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## Fredholm Factorization for Wedge Problems

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### Introduction

Recently the diffraction by arbitrary impenetrable wedges has been reduced to the factorization of matrices of order four [1]. This paper provides an efficient and general factorization technique that is based on the solution of a Fredholm integral equation of second kind.

### Wiener- Hopf solution of the problem

Figure 1 illustrates the problem of the diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in a medium with permittivity  $\epsilon$  and permeability  $\mu$ .

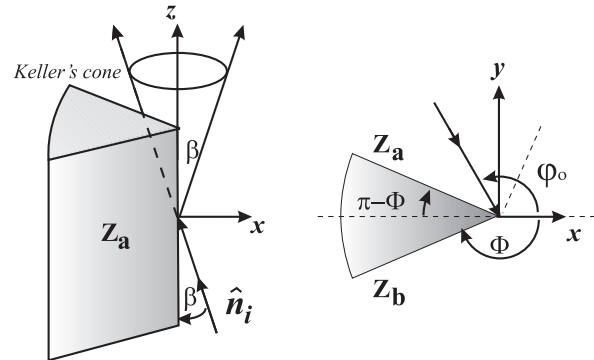


Figure 1: Geometry of the problem

The incident field is constituted by a plane wave having the following longitudinal components:

$$E_z^i = E_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad H_z^i = H_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad (1)$$

where  $\beta$  and  $\varphi_o$  are the zenithal and azimuthal angle of the direction of the plane wave  $\hat{n}_i$  and  $k = \omega\sqrt{\mu\epsilon}$ ,  $\alpha_o = k \cos \beta$ ,  $\tau_o = k \sin \beta$ .

The tangential fields are related on the boundaries of the wedge  $\varphi = +\Phi$  (a-face) and  $\varphi = -\Phi$  (b-face) through the Leontovich conditions:

$$\begin{bmatrix} E_z(\rho, \Phi) \\ E_\rho(\rho, \Phi) \end{bmatrix} = Z_a \begin{bmatrix} H_\rho(\rho, \Phi) \\ -H_z(\rho, \Phi) \end{bmatrix}, \quad \begin{bmatrix} E_z(\rho, -\Phi) \\ E_\rho(\rho, -\Phi) \end{bmatrix} = -Z_b \begin{bmatrix} H_\rho(\rho, -\Phi) \\ -H_z(\rho, -\Phi) \end{bmatrix} \quad (2)$$

where the matrices  $Z_{a,b} = Z_o \begin{bmatrix} z_{11}^{a,b} & z_{12}^{a,b} \\ z_{21}^{a,b} & z_{22}^{a,b} \end{bmatrix}$  depends on the wedge material and

$Z_o = \sqrt{\mu/\varepsilon}$  is the free space impedance.

The Wiener-Hopf formulation [1,4] of this problem yields the solution:

$$\bar{X}_+(\bar{\eta}) = \bar{G}_+^{-1}(\bar{\eta})\bar{G}_+(\bar{\eta}_o) \frac{\bar{T}_o}{\bar{\eta} - \bar{\eta}_o} \quad (3)$$

where:

$$V_{z+}(\eta, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{jn\rho} d\rho, \quad I_{z+}(\eta, \varphi) = \int_0^\infty H_z(\rho, \varphi) e^{jn\rho} d\rho \quad (4)$$

$$V_{\rho+}(\eta, \varphi) = \int_0^\infty E_\rho(\rho, \varphi) e^{jn\rho} d\rho, \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{jn\rho} d\rho \quad (5)$$

$$\bar{X}_+(\bar{\eta}) = \begin{vmatrix} V_{z+}(\eta, 0) & V_{\rho+}(\eta, 0) & Z_o I_{z+}(\eta, 0) & Z_o I_{\rho+}(\eta, 0) \end{vmatrix}^t \quad (6)$$

and  $\eta = \eta(\bar{\eta}) = -\tau_o \cos\left[\frac{\Phi}{\pi} \left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right]\right]\right]$ .

For the problem at hand the constants  $\bar{T}_o$ ,  $\bar{\eta}_o$  assume the following expressions:

$$\bar{T}_o = \frac{\pi \sin \frac{\pi}{\Phi} \varphi_o}{\sin \varphi_o} \begin{vmatrix} jE_o \\ j \frac{\alpha_o \cos \varphi_o E_o + kZ_o \sin \varphi_o H_o}{\tau_o} \\ jZ_o H_o \\ j \frac{\alpha_o Z_o \cos \varphi_o H_o - k \sin \varphi_o E_o}{\tau_o} \end{vmatrix} \quad \text{and } \bar{\eta}_o = -\tau_o \cos \frac{\pi}{\Phi} \varphi_o$$

and the matrix  $\bar{G}_+(\bar{\eta})$  is the plus factorized matrix of the matrix kernel

$$\bar{G}(\bar{\eta}) = \bar{G}_-(\bar{\eta})\bar{G}_+(\bar{\eta}), \quad \bar{G}(\bar{\eta}) = \begin{vmatrix} \frac{g_{11}}{d^a} & \frac{g_{12}}{d^a} & \frac{g_{13}}{d^a} & \frac{g_{14}}{d^a} \\ \frac{g_{21}}{d^a} & \frac{g_{22}}{d^a} & \frac{g_{23}}{d^a} & \frac{g_{24}}{d^a} \\ \frac{g_{31}}{d^b} & \frac{g_{32}}{d^b} & \frac{g_{33}}{d^b} & \frac{g_{34}}{d^b} \\ \frac{g_{41}}{d^b} & \frac{g_{42}}{d^b} & \frac{g_{43}}{d^b} & \frac{g_{44}}{d^b} \end{vmatrix} \quad (7)$$

where:

$$g_{11} = -knz_{11}^a \alpha_o \eta - m\eta \alpha_o^2 - k^2 \eta \xi + kmz_{12}^a \alpha_o \xi - kz_{22}^a \xi \tau_o^2, \quad g_{12} = -knz_{12}^a \tau_o^2 - m\alpha_o \tau_o^2,$$

$$g_{13} = kn\alpha_o \eta - m\eta z_{12}^a \alpha_o^2 - k^2 n z_{12}^a \xi - km\alpha_o \xi + z_{22}^a \eta \alpha_o \tau_o^2,$$

$$g_{14} = kn\tau_o^2 - z_{12}^a m\alpha_o \tau_o^2 + z_{22}^a \tau_o^4,$$

$$g_{21} = k(n\eta - m\xi)\alpha_o z_{11}^a + (\eta\alpha_o + kz_{21}^a \xi)\tau_o^2, \quad g_{22} = knz_{11}^a \tau_o^2 + \tau_o^4,$$

$$g_{23} = mz_{11}^a \alpha_o^2 \eta + k^2 n z_{11}^a \xi - z_{21}^a \alpha_o \eta \tau_o^2 + k\xi \tau_o^2, \quad g_{24} = mz_{11}^a \alpha_o \tau_o^2 - z_{21}^a \tau_o^4,$$

$$d^a = k^2 n^2 z_{11}^a + m^2 z_{11}^a \alpha_o^2 + kn(1 + \Delta^a)\tau_o^2 - m(z_{12}^a + z_{21}^a)\alpha_o \tau_o^2 + z_{22}^a \tau_o^4, \quad \Delta_z^a = z_{11}^a z_{22}^a - z_{12}^a z_{21}^a$$

$d^b, \Delta_z^b, g_{31}, g_{32}, g_{33}, g_{34}, g_{41}, g_{42}, g_{43}, g_{44}$  assume respectively the same expression of  $d^a, \Delta_z^a, g_{11}, g_{12}, -g_{13}, -g_{14}, g_{21}, g_{22}, -g_{23}, -g_{24}$  except for the substitution of the superscript  $a$  with the superscript  $b$ .

The functions  $\xi, m$  and  $n$  depends on  $\bar{\eta}$  and are defined by:

$$\begin{cases} \xi = \xi(\bar{\eta}) = -\tau_o \sin\left[\frac{\Phi}{\pi}\left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right]\right]\right] \\ m = m(\bar{\eta}) = \tau_o \cos\left[\frac{\Phi}{\pi}\left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right] + \Phi\right]\right] \\ n = n(\bar{\eta}) = \tau_o \sin\left[\frac{\Phi}{\pi}\left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right] + \Phi\right]\right] \end{cases} \quad (8)$$

In several important cases the matrix  $\bar{G}(\bar{\eta})$  can be factorized in closed form [2]. For instance, this property is verified for the whole class of problems that have been solved with the Malyuzhinets-Sommerfeld technique.

### Fredholm factorization of the matrix kernel $\bar{G}(\bar{\eta})$

By using the technique introduced in [3] the factorized matrix  $\bar{G}_+(\bar{\eta})$  can be expressed by:

$$\bar{G}_+(\bar{\eta}) = \frac{1}{\bar{\eta} - \bar{\eta}_p} \left[ X_{1+}(\bar{\eta}), X_{2+}(\bar{\eta}), X_{3+}(\bar{\eta}), X_{4+}(\bar{\eta}) \right]^{-1} \quad (10)$$

where  $\bar{\eta}_p$  is an arbitrary point with negative imaginary part and the functions  $X_{i+}(\bar{\eta}), \{i = 1, 2, 3, 4\}$  satisfy the following Fredholm integral equation:

$$\bar{G}(\bar{\eta})X_{i+}(\bar{\eta}) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{[\bar{G}(x) - \bar{G}(\bar{\eta})]X_{i+}(x)}{x - \bar{\eta}} dx = \frac{R_i}{\bar{\eta} - \bar{\eta}_p}, \quad \text{Im}[\bar{\eta}_p] < 0 \quad (11)$$

with the vector constant  $R_i$  given by the canonical basis for the 4D space.

Since  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\bar{G}(x) - \bar{G}(\bar{\eta})}{x - \bar{\eta}} \right|^2 dx d\bar{\eta}$  is bounded [3], (11) is a Fredholm equation of second kind where it is applicable the well-known in literature powerful solution technique.

We experienced [4] that the convergence of approximate solutions considerably increases when we solve the integral equation in the  $t$  plane defined by the mapping  $\bar{\eta} = \bar{\eta}(t) = -\tau_o \cos\left(jt - \frac{\pi}{2}\right)$ .

### Numerical validation

To ascertain the correctness of our new methodology we have chosen a well known in literature test case to compare our solution with alternative method [5]: the bistatic far field amplitude evaluation for skew incidence on an impedance half plane. Figure 2 reports the GTD Diffraction Coefficient for Ez component

( $D_E(\varphi) = s_E(\varphi - \pi) - s_E(\varphi + \pi)$ , where  $s_E(w)$  is the Sommerfeld function) for the test case with the following problem parameters:  $k=1$ , the incident field  $\varphi_0=5\pi/6$ ,  $\beta = \pi/3$ ,  $E_{z0}=1$ ,  $H_{z0}=0$ , the aperture angle  $\Phi=\pi$ , the integration parameters  $A=5$ ,  $h=1$  for the discretization of equation (11) after the transformation in the  $t$  plane. Peaks of the GTD Diffraction Coefficients are for  $\varphi = \varphi_0 - \pi$  (incident field) and for  $\varphi = 2\Phi - \varphi_0 - \pi$  (reflected field).

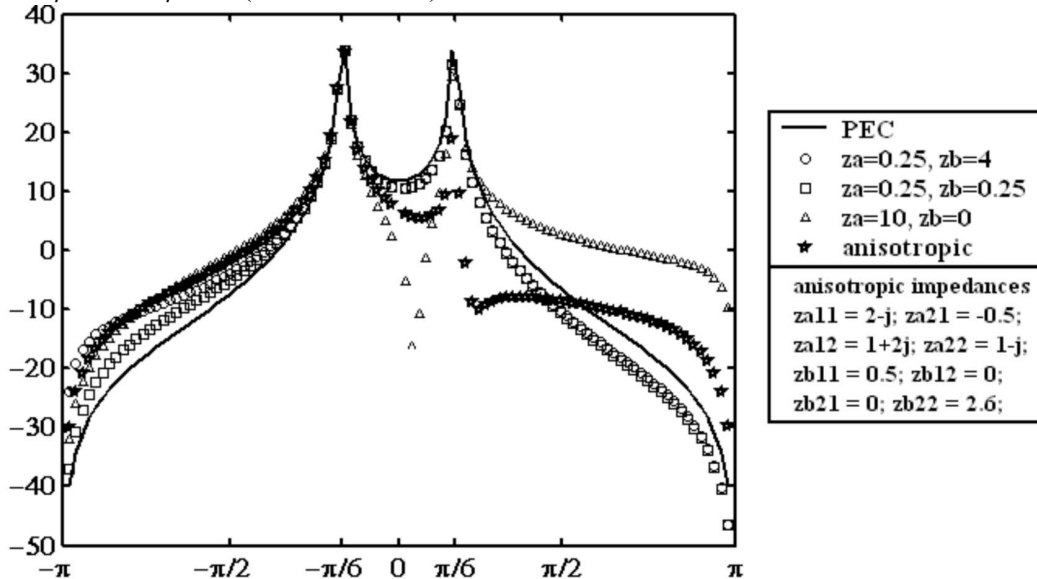


Figure 2: Amplitude of the GTD Diffraction Coefficient (dB)

Several other applications of this technique to wedge problems have been reported in [6]. New examples and convergence tests concerning with new canonical wedge problems will be illustrated in the oral presentation of the paper.

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