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## LAGRANGIAN SYSTEMS WITH LIPSCHITZ OBSTACLE ON MANIFOLDS

SERGIO LANCELOTTI — MARCO MARZOCCHI

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ABSTRACT. Lagrangian systems constrained on the closure of an open subset with Lipschitz boundary in a manifold are considered. Under suitable assumptions, the existence of infinitely many periodic solutions is proved.

### 1. Introduction

The study of Lagrangian functionals of the form

$$(1.1) \quad f(\gamma) = \int_0^1 L(s, \gamma(s), \gamma'(s)) ds$$

on a manifold  $M$ , where  $L(s, (q, v)): \mathbb{R} \times TM \rightarrow \mathbb{R}$ , constitutes a well studied topic in Mechanics and Global analysis. In particular, about the existence and multiplicity of periodic solutions  $\gamma$  of the associated Euler equation, we refer the reader to [1], where the case in which  $M$  is a compact manifold without boundary is considered. Starting from [1], some extensions have been considered in the literature, when  $M$  is embedded in an Euclidean space. In [3] the case where  $M$  is a compact submanifold with boundary in  $\mathbb{R}^n$  has been considered.

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In such a case, the associated Euler equation has the form

$$(1.2) \quad \frac{d}{ds}(D_v L(s, \gamma, \gamma')) - D_q L(s, \gamma, \gamma') \in N_{\gamma(s)} M,$$

where  $N_q M$  is the outer normal cone to  $M$  at  $q$ . The main feature is that the natural domain of the functional (1.1) is

$$(1.3) \quad X = \{\gamma \in W^{1,2}(0, 1; \mathbb{R}^n) : \gamma(0) = \gamma(1), \gamma(s) \in M \text{ for all } s\}$$

which is naturally a metric space, but not a smooth manifold (even with boundary). Moreover, solutions  $\gamma$  of (1.2) are not of class  $C^2$ , but only  $W^{2,\infty}$  and satisfy (1.2) almost everywhere. In the same direction, the case in which  $M$  is a compact  $p$ -convex subset of  $\mathbb{R}^n$  has been considered in [4]. The class of  $p$ -convex subsets [8] includes in particular subsets with corners of convex type and concave parts of class  $C^2$ . This direction of research was started by [18], where the case of an  $n$ -dimensional submanifold with boundary of class  $C^2$  in  $\mathbb{R}^n$  had been considered.

Another development has been started more recently in [11], [19], considering the case in which  $M$  is the closure of a bounded open subset of  $\mathbb{R}^n$  with Lipschitz boundary. Also in this case the set  $X$  is naturally only a metric space. Moreover, since in this case we cannot expect the solution  $\gamma$  of (1.2) to be of class  $C^1$ , the Euler equation itself requires a reformulation.

The purpose of this paper is to consider the intrinsic case in which  $M$  is the closure of a bounded open subset of a differentiable manifold  $N$ , instead of  $\mathbb{R}^n$ , and also to relax the convexity condition on  $L$ , which was in [19] of uniform quadratic type, to the mere convexity with coercivity of order  $p > 1$ .

Our approach follows the lines of [19], but it is completely intrinsic. Of course the lack of strict convexity in  $L$  causes new technical difficulties.

The paper is organized as follows: in Section 2 we state our main results, while Section 3 is devoted to some recalls of nonsmooth analysis. Finally, in Section 4 we prove the main results.

## 2. Statement of the main results

Let  $N$  be a differentiable manifold without boundary of class  $C^2$  and  $M \subseteq N$ . In the sequel, each  $\gamma \in W^{1,p}(a, b; N)$  will be identified with its continuous representative  $\tilde{\gamma}: [a, b] \rightarrow N$ . We set

$$W^{1,p}(a, b; M) := \{\gamma \in W^{1,p}(a, b; N) : \gamma(s) \in M \text{ for each } s \in [a, b]\}.$$

REMARK 2.1. Let  $g$  and  $\tilde{g}$  be two Riemannian structures on  $N$  and let  $d$  and  $\tilde{d}$  be the induced distances on  $N$ . Then there exists a continuous function  $c: N \rightarrow ]0, \infty[$  such that, for all  $q \in N$  and all  $v \in T_q N$ ,

$$g(q)(v, v) \leq c(q)\tilde{g}(q)(v, v), \quad \tilde{g}(q)(v, v) \leq c(q)g(q)(v, v).$$

In particular, for every compact subset  $K \subseteq N$  there exists  $C > 0$  such that, for all  $q_1, q_2 \in K$ ,

$$d(q_1, q_2) \leq C\tilde{d}(q_1, q_2), \quad \tilde{d}(q_1, q_2) \leq Cd(q_1, q_2).$$

Let  $1 < p < \infty$  and  $L: \mathbb{R} \times TN \rightarrow \mathbb{R}$  be a function of class  $C^1$  such that there exist two continuous functions  $c, k: M \rightarrow ]0, \infty[$  and  $d \in \mathbb{R}$  such that for every  $s \in \mathbb{R}$  and  $q \in M$  one has

$$(2.1) \quad k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN,$$

$$(2.2) \quad |D_{(q,v)}L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN,$$

$$(2.3) \quad L(s, q, \cdot) \text{ is convex} \quad \text{on } T_qN,$$

where  $|v| = \sqrt{g(q)(v, v)}$ .

In (2.1), (2.2) we mean that  $N$  is provisionally endowed with a Riemannian structure. By Remark 2.1 the above conditions do not depend on the Riemannian structure chosen on  $N$ .

In charts, (2.1), (2.2) mean that for every  $s \in \mathbb{R}$  and  $q \in M$  it is

$$k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN,$$

$$|D_qL(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN,$$

$$|D_vL(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN.$$

Let us remark that (2.1), (2.3) imply that for every  $s \in \mathbb{R}$ ,  $q \in M$  and any  $v, w \in T_qN$  we have

$$|D_vL(s, q, v)w| \leq \hat{c}(q)(1 + |v|^{p-1})|w|$$

namely, in charts,

$$|D_vL(s, q, v)| \leq \hat{c}(q)(1 + |v|^{p-1}),$$

where  $\hat{c}: M \rightarrow ]0, \infty[$  is continuous.

Define a continuous functional  $f_{a,b}: W^{1,p}(a, b; M) \rightarrow \mathbb{R}$  by

$$f_{a,b}(\gamma) = \int_a^b L(s, \gamma(s), \gamma'(s)) ds.$$

Given a Riemannian structure on  $N$ , for every  $\gamma, \eta \in W^{1,p}(a, b; M)$  we set

$$d_1(\gamma, \eta) = \int_a^b d(\gamma(s), \eta(s)) ds,$$

$$d_\infty(\gamma, \eta) = \max\{d(\gamma(s), \eta(s)) : a \leq s \leq b\},$$

where  $d$  is the distance on  $N$  associated with the Riemannian structure.

DEFINITION 2.2. We say that  $\gamma \in W^{1,p}(a, b; M)$  is *L-stationary*, if it is not possible to find  $r, c, \sigma > 0$  and a map

$$\mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \\ \rightarrow W^{1,p}(a, b; M)$$

such that:

- (a)  $\mathcal{H}$  is continuous from the product of the topology of the uniform convergence and that of  $\mathbb{R}$  to that of the uniform convergence;
- (b) for every  $\eta \in W^{1,p}(a, b; M)$  with  $d_\infty(\eta, \gamma) < r$ ,  $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$  and  $t \in [0, r]$  we have

$$\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b), \\ d_1(\mathcal{H}(\eta, t), \eta) \leq ct, \quad f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t.$$

Again we mean that the assertion holds after introducing a Riemannian structure on  $N$ . By Remark 2.1 this definition does not depend on the choice of the Riemannian structure itself.

PROPOSITION 2.3. Let  $\gamma \in W^{1,p}(a, b; M)$  be *L-stationary*. Then for every  $[\alpha, \beta] \subseteq [a, b]$  the restriction  $\gamma|_{[\alpha, \beta]}$  is *L-stationary*.

PROOF. Set  $\hat{\gamma} = \gamma|_{[\alpha, \beta]}$ . By contradiction, assume that there exist  $r, c, \sigma > 0$  and

$$\mathcal{H}: \{\eta \in W^{1,p}(\alpha, \beta; M) : d_\infty(\eta, \hat{\gamma}) < r, f_{\alpha, \beta}(\eta) < f_{\alpha, \beta}(\hat{\gamma}) + r\} \times [0, r] \\ \rightarrow W^{1,p}(\alpha, \beta; M)$$

according to Definition 2.2.

We claim that there exists  $r' \in ]0, r[$  such that if  $\eta \in W^{1,p}(a, b; M)$  with  $d_\infty(\eta, \gamma) < r'$  and  $f_{a,b}(\eta) < f_{a,b}(\gamma) + r'$ , then  $f_{\alpha, \beta}(\hat{\eta}) < f_{\alpha, \beta}(\hat{\gamma}) + r$ , where  $\hat{\eta} = \eta|_{[\alpha, \beta]}$ .

Again by contradiction, let  $(\eta_h) \subseteq W^{1,p}(a, b; M)$  with  $\eta_h$  convergent to  $\gamma$  with respect to the uniform convergence and  $\limsup_h f_{a,b}(\eta_h) \leq f_{a,b}(\gamma)$  such that  $f_{\alpha, \beta}(\hat{\eta}_h) \geq f_{\alpha, \beta}(\hat{\gamma}) + r$ . By (2.1) and (2.3) we have

$$\limsup_h f_{\alpha, \beta}(\hat{\eta}_h) \leq \limsup_h f_{a,b}(\eta_h) - \liminf_h \int_{]a, b[ \setminus ]\alpha, \beta[} L(s, \eta_h, \eta'_h) ds \\ \leq f_{a,b}(\gamma) - \int_{]a, b[ \setminus ]\alpha, \beta[} L(s, \gamma, \gamma') ds = f_{\alpha, \beta}(\hat{\gamma}),$$

whence a contradiction. Then, for any  $\eta \in W^{1,p}(a, b; M)$  define

$$\mathcal{K}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r', f_{a,b}(\eta) < f_{a,b}(\gamma) + r'\} \times [0, r'] \\ \rightarrow W^{1,p}(a, b; M)$$

by

$$\mathcal{K}(\eta, t)(s) = \begin{cases} \mathcal{H}(\widehat{\eta}, t)(s) & \text{if } s \in [\alpha, \beta], \\ \eta(s) & \text{if } s \notin [\alpha, \beta]. \end{cases}$$

It is readily seen that  $\mathcal{K}$  has all the properties required in Definition 2.2. It follows that  $\gamma$  is not  $L$ -stationary, which is absurd.  $\square$

DEFINITION 2.4. Let  $I$  be an interval in  $\mathbb{R}$  with  $\text{int}(I) \neq \emptyset$ . A continuous map  $\gamma: I \rightarrow M$  is said to be a *generalized solution* of the Lagrangian system associated to  $L$  on  $M$ , if every  $s \in \text{int}(I)$  admits a neighbourhood  $[a, b]$  in  $I$  such that  $\gamma|_{[a, b]}$  belongs to  $W^{1,p}(a, b; M)$  and is  $L$ -stationary.

DEFINITION 2.5. Given  $T > 0$ , a  $T$ -periodic *generalized solution* of the Lagrangian system associated to  $L$  on  $M$  is a generalized solution  $\gamma: \mathbb{R} \rightarrow M$  which is periodic of period  $T$ .

We now state our main existence result.

THEOREM 2.6. *Assume that  $M$  is the closure of an open subset of  $N$  with locally Lipschitz boundary. Suppose also that  $M$  is compact, 1-connected and non-contractible in itself and that*

$$(2.4) \quad L(s+1, q, v) = L(s, q, v) \quad \text{for all } s \in \mathbb{R} \text{ and all } (q, v) \in TN.$$

*Then there exists a sequence  $(\gamma_h)$  of 1-periodic generalized solutions of the Lagrangian system associated to  $L$  on  $M$  with*

$$\lim_h \int_0^1 L(s, \gamma_h(s), \gamma'_h(s)) ds = +\infty.$$

The notion of generalized solution we have introduced follows the approach of [11, Definition 3.3] and [19, Definition 2.6] and has the advantage to be intrinsically connected to  $M$ , although quite indirect. However, at least in the particular case  $p = 2$ , it is possible to deduce further informations on the generalized solutions.

For every  $q \in M$ , denote by  $N_q M$  the normal cone to  $M$  at  $q$  (see e.g. Definition 3.2 below).

THEOREM 2.7. *Let  $p = 2$  and assume that there exists a continuous function  $\omega: N \rightarrow ]0, \infty[$  such that for every  $s \in \mathbb{R}$ ,  $q \in M$  it is*

$$D_v L(s, q, v)(v - w) - D_v L(s, q, w)(v - w) \geq \omega(q)|v - w|^2 \quad \text{for all } v, w \in T_q N.$$

*Let  $\gamma \in W^{1,2}(a, b; M)$  be  $L$ -stationary. Then  $\gamma \in W^{1,\infty}(a, b; M)$ ,  $D_{(q,v)} L(s, \gamma, \gamma') \in L^\infty(a, b; T^*(TN))$  and there exist a finite Borel measure  $\mu$  on  $]a, b[$  and a bounded Borel function  $\nu: ]a, b[ \rightarrow T^*N$  such that  $\nu(s) \in N_{\gamma(s)} M$  for  $\mu$ -a.e.  $s \in ]a, b[$  and*

$$\int_a^b D_{(q,v)} L(s, \gamma, \gamma')(\delta, \delta') ds = - \int_a^b \nu(\delta) d\mu$$

for any  $\delta \in W_0^{1,1}(a, b; TN)$  with  $\delta(s) \in T_{\gamma(s)}N$  for every  $s \in [a, b]$ .

Also in this assertion we mean that  $N$  is provisionally endowed with a Riemannian structure. Since  $\gamma$  is continuous, by Remark 2.1 the assertion is independent of the choice of the structure.

PROOF OF THEOREM 2.7. By Proposition 2.3, we may assume that  $\gamma([a, b])$  is contained in a coordinated neighbourhood. Then the assertion follows from [19, Theorem 2.10].  $\square$

### 3. Some relevant results of nonsmooth analysis

In the first part of this section let  $N$  be a differentiable manifold of class  $C^2$  and  $M$  be the closure of an open set in  $N$  with locally Lipschitz boundary.

If  $X$  is a Banach space,  $E \subseteq X$  and  $x \in E$ , we denote by  $T_xE$  the tangent cone to  $E$  at  $x$ , according to [6]. We also denote by  $B_r(x)$  the open ball of center  $x$  and radius  $r$ .

DEFINITION 3.1. Let  $x \in E$  and  $v \in X$ . We say that  $v$  is *hypertangent to  $E$  at  $x$*  if there exists  $\delta > 0$  such that  $B_\delta(x) + [0, \delta]B_\delta(v) \subseteq E$ . Let us denote by  $\text{Hyp}_x E$  the set of the  $v$ 's hypertangent to  $E$  at  $x$ .

DEFINITION 3.2. Let  $q \in M$  and  $v \in T_q N$ . We say that  $v$  is *tangent to  $M$  at  $q$*  if there exists a chart  $(U, \varphi)$  at  $q$  such that  $d\varphi(q)v \in T_{\varphi(q)}\varphi(U \cap M)$ . The set of the  $v$ 's tangent to  $M$  at  $q$  is denoted by  $T_q M$  and is called the *tangent cone to  $M$  at  $q$* .

We say that  $v$  is *hypertangent to  $M$  at  $q$*  if there exists a chart  $(U, \varphi)$  at  $q$  such that  $d\varphi(q)v$  is hypertangent to  $\varphi(U \cap M)$  at  $\varphi(q)$ . The set of the  $v$ 's hypertangent to  $M$  at  $q$  is denoted by  $\text{Hyp}_q M$  and is called the *hypertangent cone to  $M$  at  $q$* . Finally, we set  $N_q M = \{\varphi \in T_q^* N : \varphi(v) \leq 0 \text{ for all } v \in T_q M\}$ .  $N_q M$  is called the *normal cone to  $M$  at  $q$* .

REMARK 3.3. For every  $q \in M$  it is  $\text{Hyp}_q M \neq \emptyset$  (see [6]) and  $\text{Hyp}_q M \subseteq T_q M$ .

THEOREM 3.4. *There exists a section  $\nu: N \rightarrow TN$  of class  $C^1$  such that*

$$\nu(q) \in \text{Hyp}_q M \quad \text{for all } q \in M.$$

PROOF. For all  $q \in N$ , let

$$\Psi(q) = \begin{cases} \text{Hyp}_q M & \text{if } q \in M, \\ T_q N & \text{if } q \in N \setminus M. \end{cases}$$

Then for every  $q \in N$ ,  $\Psi(q)$  is convex in  $T_q N$  and for every  $q \in N$  there exists a chart  $(U, \varphi)$  at  $q$  such that

$$\bigcap_{\xi \in U} (d\varphi(\xi)(\Psi(\xi))) \neq \emptyset.$$

It follows that there exists  $\nu: N \rightarrow TN$  of class  $C^1$  with  $\nu(q) \in \Psi(q)$  for every  $q \in N$ , hence the assertion.  $\square$

LEMMA 3.5. *Let  $\tilde{N}$  be a submanifold of class  $C^2$  of  $\mathbb{R}^n$ ,  $\tilde{M}$  be the closure of an open subset of  $\tilde{N}$  with locally Lipschitz boundary,  $A$  be an open subset of  $\mathbb{R}^n$  with  $\tilde{N} \subseteq A$  and  $\pi: A \rightarrow \tilde{N}$  be a retraction of class  $C^2$  such that  $\pi$  is Lipschitz continuous of constant 2. Then there exists a map  $\nu: \tilde{N} \rightarrow \mathbb{R}^n$  of class  $C^1$  such that the following facts hold:*

- (a) for any  $q \in \tilde{N}$  we have  $\nu(q) \in T_q \tilde{N}$ ;  
 (b) for any  $q \in \tilde{M}$  there exists  $\delta > 0$  such that

$$\text{if } \begin{cases} \xi \in B_\delta(q), \\ \pi(\xi) \in \tilde{M}, \\ 0 < t \leq \delta, \\ v \in B_\delta(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + tv) \in \text{int}(\tilde{M});$$

- (c) for every compact subset  $K \subseteq \tilde{M}$  there exist  $\hat{r}, \hat{c} > 0$  satisfying

$$\pi((1-t)q + t\pi(\xi + \rho\nu(\xi))) \in \tilde{M}$$

whenever  $q \in \tilde{M}$ ,  $\xi \in K$ ,  $\hat{c}|q - \xi| \leq \rho \leq \hat{r}$  and  $t \in [0, 1]$ .

PROOF. By Theorem 3.4 there exists a map  $\nu: \tilde{N} \rightarrow \mathbb{R}^n$  of class  $C^1$  such that for any  $q \in \tilde{N}$  it is  $\nu(q) \in T_q \tilde{N}$ .

To prove (b), assume by contradiction that  $q \in \tilde{M}$ ,  $\xi_h \rightarrow q$ ,  $t_h \rightarrow 0^+$  and  $v_h \rightarrow \nu(q)$  with  $\pi(\xi_h) \in \tilde{M}$  and  $\pi(\xi_h + t_h v_h) \notin \text{int}(\tilde{M})$ .

Let  $(U, \varphi)$  be the chart at  $q$  such that  $\varphi: U \rightarrow T_q \tilde{N}$ ,  $\varphi(q) = 0$  and  $\pi(q + \varphi(\xi)) = \xi$  for any  $\xi \in U$ ; in particular,  $\nu(q) \in \text{Hyp}_0 \varphi(U \cap \tilde{M})$ .

Then we have

$$\varphi(\pi(\xi_h + t_h v_h)) \notin \text{int}(\varphi(U \cap \tilde{M})).$$

Since

$$\varphi(\pi(\xi_h + t_h v_h)) = \varphi(\pi(\xi_h)) + t_h(d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h)$$

with  $\varepsilon_h \rightarrow 0$  in  $T_q \tilde{N}$ , it follows that  $d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h \in T_q \tilde{N}$  and

$$\varphi(\pi(\xi_h + t_h v_h)) \in \text{int}(\varphi(U \cap \tilde{M}))$$

for large  $h$ , which is absurd.

Now let us prove (c). By contradiction, let  $(q_h)$  in  $\tilde{M}$ ,  $(\xi_h)$  in  $K$ ,  $(t_h)$  in  $[0, 1]$ ,  $\rho_h \rightarrow 0$  with  $h|q_h - \xi_h| \leq \rho_h \leq 1/h$  and

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) \notin \tilde{M}.$$

Up to a subsequence  $\xi_h \rightarrow \xi$  in  $K$ ,  $q_h \rightarrow \xi$  in  $\tilde{M}$  and  $t_h \rightarrow t$  in  $[0, 1]$ . It is

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) = \pi\left(q_h + t_h\rho_h\left(\frac{\pi(\xi_h + \rho_h\nu(\xi_h)) - q_h}{\rho_h}\right)\right).$$

On the other hand,

$$\begin{aligned} \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} - \nu(\xi) &= \frac{\pi(\xi_h + \rho_h \nu(\xi)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} \\ &\quad + \frac{\xi_h - q_h}{\rho_h} + \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h}. \end{aligned}$$

By [11, Theorem 4.4], it is

$$\lim_h \frac{\pi(\xi_h + \rho_h \nu(\xi)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} = 0.$$

Moreover, by the lipschitzianity of  $\pi$  it is also

$$\left| \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h} \right| \leq 2|\nu(\xi_h) - \nu(\xi)|.$$

It follows that

$$\lim_h \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} = \nu(\xi),$$

hence by (a) it is

$$\pi\left(q_h + t_h \rho_h \left(\frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h}\right)\right) \in \widetilde{M}$$

for large  $h$ , which is a contradiction.  $\square$

**DEFINITION 3.6.** A subset  $E$  of  $N$  is said to be a LNR in  $N$  if there exists an open neighbourhood  $U$  of  $E$  in  $N$  and a locally Lipschitzian retraction  $r: U \rightarrow E$ .

**THEOREM 3.7.** *The set  $M$  is a LNR in  $N$ .*

**PROOF.** By [14, §2, Theorems 2.10 and 2.14], we may assume that  $N$  is a smooth submanifold of  $\mathbb{R}^n$ . By [14, §4, Theorem 5.1], there exist an open subset  $A$  of  $\mathbb{R}^n$  with  $N \subseteq A$  and a retraction  $\pi: A \rightarrow N$  of class  $C^\infty$  such that  $\pi$  is Lipschitz continuous of constant 2. Let  $\nu: N \rightarrow \mathbb{R}^n$  be as in Lemma 3.5. By (b) of Lemma 3.5, for every  $q \in M$  there exists  $\delta_q > 0$  such that

$$\text{if } \begin{cases} \xi \in B_{\delta_q}(q), \\ \pi(\xi) \in M, \\ 0 < t \leq \delta_q, \\ v \in B_{\delta_q}(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + tv) \in \text{int}(M).$$

Let  $\delta'_q \in ]0, \delta_q]$  be such that

$$\text{if } \begin{cases} \xi \in B_{\delta'_q}(q), \\ 0 \leq t \leq \delta'_q, \end{cases} \quad \text{then } \begin{cases} \xi + t\nu(\xi) \in B_{\delta_q}(q), \\ \nu(\xi) \in B_{\delta_q/2}(\nu(q)), \\ |\xi - q| + \delta_q |\nu(\xi) - \nu(q)| \leq \delta_q^2/4. \end{cases}$$

For every  $q \in M$ , define

$$U_q = \{\xi \in B_{\delta'_q}(q) : \pi(\xi + \delta'_q \nu(\xi)) \in \text{int}(M)\}, \quad U = \bigcup_{q \in M} U_q.$$

For every  $\xi \in U$ , let  $T(\xi) = \min\{t \geq 0 : \pi(\xi + t\nu(\xi)) \in M\}$ . It is easy to see that, if  $q \in M$  and  $\xi \in U_q$ , then

$$T(\xi) < \delta'_q, \quad \xi + T(\xi)\nu(\xi) \in B_{\delta'_q}(q), \quad \pi(\xi + T(\xi)\nu(\xi)) \in M$$

and

$$(3.1) \quad \text{if } \begin{cases} 0 \leq t \leq \delta_q, \\ v \in B_{\delta_q}(\nu(q)), \end{cases} \quad \text{then } \pi(\xi + T(\xi)\nu(\xi) + tv) \in M.$$

Let now  $q \in M$  and  $\xi_1, \xi_2 \in U_q$  with  $\xi_1 \neq \xi_2$ . We set

$$s = \frac{2}{\delta_q} (|\xi_1 - \xi_2| + T(\xi_1)|\nu(\xi_1) - \nu(\xi_2)|)$$

and

$$v = \nu(\xi_2) - \frac{1}{s}(\xi_1 - \xi_2 + T(\xi_1)(\nu(\xi_1) - \nu(\xi_2))).$$

We have  $s \in ]0, \delta_q]$  and  $v \in B_{\delta_q}(\nu(q))$ . If we consider  $t = T(\xi_1) + s$ , an easy calculation shows that

$$\xi_2 + t\nu(\xi_2) = \xi_1 + T(\xi_1)\nu(\xi_1) + sv.$$

By (3.1) it follows that  $\pi(\xi_2 + t\nu(\xi_2)) \in M$ , hence  $T(\xi_2) \leq t$ . Therefore we get

$$T(\xi_2) \leq T(\xi_1) + s \leq T(\xi_1) + \frac{2}{\delta_q} (|\xi_1 - \xi_2| + \delta_q |\nu(\xi_1) - \nu(\xi_2)|);$$

exchanging the role of  $\xi_1$  and  $\xi_2$  we have

$$|T(\xi_1) - T(\xi_2)| \leq \frac{2}{\delta_q} (|\xi_1 - \xi_2| + \delta_q |\nu(\xi_1) - \nu(\xi_2)|),$$

hence  $T$  is locally Lipschitzian. It follows that the map  $r: U \rightarrow M$  defined by  $r(\xi) = \pi(\xi + T(\xi)\nu(\xi))$  is a locally Lipschitzian retraction. Therefore  $M$  is an LNR in  $\mathbb{R}^n$ , in particular in  $N$ .  $\square$

In the second part of this section, we recall some abstract notions and results of nonsmooth analysis.

Let  $Y$  be a metric space endowed with the metric  $d$  and let  $f: Y \rightarrow \overline{\mathbb{R}}$  be a function. We set

$$\text{epi}(f) = \{(u, \lambda) \in Y \times \mathbb{R} : f(u) \leq \lambda\}.$$

In the following,  $Y \times \mathbb{R}$  will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = (d(u, v)^2 + (\lambda - \mu)^2)^{1/2}$$

and  $\text{epi}(f)$  with the induced metric.

DEFINITION 3.8. For every  $u \in Y$  with  $f(u) \in \mathbb{R}$ , we denote by  $|df|(u)$  the supremum of the  $\sigma$ 's in  $[0, \infty[$  such that there exist  $r > 0$  and a continuous map

$$\mathcal{H}: (B_r(u, f(u)) \cap \text{epi}(f)) \times [0, r] \rightarrow Y$$

satisfying

$$d(\mathcal{H}((v, \mu), t), v) \leq t, \quad f(\mathcal{H}((v, \mu), t)) \leq \mu - \sigma t,$$

whenever  $(v, \mu) \in B_r(u, f(u)) \cap \text{epi}(f)$  and  $t \in [0, r]$ .

The extended real number  $|df|(u)$  is called *the weak slope* of  $f$  at  $u$ .

The above notion has been introduced in [9], following an equivalent approach. When  $f$  is continuous, it has been independently introduced also in [17], while a variant appears in [15], [16]. The version we have recalled here is taken from [2].

PROPOSITION 3.9. *Let  $u \in Y$  with  $f(u) \in \mathbb{R}$ . Assume there exist  $r, c, \sigma > 0$  and a continuous map*

$$\mathcal{H}: \{v \in B_r(u) : f(v) < f(u) + r\} \times [0, r] \rightarrow Y$$

*such that for any  $v \in B_r(u)$  with  $f(v) < f(u) + r$  and any  $t \in [0, r]$  it is*

$$d(\mathcal{H}(v, t), v) \leq ct, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

*Then we have  $|df|(u) \geq \sigma/c$ .*

PROOF. See [11, Proposition 2.3]. □

Now, according to [8], we define a function  $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$  by  $\mathcal{G}_f(u, \lambda) = \lambda$ . Of course,  $\mathcal{G}_f$  is Lipschitzian of constant 1.

PROPOSITION 3.10. *For every  $u \in Y$  with  $f(u) \in \mathbb{R}$ , we have  $f(u) = \mathcal{G}_f(u, f(u))$  and*

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ \infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

PROOF. See [2, Proposition 2.3]. □

The previous proposition allows us to reduce, at some extent, the study of the general function  $f$  to that of the continuous function  $\mathcal{G}_f$ . For this purpose, the next result will be useful.

PROPOSITION 3.11. *Let  $(u, \lambda) \in \text{epi}(f)$  with  $f(u) < \lambda$ . Assume that for every  $\varepsilon > 0$  there exist  $r > 0$  and a continuous map*

$$\mathcal{H}: \{v \in B_r(u) : f(v) < \lambda + r\} \times [0, r] \rightarrow Y$$

such that for any  $v \in B_r(u)$  with  $f(v) < \lambda + r$  and any  $t \in [0, r]$  it is

$$\begin{aligned} d(\mathcal{H}(v, t), v) &\leq \varepsilon t, \\ f(\mathcal{H}(v, t)) &\leq (1 - t)f(v) + t(f(u) + \varepsilon). \end{aligned}$$

Then we have  $|d\mathcal{G}_f|(u, \lambda) = 1$ .

PROOF. See [10, Corollary 2.11].  $\square$

Definition 3.8 may be simplified, when  $f$  is continuous.

PROPOSITION 3.12. *Let  $f: Y \rightarrow \mathbb{R}$  be continuous. Then  $|df|(u)$  is the supremum of the  $\sigma$ 's in  $[0, +\infty[$  such that there exist  $r > 0$  and a continuous map*

$$\mathcal{H}: B_r(u) \times [0, r] \rightarrow Y$$

satisfying

$$(3.2) \quad d(\mathcal{H}(v, t), v) \leq t, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,$$

whenever  $v \in B_r(u)$  and  $t \in [0, r]$ .

PROOF. See [2, Proposition 2.2].  $\square$

By means of the weak slope, we can now introduce the two main notions of critical point theory.

DEFINITION 3.13. We say that  $u \in Y$  is a (lower) critical point of  $f$ , if  $f(u) \in \mathbb{R}$  and  $|df|(u) = 0$ . We say that  $c \in \mathbb{R}$  is a (lower) critical value of  $f$ , if there exists a (lower) critical point  $u \in Y$  of  $f$  with  $f(u) = c$ .

REMARK 3.14. Let  $\tilde{d}$  be another metric on  $Y$  and let  $u \in Y$ . Assume that there exist a neighbourhood  $U$  of  $u$  and  $c > 0$  such that, for all  $v, w \in U$ ,

$$d(v, w) \leq c\tilde{d}(v, w), \quad \tilde{d}(v, w) \leq cd(v, w).$$

Then one has  $|df|(u) = 0$  if and only if  $|\tilde{d}f|(u) = 0$ , where  $|\tilde{d}f|(u)$  is the weak slope with respect to  $\tilde{d}$ .

DEFINITION 3.15. Let  $c \in \mathbb{R}$ . A sequence  $(u_h)$  in  $Y$  is said to be a Palais–Smale sequence at level  $c$  ( $(PS)_c$ -sequence, for short) for  $f$ , if  $f(u_h) \rightarrow c$  and  $|df|(u_h) \rightarrow 0$ .

We say that  $f$  satisfies the Palais–Smale condition at level  $c$  ( $(PS)_c$ , for short), if every  $(PS)_c$ -sequence  $(u_h)$  for  $f$  admits a convergent subsequence  $(u_{h_k})$  in  $Y$ .

DEFINITION 3.16. A topological space  $Z$  is said to be weakly locally contractible, if every  $u \in Z$  admits a neighbourhood  $U$  which is contractible in  $Z$ .

**THEOREM 3.17.** *Let  $Y$  be weakly locally contractible with  $\text{cat}Y = \infty$ , let  $f: Y \rightarrow \mathbb{R}$  be continuous and bounded from below and assume that  $\{u \in Y : f(u) \leq c\}$  is complete and  $(\text{PS})_c$  hold for every  $c \in \mathbb{R}$ . Then there exists a sequence  $(u_h)$  of critical points of  $f$  with  $f(u_h) \rightarrow \infty$ .*

**PROOF.** See [7, Theorem 3.6] and [5, Theorem 1.4.13].  $\square$

**COROLLARY 3.18.** *Let  $Z$  be a metrizable topological space and  $f: Z \rightarrow \mathbb{R}$  a continuous function. Assume that*

- (a)  $Z$  is weakly locally contractible and  $\text{cat}Z = \infty$ ;
- (b) for every  $c \in \mathbb{R}$ , the set  $\{u \in Z : f(u) \leq c\}$  is compact.

*Then, for every compatible metric on  $Z$ , there exists a sequence  $(u_h)$  of critical points of  $f$  with  $f(u_h) \rightarrow \infty$ .*

#### 4. Proof of the main results

In the first part of this section, let  $N$  be a differentiable manifold of class  $C^2$  and  $M$  be a LNR in  $N$ . Let us consider

$$\Lambda(M) = \{\gamma \in C([0, 1]; M) : \gamma(0) = \gamma(1)\}$$

endowed with the uniform topology ( $\Lambda(M)$  is called the *free loop space* of  $M$ ) and

$$X = \{\gamma \in W^{1,p}(0, 1; M) : \gamma(0) = \gamma(1)\}.$$

Let  $L: \mathbb{R} \times TN \rightarrow \mathbb{R}$  be a function of class  $C^1$  satisfying (2.1)–(2.4) and define a lower semicontinuous functional  $f: \Lambda(M) \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$f(\gamma) = \begin{cases} \int_0^1 L(s, \gamma(s), \gamma'(s)) ds & \text{if } \gamma \in X, \\ \infty & \text{if } \gamma \in \Lambda(M) \setminus X. \end{cases}$$

In the following, we will consider the metrizable topological space  $\text{epi}(f)$ , endowed with the topology induced by  $\Lambda(M) \times \mathbb{R}$ , and the continuous function  $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$ .

Given a Riemannian structure on  $N$ , for every  $\gamma, \eta \in W^{1,p}(0, 1; M)$ , we set as before

$$d_1(\gamma, \eta) = \int_0^1 d(\gamma(s), \eta(s)) ds,$$

$$d_\infty(\gamma, \eta) = \max\{d(\gamma(s), \eta(s)) : 0 \leq s \leq 1\},$$

where  $d$  is the distance on  $N$  associated with the Riemannian structure.

LEMMA 4.1. *Consider a Riemannian structure on  $N$ . Let  $(\gamma_h)$  be a sequence in  $W^{1,p}(0,1;M)$  convergent to  $\gamma \in W^{1,p}(0,1;M)$  with respect to the topology induced by  $d_1$  and such that  $(f(\gamma_h))$  is bounded. Then  $(\gamma_h)$  is convergent to  $\gamma$  with respect to the uniform convergence.*

PROOF. Let  $U$  be an open subset of  $M$  with  $\bar{U}$  compact such that  $\gamma([0,1]) \subseteq U$ . First of all we claim that  $\gamma_h([0,1]) \subseteq U$  for  $h$  large enough. By contradiction, let  $h_k \rightarrow \infty$  and  $(s_k) \subseteq [0,1]$  such that  $\gamma_{h_k}(s_k) \notin U$ . Up to a subsequence we have that  $s_k \rightarrow s \in [0,1]$  and  $\gamma_{h_k} \rightarrow \gamma$  a.e. in  $[0,1]$ . Let  $a \in [0,1]$  be such that  $\gamma_{h_k}(a) \rightarrow \gamma(a)$ . Assume that  $a < s$ . It follows that, for  $k$  large enough, there exists  $b_k \in ]a, s_k]$  such that  $\gamma_{h_k}([a, b_k]) \subseteq \bar{U}$  and  $\gamma_{h_k}(b_k) \notin U$ . Since  $\bar{U}$  is compact, there exists  $C > 0$  such that, by (2.1),

$$\int_a^{b_k} L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq \int_a^{b_k} (k(\gamma_{h_k})|\gamma'_{h_k}|^p - d) ds \geq \int_a^{b_k} (C|\gamma'_{h_k}|^p - d) ds.$$

Moreover, again by (2.1), we have

$$\int_0^a L(s, \gamma_{h_k}, \gamma'_{h_k}) ds + \int_{b_k}^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq -d(1 - b_k + a).$$

It follows that

$$f(\gamma_{h_k}) = \int_0^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) ds \geq C \int_a^{b_k} |\gamma'_{h_k}|^p ds - d.$$

Hence for every  $\sigma, \tau \in [a, b_k]$  with  $\tau \leq \sigma$  we have

$$\begin{aligned} d(\gamma_{h_k}(\sigma), \gamma_{h_k}(\tau)) &\leq \int_\tau^\sigma |\gamma'_{h_k}(t)| dt \leq \left( \int_\tau^\sigma |\gamma'_{h_k}(t)|^p dt \right)^{1/p} |\sigma - \tau|^{1/p'} \\ &\leq \left( \int_a^{b_k} |\gamma'_{h_k}(t)|^p dt \right)^{1/p} |\sigma - \tau|^{1/p'} \leq \left( \frac{f(\gamma_{h_k}) + d}{C} \right)^{1/p} |\sigma - \tau|^{1/p'}. \end{aligned}$$

It follows that  $(\gamma_{h_k})$  is equi-uniformly continuous on  $[a, b_k]$ . Up to a further subsequence we have that  $\gamma_{h_k}(b_k) \rightarrow x \in \partial U$ . Since  $\inf\{d(\gamma(a), y) : y \in \partial U\} > 0$ , if  $a$  is sufficiently closed to  $s$  a contradiction follows.

Arguing as above, for any  $s, t \in [0,1]$  we have that

$$d(\gamma_h(s), \gamma_h(t)) \leq \left( \frac{f(\gamma_h) + d}{C} \right)^{1/p} |s - t|^{1/p'}.$$

Since  $(f(\gamma_h))$  is bounded, we deduce that  $(\gamma_h)$  is equi-uniformly continuous on  $[0,1]$ . Therefore it is easy to see that  $(\gamma_h)$  is convergent to  $\gamma$  with respect to the uniform convergence.  $\square$

THEOREM 4.2. Consider any Riemannian structure on  $N$  and define on  $\text{epi}(f)$  the metric

$$(4.1) \quad d((\gamma, \lambda), (\eta, \mu)) = \sqrt{d_1(\gamma, \eta)^2 + |\lambda - \mu|^2}.$$

Then the following facts hold:

- (a) the metric  $d$  is compatible with the topology of  $\text{epi}(f)$ ;
- (b) the set of critical points of  $\mathcal{G}_f: \text{epi}(f) \rightarrow \mathbb{R}$  does not depend on the Riemannian structure;
- (c) if  $(\gamma, \lambda) \in \text{epi}(f)$  is a critical point of  $\mathcal{G}_f$  with  $f(\gamma) = \lambda$ , then  $\gamma$  is the restriction to  $[0, 1]$  of a 1-periodic generalized solution of the Lagrangian system associated to  $L$  on  $M$ .

PROOF. (a) is an easy consequence of Lemma 4.1; (b) follows from Remarks 2.1 and 3.14. Let us consider property (c). First, let us prove that  $\gamma$  is  $L$ -stationary on  $[0, 1]$ . By contradiction, assume that there exist  $r, c, \sigma > 0$  and

$$\mathcal{H}: \{\eta \in W^{1,p}(0, 1; M) : d_\infty(\eta, \gamma) < r, f(\eta) < f(\gamma) + r\} \times [0, r] \rightarrow W^{1,p}(0, 1; M)$$

continuous from the product of the uniform convergence and that of  $\mathbb{R}$  to that of the uniform convergence such that

$$\begin{aligned} \mathcal{H}(\eta, t)(0) &= \eta(0), & \mathcal{H}(\eta, t)(1) &= \eta(1), \\ d_1(\mathcal{H}(\eta, t), \eta) &\leq ct, & f(\mathcal{H}(\eta, t)) &\leq f(\eta) - \sigma t. \end{aligned}$$

If  $r' \in ]0, r[$  is such that if  $\eta \in W^{1,p}(0, 1; M)$  with  $d_1(\eta, \gamma) < r'$  and  $f(\eta) < f(\gamma) + r'$ , then  $d_\infty(\eta, \gamma) < r$ . Then the restriction of  $\mathcal{H}$  to

$$\{\eta \in W^{1,p}(0, 1; M) : d_1(\eta, \gamma) < r', f(\eta) < f(\gamma) + r'\} \times [0, r']$$

satisfies the assumptions of Proposition 3.9. It follows that  $\gamma$  is not a critical point of  $f$ , a contradiction.

Finally, if we define

$$\widehat{\gamma}(s) = \begin{cases} \gamma\left(s + \frac{1}{2}\right) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \gamma\left(s - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

it turns out that also  $\widehat{\gamma}$  is  $L$ -stationary on  $[0, 1]$ , whence the assertion. □

LEMMA 4.3. Define  $\mathcal{E}: \Lambda([0, 1]; N) \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$\mathcal{E}(\gamma) = \begin{cases} \int_0^1 |\gamma'(s)|^p ds & \text{if } \gamma \in X, \\ \infty & \text{if } \gamma \in \Lambda([0, 1]; N) \setminus X. \end{cases}$$

Then  $\text{epi}(f)$  is homotopically equivalent to  $\text{epi}(\mathcal{E})$ .

PROOF. By (2.1), for every  $\gamma \in X$  we have

$$\mathcal{E}(\gamma) \leq \left\| \frac{1}{k \circ \gamma} \right\|_{\infty} (f(\gamma) + d), \quad f(\gamma) \leq \|c \circ \gamma\|_{\infty} (\mathcal{E}(\gamma) + 1).$$

Define  $\Phi: \text{epi}(f) \rightarrow \text{epi}(\mathcal{E})$  and  $\Psi: \text{epi}(\mathcal{E}) \rightarrow \text{epi}(f)$  by

$$\Phi(\gamma, \lambda) = \left( \gamma, \left\| \frac{1}{k \circ \gamma} \right\|_{\infty} (\lambda + d) \right), \quad \Psi(\gamma, \lambda) = (\gamma, \|c \circ \gamma\|_{\infty} (\lambda + 1)).$$

Then  $\Psi$  and, by Lemma 4.1,  $\Phi$  are continuous and it is readily seen that  $\Psi \circ \Phi$  is homotopic to the identity of  $\text{epi}(f)$  while  $\Phi \circ \Psi$  is homotopic to the identity of  $\text{epi}(\mathcal{E})$ .  $\square$

LEMMA 4.4. *Let  $U$  be an open subset of  $\mathbb{R}^n$  and let*

$$\Lambda^1(U) = \{\gamma \in W^{1,p}(0, 1; U) : \gamma(0) = \gamma(1)\}$$

*endowed with the  $W^{1,p}$ -metric. Then there exists a continuous map*

$$\mathcal{K}: \Lambda(U) \times [0, 1] \rightarrow \Lambda(U)$$

*such that*

$$\begin{aligned} \mathcal{K}(\gamma, 0) &= \gamma, \quad \mathcal{K}(\gamma, 1) \in \Lambda^1(U) \quad \text{for all } \gamma \in \Lambda(U), \\ \mathcal{K}(\cdot, 1): \Lambda(U) &\rightarrow \Lambda^1(U) \text{ is continuous,} \\ \mathcal{K}(\Lambda^1(U) \times [0, 1]) &\subseteq \Lambda^1(U), \\ \|[\mathcal{K}(\gamma, t)]'\|_p &\leq \|\gamma'\|_p \quad \text{for all } \gamma \in \Lambda^1(U) \text{ and all } t \in [0, 1]. \end{aligned}$$

PROOF. Let  $(\rho_{\varepsilon})$  be a sequence of mollifiers of class  $C_c^{\infty}$  on  $\mathbb{R}^n$ . Let  $R_0\gamma = \gamma$  and for every  $\varepsilon > 0$  let

$$R_{\varepsilon}\gamma(s) = \int_{\mathbb{R}} \rho_{\varepsilon}(s-t) \bar{\gamma}(t) dt,$$

where  $\bar{\gamma}: \mathbb{R} \rightarrow U$  is 1-periodic such that  $\bar{\gamma}|_{[0,1]} = \gamma$ . It turns out that there exists a continuous function  $\lambda: \Lambda(U) \rightarrow ]0, 1]$  such that for every  $\gamma \in \Lambda(U)$  it is

$$R_{\varepsilon}\gamma(s) \in U \quad \text{for all } \varepsilon \in ]0, \lambda(\gamma)], \text{ and all } s \in [0, 1].$$

Let  $\mathcal{K}: \Lambda(U) \times [0, 1] \rightarrow \Lambda(U)$  defined by  $\mathcal{K}(\gamma, t) = R_{t\lambda(\gamma)}\gamma$ . It is readily seen that  $\mathcal{K}$  satisfies all the properties required and the assertion follows.  $\square$

LEMMA 4.5. *The map  $\tilde{\pi}: \text{epi}(\mathcal{E}) \rightarrow \Lambda(M)$  defined by  $\tilde{\pi}(\gamma, \lambda) = \gamma$  is a homotopy equivalence ( $\text{epi}(\mathcal{E})$  is endowed with the product of the uniform topology and that of  $\mathbb{R}$ ).*

PROOF. Arguing as in the proof of Theorem 3.7, we may assume that  $N$  is a smooth submanifold of  $\mathbb{R}^n$  and we may consider an open subset  $A$  of  $\mathbb{R}^n$  with  $N \subseteq A$  and a retraction  $\pi: A \rightarrow N$  of class  $C^\infty$  such that  $\pi$  is Lipschitz continuous of constant 2. Since  $M$  is a LNR in  $N$ , there exists an open neighbourhood  $U$  of  $M$  in  $N$  and a locally Lipschitzian retraction  $r: U \rightarrow M$ . Since  $r \circ \pi: \pi^{-1}(U) \rightarrow M$  is a locally Lipschitzian retraction, then  $M$  is also a LNR in  $\mathbb{R}^n$ . Now taking into account Lemma 4.4 the proof follows the same argument of [11, Theorem 5.3].  $\square$

THEOREM 4.6. *The map  $\hat{\pi}: \text{epi}(f) \rightarrow \Lambda(M)$  defined by  $\hat{\pi}(\gamma, \lambda) = \gamma$  is a homotopy equivalence ( $\text{epi}(f)$  is endowed with the product of the uniform topology and that of  $\mathbb{R}$ ).*

PROOF. Combining Lemmas 4.3 and 4.5 the assertion follows.  $\square$

From now on, we assume that  $M$  is the closure of an open subset in  $N$  with locally Lipschitz boundary. By Theorem 3.7,  $M$  is a LNR in  $N$ .

THEOREM 4.7. *Consider a Riemannian structure on  $N$  and the metric defined in (4.1). Let  $(\gamma, \lambda)$  be in  $\text{epi}(f)$  such that  $f(\gamma) < \lambda$ . Then*

$$|d\mathcal{G}_f|(\gamma, \lambda) = 1.$$

PROOF. Arguing as in the proof of Theorem 3.7, we may assume that  $N$  is a smooth submanifold of  $\mathbb{R}^n$  and we may consider an open subset  $A$  of  $\mathbb{R}^n$  with  $N \subseteq A$  and a retraction  $\pi: A \rightarrow N$  of class  $C^\infty$  such that  $\pi$  is Lipschitz continuous of constant 2. Therefore we may also consider the function  $\tilde{L}: \mathbb{R} \times A \times \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\tilde{L}$  is a  $C^1$ -extension of  $L$  to  $\mathbb{R} \times A \times \mathbb{R}^n$  and such that there exist two continuous functions  $\tilde{c}, \tilde{k}: A \rightarrow ]0, \infty[$  and  $d \in \mathbb{R}$  such that for every  $(s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n$  one has

$$(4.2) \quad |D_q \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^p),$$

$$(4.3) \quad |D_v \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^{p-1}),$$

$$(4.4) \quad \tilde{L}(s, q, v) \geq \tilde{k}(q)|v|^p - d,$$

$$(4.5) \quad \tilde{L}(s, q, \cdot) \text{ is convex.}$$

First of all we claim that there exist  $\bar{\varepsilon} > 0$  and  $\bar{C} > 0$  such that for every  $\eta_1, \eta_2 \in X$  with  $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$  and for every  $t \in [0, 1]$  it is

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq \bar{C}t \left( 1 + \int_0^1 \tilde{L}(s, \eta_1, \eta'_1) ds + \int_0^1 \tilde{L}(s, \eta_2, \eta'_2) ds \right) \|\eta_1 - \eta_2\|_\infty. \end{aligned}$$

Let  $\varepsilon > 0$  be such that if  $\eta \in W^{1,p}(0, 1; \mathbb{R}^n)$  with  $\|\eta - \gamma\|_\infty \leq \varepsilon$  then  $\eta \in W^{1,p}(0, 1; A)$ . Since  $\pi$  is of class  $C^\infty$  and Lipschitz continuous of constant 2, there exists  $\bar{\varepsilon} \in ]0, \varepsilon]$  and  $\tilde{C} \geq 2$  such that for every  $\eta_1, \eta_2 \in W^{1,p}(0, 1; A)$  with  $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$  and for every  $\xi \in \mathbb{R}^n$  it is

$$|\pi(\eta_1) - \pi(\eta_2)| \leq \tilde{C}|\eta_1 - \eta_2|, \quad |[\pi'(\eta_1) - \pi'(\eta_2)]\xi| \leq \tilde{C}|\eta_1 - \eta_2|\|\xi\|.$$

Now let  $\eta_1, \eta_2 \in X$  with  $\|\eta_i - \gamma\|_\infty \leq \bar{\varepsilon}$  and let  $t \in [0, 1]$ . For every  $\vartheta \in [0, 1]$  we have

$$(4.6) \quad \begin{aligned} & |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)| \\ & = |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1)| \\ & \leq |\eta'_1| + \tilde{C}|\eta_2 - \eta_1|\|\eta'_1\| \leq \widehat{C}(|\eta'_1| + |\eta'_2|) \end{aligned}$$

for some  $\widehat{C} > 0$ . Unless reducing  $\bar{\varepsilon}$ , we may suppose that  $\tilde{c}, \tilde{k}$  are constants on  $\{\eta \in W^{1,p}(0, 1; A) : d_\infty(\eta, \gamma) < \bar{\varepsilon}\}$ . Furthermore, applying Lagrange's Theorem, (4.2), (4.3) and (4.6) it is, for some  $\vartheta \in [0, 1]$ ,

$$\begin{aligned} & \tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1) \\ & = D_q \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\ & \quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \cdot (\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1) \\ & \quad + D_v \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\ & \quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \cdot (\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \\ & \leq C(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^p) |\pi(\eta_1 + t(\eta_2 - \eta_1)) - \pi(\eta_1)| \\ & \quad + C(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^{p-1}) \\ & \quad \cdot |\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1| \\ & \leq C_2 t(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^p) |\eta_1 - \eta_2| \\ & \quad + C_2 t(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|^{p-1}) |\eta'_1| |\eta_1 - \eta_2| \\ & \leq C_3 t(1 + |\eta'_1|^p + |\eta'_2|^p) |\eta_1 - \eta_2| + C_3 t(1 + |\eta'_1|^{p-1} + |\eta'_2|^{p-1}) |\eta'_1| |\eta_1 - \eta_2| \\ & = C_3 t(1 + |\eta'_1|^p + |\eta'_2|^p) |\eta_1 - \eta_2| + C_3 t(|\eta'_1| + |\eta'_1|^p + |\eta'_1| |\eta'_2|^{p-1}) |\eta_1 - \eta_2| \end{aligned}$$

for some  $C_3 > 0$ . It follows that

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq C_3 t(1 + 2\|\eta'_1\|_p^p + \|\eta'_2\|_p^p + \|\eta'_1\|_1 + \|\eta'_1\|_p \|\eta'_2\|_p^{p-1}) \|\eta_1 - \eta_2\|_\infty \\ & \leq C_4 t(1 + \|\eta'_1\|_p^p + \|\eta'_2\|_p^p) \|\eta_1 - \eta_2\|_\infty \end{aligned}$$

for some  $C_4 > 0$ . Finally, applying (4.4) we may find  $\bar{C} > 0$  such that

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1)] ds \right| \\ & \leq \bar{C}t \left( 1 + \int_0^1 \tilde{L}(s, \eta_1, \eta'_1) ds + \int_0^1 \tilde{L}(s, \eta_2, \eta'_2) ds \right) \|\eta_1 - \eta_2\|_\infty \end{aligned}$$

and the claim follows. Let  $\varepsilon > 0$ ,  $K = \gamma([0, 1])$  and let  $\bar{\varepsilon}, \bar{C} > 0$  be as before. Let  $C_2 = \bar{C}(1 + 2\lambda + \varepsilon)$ . Let now  $\hat{r}$  and  $\hat{c}$  be as in (c) of Lemma 3.5, and let

$$\hat{\gamma}(s) = \gamma(s) + \rho\nu(\gamma(s)),$$

where  $\rho \in ]0, \hat{r}[$  is such that

$$\|\pi(\hat{\gamma}) - \gamma\|_\infty \leq \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{8C_2}, \bar{\varepsilon} \right\}, \quad f(\pi \circ \hat{\gamma}) \leq f(\gamma) + \frac{\varepsilon}{4}.$$

Let  $r \in ]0, \varepsilon/2[$  be such that if  $\|\eta - \gamma\|_1 < r$  with  $f(\eta) < \lambda + r$ , then  $\|\eta - \gamma\|_\infty \leq \min\{\rho/\hat{c}, \varepsilon/4, \varepsilon/8C_2, \bar{\varepsilon}\}$ . Then, again by (c) of Lemma 3.5 it is possible to define a continuous map

$$\mathcal{H}: \{\eta \in X : \|\eta - \gamma\|_1 < r, f(\eta) < \lambda + r\} \times [0, r] \rightarrow X$$

by

$$\mathcal{H}(\eta, t) = \pi((1-t)\eta + t\pi(\hat{\gamma})).$$

It is

$$\|\mathcal{H}(\eta, t) - \eta\|_\infty \leq 2t\|\pi(\hat{\gamma}) - \eta\|_\infty \leq 2t(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \varepsilon t$$

and hence also

$$\|\mathcal{H}(\eta, t) - \eta\|_1 \leq \varepsilon t.$$

Since  $\tilde{L}$  is convex with respect to the third variable, we get

$$\begin{aligned} & f(\mathcal{H}(\eta, t)) \\ & = \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\eta' + t((\pi \circ \hat{\gamma})' - \eta'))) ds \\ & \leq \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') ds \\ & \quad + t \left[ \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\pi \circ \hat{\gamma})') ds \right. \\ & \quad \left. - \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') ds \right]. \end{aligned}$$

Furthermore, it is

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))\eta') - \tilde{L}(s, \eta, \eta')] ds \right| \\ & \leq \bar{C}t(1 + f(\eta) + f(\pi \circ \hat{\gamma}))\|\pi(\hat{\gamma}) - \eta\|_\infty \\ & < \bar{C}t(1 + 2\lambda + \varepsilon)(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \frac{\varepsilon}{4}t \end{aligned}$$

and

$$\begin{aligned} & \left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\pi(\hat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\hat{\gamma}) - \eta))(\pi \circ \hat{\gamma})') \right. \\ & \quad \left. - \tilde{L}(s, \pi \circ \hat{\gamma}, (\pi \circ \hat{\gamma})')] ds \right| \\ & \leq \bar{C}t(1 + f(\eta) + f(\pi \circ \hat{\gamma}))\|\pi(\hat{\gamma}) - \eta\|_\infty \\ & < \bar{C}t(1 + 2\lambda + \varepsilon)(\|\pi(\hat{\gamma}) - \gamma\|_\infty + \|\gamma - \eta\|_\infty) \leq \frac{\varepsilon}{4}t. \end{aligned}$$

Therefore we finally get

$$f(\mathcal{H}(\eta, t)) \leq f(\eta) + \frac{\varepsilon}{4}t + \left( f(\pi \circ \hat{\gamma}) - f(\eta) + \frac{\varepsilon}{2} \right) t \leq f(\eta) + t(f(\gamma) - f(\eta) + \varepsilon)$$

and the assertion follows from Proposition 3.11.  $\square$

Finally, we can prove Theorem 2.6.

PROOF. Now assume also that  $M$  is compact, 1-connected and non-contractible in itself. By Theorem 3.7, we have that  $M$  is a LNR in  $N$ , in particular an absolute neighbourhood retract. From [13, Corollary 1.4] it follows that  $\text{cat}\Lambda(M) = \infty$ . Moreover,  $\Lambda(M)$  also is an absolute neighbourhood retract, hence weakly locally contractible. On the other hand, by Theorem 4.6  $\Lambda(M)$  is homotopically equivalent to  $\text{epi}(f)$ . Therefore  $\text{cat epi}(f) = \infty$  and  $\text{epi}(f)$  is weakly locally contractible. Let now  $c \in \mathbb{R}$  and consider the sublevel

$$\mathcal{G}_f^c = \{(\gamma, \lambda) \in \Lambda(M) \times \mathbb{R} : f(\gamma) \leq \lambda \leq c\}.$$

Since  $M$  is compact, from (2.1) and Ascoli's theorem we deduce that  $\mathcal{G}_f^c$  is compact. By Corollary 3.18, there exists a sequence  $(\gamma_h, \lambda_h)$  of critical points of  $\mathcal{G}_f^c$  with respect to the metric (4.1) with  $\lambda_h \rightarrow \infty$ . By Theorem 4.7 we have that  $\lambda_h = f(\gamma_h)$ . From (c) of Theorem 4.2 the assertion follows.  $\square$

The next two results correspond to the well-known equation  $d/ds H = -D_s L$ , where  $H$  is the Hamiltonian function associated with  $L$ .

THEOREM 4.8. *Let  $\gamma \in W^{1,p}(a, b; M)$  be  $L$ -stationary. Assume that  $L$  does not depend on  $s$ . Then the map  $\{s \mapsto D_v L(\gamma, \gamma')\gamma' - L(\gamma, \gamma')\}$  is constant a.e.*

PROOF. Arguing as in the proof of Theorem 4.7, we may assume that  $N$  is a smooth submanifold of  $\mathbb{R}^n$ ,  $A$  is an open subset of  $\mathbb{R}^n$  with  $N \subseteq A$  and

$\tilde{L}: A \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^1$ -extension of  $L$  to  $A \times \mathbb{R}^n$  satisfying (4.2)–(4.5). Assume, for a contradiction, that there exists  $\varphi \in C_c^\infty(a, b)$  such that

$$\sigma := \frac{1}{2} \int_a^b \{ [D_v \tilde{L}(\gamma, \gamma') \cdot \gamma' - \tilde{L}(\gamma, \gamma')] \varphi' \} ds > 0.$$

Let  $r > 0$  be such that  $r \|\varphi'\|_\infty < 1$  and let  $\psi: [a, b] \times [0, r] \rightarrow [a, b]$  be the smooth function such that

$$\lambda = \psi(\lambda, t) - t\varphi(\psi(\lambda, t)) \quad \text{for all } \lambda \in [a, b] \text{ and all } t \in [0, r].$$

Unless reducing  $r$  we may suppose that the functions  $c, k$  in (4.2)–(4.4) are constants on  $\{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r\}$ . Define  $\mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \rightarrow W^{1,p}(a, b; M)$  by

$$\mathcal{H}(\eta, t)(\mu) = \eta(\mu - t\varphi(\mu)) .$$

It is easy to see that  $\mathcal{H}$  is continuous from the product topology of the uniform convergence and of  $\mathbb{R}$  to that of the uniform convergence and that

$$\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b).$$

Moreover, by (4.4)

$$\begin{aligned} d_1(\mathcal{H}(\eta, t), \eta) &= \int_a^b |\eta(\mu - t\varphi(\mu)) - \eta(\mu)| d\mu \\ &= t \int_a^b |\eta'(\mu - t\varphi(\mu))| |1 - t\varphi'(\mu)| d\mu \\ &\leq t \left( \int_a^b |\eta'(\lambda)|^p \frac{1}{|1 - \theta\varphi'(\psi(\lambda, \theta))|^p} d\lambda \right)^{1/p} \left( \int_a^b |1 - t\varphi'(\mu)|^{p'} d\mu \right)^{1/p'} \\ &\leq \frac{t}{(1 - \theta\|\varphi'\|_\infty)^p} \left( \int_a^b |\eta'(\lambda)|^p d\lambda \right)^{1/p} \left( \int_a^b |1 - t\varphi'(\mu)|^{p'} d\mu \right)^{1/p'} \\ &\leq \bar{C}t \left( \int_a^b (L(\eta(\lambda), \eta'(\lambda)) + d) d\lambda \right)^{1/p} < \hat{C}t(f_{a,b}(\gamma) + r + d(b-a))^{1/p}, \end{aligned}$$

for some  $\hat{C} > 0$ . Following the same argument of the proof of [19, Theorem 5.10] we also have

$$f_{a,b}(\mathcal{H}(\eta, t)) = f_{a,b}(\eta) + t\Theta(\eta, t)$$

where

$$\begin{aligned} \Theta(\eta, t) &= \int_a^b \left[ -D_v \tilde{L}(\eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \cdot \eta'(\lambda) \varphi'(\psi(\lambda, t)) \right. \\ &\quad \left. + \tilde{L}(\eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \frac{\varphi'(\psi(\lambda, t))}{1 - t\varphi'(\psi(\lambda, t))} \right] d\lambda. \end{aligned}$$

We claim that, for  $r$  sufficiently small, we have  $\Theta(\eta, t) \leq -\sigma$  for any  $\eta \in W^{1,p}(a, b; M)$  with  $d_\infty(\eta, \gamma) < r$ ,  $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$  and  $0 \leq t \leq r$ . By contradiction, let  $(\eta_h)$  be a sequence in  $W^{1,p}(a, b; M)$  uniformly convergent to  $\gamma$  with  $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + 1/h$  and  $(t_h)$  be a non negative sequence convergent to 0 such that  $\Theta(\eta_h, t_h) > -\sigma$ . Because of (4.4) and  $f_{a,b}$  is lower semicontinuous, we have that  $f_{a,b}(\eta_h) \rightarrow f_{a,b}(\gamma)$ . Again by (4.4)  $(\eta_h)$  is bounded in  $W^{1,p}(a, b; M)$  and up to a subsequence  $\eta'_h \rightharpoonup \gamma'$  in  $L^p(a, b; M)$ . Therefore  $[1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h \rightharpoonup \gamma'$  in  $L^p(a, b; M)$ . We have that

$$\begin{aligned} & \int_a^b [\tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - \tilde{L}(\gamma(\lambda), \gamma'(\lambda))] d\lambda \\ &= \int_a^b D_v \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta'_h(\lambda)) \cdot (\eta'_h(\lambda) - \gamma'(\lambda)) d\lambda \\ & \quad + t_h \int_a^b \varphi'(\psi(\lambda, t_h)) D_v \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta'_h(\lambda) \\ & \quad + \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot \eta'_h(\lambda) d\lambda. \end{aligned}$$

By (4.3) we have that  $D_v \tilde{L}(\gamma, (1 - \tau) \gamma' + \tau \eta'_h) \in L^{p'}(a, b; M)$  and hence

$$\int_a^b D_v \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta'_h(\lambda)) \cdot (\eta'_h(\lambda) - \gamma'(\lambda)) d\lambda \rightarrow 0.$$

Again by (4.3) we have that

$$\int_a^b \varphi'(\psi(\lambda, t_h)) D_v \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta'_h(\lambda) + \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot \eta'_h(\lambda) d\lambda$$

is bounded. Therefore we have that

$$\int_a^b \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h) d\lambda \rightarrow \int_a^b \tilde{L}(\gamma(\lambda), \gamma'(\lambda)) d\lambda.$$

By [12, Lemma 3.1] applied to the function  $\mathcal{F}(\lambda, \xi) = \tilde{L}(\gamma(\lambda), \xi)$  we obtain that

$$\begin{aligned} \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) &\rightarrow \tilde{L}(\gamma, \gamma') \quad \text{in } L^1(a, b; M), \\ D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) &\rightarrow D_v \tilde{L}(\gamma, \gamma') \quad \text{in } L^{p'}(a, b; M) \end{aligned}$$

and there exists  $\Psi \in L^1(a, b; M)$  such that  $|\eta'_h|^p \leq \Psi$ . For some  $t \in ]0, 1[$  we have that

$$\begin{aligned} & \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ &= D_q \tilde{L}((1 - t) \gamma(\lambda) + t \eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \cdot (\eta_h(\lambda) - \gamma(\lambda)). \end{aligned}$$

By (4.2) we deduce that  $D_q \tilde{L}((1 - t) \gamma + t \eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \in L^{p'}(a, b; M)$  and hence

$$[\tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) - \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h)] \rightarrow 0 \quad \text{in } L^1(a, b; M).$$

It follows that

$$\tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \rightarrow \tilde{L}(\gamma, \gamma') \quad \text{in } L^1(a, b; M).$$

Fix  $\varepsilon > 0$ , let  $\delta > 0$  such that for any  $\mathcal{L}^1$ -measurable subset  $\Omega \subseteq ]a, b[$  with  $\mathcal{L}^1(\Omega) < \delta$  we have

$$\int_{\Omega} \Phi(\lambda) d\lambda < \frac{\varepsilon}{2} \quad \text{for all } \Phi \in L^1(a, b; M).$$

Let  $R > 0$  be such that  $\mathcal{L}^1(\{\lambda \in [a, b] : |\eta'_h(\lambda)| > R\}) < \delta$ . Let  $\Omega_h = \{\lambda \in [a, b] : |\eta'_h(\lambda)| > R\}$  and  $\Omega'_h = \{\lambda \in [a, b] : |\eta'_h(\lambda)| \leq R\}$ . By (4.3) we have

$$\begin{aligned} & \int_a^b |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda \\ & \leq \int_{\Omega_h} \bar{C}(1 + \Psi(\lambda)) d\lambda + \int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda \\ & < \frac{\varepsilon}{2} + \int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) \\ & \quad - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda. \end{aligned}$$

Since the map

$$\{\lambda \rightarrow [D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))]\}$$

is uniformly continuous on  $\Omega'_h$ , for  $h$  sufficiently large we have

$$\int_{\Omega'_h} |D_v \tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda)) - D_v \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta'_h(\lambda))|^{p'} d\lambda < \frac{\varepsilon}{2}.$$

It follows that

$$\|D_v \tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h)\|_{p'} \rightarrow 0.$$

Therefore

$$D_v \tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta'_h) \rightarrow D_v \tilde{L}(\gamma, \gamma') \quad \text{in } L^{p'}(a, b; M)$$

and we deduce that

$$\Theta(\eta_h, t_h) \rightarrow \int_a^b \{-D_v \tilde{L}(\gamma, \gamma') \cdot \gamma' + \tilde{L}(\gamma, \gamma')\}' d\lambda = -2\sigma,$$

a contradiction. Finally, we have  $f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t$ . It follows that  $\gamma$  is not  $L$ -stationary, a contradiction.  $\square$

THEOREM 4.9. *Let  $\gamma \in W^{1,p}(a, b; M)$  be  $L$ -stationary. Assume that for every  $s \in \mathbb{R}$  and  $q \in M$  one has*

$$(4.7) \quad |D_s L(s, q, v)| \leq c(q)(1 + |v|^p), \quad \text{for all } v \in T_q N,$$

$$(4.8) \quad L(s, q, \cdot) \text{ is strictly convex on } T_q N.$$

Then the map  $\{s \mapsto D_v L(s, \gamma, \gamma')\gamma' - L(s, \gamma, \gamma')\}$  belongs to  $W^{1,1}(a, b)$  and we have

$$\int_a^b [D_v L(s, \gamma, \gamma')\gamma' - L(s, \gamma, \gamma')]\varphi' ds = \int_a^b D_s L(s, \gamma, \gamma')\varphi ds$$

for all  $\varphi \in C_c^\infty(a, b)$ .

PROOF. Arguing as in the proof of Theorem 4.7, we may assume that  $N$  is a smooth submanifold of  $\mathbb{R}^n$ ,  $A$  is an open subset of  $\mathbb{R}^n$  with  $N \subseteq A$  and  $\tilde{L}: \mathbb{R} \times A \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^1$ -extension of  $L$  to  $\mathbb{R} \times A \times \mathbb{R}^n$  satisfying (4.2)–(4.4) and such that for every  $(s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n$  one has

$$(4.9) \quad |D_s \tilde{L}(s, q, v)| \leq \tilde{c}(q)(1 + |v|^p),$$

$$(4.10) \quad \tilde{L}(s, q, \cdot) \text{ is strictly convex.}$$

Assume, for a contradiction, that there exists  $\varphi \in C_c^\infty(a, b)$  such that

$$\sigma := \frac{1}{2} \int_a^b \{[D_v \tilde{L}(s, \gamma, \gamma') \cdot \gamma' - \tilde{L}(s, \gamma, \gamma')]\varphi' - D_s \tilde{L}(s, \gamma, \gamma')\varphi\} ds > 0.$$

Arguing as in the proof of Theorem 4.8 we may introduce the continuous map

$$\begin{aligned} \mathcal{H}: \{\eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0, r] \\ \rightarrow W^{1,p}(a, b; M) \end{aligned}$$

defined by

$$\mathcal{H}(\eta, t)(\mu) = \eta(\mu - t\varphi(\mu))$$

satisfying the following facts:

$$\begin{aligned} \mathcal{H}(\eta, t)(a) &= \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b), \\ d_1(\mathcal{H}(\eta, t), \eta) &< \widehat{C}t(f_{a,b}(\gamma) + r + d(b-a))^{1/p}, \\ f_{a,b}(\mathcal{H}(\eta, t)) &\leq f_{a,b}(\eta) + t\Theta(\eta, t) \end{aligned}$$

where  $\widehat{C} > 0$ ,

$$\begin{aligned} \Theta(\eta, t) &= \int_a^b \left[ D_s \tilde{L}(\lambda + t\vartheta(\lambda, t)\varphi(\psi(\lambda, t)), \eta, (1 - t\vartheta(\lambda, t)\varphi'(\psi(\lambda, t)))\eta')\varphi(\psi(\lambda, t)) \right. \\ &\quad - D_v \tilde{L}(\lambda + t\vartheta(\lambda, t)\varphi(\psi(\lambda, t)), \eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \cdot \eta'(\lambda)\varphi'(\psi(\lambda, t)) \\ &\quad \left. + \tilde{L}(\psi(\lambda, t), \eta(\lambda), (1 - t\varphi'(\psi(\lambda, t)))\eta'(\lambda)) \frac{\varphi'(\psi(\lambda, t))}{1 - t\varphi'(\psi(\lambda, t))} \right] d\lambda \end{aligned}$$

and  $0 < \vartheta(\lambda, t) < 1$ .

We claim that, for  $r$  sufficiently small, we have  $\Theta(\eta, t) \leq -\sigma$  for any  $\eta \in W^{1,p}(a, b; M)$  with  $d_\infty(\eta, \gamma) < r$ ,  $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$  and  $0 \leq t \leq r$ . By contradiction, let  $(\eta_h)$  be a sequence in  $W^{1,p}(a, b; M)$  uniformly convergent to  $\gamma$  with  $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + \frac{1}{h}$  and  $(t_h)$  be a non negative sequence convergent to 0 such that  $\Theta(\eta_h, t_h) > -\sigma$ . Because of (4.4) and  $f_{a,b}$  is lower semicontinuous, we have that  $f_{a,b}(\eta_h) \rightarrow f_{a,b}(\gamma)$ . Again by (4.4)  $(\eta_h)$  is bounded in  $W^{1,p}(a, b; M)$  and up to a subsequence  $\eta_h \rightharpoonup \gamma$  in  $W^{1,p}(a, b; M)$ . On the other hand, we have

$$\begin{aligned} & \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda - \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda)) d\lambda \\ &= f_{a,b}(\eta_h) - f_{a,b}(\gamma) - \int_a^b \tilde{L}(\lambda, \eta_h(\lambda), \eta'_h(\lambda)) d\lambda + \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda. \end{aligned}$$

Taking into account (4.2), we get that

$$\int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda)) d\lambda \rightarrow \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda)) d\lambda.$$

By [20, Theorem 3] applied to the function  $\Phi(\lambda, \xi) = \tilde{L}(\lambda, \gamma(\lambda), \xi)$  it follows that  $\eta'_h$  is strongly convergent to  $\gamma'$  in  $L^p(a, b; M)$ ; hence  $\eta_h \rightarrow \gamma$  in  $W^{1,p}(a, b; M)$ . Because of (4.2), (4.3) and (4.9), we have that

$$\Theta(\eta_h, t_h) \rightarrow \int_a^b \{[-D_v \tilde{L}(\lambda, \gamma, \gamma') \cdot \gamma' + \tilde{L}(\lambda, \gamma, \gamma')]\varphi' + D_s \tilde{L}(\lambda, \gamma, \gamma')\varphi\} d\lambda = -2\sigma,$$

a contradiction. Finally, we have  $f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t$ . It follows that  $\gamma$  is not  $L$ -stationary, a contradiction.  $\square$

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