ such that $\gamma_{h_k}(s_k) \notin U$. Up to a subsequence we have that $s_k \rightarrow s \in [0,1]$ and $\gamma_{h_k} \rightarrow \gamma$ a.e. in $[0,1]$. Let $a \in [0,1]$ be such that $\gamma_{h_k}(a) \rightarrow \gamma(a)$. Assume that $a < s$. It follows that, for k large enough, there exists $b_k \in]a, s_k]$ such that $\gamma_{h_k}([a, b_k]) \subseteq \bar{U}$ and $\gamma_{h_k}(b_k) \notin U$. Since \bar{U} is compact, there exists $C > 0$ such that, by (2.1),

$$\int_a^{b_k} L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq \int_a^{b_k} (k(\gamma_{h_k})|\gamma'_{h_k}|^p - d) ds \geq \int_a^{b_k} (C|\gamma'_{h_k}|^p - d) ds.$$

Moreover, again by (2.1), we have

$$\int_0^a L(s, \gamma_{h_k}, \gamma'_{h_k}) ds + \int_{b_k}^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq -d(1 - b_k + a).$$

It follows that

$$f(\gamma_{h_k}) = \int_0^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq C \int_a^{b_k} |\gamma'_{h_k}|^p ds - d.$$

Hence for every $\sigma, \tau \in [a, b_k]$ with $\tau \leq \sigma$ we have

$$\begin{aligned} d(\gamma_{h_k}(\sigma), \gamma_{h_k}(\tau)) &\leq \int_\tau^\sigma |\gamma'_{h_k}(t)| dt \leq \left(\int_\tau^\sigma |\gamma'_{h_k}(t)|^p dt \right)^{1/p} |\sigma - \tau|^{1/p'} \\ &\leq \left(\int_a^{b_k} |\gamma'_{h_k}(t)|^p dt \right)^{1/p} |\sigma - \tau|^{1/p'} \leq \left(\frac{f(\gamma_{h_k}) + d}{C} \right)^{1/p} |\sigma - \tau|^{1/p'}. \end{aligned}$$

It follows that (γ_{h_k}) is equi-uniformly continuous on $[a, b_k]$. Up to a further subsequence we have that $\gamma_{h_k}(b_k) \rightarrow x \in \partial U$. Since $\inf\{d(\gamma(a), y) : y \in \partial U\} > 0$, if a is sufficiently closed to s a contradiction follows.

Arguing as above, for any $s, t \in [0,1]$ we have that

$$d(\gamma_h(s), \gamma_h(t)) \leq \left(\frac{f(\gamma_h) + d}{C} \right)^{1/p} |s - t|^{1/p'}.$$

Since $(f(\gamma_h))$ is bounded, we deduce that (γ_h) is equi-uniformly continuous on $[0,1]$. Therefore it is easy to see that (γ_h) is convergent to γ with respect to the uniform convergence. \square

THEOREM 4.2. Consider any Riemannian structure on N and define on $\text{epi}(f)$ the metric

$$(4.1) \quad d((\gamma, \lambda), (\eta, \mu)) = \sqrt{d_1(\gamma, \eta)^2 + |\lambda - \mu|^2}.$$

Then the following facts hold:

- (a) the metric d is compatible with the topology of $\text{epi}(f)$;
- (b) the set of critical points of $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$ does not depend on the Riemannian structure;
- (c) if $(\gamma, \lambda) \in \text{epi}(f)$ is a critical point of \mathcal{G}_f with $f(\gamma) = \lambda$, then γ is the restriction to $[0, 1]$ of a 1-periodic generalized solution of the Lagrangian system associated to L on M .

PROOF. (a) is an easy consequence of Lemma 4.1; (b) follows from Remarks 2.1 and 3.14. Let us consider property (c). First, let us prove that γ is L -stationary on $[0, 1]$. By contradiction, assume that there exist $r, c, \sigma > 0$ and

$$\mathcal{H}: \{\eta \in W^{1,p}(0, 1; M) : d_\infty(\eta, \gamma) < r, f(\eta) < f(\gamma) + r\} \times [0, r] \rightarrow W^{1,p}(0, 1; M)$$

continuous from the product of the uniform convergence and that of \mathbb{R} to that of the uniform convergence such that

$$\begin{aligned} \mathcal{H}(\eta, t)(0) &= \eta(0), & \mathcal{H}(\eta, t)(1) &= \eta(1), \\ d_1(\mathcal{H}(\eta, t), \eta) &\leq ct, & f(\mathcal{H}(\eta, t)) &\leq f(\eta) - \sigma t. \end{aligned}$$

If $r' \in]0, r[$ is such that if $\eta \in W^{1,p}(0, 1; M)$ with $d_1(\eta, \gamma) < r'$ and $f(\eta) < f(\gamma) + r'$, then $d_\infty(\eta, \gamma) < r$. Then the restriction of \mathcal{H} to

$$\{\eta \in W^{1,p}(0, 1; M) : d_1(\eta, \gamma) < r', f(\eta) < f(\gamma) + r'\} \times [0, r']$$

satisfies the assumptions of Proposition 3.9. It follows that γ is not a critical point of f , a contradiction.

Finally, if we define

$$\widehat{\gamma}(s) = \begin{cases} \gamma\left(s + \frac{1}{2}\right) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \gamma\left(s - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

it turns out that also $\widehat{\gamma}$ is L -stationary on $[0, 1]$, whence the assertion. □

LEMMA 4.3. Define $\mathcal{E}: \Lambda([0, 1]; N) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}(\gamma) = \begin{cases} \int_0^1 |\gamma'(s)|^p ds & \text{if } \gamma \in X, \\ \infty & \text{if } \gamma \in \Lambda([0, 1]; N) \setminus X. \end{cases}$$

Then $\text{epi}(f)$ is homotopically equivalent to $\text{epi}(\mathcal{E})$.

PROOF. By (2.1), for every $\gamma \in X$ we have

$$\mathcal{E}(\gamma) \leq \left\| \frac{1}{k \circ \gamma} \right\|_{\infty} (f(\gamma) + d), \quad f(\gamma) \leq \|c \circ \gamma\|_{\infty} (\mathcal{E}(\gamma) + 1).$$

Define $\Phi: \text{epi}(f) \rightarrow \text{epi}(\mathcal{E})$ and $\Psi: \text{epi}(\mathcal{E}) \rightarrow \text{epi}(f)$ by

$$\Phi(\gamma, \lambda) = \left(\gamma, \left\| \frac{1}{k \circ \gamma} \right\|_{\infty} (\lambda + d) \right), \quad \Psi(\gamma, \lambda) = (\gamma, \|c \circ \gamma\|_{\infty} (\lambda + 1)).$$

Then Ψ and, by Lemma 4.1, Φ are continuous and it is readily seen that $\Psi \circ \Phi$ is homotopic to the identity of $\text{epi}(f)$ while $\Phi \circ \Psi$ is homotopic to the identity of $\text{epi}(\mathcal{E})$. \square

LEMMA 4.4. *Let U be an open subset of \mathbb{R}^n and let*

$$\Lambda^1(U) = \{\gamma \in W^{1,p}(0, 1; U) : \gamma(0) = \gamma(1)\}$$

endowed with the $W^{1,p}$ -metric. Then there exists a continuous map

$$\mathcal{K}: \Lambda(U) \times [0, 1] \rightarrow \Lambda(U)$$

such that

$$\begin{aligned} \mathcal{K}(\gamma, 0) &= \gamma, \quad \mathcal{K}(\gamma, 1) \in \Lambda^1(U) \quad \text{for all } \gamma \in \Lambda(U), \\ \mathcal{K}(\cdot, 1): \Lambda(U) &\rightarrow \Lambda^1(U) \text{ is continuous,} \\ \mathcal{K}(\Lambda^1(U) \times [0, 1]) &\subseteq \Lambda^1(U), \\ \|[\mathcal{K}(\gamma, t)]'\|_p &\leq \|\gamma'\|_p \quad \text{for all } \gamma \in \Lambda^1(U) \text{ and all } t \in [0, 1]. \end{aligned}$$

PROOF. Let (ρ_{ε}) be a sequence of mollifiers of class C_c^{∞} on \mathbb{R}^n . Let $R_0\gamma = \gamma$ and for every $\varepsilon > 0$ let

$$R_{\varepsilon}\gamma(s) = \int_{\mathbb{R}} \rho_{\varepsilon}(s-t) \bar{\gamma}(t) dt,$$

where $\bar{\gamma}: \mathbb{R} \rightarrow U$ is 1-periodic such that $\bar{\gamma}|_{[0,1]} = \gamma$. It turns out that there exists a continuous function $\lambda: \Lambda(U) \rightarrow]0, 1]$ such that for every $\gamma \in \Lambda(U)$ it is

$$R_{\varepsilon}\gamma(s) \in U \quad \text{for all } \varepsilon \in]0, \lambda(\gamma)], \text{ and all } s \in [0, 1].$$

Let $\mathcal{K}: \Lambda(U) \times [0, 1] \rightarrow \Lambda(U)$ defined by $\mathcal{K}(\gamma, t) = R_{t\lambda(\gamma)}\gamma$. It is readily seen that \mathcal{K} satisfies all the properties required and the assertion follows. \square

LEMMA 4.5. *The map $\tilde{\pi}: \text{epi}(\mathcal{E}) \rightarrow \Lambda(M)$ defined by $\tilde{\pi}(\gamma, \lambda) = \gamma$ is a homotopy equivalence ($\text{epi}(\mathcal{E})$ is endowed with the product of the uniform topology and that of \mathbb{R}).*

PROOF. Arguing as in the proof of Theorem 3.7, we may assume that N is a smooth submanifold of \mathbb{R}^n and we may consider an open subset A of \mathbb{R}^n with $N \subseteq A$ and a retraction $\pi: A \rightarrow N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Since M is a LNR in N , there exists an open neighbourhood U of M in N and a locally Lipschitzian retraction $r: U \rightarrow M$. Since $r \circ \pi: \pi^{-1}(U) \rightarrow M$ is a locally Lipschitzian retraction, then M is also a LNR in \mathbb{R}^n . Now taking into account Lemma 4.4 the proof follows the same argument of [11, Theorem 5.3]. \square

THEOREM 4.6. *The map $\hat{\pi}: \text{epi}(f) \rightarrow \Lambda(M)$ defined by $\hat{\pi}(\gamma, \lambda) = \gamma$ is a homotopy equivalence ($\text{epi}(f)$ is endowed with the product of the uniform topology and that of \mathbb{R}).*

PROOF. Combining Lemmas 4.3 and 4.5 the assertion follows. \square

From now on, we assume that M is the closure of an open subset in N with locally Lipschitz boundary. By Theorem 3.7, M is a LNR in N .

THEOREM 4.7. *Consider a Riemannian structure on N and the metric defined in (4.1). Let (γ, λ) be in $\text{epi}(f)$ such that $f(\gamma) < \lambda$. Then*

$$|d\mathcal{G}_f|(\gamma, \lambda) = 1.$$

PROOF. Arguing as in the proof of Theorem 3.7, we may assume that N is a smooth submanifold of \mathbb{R}^n and we may consider an open subset A of \mathbb{R}^n with $N \subseteq A$ and a retraction $\pi: A \rightarrow N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Therefore we may also consider the function $\tilde{L}: \mathbb{R} \times A \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that \tilde{L} is a C^1 -extension of L to $\mathbb{R} \times A \times \mathbb{R}^n$ and such that there exist two continuous functions $\tilde{c}, \tilde{k}: A \rightarrow]0, \infty[$ and $d \in \mathbb{R}$ such that for every $(s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n$ one has

$$(4.2) \quad |D_q \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^p),$$

$$(4.3) \quad |D_v \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^{p-1}),$$

$$(4.4) \quad \tilde{L}(s, q, v) \geq \tilde{k}(q)|v|^p - d,$$

$$(4.5) \quad \tilde{L}(s, q, \cdot) \text{ is convex.}$$

First of all we claim that there exist $\bar{\varepsilon} > 0$ and $\bar{C} > 0$ such that for every $\eta_1, \eta_2 \in X$ with $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$ and for every $t \in [0, 1]$ it is

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq \bar{C}t \left(1 + \int_0^1 \tilde{L}(s, \eta_1, \eta'_1) ds + \int_0^1 \tilde{L}(s, \eta_2, \eta'_2) ds \right) \|\eta_1 - \eta_2\|_\infty. \end{aligned}$$

Let $\varepsilon > 0$ be such that if $\eta \in W^{1,p}(0, 1; \mathbb{R}^n)$ with $\|\eta - \gamma\|_\infty \leq \varepsilon$ then $\eta \in W^{1,p}(0, 1; A)$. Since π is of class C^∞ and Lipschitz continuous of constant 2, there exists $\bar{\varepsilon} \in]0, \varepsilon]$ and $\tilde{C} \geq 2$ such that for every $\eta_1, \eta_2 \in W^{1,p}(0, 1; A)$ with $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$ and for every $\xi \in \mathbb{R}^n$ it is

$$|\pi(\eta_1) - \pi(\eta_2)| \leq \tilde{C}|\eta_1 - \eta_2|, \quad |[\pi'(\eta_1) - \pi'(\eta_2)]\xi| \leq \tilde{C}|\eta_1 - \eta_2|\|\xi\|.$$

Now let $\eta_1, \eta_2 \in X$ with $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$ and let $t \in [0, 1]$. For every $\vartheta \in [0, 1]$ we have

$$(4.6) \quad \begin{aligned} & |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)| \\ & = |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1)| \\ & \leq |\eta'_1| + \tilde{C}|\eta_2 - \eta_1|\|\eta'_1\| \leq \widehat{C}(|\eta'_1| + |\eta'_2|) \end{aligned}$$

for some $\widehat{C} > 0$. Unless reducing $\bar{\varepsilon}$, we may suppose that \tilde{c}, \tilde{k} are constants on $\{\eta \in W^{1,p}(0, 1; A) : d_\infty(\eta, \gamma) < \bar{\varepsilon}\}$. Furthermore, applying Lagrange's Theorem, (4.2), (4.3) and (4.6) it is, for some $\vartheta \in [0, 1]$,

$$\begin{aligned} & \tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1) \\ & = D_q \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\ & \quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \cdot (\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1) \\ & \quad + D_v \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\ & \quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \cdot (\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \\ & \leq C(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^p) |\pi(\eta_1 + t(\eta_2 - \eta_1)) - \pi(\eta_1)| \\ & \quad + C(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^{p-1}) \\ & \quad \cdot |\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1| \\ & \leq C_2 t(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^p) |\eta_1 - \eta_2| \\ & \quad + C_2 t(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^{p-1}) |\eta'_1| |\eta_1 - \eta_2| \\ & \leq C_3 t(1 + |\eta'_1|^p + |\eta'_2|^p) |\eta_1 - \eta_2| + C_3 t(1 + |\eta'_1|^{p-1} + |\eta'_2|^{p-1}) |\eta'_1| |\eta_1 - \eta_2| \\ & = C_3 t(1 + |\eta'_1|^p + |\eta'_2|^p) |\eta_1 - \eta_2| + C_3 t(|\eta'_1| + |\eta'_1|^p + |\eta'_1| |\eta'_2|^{p-1}) |\eta_1 - \eta_2| \end{aligned}$$

for some $C_3 > 0$. It follows that

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq C_3 t(1 + 2\|\eta'_1\|_p^p + \|\eta'_2\|_p^p + \|\eta'_1\|_1 + \|\eta'_1\|_p \|\eta'_2\|_p^{p-1}) \|\eta_1 - \eta_2\|_\infty \\ & \leq C_4 t(1 + \|\eta'_1\|_p^p + \|\eta'_2\|_p^p) \|\eta_1 - \eta_2\|_\infty \end{aligned}$$

for some $C_4 > 0$. Finally, applying (4.4) we may find $\bar{C} > 0$ such that

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq \bar{C}t \left(1 + \int_0^1 \tilde{L}(s, \eta_1, \eta'_1) ds + \int_0^1 \tilde{L}(s, \eta_2, \eta'_2) ds \right) \|\eta_1 - \eta_2\|_\infty \end{aligned}$$

and the claim follows. Let $\varepsilon > 0$, $K = \gamma([0, 1])$ and let $\bar{\varepsilon}, \bar{C} > 0$ be as before. Let $C_2 = \bar{C}(1 + 2\lambda + \varepsilon)$. Let now \hat{r} and \hat{c} be as in (c) of Lemma 3.5, and let

$$\hat{\gamma}(s) = \gamma(s) + \rho\nu(\gamma(s)),$$

where $\rho \in]0, \hat{r}[$ is such that

$$\|\pi(\hat{\gamma}) - \gamma\|_\infty \leq \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{8C_2}, \bar{\varepsilon} \right\}, \quad f(\pi \circ \hat{\gamma}) \leq f(\gamma) + \frac{\varepsilon}{4}.$$

Let $r \in]0, \varepsilon/2[$ be such that if $\|\eta - \gamma\|_1 < r$ with $f(\eta) < \lambda + r$, then $\|\eta - \gamma\|_\infty \leq \min\{\rho/\hat{c}, \varepsilon/4, \varepsilon/8C_2, \bar{\varepsilon}\}$. Then, again by (c) of Lemma 3.5 it is possible to define a continuous map

$$\mathcal{H}: \{\eta \in X : \|\eta - \gamma\|_1 < r, f(\eta) < \lambda + r\} \times [0, r] \rightarrow X$$

by

$$\mathcal{H}(\eta, t) = \pi((1-t)\eta + t\pi(\hat{\gamma})).$$

It is

$$\|\mathcal{H}(\eta, t) - \eta\|_\infty \leq 2t\|\pi(\hat{\gamma}) - \eta\|_\infty \leq 2t(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \varepsilon t$$

and hence also

$$\|\mathcal{H}(\eta, t) - \eta\|_1 \leq \varepsilon t.$$

Since \tilde{L} is convex with respect to the third variable, we get

$$\begin{aligned} & f(\mathcal{H}(\eta, t)) \\ & = \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\eta' + t((\pi \circ \hat{\gamma})' - \eta'))) ds \\ & \leq \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') ds \\ & \quad + t \left[\int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\pi \circ \hat{\gamma})') ds \right. \\ & \quad \left. - \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') ds \right]. \end{aligned}$$

Furthermore, it is

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') - \tilde{L}(s, \eta, \eta')] ds \right| \\ & \leq \bar{C}t(1 + f(\eta) + f(\pi \circ \hat{\gamma}))\|\pi(\hat{\gamma}) - \eta\|_\infty \\ & < \bar{C}t(1 + 2\lambda + \varepsilon)(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \frac{\varepsilon}{4}t \end{aligned}$$

and

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\pi \circ \hat{\gamma})') \right. \\ & \quad \left. - \tilde{L}(s, \pi \circ \hat{\gamma}, (\pi \circ \hat{\gamma})')] ds \right| \\ & \leq \bar{C}t(1 + f(\eta) + f(\pi \circ \hat{\gamma}))\|\pi(\hat{\gamma}) - \eta\|_\infty \\ & < \bar{C}t(1 + 2\lambda + \varepsilon)(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \frac{\varepsilon}{4}t. \end{aligned}$$

Therefore we finally get

$$f(\mathcal{H}(\eta, t)) \leq f(\eta) + \frac{\varepsilon}{4}t + \left(f(\pi \circ \hat{\gamma}) - f(\eta) + \frac{\varepsilon}{2} \right) t \leq f(\eta) + t(f(\gamma) - f(\eta) + \varepsilon)$$

and the assertion follows from Proposition 3.11. \square

Finally, we can prove Theorem 2.6.

PROOF. Now assume also that M is compact, 1-connected and non-contractible in itself. By Theorem 3.7, we have that M is a LNR in N , in particular an absolute neighbourhood retract. From [13, Corollary 1.4] it follows that $\text{cat}\Lambda(M) = \infty$. Moreover, $\Lambda(M)$ also is an absolute neighbourhood retract, hence weakly locally contractible. On the other hand, by Theorem 4.6 $\Lambda(M)$ is homotopically equivalent to $\text{epi}(f)$. Therefore $\text{cat}\text{epi}(f) = \infty$ and $\text{epi}(f)$ is weakly locally contractible. Let now $c \in \mathbb{R}$ and consider the sublevel

$$\mathcal{G}_f^c = \{(\gamma, \lambda) \in \Lambda(M) \times \mathbb{R} : f(\gamma) \leq \lambda \leq c\}.$$

Since M is compact, from (2.1) and Ascoli's theorem we deduce that \mathcal{G}_f^c is compact. By Corollary 3.18, there exists a sequence (γ_h, λ_h) of critical points of \mathcal{G}_f^c with respect to the metric (4.1) with $\lambda_h \rightarrow \infty$. By Theorem 4.7 we have that $\lambda_h = f(\gamma_h)$. From (c) of Theorem 4.2 the assertion follows. \square

The next two results correspond to the well-known equation $d/ds H = -D_s L$, where H is the Hamiltonian function associated with L .

THEOREM 4.8. *Let $\gamma \in W^{1,p}(a, b; M)$ be L -stationary. Assume that L does not depend on s . Then the map $\{s \mapsto D_v L(\gamma, \gamma')\gamma' - L(\gamma, \gamma')\}$ is constant a.e.*

PROOF. Arguing as in the proof of Theorem 4.7, we may assume that N is a smooth submanifold of \mathbb{R}^n , A is an open subset of \mathbb{R}^n with $N \subseteq A$ and

$\tilde{L}: A \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 -extension of L to $A \times \mathbb{R}^n$ satisfying (4.2)–(4.5). Assume, for a contradiction, that there exists $\varphi \in C_c^\infty(a, b)$ such that

$$\sigma := \frac{1}{2} \int_a^b \{ [D_v \tilde{L}(\gamma, \gamma') \cdot \gamma' - \tilde{L}(\gamma, \gamma')] \varphi' \} ds > 0.$$

Let $r > 0$ be such that $r \|\varphi'\|_\infty < 1$ and let $\psi: [a, b] \times [0, r] \rightarrow [a, b]$ be the smooth function such that

$$\lambda = \psi(\lambda, t) - t\varphi(\psi(\lambda, t)) \quad \text{for all } \lambda \in [a, b] \text{ and all } t \in [0, r].$$

Unless reducing r we may suppose that the functions c, k in (4.2)–(4.4) are constants on $\{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r\}$. Define $\mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \rightarrow W^{1,p}(a, b; M)$ by

$$\mathcal{H}(\eta, t)(\mu) = \eta(\mu - t\varphi(\mu)).$$

It is easy to see that \mathcal{H} is continuous from the product topology of the uniform convergence and of \mathbb{R} to that of the uniform convergence and that

$$\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b).$$

Moreover, by (4.4)

$$\begin{aligned} d_1(\mathcal{H}(\eta, t), \eta) &= \int_a^b |\eta(\mu - t\varphi(\mu)) - \eta(\mu)| d\mu \\ &= t \int_a^b |\eta'(\mu - t\varphi(\mu))| |1 - t\varphi'(\mu)| d\mu \\ &\leq t \left(\int_a^b |\eta'(\lambda)|^p \frac{1}{|1 - \theta\varphi'(\psi(\lambda, \theta))|^p} d\lambda \right)^{1/p} \left(\int_a^b |1 - t\varphi'(\mu)|^{p'} d\mu \right)^{1/p'} \\ &\leq \frac{t}{(1 - \theta\|\varphi'\|_\infty)^p} \left(\int_a^b |\eta'(\lambda)|^p d\lambda \right)^{1/p} \left(\int_a^b |1 - t\varphi'(\mu)|^{p'} d\mu \right)^{1/p'} \\ &\leq \bar{C}t \left(\int_a^b (L(\eta(\lambda), \eta'(\lambda)) + d) d\lambda \right)^{1/p} < \hat{C}t(f_{a,b}(\gamma) + r + d(b-a))^{1/p}, \end{aligned}$$

for some $\hat{C} > 0$. Following the same argument of the proof of [19, Theorem 5.10] we also have

$$f_{a,b}(\mathcal{H}(\eta, t)) = f_{a,b}(\eta) + t\Theta(\eta, t)$$

where

$$\begin{aligned} \Theta(\eta, t) &= \int_a^b \left[-D_v \tilde{L}(\eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \cdot \eta'(\lambda) \varphi'(\psi(\lambda, t)) \right. \\ &\quad \left. + \tilde{L}(\eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \frac{\varphi'(\psi(\lambda, t))}{1 - t\varphi'(\psi(\lambda, t))} \right] d\lambda. \end{aligned}$$

We claim that, for r sufficiently small, we have $\Theta(\eta, t) \leq -\sigma$ for any $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $0 \leq t \leq r$. By contradiction, let (η_h) be a sequence in $W^{1,p}(a, b; M)$ uniformly convergent to γ with $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + 1/h$ and (t_h) be a non negative sequence convergent to 0 such that $\Theta(\eta_h, t_h) > -\sigma$. Because of (4.4) and $f_{a,b}$ is lower semicontinuous, we have that $f_{a,b}(\eta_h) \rightarrow f_{a,b}(\gamma)$. Again by (4.4) (η_h) is bounded in $W^{1,p}(a, b; M)$ and up to a subsequence $\eta'_h \rightharpoonup \gamma'$ in $L^p(a, b; M)$. Therefore $[1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h \rightharpoonup \gamma'$ in $L^p(a, b; M)$. We have that

$$\begin{aligned} & \int_a^b [\tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - \tilde{L}(\gamma(\lambda), \gamma'(\lambda))] d\lambda \\ &= \int_a^b D_v \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta'_h(\lambda)) \cdot (\eta'_h(\lambda) - \gamma'(\lambda)) d\lambda \\ & \quad + t_h \int_a^b \varphi'(\psi(\lambda, t_h)) D_v \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta'_h(\lambda) \\ & \quad + \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot \eta'_h(\lambda) d\lambda. \end{aligned}$$

By (4.3) we have that $D_v \tilde{L}(\gamma, (1 - \tau) \gamma' + \tau \eta'_h) \in L^{p'}(a, b; M)$ and hence

$$\int_a^b D_v \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta'_h(\lambda)) \cdot (\eta'_h(\lambda) - \gamma'(\lambda)) d\lambda \rightarrow 0.$$

Again by (4.3) we have that

$$\int_a^b \varphi'(\psi(\lambda, t_h)) D_v \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta'_h(\lambda) + \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot \eta'_h(\lambda) d\lambda$$

is bounded. Therefore we have that

$$\int_a^b \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h) d\lambda \rightarrow \int_a^b \tilde{L}(\gamma(\lambda), \gamma'(\lambda)) d\lambda.$$

By [12, Lemma 3.1] applied to the function $\mathcal{F}(\lambda, \xi) = \tilde{L}(\gamma(\lambda), \xi)$ we obtain that

$$\begin{aligned} \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) &\rightarrow \tilde{L}(\gamma, \gamma') \quad \text{in } L^1(a, b; M), \\ D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) &\rightarrow D_v \tilde{L}(\gamma, \gamma') \quad \text{in } L^{p'}(a, b; M) \end{aligned}$$

and there exists $\Psi \in L^1(a, b; M)$ such that $|\eta'_h|^p \leq \Psi$. For some $t \in]0, 1[$ we have that

$$\begin{aligned} & \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ &= D_q \tilde{L}((1 - t) \gamma(\lambda) + t \eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot (\eta_h(\lambda) - \gamma(\lambda)). \end{aligned}$$

By (4.2) we deduce that $D_q \tilde{L}((1 - t) \gamma + t \eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \in L^{p'}(a, b; M)$ and hence

$$[\tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) - \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h)] \rightarrow 0 \quad \text{in } L^1(a, b; M).$$

It follows that

$$\tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \rightarrow \tilde{L}(\gamma, \gamma') \quad \text{in } L^1(a, b; M).$$

Fix $\varepsilon > 0$, let $\delta > 0$ such that for any \mathcal{L}^1 -measurable subset $\Omega \subseteq]a, b[$ with $\mathcal{L}^1(\Omega) < \delta$ we have

$$\int_{\Omega} \Phi(\lambda) d\lambda < \frac{\varepsilon}{2} \quad \text{for all } \Phi \in L^1(a, b; M).$$

Let $R > 0$ be such that $\mathcal{L}^1(\{\lambda \in [a, b] : |\eta'_h(\lambda)| > R\}) < \delta$. Let $\Omega_h = \{\lambda \in [a, b] : |\eta'_h(\lambda)| > R\}$ and $\Omega'_h = \{\lambda \in [a, b] : |\eta'_h(\lambda)| \leq R\}$. By (4.3) we have

$$\begin{aligned} & \int_a^b |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda \\ & \leq \int_{\Omega_h} \bar{C}(1 + \Psi(\lambda)) d\lambda + \int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda \\ & < \frac{\varepsilon}{2} + \int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda. \end{aligned}$$

Since the map

$$\{\lambda \rightarrow [D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))]\}$$

is uniformly continuous on Ω'_h , for h sufficiently large we have

$$\int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda < \frac{\varepsilon}{2}.$$

It follows that

$$\|D_v \tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h)\|_{p'} \rightarrow 0.$$

Therefore

$$D_v \tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \rightarrow D_v \tilde{L}(\gamma, \gamma') \quad \text{in } L^{p'}(a, b; M)$$

and we deduce that

$$\Theta(\eta_h, t_h) \rightarrow \int_a^b \{-D_v \tilde{L}(\gamma, \gamma') \cdot \gamma' + \tilde{L}(\gamma, \gamma')\}' d\lambda = -2\sigma,$$

a contradiction. Finally, we have $f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t$. It follows that γ is not L -stationary, a contradiction. \square

THEOREM 4.9. *Let $\gamma \in W^{1,p}(a, b; M)$ be L -stationary. Assume that for every $s \in \mathbb{R}$ and $q \in M$ one has*

$$(4.7) \quad |D_s L(s, q, v)| \leq c(q)(1 + |v|^p), \quad \text{for all } v \in T_q N,$$

$$(4.8) \quad L(s, q, \cdot) \text{ is strictly convex on } T_q N.$$

Then the map $\{s \mapsto D_v L(s, \gamma, \gamma')\gamma' - L(s, \gamma, \gamma')\}$ belongs to $W^{1,1}(a, b)$ and we have

$$\int_a^b [D_v L(s, \gamma, \gamma')\gamma' - L(s, \gamma, \gamma')]\varphi' ds = \int_a^b D_s L(s, \gamma, \gamma')\varphi ds$$

for all $\varphi \in C_c^\infty(a, b)$.

PROOF. Arguing as in the proof of Theorem 4.7, we may assume that N is a smooth submanifold of \mathbb{R}^n , A is an open subset of \mathbb{R}^n with $N \subseteq A$ and $\tilde{L}: \mathbb{R} \times A \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 -extension of L to $\mathbb{R} \times A \times \mathbb{R}^n$ satisfying (4.2)–(4.4) and such that for every $(s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n$ one has

$$(4.9) \quad |D_s \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^p),$$

$$(4.10) \quad \tilde{L}(s, q, \cdot) \text{ is strictly convex.}$$

Assume, for a contradiction, that there exists $\varphi \in C_c^\infty(a, b)$ such that

$$\sigma := \frac{1}{2} \int_a^b \{[D_v \tilde{L}(s, \gamma, \gamma') \cdot \gamma' - \tilde{L}(s, \gamma, \gamma')]\varphi' - D_s \tilde{L}(s, \gamma, \gamma')\varphi\} ds > 0.$$

Arguing as in the proof of Theorem 4.8 we may introduce the continuous map

$$\begin{aligned} \mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \\ \rightarrow W^{1,p}(a, b; M) \end{aligned}$$

defined by

$$\mathcal{H}(\eta, t)(\mu) = \eta(\mu - t\varphi(\mu))$$

satisfying the following facts:

$$\begin{aligned} \mathcal{H}(\eta, t)(a) &= \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b), \\ d_1(\mathcal{H}(\eta, t), \eta) &< \widehat{C}t(f_{a,b}(\gamma) + r + d(b-a))^{1/p}, \\ f_{a,b}(\mathcal{H}(\eta, t)) &\leq f_{a,b}(\eta) + t\Theta(\eta, t) \end{aligned}$$

where $\widehat{C} > 0$,

$$\begin{aligned} \Theta(\eta, t) &= \int_a^b \left[D_s \tilde{L}(\lambda + t\vartheta(\lambda, t)\varphi(\psi(\lambda, t)), \eta, (1 - t\vartheta(\lambda, t)\varphi'(\psi(\lambda, t)))\eta')\varphi(\psi(\lambda, t)) \right. \\ &\quad - D_v \tilde{L}(\lambda + t\vartheta(\lambda, t)\varphi(\psi(\lambda, t)), \eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \cdot \eta'(\lambda)\varphi'(\psi(\lambda, t)) \\ &\quad \left. + \tilde{L}(\psi(\lambda, t), \eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \frac{\varphi'(\psi(\lambda, t))}{1 - t\varphi'(\psi(\lambda, t))} \right] d\lambda \end{aligned}$$

and $0 < \vartheta(\lambda, t) < 1$.

We claim that, for r sufficiently small, we have $\Theta(\eta, t) \leq -\sigma$ for any $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $0 \leq t \leq r$. By contradiction, let (η_h) be a sequence in $W^{1,p}(a, b; M)$ uniformly convergent to γ with $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + \frac{1}{h}$ and (t_h) be a non negative sequence convergent to 0 such that $\Theta(\eta_h, t_h) > -\sigma$. Because of (4.4) and $f_{a,b}$ is lower semicontinuous, we have that $f_{a,b}(\eta_h) \rightarrow f_{a,b}(\gamma)$. Again by (4.4) (η_h) is bounded in $W^{1,p}(a, b; M)$ and up to a subsequence $\eta_h \rightharpoonup \gamma$ in $W^{1,p}(a, b; M)$. On the other hand, we have

$$\begin{aligned} & \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda - \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda)) d\lambda \\ &= f_{a,b}(\eta_h) - f_{a,b}(\gamma) - \int_a^b \tilde{L}(\lambda, \eta_h(\lambda), \eta'_h(\lambda)) d\lambda + \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda. \end{aligned}$$

Taking into account (4.2), we get that

$$\int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda \rightarrow \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda)) d\lambda.$$

By [20, Theorem 3] applied to the function $\Phi(\lambda, \xi) = \tilde{L}(\lambda, \gamma(\lambda), \xi)$ it follows that η'_h is strongly convergent to γ' in $L^p(a, b; M)$; hence $\eta_h \rightarrow \gamma$ in $W^{1,p}(a, b; M)$. Because of (4.2), (4.3) and (4.9), we have that

$$\Theta(\eta_h, t_h) \rightarrow \int_a^b \{[-D_v \tilde{L}(\lambda, \gamma, \gamma') \cdot \gamma' + \tilde{L}(\lambda, \gamma, \gamma')]\varphi' + D_s \tilde{L}(\lambda, \gamma, \gamma')\varphi\} d\lambda = -2\sigma,$$

a contradiction. Finally, we have $f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t$. It follows that γ is not L -stationary, a contradiction. \square

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