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HIERARCHICAL SINGULAR VECTOR BASES FOR THE FEM SOLUTION OF WEDGE PROBLEMS

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***Abstract*:** This paper presents new hierarchical, singular vector bases that incorporate the edge conditions in curved triangular elements. The bases are fully compatible with the hierarchical, high-order regular vector bases used in adjacent elements. The new bases guarantee tangential continuity along the edges of the elements allowing for the discontinuity of normal components, adequate modelling of the curl, and removal of spurious modes. Several numerical results confirming the faster convergence of these bases on wedge problems will be presented at the Conference.

INTRODUCTION

Numberless structures of practical engineering interest contain conducting or penetrable edges and, in the vicinity of these edges, the field behavior and the surface charge can be singular Van Bladel[1], Meixner[2]. This paper considers Finite Element Method (FEM) applications for the solution of two-dimensional wedge problems. When one models the singular behavior by using *regular* elements, the FEM meshes must have a higher density in the neighborhood of the edges, even when the used bases are of higher polynomial order, Graglia et al.[3]. Furthermore, it is not generally granted that *iterative mesh refinement* could provide good effective solutions to these problems, whereas iterative mesh refinement involves complex procedures and codes, uses additional unknowns and it usually results in an increase of the computational time and/or memory requirements. To solve these problems more effectively, the authors of this paper have very recently derived new *singular*, high-order vector bases on curved two-dimensional domains, Graglia et al.[4]. The bases incorporate the edge conditions and are able to approximate the unknown fields in the neighborhood of the edge of a wedge for any order of the singularity coefficient ν, that is supposed given and known *a priori*. The singularity coefficient, usually evaluated in the static case, is frequency independent and it is used to model the $\rho^{1-\nu}$ singular behavior of the transverse components of the electromagnetic fields in the wedge-neighborhood, ρ being the radial distance form the edge of the wedge. The wedge can be made of penetrable and/or impenetrable (metallic) materials.

The singular bases reported in Graglia et al.[4] contain as a subset the regular p-th order bases given in Graglia et al.[3]. Only for the elements attached to the edge of a wedge, the regular basis subset is supplemented with an irrational algebraic vector subset. This latter subset, named after Meixner because it models just the Meixner series, Meixner[2], contains singular terms as well as other non-singular irrational algebraic (that is non-polynomial) terms. The Meixner subset can be regarded as the irrational algebraic part of the *complete* bases. By using complete interpolatory polynomials of order s, in Graglia et al.[4] we make the Meixner subset *complete* to arbitrary high-order s. Since these new bases are formed by the union of p-th order regular plus an s-th order singular part, there is no need to limit the size of the mesh in the neighborhood of the edge of the wedge. Above all, these new bases permit one to deal with all cases where the singularity of the fields is not excited.

Higher-order *regular* vector bases can have hierarchical form, and in the literature one finds many papers that present different procedures to built hierarchical regular bases, as well as examples of different hierarchical regular bases for two- and three-dimensional elements (*e.g.*, Andersen et al.[5], Webb[6]). Therefore, in this work we can restrict our attention on the technique used to construct

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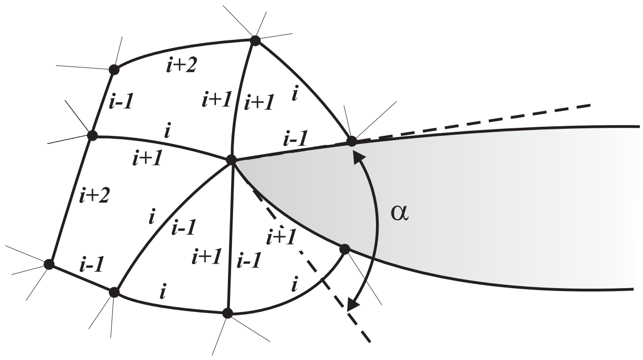

Figure 1: Cross-sectional view of the meshed region around a sharp, curved edge of aperture angle α.

Meixner vector subsets of hierarchical form. For lack of space we can provide the reader only with the hierarchical form of the Meixner subset of the triangular bases. At the Conference we will also show the hierarchical form of the Meixner subset of quadrilateral bases. We easily construct hierarchical Meixner subsets because, in Graglia et al.[4], we were able to: (1) count the number of their Degrees Of Freedom (DOF), (2) associate the DOF either to a curl-free part or to a rotational part, and (3) associate these DOF either to the element edges or to internal element-points.

ELEMENT REPRESENTATION AND HIERARCHICAL FORMS

With reference to Graglia et al.[3], Graglia et al.[4], and to Fig. 1, the edges of each element are locally numbered counter-clockwise from 1 to 3 for triangles, and from 1 to 4 for quadrilaterals. The parent variable ξ_β (with $\beta = 1, 2, ...$) varies linearly across the element and vanishes on the β-th edge, with $\xi_\beta = 1$ on the vertex opposite to edge β, for triangular elements, or opposite to the $\xi_\beta = 0$ coordinate line, for quadrilateral elements. We call sharp-edge elements those attached to the sharp-edge vertex, and Fig. 1 shows a case with five sharp-edge elements. For each sharp-edge element it is convenient to refer to these local numbers in terms of the dummy indexes $i-1, i, i+1, i+2$, with index arithmetic performed modulo three for triangles ($i+1 = 1$ if $i = 3$) and modulo four for quadrilaterals. For each triangular sharp-edge element, the two edges ($i \pm 1$) depart from the sharp-edge vertex, while the i-th edge is opposite to the sharp-edge vertex. For quadrilateral sharp-edge elements, edge i and $i+1$ are those departing from the sharp-edge vertex.
By use of the above mentioned notation, Table I reports the hierarchical form of the Meixner subset of singular triangular bases.

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TABLE I

HIERARCHICAL FORM OF THE MEIXNER PART OF TRIANGULAR BASES, FOR ARBITRARY ORDER $s \geq 0$

The total number of DOF (Degrees of Freedom) for the s-th order singular part of the triangular bases is $(s+1)(s+3)$. DOF are hierarchically associated to vector functions as it follows

$2(s+1)$ curl-free functions with edge DOF:

$$\left.\begin{array}{r} \left.\nabla \phi_{i\pm1}(\boldsymbol{r})\right\} s=0 \\ \nabla\left(\xi_i - \xi_{i\mp1}\right)\phi_{i\pm1}(\boldsymbol{r}) \end{array}\right\} s=1$$

building rule $\Rightarrow$ the $s{=}(m{+}1)$–order set is obtained by adding the following 2 vector functions to the $s{=}m$–order subset: $\nabla\left(\xi_i - \xi_{i\mp1}\right)^{m+1}\phi_{i\pm1}(\boldsymbol{r})$.

$s(s+1)/2$ curl-free functions with interior DOF:

the zeroth-order set is empty (*i.e.* no functions for $s{=}0$)

$$\left.\begin{array}{r} \left.\nabla \xi_{i+1}\phi_{i+1}(\boldsymbol{r})\right\} s=1 \\ \nabla \xi_{i-1}\xi_{i+1}\phi_{i+1}(\boldsymbol{r}) \\ \nabla \xi_{i+1}^2\phi_{i+1}(\boldsymbol{r}) \end{array}\right\} s=2$$

building rule $\Rightarrow$ the $s{=}(m{+}1)$–order set is obtained by adding the following $m{+}1$ vector functions to the $s{=}m$–order subset: $\nabla \xi_{i-1}^{q}\,\xi_{i+1}^{p+1}\,\phi_{i+1}(\boldsymbol{r})$, for $p{=}0,\ m$; with $q{=}m{-}p$.

$(s+1)(s+2)/2$ edge-less functions with interior DOF:

$$\left.\begin{array}{r} \left.{}^{\boldsymbol{\nu}}\boldsymbol{\mho}_i(\boldsymbol{r})\right\} s=0 \\ \xi_{i-1}\,{}^{\boldsymbol{\nu}}\boldsymbol{\mho}_i(\boldsymbol{r}) \\ \xi_{i+1}\,{}^{\boldsymbol{\nu}}\boldsymbol{\mho}_i(\boldsymbol{r}) \end{array}\right\} s=1$$

building rule $\Rightarrow$ the $s{=}(m{+}1)$–order set is obtained by adding the following $m{+}2$ vector functions to the $s{=}m$–order subset: $\xi_{i-1}^{q}\,\xi_{i+1}^{p}\,{}^{\boldsymbol{\nu}}\boldsymbol{\mho}_i(\boldsymbol{r})$, for $p{=}0,\ m{+}1$; with $q{=}m{+}1{-}p$.

Subscripts are counted modulo 3, with $i = 1, 2$ or 3. $\phi_{i\pm1}(\boldsymbol{r}) = \xi_{i\mp1}\left[1-(1-\xi_i)^{\nu-1}\right]$ are the two lowest order potential functions, whereas ${}^{\boldsymbol{\nu}}\boldsymbol{\mho}_i(\boldsymbol{r}) = (1-\nu)\left(\chi^{\nu}-1\right)\boldsymbol{\Omega}_i(\boldsymbol{r})$ is the lowest order edge-less function, with $\boldsymbol{\Omega}_i(\boldsymbol{r}) = \xi_{i+1}\nabla\xi_{i-1} - \xi_{i-1}\nabla\xi_{i+1}$.