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FOUR-DIMENSIONAL ELASTICITY AND GENERAL RELATIVITY

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It has been shown that an extension of the elasticity theory to more than three dimensions enables one to describe the space-time as a properly stressed medium, even recovering the Minkowski metric in the case of uniaxial stress. A fundamental equation for the metric in the theory is shown to be the equilibrium equation for the medium. Examples of spherical and cylindrical symmetries in four dimensions are considered, indicating convergencies and divergencies with classical general relativity. Finally, a possible meaning of the dynamics of four-dimensional elastic medium is discussed.

1. Introduction

The tensor theory of elasticity in three dimensions has some apparent similarities with classical general relativity. The question I have addressed in this paper is, whether or not this formal analogy may correspond to something more profound than a mere use of symmetric tensors in both cases. In fact, many authors have tried and introduced elasticity into general relativity, casting the general equations into a relativistic form [1–6]. This was usually made for "practical" purposes, in order to describe the dynamic behaviour of astrophysical bodies in relativistic conditions, the interaction of gravitational waves with bar antennas, the propagation of shock waves in viscoelastic media and the like.

Here the approach is different, because the space-time itself will be looked at as an elastic medium. Some hint in that direction can be found in the literature, for instance, in [7] where Gerlach and Scott introduce a sort of "elasticity of the metric", though in connection with the presence of matter.

The guiding idea of this paper is as follows: suppose the space-time is a four dimensional elastic medium. The latter, when unstrained, is perfectly homogeneous and isotropic. The fundamental symmetry around any point inside it is $GL(4,R)$. Apply now some stresses to the medium: the symmetry will be broken and reduced. In particular, if the applied stress is one-dimensional, the consequence is that one particular direction is specialized: could this be time? Otherwise stated: is it possible that a uniaxial stress reduces the $GL(4,R)$ symmetry to $SO(3,1)$?

This approach, as we shall see, is indeed viable at a first level but leads (I would say 'of course') to final equilibrium equations which are different from those of general relativity. First of all, the linear theory of elasticity in any number of dimensions leads inevitably to linear equations. The description of space-time that comes out as a result is "static", i.e., perfectly deterministic. Any dynamics of a four-dimensional medium needs a fifth evolution parameter and the evolution itself would bring about certain modifications both in the past and the future of a given event.

In what follows I shall review the theory of elasticity and show how it can be brought to describe a reasonable space-time.

2. Instant review of elasticity theory

2.1. Strain

Suppose you have an n-dimensional elastic medium. In the absence of any strain the geometry inside it is Euclidean (or at least, assume it is). The squared distance between two nearby points is

$$dl_o^2 = \epsilon_{\alpha\beta} dx^\alpha dx^\beta \tag{1}$$

where $\epsilon_{\mu\nu}$ is the metric tensor of the unstrained medium.

Introduce now an infinitesimal strain. Any point will be, in general, displaced by a small vector $\overrightarrow{w}$, varying from place to place. As a consequence the new squared distance between two points will be

$$dl^2 = (\epsilon_{\alpha\beta} + d\epsilon_{\alpha\beta})(dx^\alpha + dw^\alpha)\left(dx^\beta + dw^\beta\right). \tag{2}$$

The quantities x still refer to the unperturbed background and the $\overrightarrow{w}$, as well as ϵ, are functions of the

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x^α. In an explicit form one has:

$$\begin{aligned}\delta\epsilon_{\alpha\beta} &= \epsilon_{\alpha\beta,\mu}w^\mu,\\ \delta w^\alpha &= w^\alpha_{,\mu}dx^\mu\end{aligned} \tag{3}$$

where commas denote partial derivatives.

Now Eq. (2) may be written in the form

$$\begin{aligned} dl^2 = dx^\alpha dx^\beta (&\epsilon_{\alpha\beta} + \epsilon_{\alpha\mu}w^\mu_{,\beta} + \epsilon_{\mu\beta}w^\mu_{,\alpha}\\ &+ \epsilon_{\mu\nu}w^\mu_{,\alpha}w^\nu_{,\beta} + \epsilon_{\alpha\beta,\mu}w^\mu + \epsilon_{\alpha\mu,\nu}w^\nu w^\mu_{,\beta}\\ &+ \epsilon_{\mu\beta,\nu}w^\nu w^\mu_{,\alpha} + \epsilon_{\mu\nu,\lambda}w^\lambda w^\mu_{,\alpha}w^\nu_{,\beta}).\end{aligned} \tag{4}$$

Usually a part of the content of the brackets in (4) is identified with the strain tensor $u_{\mu\nu}$, which is manifestly symmetric.

Now (4) becomes

$$dl^2 = (\epsilon_{\alpha\beta} + u_{\alpha\beta})dx^\alpha dx^\beta. \tag{5}$$

The almost obvious identification

$$g_{\mu\nu} \equiv \epsilon_{\mu\nu} + u_{\mu\nu} \tag{6}$$

leads to

$$dl^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{7}$$

The symmetric tensor $g_{\mu\nu}$ is now the metric tensor of the strained medium.

All this, as said, has been written using the unperturbed Euclidean coordinates. It is more natural to have recourse to internal or intrinsic coordinates (those attached to the medium); these (let us call them ξ^μ will in general be functions of x^μ. The geometric nature of the objects used in the theory is such that it is possible to recast everything in terms of ξ^μ by standard coordinate transformations for tensors. In practice we can simply rewrite formulas from (5) to (7) as if the x^μ were ξ^μ and nothing changes, consequently, we shall continue to use x^μ in the new meaning [8]. By the way, in the base situation (unstrained medium) x^μ and ξ^μ coincide.

2.2. Stress

In the classical theory of elasticity, a stress tensor, whose element $\sigma^{\alpha\beta}$ has the meaning of the α-component of the force per unit surface acting on a surface element orthogonal to the β-direction, is introduced. Assuming the range of stresses to be zero (propagation only by surface interaction), $\sigma^{\mu\nu}$ is symmetric. The next step is to link stresses and strains. This can be done by the so-called Hooke's law, which is indeed a linearization stating the proportionality between stresses and strains. The theory may be found in any textbook on elasticity, such as, for instance, [9]. Hooke's law in n dimensions is expressed by the equivalent formulas

$$\begin{aligned}\sigma_{\alpha\beta} &= \left(K - \frac{2\mu}{n}\right)\epsilon_{\alpha\beta}\epsilon^{\nu\lambda}u_{\lambda\nu} + 2\mu u_{\alpha\beta},\\ u_{\alpha\beta} &= \left(\frac{1}{n^2K} - \frac{1}{2\mu n}\right)\epsilon_{\alpha\beta}\epsilon^{\nu\lambda}\sigma_{\nu\lambda} + \frac{1}{2\mu}\sigma_{\alpha\beta}\end{aligned} \tag{8}$$

where K is the uniform compression modulus and μ is the shear modulus. Reasonable restrictions upon the values of K and μ are

$$K > 0, \qquad \mu > 0. \tag{9}$$

In general, in a linearized theory of elasticity in any dimensions two independent parameters are sufficient for describing the properties of the medium. The parameters used may be variously combined to produce others such as the first Lamé coefficient λ (the second is μ), the Young modulus E and the Poisson coefficient σ.

3. Equilibrium conditions

In our homogeneous stressed medium an equilibrium is attained when the following equation holds:

$$\sigma_{\alpha\beta,\beta} + f_\alpha = 0 \tag{10}$$

Now f_α represents the α-component of any force per unit volume; due to the linearity of Hooke's law, indices are raised and lowered using $\epsilon_{\mu\nu}$.

Combining Eqs (8), (10) and (6), one directly obtains:

$$\begin{aligned}&\left[(K - 2\mu/n)(\epsilon^{\nu\lambda}g_{\lambda\nu} - n) - 2\mu\right]\epsilon_{\alpha\beta,\beta}\\ &+ (K - 2\mu/n)\epsilon_{\alpha\beta}(\epsilon^{\mu\nu}g_{\mu\nu})_{,\beta} + 2\mu g_{\alpha\beta,\beta} = -f_\alpha.\end{aligned} \tag{11}$$

In Cartesian coordinates (with the Euclidean background geometry), (11) is simplified to

$$(K - 2\mu/n)g_{\alpha\alpha,\mu} + 2\mu g_{\mu\nu,\nu.} = -f_\mu \tag{12}$$

Eqs. (11) or (12) are n equations for $n(n+1)/2$ unknowns, consequently, the problem is underdetermined. Suitable boundary conditions are needed.

4. Uniaxial stress

Let us now suppose that in our homogeneous n-dimensional medium a uniform stress is applied along an arbitrary direction and call the corresponding axis the τ axis. The stress tensor (as referred to a Cartesian coordinates system) is in our conditions

$$\begin{aligned}\sigma_{00} &= p,\\ \sigma_{\alpha\beta} &= 0, \qquad \alpha \neq \beta,\\ \sigma_{ii} &= \Sigma.\end{aligned} \tag{13}$$

The index number 0 corresponds to τ, Latin indices run from 1 to $n-1$; Σ and p are constants; $p > 0$ means traction and $p < 0$ means compression.

Looking at Eq. (12), we see that any $g^{\mu\nu} = \text{const}$ is a solution of the equilibrium equation. To actually solve the problem, we have to directly deduce $g^{\mu\nu}$ from (6) and Hooke's law (8):

$$
\begin{aligned}
u_{00} &= \frac{1}{n}\left[\left(\frac{1}{nK}+\frac{n-1}{2\mu}\right)p+(n-1)\left(\frac{1}{nK}-\frac{1}{2\mu}\right)\Sigma\right],\\
u_{\alpha\beta} &= 0, \qquad\qquad \alpha \neq \beta,\\
u_{ii} &= \frac{1}{n}\left[\left(\frac{1}{nK}-\frac{1}{2\mu}\right)p+\left(\frac{n-1}{nK}+\frac{1}{2\mu}\right)\Sigma\right]. \qquad (14)
\end{aligned}
$$

Now applying (6), we see that $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric tensor, whenever

$$
\begin{aligned}
p &= [(n-1)/n](2\mu - nK),\\
\Sigma &= -[2\mu + nK(n-1)]n. \qquad (15)
\end{aligned}
$$

In four dimensions that is:

$$
\begin{aligned}
p &= \tfrac{3}{2}(\mu - 2K),\\
\Sigma &= -\tfrac{1}{2}(\mu + 6K). \qquad (16)
\end{aligned}
$$

In view of the conditions (9), it comes out that $\Sigma < 0$ in any case, which means transverse compression. This is consistent with what we know from three-dimensional elasticity if $p > 0$, i.e., if there is traction along the τ axis. The parameter p is actually positive when

$$\mu > 2K. \qquad (17)$$

The Minkowski space-time looks like a four-dimensional medium with suitable elastic properties stretched along the time axis. Before the stress is applied, there is no difference among various coordinates, so there is no "time"; once there is a stress, one of the coordinates, measured along any axis within the light cone about τ, becomes no longer interchangeable with the others: this, from the intrinsic viewpoint, is time.

5. Spatially flat expanding universe

Another case of interest is that of an open expanding universe. The corresponding conformally flat metric, in Cartesian coordinates, may be written as

$$ds^2 = \alpha^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2). \qquad (18)$$

Introducing the metric (18) into (12), it is easily verified that a nontrivial solution is found for $\alpha(\tau)$ if

$$f_i = 0, \qquad f_o = F = \text{const.}$$

The solution is

$$\alpha^2(\tau) = \frac{F}{2K - 3\mu}\tau + \text{const.} \qquad (19)$$

and is consistent with the existence of a uniform volume field orthogonal to any space section of the four-dimensional elastic medium.

After time rescaling according to

$$\alpha(\tau)\, d\tau = dt,$$

the line element acquires the synchronous form

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \qquad (20)$$

with

$$a^2(t) = \left|\frac{F}{3\mu - 2K}\right|^{2/3} t^{2/3} \qquad (21)$$

As can be seen, the time dependence of the space scale factor is the same as that for a matter-dominated Friedmann universe [10].

The solution we found corresponds to a spherically symmetric situation in four dimensions. There is one center of symmetry (the big bang) and any radial axis may be used as a time axis. Unlike that, a more general positive or negative space curvature Robertson-Walker metric does not comply with this symmetry and is not a solution to Eq. (11).

6. Rotation symmetry about an axis

This is a typical situation which in general relativity leads, in the static case, to the Schwarzschild solution. A general form for a metric with this symmetry is

$$g = \begin{pmatrix} f(r,\tau) & 0 & 0 & 0 \\ 0 & -h(r,\tau) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\vartheta \end{pmatrix} \qquad (22)$$

The cylindrical coordinates $\tau, r, \vartheta, \varphi$ have been used. The corresponding expression for the ϵ is

$$\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix} \qquad (23)$$

Inserting (22) and (23) into (11) leads to a pair of independent equations:

$$
\begin{aligned}
(K - 2\mu)(\dot f - \dot h) + 2\mu\dot f &= -f_0,\\
(K - 2\mu)(f' - h') - 2\mu f' &= -f_r. \qquad (24)
\end{aligned}
$$

Dots stand for partial derivatives with respect to τ and primes for partial derivatives with respect to r.

The static case (τ-independence of f and h) requires $f_0 = 0$, whereas nontrivial solutions exist only if $f_r \neq 0$. To actually solve the problem, it is necessary to impose the distribution of strains or stresses on a suitable surface, remembering also that f and h should be positive.

One can arrive at the same results starting from the solution for the uniaxial stress case and allowing p and Σ to depend on τ and r.

7. Discussion

It has been shown that solutions of the equilibrium conditions inside a four-dimensional elastic medium stressed in any way may provide reasonable forms for space-time metric under various symmetry conditions. There are, however, some problems; one of them is that of the signature.

Treating the case of a uniaxial stress, we saw that it is possible to recover the Minkowski metric. In four dimensions (14) and (16) lead to the strain tensor components

$$u_{00} = 0, \qquad u_{ii} = -2. \tag{25}$$

However, we know that the strain tensor is defined starting from a strain vector field according to (4). In the case of uniaxial symmetry the explicit form of the strain tensor in the background Euclidean coordinates is

$$u_{\mu\nu} = w_{\mu,\nu} + w_{\nu,\mu} + w^{\alpha}_{,\mu} w_{\alpha,\nu}. \tag{26}$$

Substituting (25) into (26) and solving for the w^{α}, one obtains:

$$w^0 = \text{const} - 2\tau, \qquad w^r = (-1 \mp i)r. \tag{27}$$

Thus, while the strain tensor is real, the strain vector field is complex: this is the price to be paid for the Minkowski signature.

Another important point to remind is that the theory, due to Hooke's law, is linear. This implies that only weak field regions may be described in this way. It is possible to obtain a better approximation for stronger fields considering nonlinear elasticity. The starting point is the development of the Helmholz free energy F of the medium in powers of the strains, where Hooke's law comes from. The next approximation after the linear one is

$$F = F_0 + \frac{\lambda}{2}(u^{\alpha}_{\alpha})^2 + \mu u_{\alpha\mu}u^{\mu\alpha} + \frac{\nu}{3}(u^{\alpha}_{\alpha})^3 + \pi u^{\alpha}_{\alpha}u^{\nu}_{\mu}u^{\mu}_{\nu} + \rho u^{\beta}_{\alpha}u^{\mu}_{\beta}u^{\alpha}_{\mu} + \ldots \tag{28}$$

Three new parameters (ν, π, ρ) have been introduced to characterize the behaviour of the medium. It is now no longer allowed to raise and lower indices using simply the ϵ's; instead, $g_{\mu\nu}$ must be used after developing them to the first order in u^{μ}. Everything is much more complicated but it may be managed.

Finally we may remark that our treatment of an equilibrium condition corresponds to a perfectly static situation, i.e., to an entirely deterministic universe where the histories coincide with the flux lines of the strain vector field. However, any elastic medium has not only statics but also dynamics: it may vibrate and has characteristic internal frequencies. If we consider a four-dimensional elastic medium, vibrations evidently have a meaning only with respect to some appropriate evolution parameter, let us call it T: something like the good old Newtonian time. Four-dimensional observers have of course no means to measure T: their clocks actually measure what we called τ (or t), though now τ, as well as the other space coordinates, parametrically depends on T. An influence of the vibrations in four plus one dimensions may, however, be seen from inside the four-dimensional world.

Suppose, for instance, that the medium contains a couple of points held fixed, whereas the rest undergoes elastic vibrations. Any strain flux line (or history) going from one point to the other is continuously modified by the vibrations in T. An internal observer, unable to perceive T, will notice that there are many nearby histories, and what in four plus one dimensions is a T evolution for him may well be transformed into different probabilities to be attached to various histories. If the four-dimensional observer wants to forecast the future, he will be led to an average over histories. A remarkable feature is the fact that for a given vibrating point both the future and the past (in τ or t) vary in T.

I think that this viewpoint may provide a new approach to quantum mechanics and in any case is worth further investigation.

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