Kähler submanifolds of Wolf spaces

D.V. ALEKSEEVSKY, A.J. DI SCALA and S. MARCHIAFAVA

November 3, 2003

Abstract

We report on several results about Kähler submanifolds of a symmetric quaternionic Kähler manifold, i.e. a Wolf space or its non compact dual: a twistor construction of such Kähler submanifolds, the classification of Kähler manifolds which can be immersed as parallel Kähler submanifolds, the classification of parallel Kähler submanifolds. An alternative elementary proof of the non existence of non totally geodesic parallel maximal Kähler submanifolds of the Wolf space $G_2(\mathbb{C}^{n+2})$ is also given.

1 Introduction

Nowadays it is well known that quaternionic Kähler manifolds are interesting Einstein manifolds, [Bes], Chapter 14, where there is a rich interplay between several basic structures, such as complex and quaternionic, and also between mathematics and physics, [QSMP].

This report concerns a research on submanifolds of a quaternionic Kähler manifold undertaken by the three authors in the last years. See in particular the papers [AM2], [AM3], [ADM]. Recent results on Kähler submanifolds of quaternionic *symmetric* spaces are presented.

2 Basic facts about Kähler submanifolds of a quaternionic Kähler manifold

Let $(\tilde{M}^{4n}, Q, \tilde{g})$ be a **quaternionic Kähler** manifold. We recall that $(\tilde{M}^{4n}, \tilde{g})$ is a Riemannian manifold of dimension 4n, with Levi-Civita connection $\nabla^{\tilde{g}}$; Q is a $\nabla^{\tilde{g}}$-parallel quaternionic structure, i.e. a rank-3 subbundle of the bundle of endomorphisms locally spanned by a triple of locally defined anticommuting $\tilde{g}$-orthogonal almost complex structures $H = (J_1, J_2, J_3 = J_1 J_2)$; H is called a *local basis* of Q. A typical example is the quaternionic projective space $\mathbb{H}P^n$ with a standard metric and the natural quaternionic structure.

We recall that $(\tilde{M}^{4n}, Q, \tilde{g})$ is an Einstein manifold and its curvature tensor has a decomposition

$$\widetilde{R} = \nu R_{\mathbb{H}P^n} + \widetilde{W} \tag{1}$$

where $R_{\mathbb{H}P^n}$ is the curvature tensor of the quaternionic projective space $\mathbb{H}P^n$, i.e.

$$R_{\mathbb{H}P^n}(X,Y) = \frac{1}{4}\Big(X \wedge Y + \sum_\alpha J_\alpha X \wedge J_\alpha Y - \sum_\alpha 2\langle J_\alpha X, Y\rangle J_\alpha\Big),$$

$\langle\, ,\, \rangle = \widetilde{g}(\, ,\,)$ is the Riemannian scalar product of $\widetilde{M}$, ν is a constant which is called the **reduced scalar curvature**, such that $K = 4n(n+2)\nu$ is the scalar curvature, and W is the **quaternionic Weyl tensor**, which verifies the identity $[W(X,Y), Q] = 0$ and has all contractions equal to zero.

A **Wolf space** is a compact, simply connected, quaternionic Kähler symmetric space. It has the form $W = G/K$ where G is a compact centerless Lie group and K is a certain subgroup (unique up to conjugacy) which is a local direct product $K = K_1 \cdot Sp(1)$, where $Sp(1)$ is the multiplicative group of unit quaternions, see [Bes], pag. 408 . Main examples are

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n)\cdot Sp(1)} \qquad ,$$

$$G_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n)\times U(2))} \qquad , \qquad G_4^+(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{S(O(n)\times O(4))}$$

Moreover there are 5 exceptional spaces like $G_2/SO(4), F_4/Sp(1)\cdot Sp(3)$, etc. .

There are two kinds of submanifolds of a quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$ which are of primary interest:

quaternionic submanifolds, i.e. Q-invariant submanifolds (M^{4m}, Q', g), where $Q' = Q_{|TM^{4m}}, g = \widetilde{g}_{|TM^{4m}}$. It is a classical result that they are totally geodesic.

(almost-)complex submanifolds with respect to an (almost-)complex structure compatible with Q, i.e. J_1-invariant submanifolds (M^{2m}, J_1, g) where $J_1 \in \Gamma(Q)_{|M}$ is a section such that $J_1^2 = -Id$ and $J_1 TM = TM, g = \widetilde{g}_{|TM}$.

The almost-complex submanifold (M^{2m}, J_1, g) is called **Kähler** if J_1 is parallel with respect to the Levi-Civita connection of $\widetilde{g}$; in such a case $(M^{2m}, J = J_{1|TM}, g)$ is a (complex) Kähler manifold.

One of the main reasons of interest into *Kähler submanifolds* is that they are *minimal*, in fact *pluriminimal* (or (1-1)-geodesic), i.e. the complexification of h, the second fundamental form, has type $(2,0)+(0,2)$, i.e. $h^{(1,1)}=0$), see Ohnita [O]. The stronger condition of pluriminimality is a consequence of the following identity which is satisfied by the second fundamental form h, see Funabishi [Fu],

$$h(X, J_1 Y) - J_1 h(X,Y) = 0 \qquad\qquad \forall X, Y \in TM.$$

In fact it implies that

$$h(X,Y) + h(J_1 X, J_1 Y) = 0 \qquad\qquad \forall X, Y \in TM$$

which is equivalent to the fact that any almost complex surface (M', J') of M is minimal. In turn the pluriminimality implies that the mean curvature vector of M vanishes.

3 Maximal Kähler submanifolds

In the following we will be interested in submanifolds of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ of *non zero scalar curvature*, $K \neq 0$, and we will assume always to be in such a case.

Then it results that, [Fu], [AM2],

the almost complex submanifold (M^{2m}, J_1) is Kähler iff it is totally complex, i.e. $J_2(TM) \perp TM$ where $J_2 \in Q$ is a complex structure which anticommutes with J_1.

As an immediate consequence, a Kähler submanifold (M^{2m}, J_1, g) has at most dimension $2n$ and in this case is called **maximal**.

Remark 3.1 *R.C. Mc Lean, [McL], and F.E. Burstall, [Ko2], called* **complex-Lagrangian** *such a maximal Kähler submanifold.*

A twistor construction of maximal Kähler submanifolds, which generalizes a famous Bryant's construction of superminimal surfaces of S^4, was given in [Tak], [AM3]. Let say some words on this subject.

In 1982 R. Bryant [Br] gave an explicit construction of superminimal conformal immersions of compact Riemann surfaces into the 4-sphere S^4 which led to several developments. The key idea of the construction is to use the twistor fibration $\pi : \mathbb{C}P^3 \to S^4 = \mathbb{H}P^1$ which is a Riemannian submersion with fiber $S^2 \equiv \mathbb{C}P^1$, and whose horizontal distribution $\mathcal{H}$ is a holomorphic contact structure. An oriented surface $M^2 \subset S^4$ has a natural lift $L = J(M^2)$ into $\mathbb{C}P^3$ defined by the complex structure of M^2. The surface M^2 is superminimal if an only if its lift is a holomorphic Legendrian submanifold of $\mathbb{C}P^3$, that is a horizontal holomorphic 2-dimensional submanifold. This reduces a description of superminimal surfaces $M^2 \subset S^4$ to a description of holomorphic Legendrian submanifolds of $\mathbb{C}P^3$. R. Bryant constructed the holomorphic local coordiantes u, p, q such that the contact distribution $\mathcal{H}$ is the kernel of the 1-form $\theta = du - pdq$. In terms of these coordinates a legendrian submanifold has the form $L = L_f = \{u = f(q), p = \frac{\partial f}{\partial q}\}$ where $u = f(q)$ is a holomorphic function. Then the projection $M^2 = \pi(L_f)$ is a superminimal surface of S^4.

The generalization of the Bryant construction for a quaternionic Kähler $(M^{4n}, Q, \widetilde{g})$ goes as follows.

Let $\mathcal{Z} \xrightarrow{\pi} \widetilde{M}$ be the twistor fibration, where $\mathcal{Z} = S(Q)$ is the sphere bundle of compatible complex structures of Q (for details see [Sal]). It carries a natural complex contact structure and the maximal Kähler submanifolds of $\widetilde{M}^{4n}$ come by projection π from the Legendrian submanifolds of such complex contact structure.

The construction of the correspondence between maximal Kähler submanifolds and Legendrian submanifolds works as follows. If the scalar curvature of $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is non zero (like in the case of 4-sphere) there is a correspondence between holomorphic, horizontal submanifolds N of the twistor space equipped with the holomorphic contact structure and Kähler submanifolds M^{2m} of $\widetilde{M}^{4m}$: the natural lift $N = J_1(M)$ of a Kähler submanifold M^{2m} of $\widetilde{M}^{2m}$ to the twistor space $\mathcal{Z}$ and conversely the projection $M^{2m} = \pi(N)$ to $\widetilde{M}^{4n}$ of a holomorphic horizontal submanifold $N \subset \mathcal{Z}$ is a Kähler submanifold. Like in the case of S^4, the explicit construction of Legendrian submanifolds $L \subset \mathcal{Z}$ (i.e. maximal holomorphic horizontal submanifolds) and hence maximal Kähler submanifolds $M^{2n} \subset \widetilde{M}^{4n}$ reduces to the construction of Darboux coordinates u, p_k, q^k such that a (local) contact 1-form θ with $\mathcal{H} = \mathrm{Ker}\theta$ has the form $\theta = du - \sum_k p_k dq^k$. A direct generalization of Bryant's formulas provides such Darboux coordinates for the twistor space $\mathcal{Z} = \mathbb{C}P^{2n+1}$ of the quaternionic projective space $\widetilde{M}^{4n} = \mathbb{H}P^n$. This allows the construction of Kähler submanifolds of $\mathbb{H}P^n$; hence, due to results of F.E. Burstall, [Bu], which proved that *quaternionic twistor spaces of Wolf spaces of the same dimension are birationally equivalent as complex contact manifolds* and explicit computations of P. Kobak, [Ko1], **one can construct the Kähler submanifolds of any Wolf space explicitly**.

4 Kähler submanifolds of Wolf spaces (and their duals)

4.1 Totally geodesic maximal Kähler submanifolds

The maximal totally geodesic Kähler submanifolds of symmetric quaternionic spaces and their duals were classified by Takeuchi for cases $\mathbb{H}P^n, G_2(\mathbb{C}^{n+2})$ and $G_4^+(\mathbb{R}^{n+4})$, [Tak]. The cases of exceptional spaces could be obtained by putting together results in a recent paper of Wolf, [W], and the paper [ADM].

4.2 Parallel Kähler submanifolds

In [Tsu2], K. Tsukada classified all parallel (non totally geodesic) maximal Kähler submanifolds of the quaternionic projective space $\mathbb{H}P^n$ and proved also that in its dual, the hyperbolic quaternionic space $\mathcal{H}\mathbb{H}^n$ there are no such submanifolds.

The remarkable fact concerning a maximal Kähler, or equivalently, totally complex submanifold (M^{2n}, J_1) is that locally its normal bundle can be identified with the tangent bundle by means of a section J_2 orthogonal to J_1 in the bundle Q:

$$J_2 : T^{\perp}M \overset{\cong}{\to} TM$$

and the Gauss-Codazzi equations can be expressed only in terms of the tangent space TM. In particular we associate with the second fundamental form h a (local) tensor field C on M, called *shape tensor*, by

$$C(X,Y,Z) := \langle J_2 h(X,Y), Z\rangle$$

which is symmetric with respect to X, Y, Z and satisfies the following identities:

$$C(X,Y,JZ) = C(JX,Y,Z) = C(X,JY,Z)$$

Passing to the complexification, it results that C defines a parallel line bundle which is independent from the local section J_2 and (M, J_1) is a *Kähler manifold with parallel cubic line bundle of type* ν, where ν is the reduced scalar curvature of $(\widetilde{M}^{4n}, Q, \widetilde{g})$.

Looking to this property allowed to classify the Kähler manifolds M^{2n} which can be immersed into a quaternionic Kähler manifold $\widetilde{M}^{4n}$ as maximal parallel Kähler submanifolds, [AM2]. They are the following Hermitian symmetric manifolds or their duals:

reducible : $Q_{n-1} \times \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$

irreducible : $Sp_2/U_2 \times \mathbb{C}P^1, \mathbb{C}P^1, Sp_3/U_3, SU_6/S(U_3 \times U_3), SO_{12}/U_6, E_7/T^1 \cdot E_6$

4.3 Parallel maximal Kähler submanifolds of Wolf spaces

We recall the following definition.

A submanifold $M \subset \widetilde{M}$ of a Riemannian manifold $\widetilde{M}$ is called **curvature invariant** (respectively, **normal curvature invariant**) if each tangent space T_xM, $x \in M$, is curvature invariant with respect to the curvature tensor $\widetilde{R}$ of $\widetilde{M}$, i.e.

$$\widetilde{R}(T_xM, T_xM)T_xM \subset T_xM$$

(respectively, if each normal space $T_xM, x \in M$, is curvature invariant, i.e.

$$\widetilde{R}(T_x^\perp M, T_x^\perp M)T_x^\perp M \subset T_x^\perp M \qquad).$$

It is a straightforward and well known consequence of the Codazzi equation that a parallel submanifold is curvature invariant. Moreover, the Gauss equation implies that a parallel submanifold of a locally symmetric space is itself a locally symmetric space. Nevertheless, it is not true that a curvature invariant locally symmetric submanifold of a symmetric space is parallel e.g. a flat surface in $\mathbb{R}^3$ different from a plane or cylinder. However, for maximal Kähler submanifolds of Wolf spaces the following result holds.

Theorem 4.1 *([ADM]) A locally symmetric maximal and curvature invariant Kähler submanifold M^{2n} of a quaternionic Kähler symmetric space is parallel.*

For a parallel maximal Kähler submanifold of a quaternionic Kähler manifold $\widetilde{M}$ a stronger result holds: it is also normal curvature invariant, due to the fact that the curvature identities of $\widetilde{M}^{4n}$ imply that

$$\langle R(J_2X, J_2Y)J_2Z, J_2T\rangle = \langle R(X,Y)Z,T\rangle$$

for any complex structure $J_2 \in Q$ (see [AM2]).

This remark allows to apply some results of Naitoh and prove the following theorem.

Theorem 4.2 *([ADM]) Any curvature invariant (in particular, any parallel) Kähler submanifold M^{2n} of a quaternionic Kähler symmetric space $\widetilde{M}^{4n} \neq \mathbb{H}P^n$ is totally geodesic.*

The Naitoh results used in the proof are the following.

Theorem 4.3 *([Na2]) Let $\widetilde{M}$ be a simply connected Riemannian symmetric space. A submanifold M of $\widetilde{M}$ is parallel and normal curvature invariant if and only if it is extrinsically symmetric.*

Theorem 4.4 *([Na3]) Let $\widetilde{M} = G/K$ be a compact (also non compact) simply connected symmetrixc space with simple isometry group G, and $\mathcal{V}$ is an orbit of G in $Gr_k(T\widetilde{M})$ which is curvature invariant and normal curvature invariant. Then any $\mathcal{V}$-submanifold is totally geodesic with the exception of the following cases:*

a) $\widetilde{M} = S^n = SO(n+1)/SO(n), 1 \leq k \leq n$,

b) $\widetilde{M} = \mathbb{C}P^n, \mathcal{V}$ *is the set of complex $2k$-subspaces*

c) $\widetilde{M} = \mathbb{C}P^n, \mathcal{V}$ *is the set of totaly real n-spaces*

d) $\widetilde{M} = \mathbb{H}P^n, \mathcal{V}$ *is the set of totally complex $2n$-subspaces*

e) $\widetilde{M} = G/K$ *is an irreducible Hermitian symmetric R-space (or its non compact dual) (real flag manifold).*

In the last paragraph we will give a more direct proof of the result stated in Theorem 4.2 in the case of the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ and its non compact dual. It could be interesting to find an unified proof of this type for all Wolf spaces, at least for the other non exceptional spaces $G_4^+(\mathbb{R}^{n+4})$.

4.4 Classification of non-maximal parallel Kähler submanifolds of Wolf spaces

The classification of parallel Kähler submanifolds of a quaternionic Kähler symmetric manifold reduces to the classification of full parallel Kähler submanifolds in Hermitian or quaternionic Kähler symmetric spaces.

Proposition 4.5 *([ADM] Let (M^{2m}, J) be a geodesically complete parallel submanifold of a quaternionic Kähler symmetric space $\widetilde{M}^{4n}$ and $\overline{M}$ the minimal totally geodesic submanifold of $\widetilde{M}$ which contains M.*

1) If the shape tensor C of M vanishes at one point then $\overline{M}$ is an Hermitian symmetric space and M is a full parallel Kähler submanifold of $\overline{M}$.

2) If $C \neq 0$ then $\overline{M} = \mathbb{H}P^n$ and (M^{2m}, J) is a Hermitian symmetric manifold with parallel cubic line bundle, with canonical Tsukada imbedding into $\mathbb{H}P^m$ (as described in [Tsu2]).

If $x \in M$ then

$$\overline{M} = exp(O_x^1 M) \qquad , \qquad O_x^1 M = T_x M + h(T_x M, T_x M).$$

The proof basis on the fact that along the submanifold Mwe have a decomposition of the curvature operator $\widetilde{R}_{XY}$, $X, Y \in T_x M$ of the manifold $\widetilde{M}$ according to the decomposition (which yields to the Gauss-Codazzi-Ricci equations)

$$\begin{array}{llllllll} \mathrm{End}(T_x\widetilde{M}) & = \mathrm{End}(T_xM) & + & \mathrm{Hom}(T_xM, T_x^\perp M) & + & \mathrm{Hom}(T_{x x}^\perp M, T_xM) & + & \mathrm{End}(T_x^\perp M) \\ \widetilde{R}_{XY} & = \quad R_{XY}^{TT} & + & R_{XY}^{T\perp} & + & R_{XY}^{\perp T} & + & R_{XY}^{\perp\perp} \end{array}$$

$\forall X, T \in T_x M$

and for a locally symmetric space the covariant derivatives of the tangential part R^{TT}, the normal part $R^{\perp\perp}$ and the mixed parts $R^{\perp TT}, R^{T\perp}$ of the curvature tensor $\widetilde{R}_{|TM}$ can be expressed in terms of these same tensors and the shape operatyor $C = J_2 \circ h$, see [AM2].

5 A direct proof of the non existence of non totally geodesic parallel maximal Kähler submanifolds of the complex Grassmannian

5.1 Curvature invariant maximal Kähler submanifolds of the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ and of its non compact dual $\mathcal{H}G_2(\mathbb{C}^{n+2})$

.

We know that any parallel submanifold M of a Riemannian manifold $\widetilde{M}$ is curvature invariant. Assume that the manifold $\widetilde{M} = G/H$ is symmetric. Then the condition

$$\widetilde{R}(T_xM, T_xM)T_xM \subset T_xM$$

that M is curvature invariant at the point x means that the subspace T_xM corresponds to a triple system in the symmetric decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. This is equivalent to the existence of a totally geodesic submanifold $\overline{M}(x) = exp(T_xM)$ with the tangent space T_xM. We call $\overline{M}(x)$ *the tangent totally geodesic submanifold of* M *at* x .

Remark 5.1 *Condition that M is curvature invariant means that it admits the tangent totally geodesic submanifold $\overline{M}(x)$ at any point x.*

In this section we study curvature invariant Kähler submanifolds of the classical Wolf space $G_2(\mathbb{C}^{n+2})$ and its non compact dual. In particular we prove the following theorem.

Proposition 5.2 *Any curvature invariant, maximal Kähler submanifold M^{2n} of the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ or of its non compact dual $\mathcal{H}G_2(\mathbb{C}^{n+2})$ is totally geodesic.*

It should be interesting to prove the analogous proposition for $G_4^+(\mathbb{R}^{n+4})$ by the same methods.

5.2 Basic facts of the geometry of $G_2(\mathbb{C}^{n+2})$

Let $\widetilde{M}^{4n}$ be the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ with reduced scalar curvature ν or its non compact dual $\mathcal{H}G_2(\mathbb{C}^{n+2})$.

We denote by $\mathcal{J}$ the standard complex structure. It is known that for any admissible basis (J_1, J_2, J_3) of the quaternionic structure of $\widetilde{M}^{4n}$ one has

$$\mathcal{J}J_\alpha = J_\alpha\mathcal{J} \qquad , \qquad (\mathcal{J}J_\alpha)^2 = I \qquad , \qquad \mathrm{Tr}\mathcal{J}J_\alpha = 0 \qquad (\alpha = 1, 2, 3)$$

Moreover

$$\langle \mathcal{J}X, Y\rangle + \langle X, \mathcal{J}Y\rangle = 0 \qquad , \qquad \langle J_\alpha X, Y\rangle + \langle X, J_\alpha Y\rangle = 0\,,\ (\alpha = 1, 2, 3) \quad \forall X, Y \in T\widetilde{M}^{4n} \qquad .$$

The curvature tensor of $\widetilde{M}^{4n}$ with reduced scalar curvature ν is a multiple by $\frac{\nu}{4}$ of the curvature tensor of $G_2(\mathbb{C}^{n+2})$ with reduced scalar curvature 4, which has the following expression

(2)
$$\begin{aligned}\widetilde{R}(X,Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y + \textstyle\sum_\alpha \Big[\langle J_\alpha Y,Z\rangle J_\alpha X - \langle J_\alpha X,Z\rangle J_\alpha Y - 2\langle J_\alpha X,Y\rangle J_\alpha Z\Big] \\ &+\langle \mathcal{J}Y,Z\rangle \mathcal{J}X - \langle \mathcal{J}X,Z\rangle \mathcal{J}Y - 2\langle \mathcal{J}X,Y\rangle \mathcal{J}Z \\ &+\textstyle\sum_\alpha \Big[\langle J_\alpha \mathcal{J}Y,Z\rangle J_\alpha \mathcal{J}X - \langle J_\alpha \mathcal{J}X,Z\rangle J_\alpha \mathcal{J}Y\Big]\end{aligned}$$

The quaternionic Weyl tensor of the Grassmannian is

$$\begin{aligned}\widetilde{W}(X,Y)Z &= \langle \mathcal{J}Y,Z\rangle \mathcal{J}X - \langle \mathcal{J}X,Z\rangle \mathcal{J}Y - 2\langle \mathcal{J}X,Y\rangle \mathcal{J}Z \\ &+\textstyle\sum_\alpha \Big[\langle J_\alpha \mathcal{J}Y,Z\rangle J_\alpha \mathcal{J}X - \langle J_\alpha \mathcal{J}X,Z\rangle J_\alpha \mathcal{J}Y\Big] \quad .\end{aligned} \tag{3}$$

In the following we will refer to the manifold $\widetilde{M}^{4n} = G_2(\mathbb{C}^{n+2})$ with reduced scalar curvature 4. The extension of results to the other Grassmannians and their non compact duals will be immediate.

Definition 5.3 *A complete totally geodesic submanifold M of a Riemannian manifold $\widetilde{M}^{4n}$ is called* **maximal** *if it is not contained in a proper totally geodesic submanifold.*

We recall the following theorem by [Tak].

Proposition 5.4 *Up to isometries, any maximal totally geodesic Kähler submanifold of $G_2(\mathbb{C}^{n+2})$ belongs to one of the following submanifolds:*

A) the submanifolds

$$\mathbb{C}P^p \times \mathbb{C}P^q \longrightarrow G_2(\mathbb{C}^{p+q+2}) \qquad\qquad (p+q=n)$$

immersed by using the identification $\mathbb{C}^{p+1}\oplus\mathbb{C}^{q+1}\equiv\mathbb{C}^{p+q+2}$ and sending a pair of lines (ℓ,ℓ'), $\ell\subset\mathbb{C}^{p+1},\ell'\subset\mathbb{C}^{q+1}$, one in each factor, to the corresponding plane $\ell\oplus\ell'\subset\mathbb{C}^{p+q+2}$. In particular, if for example $q=0$, the submanifolds

$$\mathbb{C}P^n \equiv \mathbb{C}P^n \times \mathbb{C}P^0 \longrightarrow G_2(\mathbb{C}^{n+2})\,.$$

These submanifolds are complex with respect to the structure $\mathcal{J}$, i.e. $\mathcal{J}TM = TM$.

B) the submanifolds

$$G_2(\mathbb{R}^{n+2}) \longrightarrow G_2(\mathbb{C}^{p+q+2}) \qquad\qquad (p+q=n)$$

immersed by using the identification $\mathbb{C}^{n+2}\equiv\mathbb{R}^{n+2}\otimes\mathbb{C}$ by sending a real 2-plane into its complexified.

These submanifolds are totally real with respect to the structure $\mathcal{J}$, i.e. $\mathcal{J}TM = TM^{\perp}$.

Proposition 5.5 *Assume that T' is a curvature invariant, totally complex, 2n-dimensional subspace of the tangent space $V = T_xG_2(\mathbb{C}^{p+q+2})$. Then there are two possibilities:*

$$\begin{array}{ll} 1) & \mathcal{J}T' = T' \quad , \\ 2) & \mathcal{J}T' \perp T' \quad . \end{array}$$

Up to transformations of the isotropy group, in case 2) the subspace T' is unique, in case 1) there exist n+1 different subspaces T'.

The proof follows from the previous remark 5.1 and proposition 5.4.

Remark 5.6 *The complex Grassmannians $G_2(\mathbb{C}^{m+2})$, $(m < n)$ which are naturally imbedded into $G_2(\mathbb{C}^{n+2})$ are Kähler manifolds with respect to the complex structure $\mathcal{J}$ but they are not Kähler submanifolds with respect to the quaternionic structure Q. In fact they are quaternionic submanifolds.*

Problem. It would be nice to give examples of maximal Kähler submanifolds of $G_2(\mathbb{C}^{m+2})$ which belong to some $\mathcal{V}$-geometry without being curvature invariant.

5.3 Proof of Proposition 5.2

We recall that for a curvature invariant, maximal Kähler submanifold of a locally symmetric quaternionic Kähler manifold the following identity holds (which follows from [AM2, 2.5.2], since $\nabla R^{\perp T} = 0$, and (1):

$$(4) \quad C_X\widetilde{W}(Y,Z)U + J_2\widetilde{W}(J_2C_XY,Z)U + J_2\widetilde{W}(Y,J_2C_XZ)U - \widetilde{W}(Y,Z)C_XU = 0\,.$$

According to the result of previous proposition we consider separately the two possible cases.

1) $\underline{\mathcal{J}TM = TM}$. By assuming $Z = \mathcal{J}Y = U$ in (4) we get the identity

$$\begin{array}{l} -3\|Z\|^2[C_X,\mathcal{J}]Z + \langle J_1\mathcal{J}Z,Z\rangle J_1[C_X,\mathcal{J}]\mathcal{J}Z \\ -\langle [C_X,\mathcal{J}]\mathcal{J}Z,Z\rangle \mathcal{J}Z + \langle J_1[C_X,\mathcal{J}]Z,Z\rangle J_1Z + \langle J_1[C_X,\mathcal{J}]\mathcal{J}Z,Z\rangle J_1\mathcal{J}Z = 0\,. \end{array}$$

Note that $[C_X,\mathcal{J}]$ is symmetric and anti-commutes with J_1 and $\mathcal{J}$. Hence, for any eigenvector Z of $[C_X,\mathcal{J}]$, that is $[C_X,\mathcal{J}]Z = \lambda Z$, it results

$$-\lambda[3\|Z\|^2Z + 2\langle J_1\mathcal{J}Z,Z\rangle J_1\mathcal{J}Z] = 0 \qquad .$$

From $\lambda \neq 0$ it would follow that Z is an eigenvector of $J_1\mathcal{J}$; since the eigenvalues of $J_1\mathcal{J}$ are ± 1 it is excluded and, hence, $\lambda = 0$. In conclusion $[C_X,\mathcal{J}] = 0$.

Let consider the splitting $T_pM = T_pM_1 \oplus T_pM_{-1}$ where

$$T_pM_1 = \{X | \mathcal{J}J_1X = X\} \quad , \quad T_pM_{-1} = \{Y | \mathcal{J}J_1Y = -Y\}$$

One has $J_1T_pM_a = T_pM_a, \mathcal{J}T_pM_a = T_pM_a, a = 1, -1$. Moreover, let show that the subspaces $T_pM_a, a = 1, -1$ are left invariant by any operator C_X, $X \in T_pM$, i.e.

$$C_XT_pM_a \subset T_pM_a, \quad a = 1, -1, \forall X \in T_pM .$$

In fact, if $X_+ \in T_pM_1, X_{-1} \in T_pM_{-1}$ it follows that the sectional curvature of the plane $\pi = \{X_+, X_-\}$ in the ambient space vanishes, as it is straightforward to verify; hence, since $T_pM_a, a = 1, -1$, are the eigendistributions of the parallel operator $\mathcal{J}J_1$, it follows that both are parallel. Then M^{2n} splits as $M^{2n} = M_1 \times M_{-1}$. By the Gauss equation, we obtain that $C_{X_+}X_- = 0$, which implies the claim. On the other hand, if $X \in T_pM = T_pM_a$, $a = 1 \,\text{or}\, a = -1$, belongs to one factor then one has $[C_X, \mathcal{J}] = a[C_X, J_1] = 0, \{C_X, J_1\} = 0$. Then $C_X = 0$ on each factor and the proof of the first case is complete.

2) $\underline{\mathcal{J}TM = TM^\perp}$. In this case (4) takes the form

$$\begin{aligned}
&\langle J_2\mathcal{J}Z, U\rangle[C_X, J_2\mathcal{J}]Y - \langle J_3\mathcal{J}Z, U\rangle J_1[C_X, J_2\mathcal{J}]Y + \langle J_3\mathcal{J}Y, U\rangle J_1[C_X, J_2\mathcal{J}]Z \\
&-\langle J_2\mathcal{J}Y, U\rangle[C_X, J_2\mathcal{J}]Z + \langle[C_X, J_2\mathcal{J}]Y, U\rangle J_2\mathcal{J}Z - \langle[C_X, J_2\mathcal{J}]Z, U\rangle J_2\mathcal{J}Y \\
&+2\langle[C_X, \mathcal{J}J_2]Y, Z\rangle J_2\mathcal{J}U - \langle J_1[C_X, J_2\mathcal{J}]Y, U\rangle J_3\mathcal{J}Z + \langle J_1[C_X, J_2\mathcal{J}]Z, U\rangle J_3\mathcal{J}Y = 0 .
\end{aligned}$$

By assuming $U = J_2\mathcal{J}Z, Y = J_1Z$ in (4), and taking into account that $[C_X, J_2\mathcal{J}]$ and J_1 commute, we get the identity

$$\begin{aligned}
&2\|Z\|^2[C_X, J_2\mathcal{J}]J_1Z + 2\langle[C_X, J_2\mathcal{J}]J_1Z, J_2\mathcal{J}Z\rangle J_2\mathcal{J}Z \\
&+2\langle[C_X, J_2\mathcal{J}]Z, J_2\mathcal{J}Z\rangle J_3\mathcal{J}Z + 2\langle[C_X, \mathcal{J}J_2]J_1Z, Z\rangle Z = 0 .
\end{aligned}$$

By computing on an eigenvector Z of the symmetric operator $[C_X, J_2\mathcal{J}]J_1$, for which say $[C_X, J_2\mathcal{J}]J_1Z = \lambda Z$, one gets

$$2\lambda[2\|Z\|^2Z + \langle Z, J_2\mathcal{J}Z\rangle J_2\mathcal{J}Z + \langle Z, J_3\mathcal{J}Z\rangle J_3\mathcal{J}Z] = 0$$

From $\lambda \neq 0$ (by taking the scalar products by $J_2\mathcal{J}Z, J_3\mathcal{J}Z$ respectively) it would follows that $\langle Z, J_2\mathcal{J}Z\rangle = \langle Z, J_3\mathcal{J}Z\rangle = 0$, which contradicts $Z \neq 0$. Hence

$$[C_X, J_2\mathcal{J}] = 0 . \tag{5}$$

Now let remark that J_1, which anticommutes with $\mathcal{J}J_2$, interchanges the $+1$, -1 eigenspaces of the symmetric operator $\mathcal{J}J_2$. On the other hand, for any $X \in TM$ the operator C_X leaves invariant each eigendistribution, as it follows from the identity (5). So $C_X \circ J_1 \equiv 0$. This completes the proof of the theorem. □

5.4 Totally geodesic maximal Kähler submanifolds of $G_2(\mathbb{C}^{n+2})$ and of its non compact dual $\mathcal{H}G_2(\mathbb{C}^{n+2})$

In this subsection we give an elementary proof of the first statement of Proposition 5.5, which evoids to refer to the classification of totally geodesic maximal Kähler submanifolds of $G_2(\mathbb{C}^{n+2})$ made by Takeuchi.

Let M^{2n} be a totally complex submanifold of $\widetilde{M}^{4n} = G_2(\mathbb{C}^{n+2})$ and $J_1 T_pM = T_pM$, $T_pG_2(\mathbb{C}^{n+2}) = T_pM \oplus J_2 T_pM$.

Lemma 5.7 *If $\mathcal{J}T_pM = T_pM$ or $\mathcal{J}T_pM = T_pM^\perp$ then T_pM is curvature invariant.*

Proof. It is a straightforward verification by using (1) and [AM2, (3), prop. 2.14] . □

In fact these are the only possible cases for a curvature invariant, maximal Kähler submanifold.

Proposition 5.8 *T_pM is curvature invariant if and only if $\mathcal{J}T_pM = T_pM$ or $\mathcal{J}T_pM = T_p^\perp M$.*

Proof. The proof of the necessity of the condition is done into three steps.

Step 1) If $\mathcal{J}T_pM \cap T_pM \neq 0$ then $\mathcal{J}T_pM = T_pM$. Let assume that there exists a non-zero $Z \in T_pM$ such that $Y = \mathcal{J}Z \in T_pM$. Then

$$\widetilde{W}(X, \mathcal{J}Z)Z = -\langle Z, Z\rangle \mathcal{J}X + \langle \mathcal{J}X, Z\rangle Z - 2\langle \mathcal{J}X, Y\rangle Y + \langle J_1\mathcal{J}X, Z\rangle J_1 Z$$

Hence $-\langle Z, Z\rangle \mathcal{J}X \in T_pM$, that is $\mathcal{J}X \in T_pM$ for any $X \in T_pM$.

Step 2) If $\mathcal{J}T_pM \cap T_pM^\perp \neq 0$ then $\mathcal{J}T_pM = T_pM^\perp$. Let assume that $\mathcal{J}Y \in T_pM^\perp$, $\|Y\| = 1$. Then, since

$$\begin{aligned}\widetilde{W}(X, Y)Z &= -\langle \mathcal{J}X, Z\rangle \mathcal{J}Y + \langle J_2\mathcal{J}Y, Z\rangle \mathcal{J}J_2X + \langle J_3\mathcal{J}Y, Z\rangle \mathcal{J}J_3X \\ &\quad -\langle J_1\mathcal{J}X, Z\rangle J_1\mathcal{J}Y - \langle J_2\mathcal{J}X, Z\rangle J_2\mathcal{J}Y - \langle J_3\mathcal{J}X, Z\rangle J_3\mathcal{J}Y\end{aligned}$$

it follows that

$$\langle \mathcal{J}X, Z\rangle \mathcal{J}Y - \langle \mathcal{J}X, J_1Z\rangle J_1\mathcal{J}Y + \Big[\langle \mathcal{J}Y, J_2Z\rangle \mathcal{J}J_2X + \langle \mathcal{J}Y, J_3Z\rangle \mathcal{J}J_3X\Big]^\perp = 0\,.$$

Let consider on T_pM an orthonormal base $(E_1, E_2, \ldots, E_{2n})$ such that $\mathcal{J}Y = J_2E_1$ and $E_2 = J_1E_1$. Then

$$\begin{aligned}\langle \mathcal{J}X, Z\rangle J_2E_1 \quad &+\langle \mathcal{J}X, J_1Z\rangle J_2J_1E_1 \\ &+ \textstyle\sum_{i=1}^{2n} \Big[\langle E_1, Z\rangle \langle \mathcal{J}X, E_i\rangle J_2E_i + \langle E_1, J_1Z\rangle \langle \mathcal{J}J_1X, E_i\rangle J_2E_i\Big] = 0\end{aligned}$$

Equivalently we have the following identities:

$$\langle \mathcal{J}X, Z\rangle + \langle E_1, Z\rangle\langle \mathcal{J}X, E_1\rangle + \langle E_1, J_1Z\rangle\langle \mathcal{J}J_1X, E_1\rangle = 0$$

and

$$\langle E_1, Z\rangle\langle \mathcal{J}X, E_i\rangle + \langle E_1, J_1Z\rangle\langle \mathcal{J}J_1X, E_i\rangle = 0 \qquad (i = 3, \dots, 2n)$$

For $Z = E_1$ and $Z = J_1E_1$ we get respectively

$$\langle \mathcal{J}X, E_1\rangle + \|E_1\|^2\langle \mathcal{J}X, E_1\rangle = 0 \quad , \quad \langle \mathcal{J}X, J_1E_1\rangle + \|E_1\|^2\langle \mathcal{J}X, J_1E_1\rangle = 0$$

and

$$\|E_1\|^2\langle \mathcal{J}X, E_i\rangle = 0 \quad , \quad i = 3, \dots, 2n$$

Hence $\mathcal{J}X \in T_pM^\perp, \forall X \in T_pM$.

Step 3) There are only two possibilities: $\mathcal{J}T_pM = T_pM$ or $\mathcal{J}T_pM = T_pM^\perp$. One has

$$\begin{aligned}\widetilde{W}(X,Y)Y = -3\langle \mathcal{J}X, Y\rangle\mathcal{J}Y \quad & -\langle \mathcal{J}Y, J_1Y\rangle\mathcal{J}J_1X - \langle \mathcal{J}Y, J_2Y\rangle\mathcal{J}J_2X - \langle \mathcal{J}Y, J_3Y\rangle\mathcal{J}J_3X \\ & +\langle \mathcal{J}X, J_1Y\rangle\mathcal{J}J_1Y + \langle \mathcal{J}X, J_2Y\rangle\mathcal{J}J_2Y + \langle \mathcal{J}X, J_3Y\rangle\mathcal{J}J_3Y\,.\end{aligned}$$

Then one finds

$$\begin{aligned}4\big[\langle \mathcal{J}X, Y\rangle^2 \quad & +\langle \mathcal{J}X, J_1Y\rangle^2\big]\mathcal{J}Y = \\ & -\langle \mathcal{J}X, Y\rangle\Big[\widetilde{W}(X,Y)Y - \widetilde{W}(X, J_1Y)J_1Y\Big] \\ & +\langle \mathcal{J}X, J_1Y\rangle\Big[\widetilde{W}(J_1X,Y)Y - \widetilde{W}(J_1X, J_1Y)J_1Y\Big] \\ & +2\Big[\langle \mathcal{J}X, J_1Y\rangle\langle \mathcal{J}Y, J_3Y\rangle - \langle \mathcal{J}X, Y\rangle\langle \mathcal{J}Y, J_2Y\rangle\Big]\mathcal{J}J_2X \\ & -2\Big[\langle \mathcal{J}X, Y\rangle\langle \mathcal{J}Y, J_3Y\rangle + \langle \mathcal{J}X, J_1Y\rangle\langle \mathcal{J}Y, J_2Y\rangle\Big]\mathcal{J}J_3X\,.\end{aligned} \tag{6}$$

Let assume that $\mathcal{J}T_pM \cap T_p^\perp M = 0$. Then for any $Y \neq 0$ $\langle \mathcal{J}X, Y\rangle^2 + \langle \mathcal{J}X, J_1Y\rangle^2$ is not identically zero (otherwise $\mathcal{J}T_pM = T_p^\perp M$). Let X be a tangent vector such that $\langle \mathcal{J}X, Y\rangle^2 + \langle \mathcal{J}X, J_1Y\rangle^2 \neq 0$. Then (if $n \neq 1$) there always exists a tangent vector Z such that $\langle \mathcal{J}Z, Y\rangle^2 + \langle \mathcal{J}Z, J_1Y\rangle^2 \neq 0$ with $\mathbb{C}X \neq \mathbb{C}Z$: in fact let $Z = X + h$ where h is a small vector $\mathbb{C}$-independent from X. Then by comparing the two identities giving $\mathcal{J}Y$ by means of X or Z we obtain that

$$\begin{aligned}\langle \mathcal{J}X, J_1Y\rangle\langle \mathcal{J}Y, J_3Y\rangle - \langle \mathcal{J}X, Y\rangle\langle \mathcal{J}Y, J_2Y\rangle = 0 \\ \langle \mathcal{J}X, Y\rangle\langle \mathcal{J}Y, J_3Y\rangle + \langle \mathcal{J}X, J_1Y\rangle\langle \mathcal{J}Y, J_2Y\rangle = 0\end{aligned}$$

and also

$$\begin{aligned}\langle \mathcal{J}Z, J_1Y\rangle\langle \mathcal{J}Y, J_3Y\rangle - \langle \mathcal{J}Z, Y\rangle\langle \mathcal{J}Y, J_2Y\rangle = 0 \\ \langle \mathcal{J}Z, Y\rangle\langle \mathcal{J}Y, J_3Y\rangle + \langle \mathcal{J}Z, J_1Y\rangle\langle \mathcal{J}Y, J_2Y\rangle = 0\end{aligned}$$

It then follows that

$$\langle \mathcal{J}Y, J_2Y\rangle = \langle \mathcal{J}Y, J_3Y\rangle = 0\,.$$

But, since $\mathcal{J}J_2$ is a symmetric operator, the identity

$$\langle \mathcal{J}J_2Y, Y\rangle = 0$$

implies that

$$\langle \mathcal{J}J_2Y, X\rangle = 0 \qquad \forall X, Y \in T_pM, p \in M$$

that is

$$\langle \mathcal{J}Y, J_2X\rangle = 0 \qquad \forall X \in T_pM$$

hence $\mathcal{J}Y \perp T_pM^{\perp}$ and it results $\mathcal{J}T_pM = T_pM$. □

References

[ADM] D.V. Alekseevsky - A.J. Di Scala - S. Marchiafava *Parallel Kähler submanifolds of quaternionic Kähler symmetric spaces*, Preprint, 2003

[AM1] D.V. Alekseevsky - S. Marchiafava *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl., **171**, 1996, 205-273

[AM2] D.V. Alekseevsky - S. Marchiafava *Hermitian and Kähler submanifolds of a quaternionic Kähler manifold*, Osaka Math. J., **38** (2001), 869-904

[AM3] D.V. Alekseevsky - S. Marchiafava *A twistor construction of (minimal) Kähler submanifolds of a quaternionic Kähler manifold* Prépublication de Institut Elie Cartan, Nancy, n. 2001/33, to appear on Ann. Mat. Pura Appl., 2003

[Ber] J. Berndt *Riemannian geometry of complex two-plane Grassmannian*, Rend. Sem. mat. Univ. Politec. Torino **55**, n. 1, 1997, 19-83

[Ber2] J. Berndt *Symmetric submanifolds of symmetric spaces,* Proceeedings of The Seventh International Workshop on Diff. Geom. **7** (2003), 1-15.

[BENT] J. Berndt - J-H Eshenburg - H. Naitoh - K. Tsukada *Symmetric submanifolds associated with the irreducible symmetric R-spaces*, Preprint (2003)

[Bes] A. Besse *Einstein manifolds*, Springer-Verlag, 1987

[Br] R.L. Bryant *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differ. Geom. **17** , 1982, 455-473

[Bu] F.E. Burstall *Minimal surfaces in quaternionic symmetric spaces*, *Geometry of low-dimensional manifolds, 1 (Durham, 1989)*, 231-235, London Math. Soc. Lecture Note Ser., **150**, Cambridge Univ. Press, Cambridge, 1990

[ES] J. Eells - S. Salamon *Twistorial construction of harmonic maps of surfaces into four manifolds*, Ann. Sc. Norm. Sup. Pisa, **12**, 1985, 589-640

[Fr4] T. Friedrich *On superminimal surfaces* , Archivum math., **33** , n 1/2, 1997, 41-56

[Fu] S. Funabishi *Totally complex submanifolds of a quaternionic Kaehlerian manifold*, Kodai Math. J. **2**, 1979, 314-336

[Ko1] P.Z. Kobak *Contact birational correspondences between twistor spaces of Wolf spaces*, pp. 66-74, in L.J. Mason et al. "Further advances in twistor theory: Vol. II: Integrable systems, conformal geometry and gravitation". Longman Scientific & Technical, Pitman Research Notes in Mathematics Series n. 232, 1995

[Ko2] P.Z. Kobak *A twistorial construction of (1,1)-geodesic maps*, pp. 75-81, in L.J. Mason et al "Further advances in twistor theory: Vol. III: Curved twistor spaces". Chapman & Hall/CRC Research Notes in Mathematics n. 424, 2001

[L] H.B. Lawson *Surfaces minimales et la construction de Calabi-Penrose*, Séminaire BOURBAKI, 36e année, 1983/84, n 624, 197-211

[M] S. Marchiafava *Sulla geometria locale delle varietà Kähleriane quaternionali*, Bollettino U.M.I. (7) 5-B, 1991, 417-447

[McL] R. C. McLean *Twistors and complex Lagrangian submanifolds of* $\mathbb{H}P^n$, Preprint, 1992

[QSMP] *Mathematical structures in Mathematics and in Physics*, Proc. of the "2nd Meeting on Quaternionic Structures in Mathematics and Physics". Ed. World. Scientific, 2000

[Na1] H. Naitoh *Isotropic submanifolds with parallel second fundamental forms in symmetric spaces,* Osaka J. Math. **17** (1980), 95-110

[Na2] H. Naitoh *Symmetric submanifolds of compact symmetric spaces,* Tsukuba J. Math. **10** (1986), 215-242.

[Na3] H. Naitoh *Grassmann geometries on compact symmetric spaces of exceptional type,* Japanese J. Math. **26** (2000), 157-206.

[Na4] H. Naitoh *Parallel submanifolds of complex space forms I,* Nagoya Math. J. **90** (1983), 85-117.

[O] Y. Ohnita *On pluriharmonicity of stable harmonic maps*, J. London math. Soc. **35**, 1987, 503-568

[Sal] S. Salamon *Quaternionic Kähler manifolds*, Invent. Math. **67**, 1982, 143-171

[Tak] M. Takeuchi *Totally complex submanifolds of quaternionic symmetric spaces*, Japan J. Math. (N.S.) **12**, 1986, 161-189

[Tsu1] K. Tsukada *Parallel Kaehler Submanifolds of Hermitian Symmetric Spaces*, Math. Z. **190** (1985), 129-150

[Tsu2] K. Tsukada *Parallel submanifolds in a quaternion projective space*, Osaka J. Math. **22**, 1985, 187-241

[W] J.A. Wolf *Complex Forms of Quaternionic Symmetric Spaces*, Preprint math.DG/0308283, 2003