# The Wiener-Hopf technique for impenetrable wedge problems

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This paper presents how to define and solve Generalized Wiener-Hopf (GWHE) equations for angular region problems. In particular it illustrates the techniques to deal efficiently with arbitrary impenetrable wedge problems at skew incidence. Very accurate and efficient numerical results are obtained numerically by reducing the factorization problem to a Fredholm integral equation of second kind. Asymptotic evaluation of far fields has been performed.

## 1 INTRODUCTION

In the last five years Daniele has developed a general theory based on the Wiener-Hopf technique to study electromagnetic problems in arbitrary angular regions [1]-[4]. In general this technique yields a new class of functional equations called Generalized Wiener-Hopf equations (GWHE). The GWHEs differ from the classical Wiener-Hopf equations (CWHE) since the involved plus and minus functions are defined into two different complex planes. It is remarkable that in some cases a suitable mapping reduces the Generalized Wiener-Hopf equations to the classical ones. For instance it happens in all the impenetrable wedge problems that have been solved by the Sommerfeld-Malyuzhinets (SM) method. Closed form W-H solutions for these problems and also for other problems, which are not solved with the SM method, have been obtained by using an explicit factorization of the matrix kernels [2]-[4]. For arbitrary impenetrable wedges problems, the W-H formulation involves the factorization of matrix kernels of order four. With the exception of some classes of problems, including the all ones solved by the SM method, closed form factorizations of kernels are not available and we need to resort to approximate factorization techniques. Several techniques to obtain approximate factorizations of arbitrary matrices are available in literature [5]. New different approximate methods are defined with respect to reference [6]. In particular the W-H factorization problem provides its immediate reduction to Fredholm equations without applying regularizations to the kernel operator. In general the powerfulness of the approximate W-H factorization technique depends on the kernel's spectrum of the related Fredholm equation. When we are dealing with impenetrable wedge problems, we experienced that a particular mapping makes the Fredholm equation suitable to be solved numerically. The aim of this work is to present an efficient method based on the W-H technique to solve the all impenetrable wedge problems which are still unsolved in literature. The second section describes how to obtain functional equations to examine wedge problems immersed in arbitrary media, the third section describes the Generalized Wiener-Hopf equations for impenetrable wedge problems. The fourth section deals with the procedure for solving Generalized Wiener-Hopf equations and

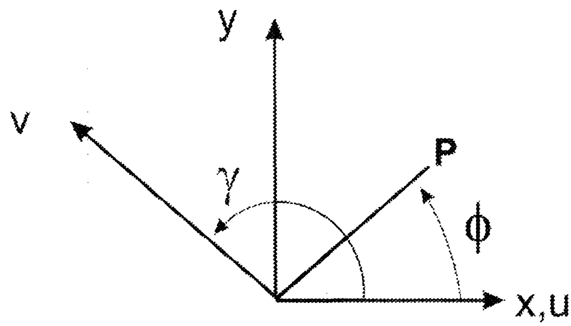


Figure 1: Angular region immersed in arbitrary homogeneous medium.

the following section is concentrated in the solution's procedure for impenetrable wedge problems. The last section shows numerical results to validate the method.

## 2 FUNCTIONAL EQUATIONS

We consider time harmonic electromagnetic fields with a time dependence specified by the factor $e^{j\omega t}$ (which is omitted) to derive functional equations for wedge problems immersed in arbitrary homogeneous medium. First, we consider the electromagnetic field in the angular region indicated as reported in Fig. 1 with aperture angle $\gamma$. For this region the following constitutive relations hold:

$$\begin{aligned} \mathbf{D} &= \varepsilon \cdot \mathbf{E} + \xi \cdot \mathbf{H} \\ \mathbf{B} &= \zeta \cdot \mathbf{E} + \mu \cdot \mathbf{H} \end{aligned}$$

In eq. (1) **D**, **B** are the induction fields; **E**, **H** are the electromagnetic fields and the parameters $\varepsilon$, $\xi$, $\zeta$, $\mu$ are known dyadic terms.

Without loss of generality we assume the $z$ dependence of the electromagnetic fields $e^{j\alpha_o z}$ which is omitted in the next formula. The reference systems used in this paper are: the Cartesian coordinates $x, y, z$, the polar coordinates $\rho, \varphi, z$ with $(0 \leq \varphi \leq \gamma)$ and the oblique Cartesian coordinates $u, v, z$ which are defined by:

$$\begin{aligned} u &= x - y \cot\gamma, \quad v = \tfrac{y}{\sin\gamma} \\ x &= u + v \cos\gamma, \quad y = v \sin\gamma \end{aligned} \tag{1}$$

By applying the abstract Bresler-Marcuvitz field trasversalization to the Maxwell equations [7] we obtain:

$$-\frac{\partial}{\partial y}\psi_t = M(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) \cdot \psi_t$$

where $\psi_t(z,x) = \begin{vmatrix} E_z(z,x) \\ E_x(z,x) \\ H_z(z,x) \\ H_x(z,x) \end{vmatrix}$ and $M$ is a known operator. By considering the $z$

dependence $e^{j\alpha_o z}$:

$$M(\frac{\partial}{\partial z},\frac{\partial}{\partial x}) = M_o(-j\alpha_o) + M_1(-j\alpha_o)\frac{\partial}{\partial x} + M_2(-j\alpha_o)\frac{\partial^2}{\partial x^2}$$

The known $M_i(j\alpha_o)$ $(i = 0, 1, 2)$ matrices are related to the complexity of the medium filling the angular region. The introduction of oblique Cartesian coordinates (1) yields:

$$-\frac{\partial}{\partial v}\psi_t = (M_{oe} + M_{1e}\frac{\partial}{\partial u} + M_{2e}\frac{\partial^2}{\partial u^2})\cdot\psi_t \tag{2}$$

where $M_{oe} = M_o \sin\gamma$, $M_{1e} = M_1\sin\gamma - I_t\cos\gamma$, $M_{2e} = M_2\sin\gamma$ and $I_t$ is the identity matrix. By introducing the Laplace transform of the unknown $\psi_t$ we obtain from (2) the following functional equation:

$$-\frac{d}{dv}\tilde{\psi}_t = M_e\cdot\tilde{\psi}_t + \psi_s$$

where $M_e = M_{oe} - j\eta\, M_{1e} - \eta^2 M_{2e}$, $\tilde{\psi}_t(\eta, v) = \int_0^\infty e^{j\eta u}\psi_t(u,v)du$ and

$$\psi_s(v) = -M_{1e}\cdot\psi_t(0_+,v) + j\eta\, M_{2e}\cdot\psi_t(0_+,v) - M_{2e}\cdot\frac{\partial}{\partial u}\psi_t(u,v)\Big|_{u=0+} = \begin{vmatrix}\psi_{s1}(v)\\ \psi_{s2}(v)\\ \psi_{s3}(v)\\ \psi_{s4}(v)\end{vmatrix}$$

These functional equations relate the Laplace transforms of the transversal (along $y$) field components of an homogeneous angular region. Fig. 2 reports the domain definition for the quantities involved in the previous equations.

By using the characteristic Green's function procedure, it yields for $v = 0$ :

$$\begin{aligned}\tilde{\psi}_t(\eta,v) \;=\; & l_1 l^1\cdot\tilde{\psi}_t(\eta,0)e^{-\lambda_1 v} + l_2 l^2\cdot\tilde{\psi}_t(\eta,0)e^{-\lambda_2 v} + \\ & + l_1 l^1\cdot\int_0^v e^{-\lambda_1(v-v')}\psi_s(v')dv' + l_2 l^2\cdot\int_0^v e^{-\lambda_2(v-v')}\psi_s(v')dv' + \\ & - l_3 l^3\cdot\int_v^\infty e^{-\lambda_3(v-v')}\psi_s(v')dv' - l_4 l^4\cdot\int_v^\infty e^{-\lambda_4(v-v')}\psi_s(v')dv'\end{aligned} \tag{3}$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ are four eigenvalues, $l_1$, $l_2$, $l_3$ and $l_4$ are four eigenvectors of the 4x4 $M_e$ matrix and $l^1$, $l^2$, $l^3$ and $l^4$ are the reciprocal eigenvectors $(l^j\cdot l_i = \delta_{ji})$. For a general linear passive medium two eigenvalues $(\lambda_1, \lambda_2)$ have positive real parts and the other two eigenvalues $(\lambda_3, \lambda_4)$ have negative real parts. We define the tangential vector along the $v$ axis:

$$\psi_\tau(v,\gamma) = \begin{vmatrix}E_z(\rho,\varphi)\\ E_v(\rho,\varphi)\\ H_z(\rho,\varphi)\\ H_v(\rho,\varphi)\end{vmatrix}_{\varphi=\gamma} = \begin{vmatrix}E_z(u,v)\\ E_v(u,v)\\ H_z(u,v)\\ H_v(u,v)\end{vmatrix}_{u=0} = \psi_\tau(v,\gamma)$$

Using the constitutive relations and the Maxwell's equations the vector $\psi_s(v)$ is related to the tangential vector $\psi_\tau(v,\gamma)$:

$$\psi_s(v) = T(\gamma)\cdot\psi_\tau(v,\gamma) \tag{4}$$

where

$$T(\gamma) = \left|\begin{array}{cccc}
\frac{-\alpha_o\xi_{yy}+\varepsilon_{yy}\mu_{xy}\omega-\zeta_{xy}\xi_{yy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma-\cos\gamma & 0 & -\frac{\alpha_o\mu_{yy}-\zeta_{yy}\mu_{xy}\omega+\zeta_{xy}\mu_{yy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 0 \\
\frac{\eta\xi_{yy}+\varepsilon_{yy}\mu_{zy}\omega-\zeta_{zy}\xi_{yy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 1 & \frac{\eta\mu_{yy}-\zeta_{zy}\mu_{yy}\omega+\zeta_{yy}\mu_{zy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 0 \\
-\frac{\alpha_o\varepsilon_{yy}-\varepsilon_{yy}\xi_{xy}\omega+\varepsilon_{xy}\xi_{yy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 0 & \frac{\alpha_o\zeta_{yy}+\varepsilon_{xy}\mu_{yy}\omega-\zeta_{yy}\xi_{xy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma-\cos\gamma & 0 \\
\frac{\eta\varepsilon_{yy}-\varepsilon_{zy}\xi_{yy}\omega+\varepsilon_{yy}\xi_{zy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 0 & \frac{\eta\zeta_{yy}-\varepsilon_{zy}\mu_{yy}\omega+\zeta_{yy}\xi_{zy}\omega}{(\varepsilon_{yy}\mu_{yy}-\zeta_{yy}\xi_{yy})\omega}\sin\gamma & 1
\end{array}\right|$$

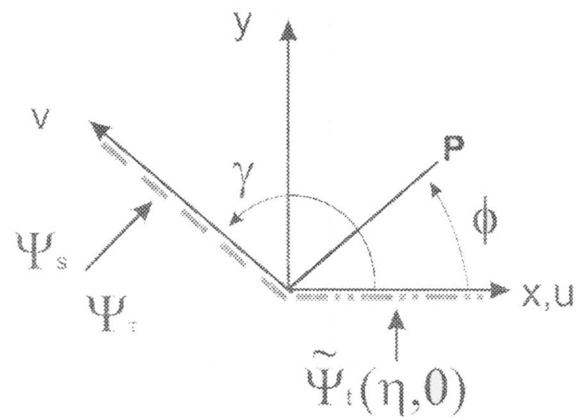


Figure 2: Domain definition for $\tilde{\psi}_t(\eta,0)$, $\psi_s(v)$ and $\psi_\tau(v,\gamma)$ .

Multiplying equation (3) by $l^j$ (j=1,2,..4) and taking into account (4), we obtain the following four equations:

$$\left\{\begin{array}{l}
l^1\cdot\tilde{\psi}_t(\eta,0) = l^1\cdot\tilde{\psi}_t(\eta,0) \\
l^2\cdot\tilde{\psi}_t(\eta,0) = l^2\cdot\tilde{\psi}_t(\eta,0) \\
l^3\cdot\tilde{\psi}_t(\eta,0) = -l^3\cdot T(\gamma)\cdot\tilde{\psi}_\tau(-j\lambda_3,\gamma) \\
l^4\cdot\tilde{\psi}_t(\eta,0) = -l^4\cdot T(\gamma)\cdot\tilde{\psi}_\tau(-j\lambda_4,\gamma)
\end{array}\right. \tag{5}$$

where $\tilde{\psi}_\tau(\alpha,\gamma) = \int_0^\infty e^{j\alpha\, v}\psi_\tau(v,\gamma)dv$. The first two functional equations are identities, conversely the last two equations relate suitable Laplace transforms of the electromagnetic field components that are tangential to the two boundaries $\varphi = 0$ (or $v = 0$) and $\varphi = \gamma$ (or $u = 0$).

### *2.1 Isotropic medium filling the angular region*

In general the last two functional equations of (5) show complicated expressions for arbitrary media but for an isotropic homogeneous medium with scalar $\varepsilon$ and $\mu$ values and vanishing $\xi$ and $\zeta$ values they reduce to simpler forms (6). In fact in this case the eigenvalues degenerate: $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_3 = -jm$.

$$\begin{cases} \xi\, V_{z+}(\eta,0) - \dfrac{\tau^2}{\omega\,\varepsilon} I_{\rho+}(\eta,0) - \dfrac{\alpha_o\,\eta}{\omega\,\varepsilon} I_{z+}(\eta,0) = \\ \qquad - n\; V_{z+}(-m,\gamma) - \dfrac{\tau^2}{\omega\,\varepsilon} I_{\rho+}(-m,\gamma) + \dfrac{\alpha_o\, m}{\omega\,\varepsilon} I_{z+}(-m,\gamma) \\ \xi\; I_{z+}(\eta,0) + \dfrac{\tau^2}{\omega\,\mu} V_{\rho+}(\eta,0) + \dfrac{\alpha_o\,\eta}{\omega\,\mu} V_{z+}(\eta,0) = \\ \qquad - n\, I_{z+}(-m,\gamma) + \dfrac{\tau^{\,2}}{\omega\,\mu} V_{\rho+}(-m,\gamma) - \dfrac{\alpha\, m}{\omega\,\mu} V_{z+}(-m,\gamma) \end{cases} \tag{6}$$

where

$$V_{z+}(\alpha,\varphi) = \int_0^\infty E_z(\rho,\varphi)e^{j\alpha\rho}d\rho, \quad I_{z+}(\alpha,\varphi) = \int_0^\infty H_z(\rho,\varphi)e^{j\alpha\rho}d\rho$$
$$V_{\rho+}(\alpha,\varphi) = \int_0^\infty E_\rho(\rho,\varphi)e^{j\alpha\rho}d\rho, \quad I_{\rho+}(\alpha,\varphi) = \int_0^\infty H_\rho(\rho,\varphi)e^{j\alpha\rho}d\rho$$

and where $\xi = \sqrt{\tau_o^2 - \eta^2}$, $m = -\ \eta\cos\gamma + \sqrt{\tau_o^2 - \eta^2}\sin\gamma$, $\tau_o = \sqrt{\omega^2\mu\varepsilon - \alpha_o^2}$ and $n = -\xi\cos\gamma - \eta\sin\gamma = \sqrt{\tau_o^2 - m^2}$.

## 3 Generalized Wiener-Hopf equations for impenetrable wedge problems

Fig. 3 illustrates the problem of the diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in a medium with permittivity $\varepsilon$ and permeability $\mu$ with anysotropic surface impedances $\mathbf{Z_a}$ and $\mathbf{Z_b}$.

$$\begin{bmatrix} E_z(\rho,+\Phi) \\ E_\rho(\rho,+\Phi) \end{bmatrix} = \mathbf{Z_a} \begin{bmatrix} H_\rho(\rho,+\Phi) \\ -H_z(\rho,+\Phi) \end{bmatrix} \quad \begin{bmatrix} E_z(\rho,-\Phi) \\ E_\rho(\rho,-\Phi) \end{bmatrix} = -\mathbf{Z_b} \begin{bmatrix} H_\rho(\rho,-\Phi) \\ -H_z(\rho,-\Phi) \end{bmatrix} \tag{7}$$

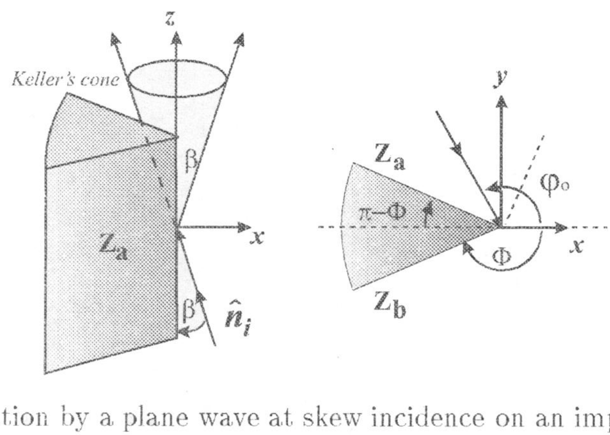


Figure 3: Diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in an homogeneous medium.

By using the same procedure of equations (6) we obtain functional equations that hold in the angular region for negative value of $\varphi$ ($-\gamma_1 \leq \varphi \leq 0$; in general $\gamma_1 \neq \gamma$). For the problem reported in Fig. 3 we have $\gamma = -\gamma_1 = \Phi$ thus:

$$\begin{cases} \xi\, V_{z+}(\eta,0) - \dfrac{\tau_o^2}{\omega\,\varepsilon} I_{\rho+}(\eta,0) - \dfrac{\alpha_o\,\eta}{\omega\,\varepsilon} I_{z+}(\eta,0) = \\ \qquad - n\ V_{z+}(-m,\Phi) - \dfrac{\tau_o^2}{\omega\,\varepsilon} I_{\rho+}(-m,\Phi) + \dfrac{\alpha_o\,m}{\omega\,\varepsilon} I_{z+}(-m,\Phi) \\ \xi\, I_{z+}(\eta,0) + \dfrac{\tau_o^2}{\omega\,\mu} V_{\rho+}(\eta,0) + \dfrac{\alpha_o\,\eta}{\omega\,\mu} V_{z+}(\eta,0) = \\ \qquad - n\, I_{z+}(-m,\Phi) + \dfrac{\tau_o^2}{\omega\,\mu} V_{\rho+}(-m,\Phi) - \dfrac{\alpha\,m}{\omega\,\mu} V_{z+}(-m,\Phi) \\ \xi\, V_{z+}(\eta,0) + \dfrac{\tau_o^2}{\omega\,\varepsilon} I_{\rho+}(\eta,0) + \dfrac{\alpha_o\,\eta}{\omega\,\varepsilon} I_{z+}(\eta,0) = \\ \qquad - n\ V_{z+}(-m,-\Phi) + \dfrac{\tau_o^2}{\omega\,\varepsilon} I_{\rho+}(-m,-\Phi) - \dfrac{\alpha_o\,m}{\omega\,\varepsilon} I_{z+}(-m,-\Phi) \\ \xi\, I_{z+}(\eta,0) - \dfrac{\tau_o^2}{\omega\,\mu} V_{\rho+}(-\eta,0) - \dfrac{\alpha_{o,}\,\eta}{\omega\,\mu} V_{z+}(\eta,0) = \\ \qquad - n\, I_{z+}(-m,-\Phi) - \dfrac{\tau_o^2}{\omega\,\mu} V_{\rho+}(-m,-\Phi) + \dfrac{\alpha_o m}{\omega\,\mu} V_{z+}(-m,-\Phi) \end{cases} \tag{8}$$

By considering the Leontovich boundaries conditions on the wedge's faces (7) and the functional equations for the two angular regions ($0 \leq \varphi \leq \Phi$ and $-\Phi \leq \varphi \leq 0$) (8), we obtain the following system of equations:

$$G(\eta)X_+(\eta) = X_-(m) \tag{9}$$

where $G(\eta) = D^{-1}(m)S(\eta)$ and

$$X_+(\eta) = \begin{vmatrix} V_{z+}(\eta,0) \\ V_{\rho+}(\eta,0) \\ Z_o I_{z+}(\eta,0) \\ Z_o I_{\rho+}(\eta,0) \end{vmatrix}, \quad X_-(m) = \begin{vmatrix} Z_o I_{\rho+}(-m,\Phi) \\ -Z_o I_{z+}(-m,\Phi) \\ -Z_o I_{\rho+}(-m,-\Phi) \\ Z_o I_{z+}(-m,-\Phi) \end{vmatrix} \tag{10}$$

and

$$S(\eta) = \begin{vmatrix} \xi & 0 & -\dfrac{\alpha_o\eta}{k} & -\dfrac{\tau_o^2}{k} \\ \dfrac{\alpha_o\eta}{k} & \dfrac{\tau_o^2}{k} & \xi & 0 \\ \xi & 0 & \dfrac{\alpha_o\eta}{k} & \dfrac{\tau_o^2}{k} \\ -\dfrac{\alpha_o\eta}{k} & -\dfrac{\tau_o^2}{k} & \xi & 0 \end{vmatrix}, \quad D(m) = \begin{vmatrix} D_1(m) & 0_2 \\ 0_2 & D_2(m) \end{vmatrix} \tag{11}$$

with

$$D_1(m) = \begin{vmatrix} -n\, z_{11}^a - \dfrac{\tau_o^2}{k} & -n\, z_{12}^a - \dfrac{m\alpha_o}{k} \\ -\dfrac{m\, z_{11}^a \alpha_o}{k} + \dfrac{z_{21}^a \tau_o^2}{k} & n - \dfrac{m\, z_{12}^a \alpha_o}{k} + \dfrac{z_{22}^a \tau_o^2}{k} \end{vmatrix}$$

$$D_2(m) = \begin{vmatrix} -n\, z_{11}^b - \dfrac{\tau_o^2}{k} & -n\, z_{12}^b - \dfrac{m\alpha_o}{k} \\ \dfrac{m\, z_{11}^b \alpha_o}{k} - \dfrac{z_{21}^b \tau_o^2}{k} & -n + \dfrac{m\, z_{12}^b \alpha_o}{k} - \dfrac{z_{22}^b \tau_o^2}{k} \end{vmatrix}$$

Eq. (9) constitutes a system of Generalized Wiener-Hopf Equations (GWHE) which differ from Classical Wiener-Hopf Equations (CWHE) because the unknowns are defined into two different complex planes ($\eta$ and $m$).

## 4 Procedures for solving the GWHE

In general in an arbitrary wedge problem several homogeneous angular regions are present and for each of them functional equations of the previous kind hold. By enforcing the boundary conditions relative to the different angular regions, we obtain GWHE of the following form (12) where the number of $m_i$ spectral variables depends on the number of the angular regions. In general GWHE for wedge problem presents the following form:

$$\sum_{j=1}^{n} G_{ij}(\eta) X_{j+}(\eta) = -Y_{i+}[-m_i(\eta)] \tag{12}$$

where the unknowns $X_{j+}(\alpha)$ and $Y_{i+}(\alpha)$ are plus functions i.e functions regular in the half-plane $\mathrm{Im}[\alpha] \geq 0$.

We recall that the all problems, that have been solved in closed form by the Malyuzhinets-Sommerfeld method, yield matrix kernels which can be factorized in closed forms [2]-[5]. From closed form factorization we obtain closed form solution of the WH problem. In particular these kernels assume the Daniele-Khrapkov form [2].

However, no analytical solution technique exists for problems involving different wave numbers $m_i$.

In the following subsections we analyze the different procedures to solve GWHE, in particular we focus our attention on the solution of wedge problems.

### *4.1 Reduction to Classical Wiener-Hopf equations*

For impenetrable wedge problems immersed in an homogeneous medium eq. (12) reduce to eq. (9) with unknowns defined in only two spectral domains ($\eta$, $m$). By using the special mapping

$$\eta = \eta(\bar{\eta}) = -\tau_o \cos\left[\frac{\Phi}{\pi} \arccos\left[-\frac{\bar{\eta}}{\tau_o}\right]\right]$$

we obtain that

$$m = m(\bar{\eta}) = \tau_o \cos\left[\frac{\Phi}{\pi} \arccos\left[-\frac{\bar{\eta}}{\tau_o}\right] + \Phi\right]$$

and eqs. (9) reduce to Classical Wiener-Hopf equations:

$$\bar{G}(\bar{\eta})\bar{X}_+(\bar{\eta}) = \bar{X}_-(\bar{\eta}) \tag{13}$$

where $\bar{X}_+(\bar{\eta}) = X_+(\eta(\bar{\eta}))$ and $\bar{X}_-(\bar{\eta}) = X_-(m(\bar{\eta}))$. When the matrix $\bar{G}(\bar{\eta})$ has closed form factorization the solution presents the following classical form where $\bar{T}_o$, $\bar{\eta}_o$ depend on the plane wave source:

$$\bar{X}_+(\bar{\eta}) = \bar{G}_+^{-1}(\bar{\eta}))\bar{G}_+(\bar{\eta}_o)\frac{\bar{T}_o}{\bar{\eta}-\bar{\eta}_o}$$
$$\bar{X}_-(\bar{\eta}) = \bar{G}_-(\bar{\eta}))\bar{G}_+(\bar{\eta}_o)\frac{\bar{T}_o}{\bar{\eta}-\bar{\eta}_o}$$

### *4.2 Moment method*

An approximate method to solve the GWHE equations is the moment method. The unknowns $X_{j+}(\eta)$ are written in term of expansion functions $\Psi^j_{r+}(\eta)$ $(r = 1, 2, ...)$

$$X_{j+}(\eta) = \sum_r C^j_r \Psi^j_{r+}(\eta)$$

By applying the Parseval theorem it is possible to eliminate the unknowns $Y_{i+}(-m)$ from eq. (12) with a suitable choice of test functions $\Phi^j_{s+}(m_i)$:

$$\int_{-\infty}^{\infty} \Phi^j_{s+}(-m_i) \cdot Y_{i+}(-m_i) dm_i = 0$$

The application of the Moment Method yields the following system of linear equation:

$$\sum_{j=1}^{n} \sum_r M^{ij}_{sr} C^j_r = 0$$

where

$$M^{ij}_{sr} = \int_{-\infty}^{\infty} \Phi^j_{s+}(-m_i) \cdot G_{ij}[(\eta(m_i)] \Psi^j_{r+}[(\eta(m_i)] dm_i$$

and $C^j_r$ are the expansion coefficients.

The success of the moment method depends on a suitable choice of the expansion and test functions.

### *4.3 Use of the approximate decomposition technique*

The presence of a rational kernel yields to closed form solution of GWHE. This property suggests the introduction of rational approximate kernels. In particular this method is efficient when the rational approximate kernels do not lose the physical properties of the original kernels and the mathematical consistence of the problem. Padé approximants and interpolation methods are very convenient to obtain rational expressions of the elements of the matrix kernels. However we experienced that the use of rational approximants produces very good results only when the kernel presents a dominant discrete spectrum (Waveguide problems). In wedge problems the spectrum is continuous, thus the use of rational approximant seems to be less convenient with respect to the Fredholm equation approximation described in the next subsection.

### 4.4 Reduction to Fredholm integral equation

For impenetrable wedge problems immersed in an homogeneous medium we obtain eq. (13) with unknowns defined in only one spectral domain $(\bar{\eta})$.

Using the Cauchy decomposition formula we obtain that the factorization problem for the plus unknown and for the plus factorized matrix can be reduced to a Fredholm integral equation of second kind [4]:

$$\bar{G}(\bar{\eta})X_{i+}(\bar{\eta}) + \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\left[\bar{G}(x) - \bar{G}(\bar{\eta})\right] X_{i+}(x)}{x - \bar{\eta}} dx = \frac{R_i}{\bar{\eta} - \bar{\eta}_p}$$

where $\frac{R_i}{\bar{\eta}-\bar{\eta}_p}$ is related to the source term (plane wave).

## 5 Solution

### 5.1 The Fredholm equation in the w-plane

We introduce the $w$-plane and the $\bar{w}$-plane defined through the following equations:

$$\bar{\eta} = -\tau_o \cos\bar{w},\ \eta = -\tau_o \cos w,\ \ w = \frac{\Phi}{\pi}\bar{w}$$

The introduction of these new planes is important to evaluate the far field in terms of the Wiener-Hopf solution $X_{i+}(\bar{\eta})$ by using the standard procedure based on Sommerfeld functions, see for example [8]. We use the spectral transformation $\bar{w} = -\pi/2 + j\,t$ to obtain a new Fredholm equation (14), which shows good convergence properties by using simple quadrature scheme (step approximations) [4], [5], [9].

$$H(t)Y_i(t) + \frac{1}{2\pi j}\int_{-\infty}^{+\infty} M(t,u)Y_i(u)du = -\frac{R_i}{\tau_o(j\sinh t - \cos\bar{w}_p)} \tag{14}$$

where

$$\bar{\eta}_p = -\tau_o\cos\bar{w}_p,\quad \tilde{G}(\bar{w}) = \tilde{G}\left(j\,t - \frac{\pi}{2}\right) = H(t),\quad \tilde{X}_{i+}(\bar{w}) = \tilde{X}_{i+}\left(j\,u - \frac{\pi}{2}\right) = Y_i(u),$$

$$M(t,u) = If\left\{u == t, \lim_{u\to t}\left[\frac{H(u)-H(t)}{-\sinh u + \sinh t}\cosh u\right], \frac{H(u)-H(t)}{-\sinh u + \sinh t}\cosh u\right\}$$

with $If\{condition, t, f\}$ that gives $t$ if condition evaluates to True, and $f$ if it evaluates to False.

### 5.2 Far field evaluation

The W-H solution provides the Laplace transforms of the electromagnetic field components $X_+(\bar{\eta})$. The definition of the Sommerfeld functions (15) in terms of the

W-H solution allow asymptotic evaluation of the field's components (16).

$$\begin{aligned} s_E(w) &= \frac{j}{2}\Bigg[-\tau_o \sin w V_{z+}(-\tau_o\cos w, 0) + \frac{\tau_o^2}{\omega\varepsilon} I_{\rho+}(-\tau_o\cos w, 0) \\ &\qquad - \frac{\alpha_o\tau_o\cos w}{\omega\varepsilon} I_{z+}(-\tau_o\cos w, 0)\Bigg] \\ s_H(w) &= \frac{j}{2}\Bigg[-\tau_o \sin w I_{z+}(-\tau_o\cos w, 0) - \frac{\tau_o^2}{\omega\mu} V_{\rho+}(-\tau_o\cos w, 0) \\ &\qquad + \frac{\alpha_o\tau_o\cos w}{\omega\mu} V_{z+}(-\tau_o\cos w, 0)\Bigg] \end{aligned} \tag{15}$$

$$\begin{aligned} E_z(\rho,\varphi) &= \frac{1}{2\pi j}\left[\int_\gamma s_E[w+\varphi]e^{+j\tau_o\cos[w]\rho}dw\right] \\ H_z(\rho,\varphi) &= \frac{1}{2\pi j}\left[\int_\gamma s_H[w+\varphi]e^{+j\tau_o\cos[w]\rho}dw\right] \end{aligned} \tag{16}$$

Using the saddle point method on eqs. (16) we obtain the far field decomposed in three components:

$$\begin{aligned} E_z(\rho,\varphi) &= E_z^g(\rho,\varphi) + E_z^d(\rho,\varphi) + E_z^s(\rho,\varphi) \\ H_z(\rho,\varphi) &= H_z^g(\rho,\varphi) + H_z^d(\rho,\varphi) + H_z^s(\rho,\varphi) \end{aligned}$$

where $E(H)_z^g(\rho,\varphi)$, $E(H)_z^d(\rho,\varphi)$, $E(H)_z^s(\rho,\varphi)$ represent the geometrical optics contribution, the diffracted fields and the possible contribution of the surface waves. The diffracted field is evaluated through the Uniform Theory of Diffraction [10].

## 6 Numerical results

### *6.1 The non symmetric wedge at normal incidence with surface wave contribution*

This case analyzes a well-known in literature test case reported in [8] by using our methodology. In particular we evaluate the total field $H_z$ due to an $H$-polarized plane wave at $k\rho = 10$ from the edge of an impedance wedge with reference to Fig. 3 and with the following parameters: $E_o = 0$, $H_o = 1$, $\varphi_o = \pi/2$, $\Phi = 7\pi/8$, $z_a = 0.01$ (quasi-perfect conducting face), $z_b = sin(\theta_b)$, $\theta_b = 0.01 + j$ (inductive impedance), $Z_{a,b} = Z_o z_{a,b}$, $Z_o = 377\Omega$. Fig. 4 presents the total field and its decomposition into three components: GO field component, UTD field component, Surface Wave (SW) field component.

### *6.2 The non symmetric impedance wedge at skew incidence*

Fig. 5 shows the copolar GTD Diffraction Coefficient (we observe $E_z$ with an $E_{zo}$ incident field) for the arbitrary impenetrable wedge at skew incidence. Two different test cases are presented with the following common parameters: $\varphi_o = \pi/2$, $E_{zo} = 1$, $H_{zo} = 0, = 7\pi/8$, $\beta = \pi/3$. The two test cases differ on the face impedances: 1) $z_a = 0$, $z_b = 0$ (PEC wedge) and 2) $z_a = 10$, $z_b = 0$.

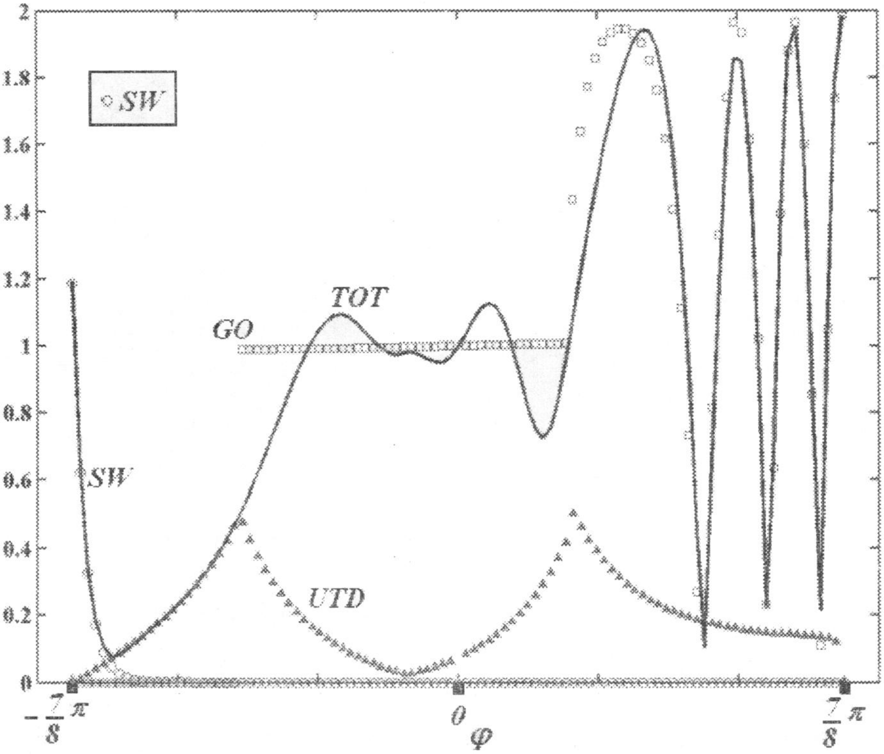


Figure 4: Total field, GO field component, UTD field component and SW field component for $k\rho = 10$, $\varphi_o = \pi/2$, $E_o = 0$, $H_o = 1$, $\Phi = 7\pi/8$, $z_a = 0.01$, $z_b = \sin(\theta_b)$, $\theta_b = 0.01 + \jmath$.

## References


[1] Daniele, V.G., 2000, Generalized Wiener-Hopf technique for wedge shaped regions of arbitrary angles, *2000 International Conference on Mathematical Methods in Electromagnetic Theory (MMET 2000)*, Kharkov, Ukraine, pp. 432-434.

[2] Daniele, V.G., 2003, The Wiener-Hopf technique for impenetrable wedges having arbitrary aperture angle, *SIAM Journal of Applied Mathematics*, **vol.63**, n.4, pp. 1442-1460.

[3] Daniele, V.G., Lombardi, G., 2003, Generalized Wiener-Hopf Equations for Wedge problems involving arbitrary linear media, *Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA)*, Torino, Italy, pp. 761-764.

[4] Daniele, V.G., 2004, The Wiener-Hopf technique for wedge problems, Rep. 2-2004, Dipartimento Elettronica, Politecnico Torino.

[5] Daniele, V.G., 2004, An introduction to the Wiener-Hopf technique for the solution of electromagnetic problems, Rep.1-2004, Dipartimento Elettronica, Politecnico Torino.

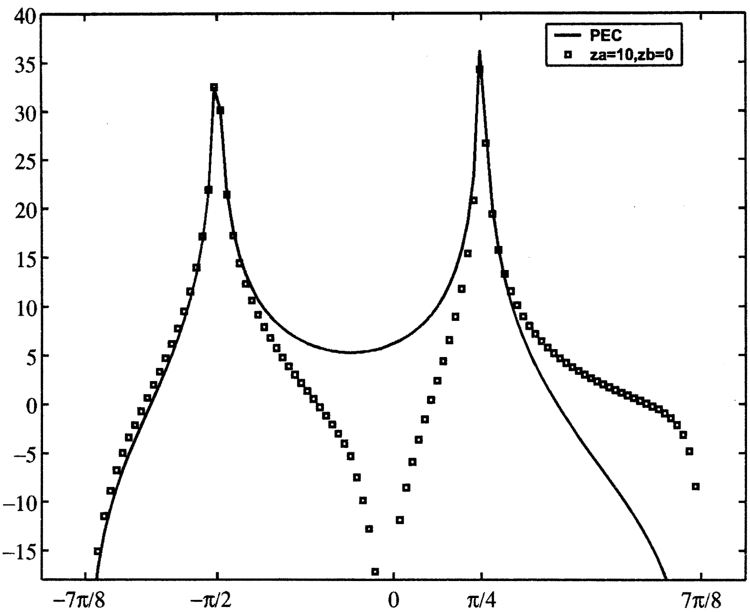


Figure 5: The copolar GTD Diffraction Coefficient.

[6] Buldyrev, V.S., Lyalinov, M.A., 2001, Mathematical Methods in Modern Diffraction Theory, Science House Inc, Tokio.

[7] Felsen, L.B., Marcuvitz, N., 1973, Radiation and Scattering of Waves, Englewood Cliffs: Prentice-Hall.

[8] Norris, A.N, Osipov, A.V., 1999, Far field analysis of the Malyuzhinets solution for plane and surface waves diffraction by an impedance wedge, *Wave Motion,* **vol. 30**, pp. 69-89.

[9] Lombardi, G., 2005, Wiener-Hopf formulation for wedge problems, *Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA),* Torino, Italy, pp. 693-696.

[10] Kouyoumjian, R.G., Pathak, P. H., 1974, A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface, *Proc. IEEE,* **vol. 62**, pp. 1448-1461.