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V. G. Daniele*   G. Lombardi*

Abstract – **This paper deals with the Wiener-Hopf technique for solving arbitrary impenetrable wedge problems. A general method for obtaining efficient solutions is presented.**

1 INTRODUCTION

Fig. 1 illustrates the problem of the diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in a medium with permettivity ε and permeability μ.

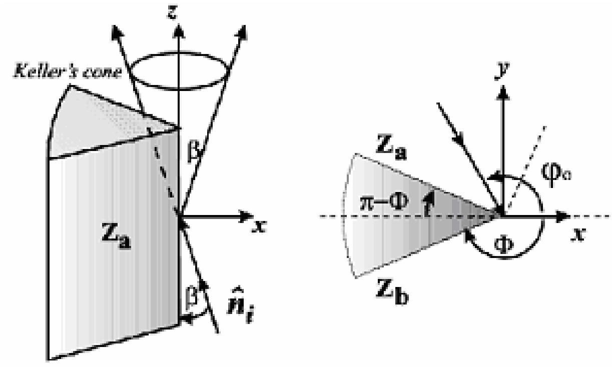

fig.1: geometry of the problem

The incident field is constituted by a plane wave having the following longitudinal components:

$$E_z^i = E_o e^{j\tau_o \rho \cos(\varphi-\varphi_o\}} e^{-j\alpha_o z} \qquad (1a)$$

$$H_z^i = H_o e^{j\tau_o \rho \cos(\varphi-\varphi_o)} e^{-j\alpha_o z} \qquad (1b)$$

where, by indicating with β and φ_o the zenithal and azimuthal angle of the direction of the plane wave $\hat{n}_i$, it is: $k = \omega\sqrt{\mu\varepsilon}$, $\alpha_o = k\cos\beta$, $\tau_o = k\sin\beta$. On the boundaries of the wedge $\varphi = +\Phi$ (a-face) and $\varphi = -\Phi$ (b-face), the tangential fields are related through the Leontovich conditions:

$$\begin{bmatrix} E_z(\rho,\Phi) \\ E_\rho(\rho,\Phi) \end{bmatrix} = Z_a \begin{bmatrix} H_\rho(\rho,\Phi) \\ -H_z(\rho,\Phi) \end{bmatrix}, \qquad (2)$$

$$\begin{bmatrix} E_z(\rho,-\Phi) \\ E_\rho(\rho,-\Phi) \end{bmatrix} = -Z_b \begin{bmatrix} H_\rho(\rho,-\Phi) \\ -H_z(\rho,-\Phi) \end{bmatrix}$$

where, by indicating with $Z_o = \sqrt{\dfrac{\mu}{\varepsilon}}$ the free space impedance, the matrices $Z_{a,b} = Z_o \begin{bmatrix} z_{11}^{a,b} & z_{12}^{a,b} \\ z_{21}^{a,b} & z_{22}^{a,b} \end{bmatrix}$ depending on the wedge material.

The Wiener-Hopf formulation of the problem is reported in [1]. It yields the following systems of four equations:

$$G(\eta) X_+(\eta) = X_-(m) \qquad (3)$$

where, introducing the following Laplace transform:

$$V_{z+}(\eta,\varphi) = \int_0^\infty E_z(\rho,\varphi) e^{j\eta\rho} d\rho,$$

$$I_{z+}(\eta,\varphi) = \int_0^\infty H_z(\rho,\varphi) e^{j\eta\rho} d\rho$$

$$V_{\rho+}(\eta,\varphi) = \int_0^\infty E_\rho(\rho,\varphi) e^{j\eta\rho} d\rho,$$

$$I_{\rho+}(\eta,\varphi) = \int_0^\infty H_\rho(\rho,\varphi) e^{j\eta\rho} d\rho$$

we have:

$$X_+(\eta) = \begin{vmatrix} V_{z+}(\eta,0) \\ V_{\rho+}(\eta,0) \\ Z_o I_{z+}(\eta,0) \\ Z_o I_{\rho+}(\eta,0) \end{vmatrix}, \quad X_-(m) = \begin{vmatrix} Z_o I_{\rho+}(-m,\Phi) \\ -Z_o I_{z+}(-m,\Phi) \\ -Z_o I_{\rho+}(-m,-\Phi) \\ Z_o I_{z+}(-m,-\Phi) \end{vmatrix}$$

with $m = m(\eta) = -\eta\cos\Phi + \xi\sin\Phi$.

The kernel matrix $G(\eta)$ is defined by:

$$G(\eta) = D^{-1}(m) S(\eta) \qquad (4)$$

where:

$$S(\eta) = \begin{vmatrix} \xi & 0 & -\dfrac{\alpha_o\eta}{k} & -\dfrac{\tau_o^2}{k} \\ \dfrac{\alpha_o\eta}{k} & \dfrac{\tau_o^2}{k} & \xi & 0 \\ \xi & 0 & \dfrac{\alpha_o\eta}{k} & \dfrac{\tau_o^2}{k} \\ -\dfrac{\alpha_o\eta}{k} & -\dfrac{\tau_o^2}{k} & \xi & 0 \end{vmatrix}$$

$$D(m) = \begin{vmatrix} D_1(m) & 0_2 \\ 0_2 & D_2(m) \end{vmatrix}$$

$$D_1(m)=\begin{vmatrix} -n\,z_{11}^a-\frac{\tau_o^2}{k} & -n\,z_{12}^a-\frac{m\alpha_o}{k} \\ -\frac{m\,z_{11}^a\alpha_o}{k}+\frac{z_{21}^a\tau_o^2}{k} & n-\frac{m\,z_{12}^a\alpha_o}{k}+\frac{z_{22}^a\tau_o^2}{k} \end{vmatrix}$$

$$D_2(m)=\begin{vmatrix} -n\,z_{11}^b-\frac{\tau_o^2}{k} & -n\,z_{12}^b-\frac{m\alpha_o}{k} \\ \frac{m\,z_{11}^b\alpha_o}{k}-\frac{z_{21}^b\tau_o^2}{k} & -n+\frac{m\,z_{12}^b\alpha_o}{k}-\frac{z_{22}^b\tau_o^2}{k} \end{vmatrix}$$

with:

$\xi=\sqrt{\tau_o^2-\eta^2}$, $n=n(\eta)=-\eta\sin\Phi-\xi\cos\Phi$

By using the mappings:

$$\eta=\eta(\overline{\eta})=-\tau_o\cos[\frac{\Phi}{\pi}[\arccos[-\frac{\overline{\eta}}{\tau_o}]]$$

$$m=m(\overline{\eta})=\tau_o\cos[\frac{\Phi}{\pi}[\arccos[-\frac{\overline{\eta}}{\tau_o}]+\Phi],$$

equations (3) reduce to the matrix W-H system of order four [1,2]:

$$\overline{G}(\overline{\eta})\overline{X}_+(\overline{\eta})=\overline{X}_-(\overline{\eta}) \qquad (5)$$

where : $\overline{G}(\overline{\eta})=G(\eta(\overline{\eta}))$, and the plus and minus unknown functions are defined by:

$$\overline{X}_+(\overline{\eta})=X_+(\eta(\overline{\eta})), \quad \overline{X}_-(\overline{\eta})=X_-(m(\overline{\eta})).$$

2 SOLUTION OF W-H EQUATION (5)

2.1 Formal Solution

The genial technique ideated by Wiener and Hopf in 1931 yields the following formal solutions of the system (5):

$$\overline{X}_+(\overline{\eta})=\overline{G}_+^{-1}(\overline{\eta}))\overline{G}_+(\overline{\eta}_o)\frac{\overline{T}_o}{\overline{\eta}-\overline{\eta}_o} \qquad (6a)$$

$$\overline{X}_-(\overline{\eta})=\overline{G}_-(\overline{\eta}))\overline{G}_+(\overline{\eta}_o)\frac{\overline{T}_o}{\overline{\eta}-\overline{\eta}_o} \qquad (6b)$$

where $\overline{G}_\pm(\overline{\eta})$ are the factorized matrices of $\overline{G}(\overline{\eta})$:

$$\overline{G}(\overline{\eta})=\overline{G}_-(\overline{\eta}))\overline{G}_+(\overline{\eta}), \qquad (7)$$

and : $\overline{\eta}_o=-\tau_o\cos\frac{\pi}{\Phi}\varphi_o$,

$$\overline{T}_o=\frac{\pi}{\Phi}\frac{\sin\frac{\pi}{\Phi}\varphi_o}{\sin\varphi_o}\begin{vmatrix} jE_o \\ j\frac{\alpha_o\cos\varphi_o E_o+kZ_o\sin\varphi_o H_o}{\tau_o} \\ jZ_oH_o \\ j\frac{\alpha_o Z_o\cos\varphi_o H_o-k\sin\varphi_o E_o}{\tau_o} \end{vmatrix}$$

2.2 Exact and approximate factorization of the matrix kernel .

The matrix kernel of order four $\overline{G}(\overline{\eta})$ can be factorized in closed form in many classes of wedge problems [2]. They include all the ones that have been solved with the Sommerfeld-Malyuzhinets method [3]. However in the more general case $\overline{G}(\overline{\eta})$ does not seem to be possible an explicit factorization. Consequently an approximate factorization technique appears to be necessary. We attempted and experienced many approximate factorization techniques [2],[4]. For wedge problems the best one is that described in the following section. It is based to the reduction of factorization problem to a Fredholm equation. This method provides very accurate results and it is simple to manage.

3 APPROXIMATE FACTORIZATION OF $\overline{G}(\overline{\eta})$

3.1 The Fredholm equation for the plus factorized function $\overline{G}_+(\overline{\eta})$

It can be shown [4] that the plus factorized matrix $\overline{G}_+(\overline{\eta})$ can be evaluated by the equation:

$$\overline{G}_+(\overline{\eta})=\frac{1}{\overline{\eta}-\overline{\eta}_p}\left|X_{1+}(\overline{\eta}),X_{2+}(\overline{\eta}),X_{2+}(\overline{\eta}),X_{4+}(\overline{\eta})\right|^{-1}$$

where $\overline{\eta}_p$ is an arbitrary point with negative imaginary part and the four $X_{i+}(\overline{\eta})$ are vanishing vectors as $\overline{\eta}\to\infty$ and separately satisfy the same Fredholm equation:

$$\overline{G}(\overline{\eta})X_{i+}(\overline{\eta})+\frac{1}{2\pi j}.\int_{-\infty}^{\infty}\frac{\left[\overline{G}(x)-\overline{G}(\overline{\eta})\right]X_{i+}(x)}{x-\overline{\eta}}dx=\frac{R_i}{\overline{\eta}-\overline{\eta}_p}, \qquad (8)$$

with:

$$R_1=\begin{vmatrix}1\\0\\0\\0\end{vmatrix},\quad R_2=\begin{vmatrix}0\\1\\0\\0\end{vmatrix},\quad R_3=\begin{vmatrix}0\\0\\1\\0\end{vmatrix},\quad R_4=\begin{vmatrix}0\\0\\0\\1\end{vmatrix},$$

3.2 Solution of the Fredholm equation (8) in the *w*-plane

In order to make function–theoretic manipulations easier we introduce the $w-plane$ and the $\overline{w}-plane$ defined through the equations:

$$\overline{\eta}=-\tau_o\cos\overline{w},\eta=-\tau_o\cos w,\ w=\frac{\Phi}{\pi}\overline{w} \qquad (9)$$

Note that we will use the following notations for quantities $F(\eta)$ defined in different spectral domains:

$$F(\eta)=F(-\tau_o\cos w)=\hat{F}(w)=\overline{F}(\overline{\eta})=$$
$$=\overline{F}(-\tau_o\cos\overline{w})=\tilde{F}(\overline{w})$$

In the $\overline{w}-plane$ the Fredholm equation (8) can be rewritten in the form [4]:

$$H(t)Y_i(t)+\frac{1}{2\pi j}\int_{-\infty}^{+\infty}m(t,u)Y_i(u)du=$$
$$=-\frac{R_i}{\tau_o(j\sinh t-\cos\overline{w}_p)},$$

where *t* (and *u*) variable is defined by $\overline{w}=-\frac{\pi}{2}+jt$

and where $\overline{\eta}_p=-\tau_o\cos\overline{w}_p$

$$\widetilde{G}(\overline{w})=\widetilde{G}(-\frac{\pi}{2}+jt)=H(t)$$

$$\widetilde{X}_{i+}(\overline{w})=\widetilde{X}_{i+}(-\frac{\pi}{2}+ju)=Y_i(u),$$

$$m(t,u)=If[u==t,\lim\frac{[H(u)-H(t)]}{-\sinh u+\sinh t}\cosh u,$$
$$\frac{[H(u)-H(t)]}{-\sinh u+\sinh t}\cosh u]$$

with the limit evaluated as $u=t$.

The sampled form of the previous Fredholm equation is:

$$H(hr)Y_{ia}(hr)+\frac{h}{2\pi j}\sum_{s=-A/h}^{A/h}m(hr,hi)Y_{ia}(hs)=$$
$$-\frac{R_i}{\tau_o(j\sinh hr-\cos\overline{w}_p)},$$

where $r=0,\pm1,\pm2,...,\pm\frac{A}{h}$ and *h* has to be chosen as small as possible and A has to be chosen as large as possible.

The solution of the previous equations in $2\frac{A}{h}+1$ unknowns $Y_{ia}(h\,s),(s=0,\pm1,\pm2,...,\pm\frac{A}{h})$ yields the following approximate representation of the elements $X_{i+}(\overline{\eta})=\hat{X}_{i+}(w)\approx\hat{X}_{ia+}(w)$ of the factorized $\hat{G}_+(w)=\overline{G}_+(\overline{\eta})$:

$$\hat{X}_{ia+}(w)=-\frac{\hat{G}(w)^{-1}}{2\pi j}h\sum_{s=-A/h}^{A/h}m[-j(\frac{\pi}{\Phi}w+\frac{\pi}{2}),hs]\,Y_{ia}(hs)+$$
$$+\hat{G}(w)^{-1}\frac{R_i}{\tau_o(\cos\overline{w}_p-\cos\frac{\pi}{\Phi}w)}, \qquad (10)$$

Since it is approximate, the representation (10) is valid only on the strip $-\Phi\le\mathrm{Re}[w]\le0$ [4]. To obtain correct approximations of $\hat{X}_{i+}(w)$ for every value of w must use analytical continuations. To provide them we must take into account that in the w-plane, all the plus functions are even functions :

$$\hat{X}_+(w)=\hat{X}_+(-w) \qquad (11)$$

consequently the following relationship holds[2,4]:

$$\hat{X}_+(\pm w)=\hat{G}^{-1}(\mp w)\hat{G}(\pm w-2\Phi)\hat{X}_+(\pm w-2\Phi) \qquad (12)$$

4 THE ELECTROMAGNETIC FIELD IN THE WEDGE PROBLEM

4.1 Evaluation of the Sommerfeld functions

The W-H solution (6a) provides the Laplace transforms of the electromagnetic fields: $V_{z,\rho+},I_{z,\rho+}(\eta,0)=\overline{V}_{z,\rho+},\overline{I}_{z,\rho+}(\overline{\eta},0)$, only along the $\varphi=0$ direction.

Using equivalence theorems or functional equations [1,2] we can evaluate $V_{z,\rho+},I_{z,\rho+}(\eta,\varphi)$ for every value of φ. However we wish maintain the procedures used in the current literature [3] and alternatively we introduce the Sommerfeld functions $s_E(w)$ and $s_H(w)$ of the problem.

It can be shown [5] that $s_E(w)$ and $s_H(w)$ are given by:

$$s_E(w)=\frac{j}{2}[-\tau_o\sin wV_{z+}(-\tau_o\cos w,0)+\frac{\tau_o^2}{\omega\varepsilon}I_{\rho+}(-\tau_o\ \cos w,0)+$$
$$-\frac{\alpha_o\tau_o\cos w}{\omega\varepsilon}I_{z+}(-\tau_o\ \cos w,0)] \qquad (13)$$

$$s_H(w)=\frac{j}{2}[-\tau_o\ \sin wI_{z+}(-\tau_o\ \cos w,0)-\frac{\tau_o^2}{\omega\mu}V_{\rho+}(-\tau_o\cos w,0)+$$
$$+\frac{\alpha_o\tau_o\cos w}{\omega\mu}V_{z+}(-\tau_o\cos w,0)] \qquad (14)$$

It yields the following representation of the longitudinal components valid for every value of φ [3]:

$$E_z(\rho,\varphi) = \frac{1}{2\pi j}\left[\int_\gamma s_E[w+\varphi]e^{+j\tau_o\cos[w]\rho}dw\right] \quad (15)$$

$$H_z(\rho,\varphi) = \frac{1}{2\pi j}\left[\int_\gamma s_H[w+\varphi]e^{+j\tau_o\cos[w]\rho}dw\right] \quad (16)$$

where γ is the Sommerfeld contour .

4.2 . Far Field evaluation

Using the saddle point method on eqs.(15-16) yields the far field evaluation:

$$E_z(\rho,\varphi) = E_z^g(\rho,\varphi) + E_z^d(\rho,\varphi) + E_z^s(\rho,\varphi)$$

$$H_z(\rho,\varphi) = H_z^g(\rho,\varphi) + H_z^d(\rho,\varphi) + H_z^s(\rho,\varphi)$$

where E_z^g, H_z^g represent the geometrical optics contribution, E_z^d, H_z^d the diffracted fields and E_z^s, H_z^s the eventually contribution of the surface waves.

The contribution of the geometrical optical field arises from the residues of the poles w_n that satisfy the equation: $\cos\left(\frac{\pi}{\Phi}w_n\right) - \cos\left(\frac{\pi}{\Phi}\varphi_o\right) = 0$ and that are present in Π^{RES} that is the region enclosed by the SDP in $\pm\pi$ and the Sommerfeld contour γ. After algebraic manipulations, it is remarkable to observe that always the evaluation of these residue do not require the explicit factorization (7). It is in accord with the fact that the geometrical optical field can be evaluated by solving the simple problems of reflection of a plane wave by flat indefinite impedance surfaces.

The diffracted fields arise from the saddle points in $w = \pm\pi$ and have the form [3]:

$$E_z^d(\rho,\varphi,z) = e^{-j\alpha_o z}\frac{e^{-j\left(\tau_o\rho+\frac{\pi}{4}\right)}}{\sqrt{2\pi\,\tau_o\,\rho}}\left[s_E(\varphi-\pi) - s_E(\varphi+\pi)\right]$$

$$H_z^d(\rho,\varphi,z) = e^{-j\alpha_o z}\frac{e^{-j\left(\tau_o\rho+\frac{\pi}{4}\right)}}{\sqrt{2\pi\,\tau_o\,\rho}}\left[s_H(\varphi-\pi) - s_H(\varphi+\pi)\right]$$

Let us remember that a diffracted rays constitutes a generatrix of the Keller cone and it is defined by the angular spherical coordinates β and φ (fig.1). Taking into account that the incident field has angular spherical coordinates β and φ_o, it is convenient to relate the transversal component E_β^d, E_φ^d of the diffracted ray to the transversal component $E_\beta^i, E_{\varphi_o}^i$ of the incident ray.

By geometrical consideration we have [2]:

$$E_\beta^d = -\frac{1}{\sin\beta}E_z^d, \qquad E_\varphi^d = -\frac{1}{\sin\beta}Z_o H_z^d,$$

$$E_z^i = \sin\beta\, E_\beta^i, \qquad H_z^i = \frac{1}{Z_o}\sin\beta\, E_{\varphi_o}^i$$

The previous equations cannot be used when the observation point approaches shadow boundaries of the incident and the reflected waves. In this case the poles are near the saddle points and uniform diffracted fields appears necessary [2,6].

By residue theorem, we obtain the following contribution of the surface waves:

$$E_z^s(\rho,\varphi,z) = \sum_i \mathrm{Res}[s_E(w+\varphi)]_{w=w_i}\, e^{+j\tau_o\cos w_i\rho}\; e^{-j\alpha_o z}$$

$$H_z^s(\rho,\varphi,z) = \sum_i \mathrm{Res}[s_H(w+\varphi)]_{w=w_i}\, e^{+j\tau_o\cos w_i\rho}\; e^{-j\alpha_o z}$$

where w_i are the structural poles $s_{E,H}(w+\varphi)$ present in Π^{RES}. Taking into account equation (12), these poles satisfy the equation [2] $\hat{d}_{a,b}[\pm(w_i+\varphi)-\Phi] = 0$

where:

$$\hat{d}_{a,b}[\psi] = \left(\tau_o^2\sin^2\psi + \alpha_o^2\right)z_{11}^{a,b} + k\tau_o\sin\psi(1+\det[z^{a,b}])$$
$$-\alpha_o\tau_o\cos\psi(z_{12}^{a,b} + z_{21}^{a,b}) + \tau_o^2 z_{22}^{a,b}$$

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