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PHYSICAL REVIEW B VOLUME 50, NUMBER 16 15 OCTOBER 1994-II

Phonons in the quantum Hall effect: A nonlinear-dynamics picture

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(Received 16 February 1994)

A model describing a system in which Landau electronic modes are coupled with phonons is proposed and discussed. A simplified version of the model is further analyzed, with special attention to the dynamical symmetries that characterize it. In particular, the corresponding equations of motion are thoroughly examined: they provide a variety of behaviors, ranging from completely integrable (in both the classical and quantum case) to chaotic (in the semiclassical approximation). The chaotic regime is believed to be suitable to eventually represent the stochastic behavior of the longitudinal voltage versus time, recently observed in several quantum Hall effect experiments.

I. INTRODUCTION

A number of experiments,[1,2] measuring the time behavior of the longitudinal voltage V_1 in a typical quantum Hall effect (QHE) experiment in regions of dissipative instability (namely, for high currents and for values of the magnetic field B corresponding to the edges of a Hall plateau) and exhibiting stochasticity, have been accounted for[2] with the conjecture that the underlying dynamics is chaotic. In this paper we aim to show, by constructing a model which includes, by the conventional Hall dynamics, electron-phonon interactions [mimicking the so-called QUILLS (quasielastic inter-Landau-level scattering) mechanism earlier proposed in the literature[3]], that—even in a simplified version suitable for a detailed analysis—such deterministic chaotic behavior can indeed be ascribed to the microscopic structure of the system.

The normal QHE is manifested by the characteristic plateaus in the Hall resistivity ρ_H vs the magnetic field B. At these plateaus the longitudinal current carried by the electrons appears to be dissipationless, in that the corresponding resistivity ρ_L is essentially vanishing.[4]

If the longitudinal current density j_L is increased, the dissipationless regime occupies a smaller and smaller domain in B until, beyond a certain critical value of j_L, it suddenly disappears, while ρ_L rapidly increases (this is referred to as QHE breakdown[5]). However, for values of j_L below the critical value but sufficiently large, ρ_L already exhibits sharp discontinuities at specific values B_c of the magnetic field. This is clearly shown in Ref. 2, where a plot of $V_1 = \rho_L j_L L_1$ vs B is reported, for several different values of $I_1 = j_L L_2$ (where L_1, L_2 denote, respectively, the sample length in the current direction and transverse to it). In the same reference[2] it is shown that, when the longitudinal voltage V_1 is measured vs time for $B = B_c$, it shows a characteristic intermittent stochastic behavior, reminiscent of what one has in the dynamical regimes of transition to chaos, and random oscillations take place among a finite number of discrete levels of V_1.

Inspired by this experimental scenario, in the present paper we study a microscopic model, constructed along similar lines to the QUILLS model,[1] showing how a special choice of the dynamical algebra as well as of the representation of such algebra does indeed lead to a dynamics in some way comparable with the relevant features of the above phenomenology. It should be stressed here that the purpose of our discussion is to show how even a relatively simple microscopic model, describing in a mean-field way the interactions of the electrons with phonons, might generate a chaotic dynamics, not to reproduce quantitatively the experimental results or to give a treatment of the system's macroscopic features.

To this effect we point out, first, that the experimental regime (QHE breakdown) we consider is dissipative in that it refers to a regime of resistive transverse conduction. Therefore dissipative processes in the customary sense should appear in the description of transport phenomena at a macroscopic level. However, there is no time decay in the processes measured in the experiments we refer to, but rather a form of instability whereby a measured quantity (V_1) which should have a fixed value, when sampled at very high frequency and with just the magnetic field for which macroscopic dissipation occurs, exhibits a chaotic distribution among several different possible values.[1] In this sense one cannot speak here of a nonstationary regime meaning that the observed phenomenon is transient, because the time scales of resistive dissipation and of dynamical equilibrium are quite different: the former are much larger than the latter, and do not enter into play in our considerations. For the chaotic nonstationarity predicted by our model, one should rather imagine a dynamical—not thermodynamic—effect, arising from the competition (and corresponding bifurcations) among the various states whose existence the macroscopic nonequilibrium situation requires, made available by the special value of the magnetic field, but on a time scale much shorter than the relaxation time. This sort of nonstationarity connected with chaos is not nonequilibrium, but rather equilibrium for a dynamical system which has a strange attractor.

Dealing with the macroscopic features of the system would require, for example, writing the equations of motion for its density matrix, introducing into play some Liouville operator consistent with the appropriate master equation. In a semiclassical scheme of coherent states, such an approach would lead to a Fokker-Planck equation rather than to a classical dynamical system. The scheme adopted in this paper has, we repeat, a quite different philosophy: our model is a microscopic model in which the electronic modes can exchange energy and momentum with the phonons of the solid in a time-reversal-invariant way. In other words, such phonons are considered not as a thermal bath, but just as a microscopic reservoir coupled with the electrons in such a way that the latter can jump from one level to the other, possibly losing (or even gaining) energy and momentum, but in a way that is not globally dissipative, but becomes so when one considers things from the point of view of the electron degrees of freedom only (for example, after tracing over the phonon degrees of freedom). In fact, our model is even more crude than this last scheme, in that we actually assume the existence of a unique phonon mode, which suitably represents the whole reservoir.

In order to understand the meaning of our construction, one should first recall that the spectral structure implied by the QHE Hamiltonian in the absence of phonons can be described by attributing to the electrons two main quantum numbers, say, n_1 and n_2, with the following physical meaning. The electron motion is characterized by two modes. The mode associated with n_1 (which describes the inner degeneracy of Landau levels) is endowed with the information concerning the average position of the electron along the x_2 (transverse) direction: one can actually imagine the sample as subdivided into a set of horizontal parallel stripes whose center has ordinate proportional to n_1, in which the electron motion is confined. The width of the stripes is instead connected with the second mode, and turns out to be proportional to $(2n_2+1)^{1/2}$. In fact, the other mode, whose quantum number is n_2, describes the (cycloidal) motion of the electron within each of such stripes, drifting along the x_1 (longitudinal) direction, when its stage is that specified by the n_2th Landau level.

When the phonon field is accounted for, we assume that the above two modes interact independently with the phonons, exchanging energy. The dynamical algebra is selected according to the requirement of realizing a sort of minimal coupling scheme whose features are (i) that of providing the maximum-simplicity coupling, associated with (ii) the aim of generating a nontrivial dynamics, and (iii) that the quantum number n_1 may assume values ranging between $-\infty$ and $+\infty$, consistent with the (cylindrical) configuration-space geometry. We shall show in Sec. II that this leads to a noncompact dynamical algebra isomorphic with su(1,1). As for the representation, the very fact that $-\infty < n_1 < +\infty$, with no gaps, brings us to select the continuous principal series C_l.[6]

A detailed analysis of the model Hamiltonian and the corresponding equations of motion shows that, in a completely quantum description, one can distinguish between an integrable situation (discussed in Sec. III) and a nonintegrable one (Sec. IV), depending on the value of the electron-phonon coupling constants Γ_1 and Γ_2. The former, integrable, case, $\Gamma_2=0$, namely, vanishing interaction between phonons and the Landau mode ($n_2=$const), is formally dealt with quantum mechanically in Sec. III A by identifying the corresponding infinite-dimensional spectrum-generating algebra. The same case is successively thoroughly studied, in Sec. III B, in the semiclassical approximation obtained by resorting to the representation of the electron-phonon states in terms of coherent states. There emerges a very interesting dynamics which leads to an equation of motion reducible to the differential equation for the Weierstrass $\wp$ function. This will constitute the integrable background on which deterministic chaos occurs when the full coupling of the Landau electrons with phonons is taken into account ($\Gamma_2 \neq 0$).

Section IV deals just with the last dynamical situation, even though, naturally, only in the semiclassical approximation. We show there how, as one might expect from the Kolmogorov-Arnol'd Moser (KAM) theorem, when the motion is far enough from resonance and the coupling which was vanishing in the integrable case is switched on, some onset of chaos indeed takes place, which may exhibit the features first of intermittency, then of fully developed chaos, depending on the model parameters. In the same section a detailed analysis of the system phase portrait is reported.

II. CONSTRUCTION OF THE MODEL

As is usually done in the standard description of the Hall effect, we consider a two-dimensional (2D) system of electrons of mass m and charge $q=-e$ flowing through a thin rectangular slab of length L_1 and width L_2, perpendicular to which is applied a uniform time-independent magnetic field $\mathbf{B}$. All dynamical variables are then naturally referred to a Cartesian reference frame attached to the sample. The electron current along axis 1 is sustained by a voltage in the same direction, whereas along axis 2, due to the customary (classical) Hall effect, in the stationary regime a transverse electric field $\mathbf{E}$ is generated.[7,8]

The electron dynamics is governed in this system by the Hall Hamiltonian

$$H=\frac{1}{2m}\left[\mathbf{p}-\frac{q}{c}\mathbf{A}\right]^2-q\mathbf{x}\cdot\mathbf{E}\ , \tag{1}$$

where $\mathbf{A}$ is the potential vector ($\text{curl}\,\mathbf{A}=\mathbf{B}$). In the Landau gauge $\mathbf{A}=-Bx_2\hat{\mathbf{u}}_1$, and (1) is written in the form

$$H_{\mathcal{L}}=\frac{1}{2m}[p_2^2+(p_1-m\omega_c x_2)^2]+eEx_2\ , \tag{2}$$

$\omega_c=eB/mc$ denoting the usual cyclotron frequency. Of course, x_j,p_j are canonically conjugated. The Hamiltonian (2) has well-known interesting features. Its spectrum, labeled by two integer quantum numbers n_1 and n_2, is given by

$$\begin{aligned}\mathcal{E}_{n_1,n_2}&=\hbar\omega_c(n_2+\tfrac{1}{2})+eEX_2+\tfrac{1}{2}mv_1^2\\&=\hbar\omega_c(n_2+\tfrac{1}{2})+\hbar k_1 v_1-\tfrac{1}{2}mv_1^2\ ,\end{aligned} \tag{3}$$

where $X_2=l_B^2k_1-v_1/\omega_c$, $k_1=2\pi n_1/L_1\equiv\chi n_1$, $l_B^2=\hbar c/eB$ indicating the (square) magnetic length, while $v_1=cE/B$ is the electron drift velocity. The corresponding eigenfunctions are

$$\Phi_{n_1,n_2}(\mathbf{x})=\left[\frac{\sigma}{h^{1/2}L_1(2n_2)!!}\right]^{1/2}\times\exp\left[ik_1x_1-\left[\frac{x_2-X_2}{l_B}\right]^2\right]\times H_{n2}\left[\frac{(x_2-X_2)}{l_B}\right], \quad (4)$$

$H_n(x)$ denoting the Hermite polynomial of order n. Notice that n_1 is embodied in X_2. One can easily check how the eigenfunctions describe the superposition of a free motion of drift in direction 1 (plane wave with wave number k_1) and of the harmonic-oscillator-like behavior, due to the magnetic field, along 2. The displacement X_2 of the oscillator depends on both B and E. As for the form of the $\mathcal{E}_{n_1,n_2}$'s, it is naturally related to the geometrical interpretation of motion as composed of the same three contributions: the electrons move by drifting along direction 1 with velocity v_1 and are confined by the magnetic field to horizontal stripes of width $\sim 2\sqrt{2n_2+1}l_B$ centered at $x_2=X_2$ (potential energy eEX_2).

In order to prepare the appropriate frame for the construction of our model, it is convenient to bring the Hamiltonian $H_{\mathcal{L}}$, by means of the unitary transformation $\mathcal{U}=\mathcal{U}_1\mathcal{U}_2$, with

$$\mathcal{U}_1=\exp\left[-i\frac{p_1p_2}{m\omega_c\hbar}\right],\quad \mathcal{U}_2=\exp\left[i\frac{\lambda p_2}{\hbar}\right], \quad (5)$$

where $\lambda=mc^2E/eB^2$, into the form

$$\mathcal{H}=\mathcal{U}_2^\dagger\mathcal{U}_1^\dagger H_{\mathcal{L}}\mathcal{U}_1\mathcal{U}_2 =\frac{1}{2m}(p_2^2+m^2\omega_c^2x_2^2)+\frac{cE}{B}p_1-\tfrac{1}{2}mv_1^2, \quad (6)$$

in which the degrees of freedom 1 and 2 are completely decoupled. It is straightforward to check that the above form of the Hamiltonian directly provides the eigenvalues $\mathcal{E}_{n_1,n_2}$ in the form at the right-hand side of (3).

One observes now that when in (3) v_1 is of the order of the sound velocity v_s in the slab, the variation of one unit of n_1 implies a variation of the electron energy comparable to that of a single phonon. This suggests how, provided energy and momentum are conserved, if phonons (e.g., piezoelectric[5]) are present in the system, they can interact with the electrons, mutually exchanging energy and momentum. Energy and momentum conservation, on the other hand, impose selection rules on the quantum-number variations allowed when an electron undergoes a transition by absorbing or emitting a phonon. Let us consider the phonon wave-vector variation along direction 1,$\Delta k_{\rm ph}$, for transitions from, say, state (n_1,n_2) to state (n_1',n_2'). In view of momentum conservation, $\Delta k_{\rm ph}$ must be equal to $2\pi(n_1-n_1')/L_1$, which is the corresponding variation in the drift wave vector of the electron. On the other hand, due to energy conservation, $\Delta k_{\rm ph}=\omega_c(n_2'-n_2)/(v_1-v_s)$ is related to the energy change for an electron moving from the Landau level n_2 to n_2'.

Naturally, the interaction can take place provided[1] the wave functions of the states defined by (n_1,n_2) and (n_1',n_2'), respectively, have sufficient overlap. The corresponding condition furnishes—for each (n_1,n_2)—a boundary for the domain of accessible states in the Hilbert state space. On the other hand, whatever the difference (n_1-n_1') [or $(n_2'-n_2)$], one can always realize it by selecting an appropriate field E; in particular, if E is close to the value $E_*\equiv\chi B(v_1-v_s)/c$, any value of such difference can be achieved.

In a recent analysis of data on spatially localized breakdown of (near-)dissipationless quantum Hall effect into a set of dissipative states, Cage *et al.*[1] propose a suggestive way for modeling such an electron-phonon interaction, which consists in including the resulting transitions between noncontiguous Landau levels in the known quasi-elastic inter-Landau-level scattering picture of Eaves and Sheard.[3] The phenomenological analysis in Ref. 1 gives promising results and we adopt the mechanism proposed in our dynamical picture, by explicitly including in the Hamiltonian terms accounting for the above processes of phonon-electron interaction. This is, of course, straightforwardly done in the language of second quantization. Upon defining the customary creation (and annihilation) operators for the harmonic-oscillator degrees of freedom x_2 and p_2,

$$a^\dagger=\frac{1}{\sqrt{2m\omega_c\hbar}}(-ip_2+m\omega_cx_2),$$

Hamiltonian (6) is written

$$\mathcal{H}=\hbar\omega_c(a^\dagger a+\tfrac{1}{2})+v_1p_1-\frac{1}{2m}v_1^2. \quad (7)$$

The construction of the model proceeds quite naturally starting from this Hamiltonian, which is diagonal, in that the degrees of freedom here are in the normal form, with which one expects intuitively that the phonons should couple. By inclusion of the phonon field, the system total Hamiltonian becomes

$$\mathcal{H}_{ep}=\mathcal{H}+\mathcal{H}_{\rm ph}+\mathcal{H}_I, \quad (8)$$

where, naturally,

$$\mathcal{H}_{\rm ph}=\sum_j\hbar\Omega_j(c_j^\dagger c_j+\tfrac{1}{2}),$$

the sum running over all possible phonon modes labeled by j, whose creation and annihilation operators are $c_j^\dagger$ and c_j. $\mathcal{H}_I$, in a minimal-coupling scheme (i.e., neglecting higher-order multiphonon processes), is expected to have the general form

$$\mathcal{H}_I=\sum_j\hbar\Gamma_1^{(j)}(c_j^\dagger K_-+K_+c_j)+\sum_j\hbar\Gamma_2^{(j)}(a^\dagger c_j+c_j^\dagger a), \quad (9)$$

where the second term on the right-hand side manifestly describes energy exchange processes between phonons and the electronic Landau modes in direction 2, whereas the first term represents momentum exchange processes in direction 1 by means of so far undefined raising and lowering operators $K_\pm$ for the electron momentum degree of freedom along 1. The latter do not have a unique definition in terms of p_1, and several unequivalent choices are allowed. What is required here is a couple of operators whose action on the Landau states is defined by

$$K_\pm : |n_2,k_1\rangle \mapsto |n_2,k_1 \pm \chi\rangle \ , \tag{10}$$

where $|n_2,k_1\rangle$ is the ket associated with the state $\Phi_{n_1,n_2}(\mathbf{x})$ defined in (4). Operators of this sort are the usual ladder operators of any simple Lie algebra. The nonuniqueness comes from the fact that, while creation and annihilation operators for the transverse Landau mode $(a,a^\dagger)$ are naturally present in the Hamiltonian, creation or destruction of quanta $\hbar\chi$ of momentum in direction 1 has not appeared yet among the ingredients for the energy. Two possible choices for such raising and lowering operators have features suitable for a sound physical interpretation, both leading to an algebraic structure whereby successive calculations can be carried on with ease. The first is the most obvious choice $K_3 \propto p_1$, which, upon setting

$$K_3 = \frac{p_1}{\hbar\chi}, \quad K_\pm = e^{\pm i\chi x_1} \ , \tag{11}$$

generates an algebra [isomorphic with E(2)] related to the algebra of the magnetic translators group:[8,7]

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = 0 \ . \tag{12}$$

The second, less intuitive, possibility is resorting to the su(1,1) algebra, namely, choosing

$$K_3 = \frac{p_1}{\hbar\chi}, \quad K_\pm = e^{\pm i\chi x_1}(K_3 \mp l) \ , \tag{13}$$

where $l = -\frac{1}{2} + is$, $s \in \mathbb{R}$, and K_0 is the Casimir operator

$$K_0 = K_3^2 - \tfrac{1}{2}(K_- K_+ + K_+ K_-) = -(\tfrac{1}{4} + s^2) \ . \tag{14}$$

The commutation relations are

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3 \ . \tag{15}$$

In the first of the above schemes the Heisenberg equations of motion have the form

$$\begin{aligned} i\dot{a} &= \omega_c a + \sum_j \Gamma_2^{(j)} c_j \ , \\ i\dot{c}_j &= \Omega_j c_j + \Gamma_2^{(j)} a + \Gamma_1^{(j)} K_- \ , \\ i\dot{K}_- &= \epsilon K_- \ , \\ i\dot{K}_3 &= \sum_j \Gamma_1^{(j)} (c_j K_+ - K_- c_j^\dagger) \ , \end{aligned} \tag{16}$$

whereas for the su(1,1) scheme they read

$$\begin{aligned} i\dot{a} &= \omega_c a + \sum_j \Gamma_2^{(j)} c_j \ , \\ i\dot{c}_j &= \Omega_j c_j + \Gamma_2^{(j)} a + \Gamma_1^{(j)} K_- \ , \\ i\dot{K}_- &= \epsilon K_- + 2\sum_j \Gamma_1^{(j)} c_j K_3 \ , \\ i\dot{K}_3 &= \sum_j \Gamma_1^{(j)} (c_j K_+ - K_- c_j^\dagger) \ . \end{aligned} \tag{17}$$

The last equation in (17) is consistent with (14), provided K_0 is independent of time, namely, s is a constant.

One can easily check that the dynamical system (16) is integrable: the dynamics it describes is simply that of a collection of $n+2$ harmonic oscillators (a; c_j, $j=1,\ldots,n$, where n is the number of phonon modes; K_-), whose normal frequencies ν_k, $k=1,\ldots,n+2$ are the solutions of the secular equation

$$(\epsilon-\nu)\sum_{j=0}^{n}\{(\omega_c-\nu)(\Omega_j-\nu) - n[\Gamma_2^{(j)}]^2\}\prod_{\substack{m=1\\ m\neq j}}^{n}(\Omega_m-\nu)=0 \ , \tag{18}$$

whereas the motion of the variable K_3 can be obtained by quadrature from the last equation in (16). In this case, one cannot therefore expect any irregular dynamical behavior except at most resonances.

In the second case, described by Eq. (17), on the other hand, the equations of motion are nonlinear, due to the presence not only of the terms $c_j K_3$, but to two new features:

(i) $K_\pm$ generate K_3 by commutation,

(ii) K_3 is itself nonlinear in that it is related to $K_\pm$ by the conservation law expressed by the Casimir operator: $K_3^2 = K_0 + \frac{1}{2}\{K_-, K_+\}$.

It should be emphasized that Eqs. (17) are not only nonlinear, but nonintegrable as well. This can be directly checked, when a single phonon mode is present. In this case the equations of motion (the notation is self-explanatory) become

$$\begin{aligned} i\dot{a} &= \omega_c a + \Gamma_2 c \ , \\ i\dot{c} &= \Omega c + \Gamma_2 a + \Gamma_1 K_- \ , \\ i\dot{K}_- &= \epsilon K_- + 2\Gamma_1 c K_3 \ , \\ i\dot{K}_3 &= \Gamma_1 (c K_+ - K_- c^\dagger) \ , \end{aligned} \tag{19}$$

corresponding to the single-mode Hamiltonian

$$\begin{aligned} \mathcal{H}_{\rm SM} &= \mathcal{H} + \hbar\Omega(N_c + \tfrac{1}{2}) + \hbar\Gamma_1(c^\dagger K_- + K_+ c) \\ &\quad + \hbar\Gamma_2(a^\dagger c + c^\dagger a) \ , \\ \mathcal{H} &= \hbar\omega_c(N_a + \tfrac{1}{2}) + \hbar\epsilon K_3 - \frac{1}{2m}v_1^2 \ , \end{aligned} \tag{20}$$

where $N_a = a^\dagger a$, $N_c = c^\dagger c$, and $\epsilon = v_1\chi$. Equations (19) apparently couple seven degrees of freedom (recall that the variables a, c, K_- are not Hermitian, while K_3 is Hermitian). However, in view of (14), the last equation is not independent of the previous ones, and can be removed. If one does so, out of the three manifest Hermitian constants of motion in involution of (19),

$$\mathcal{H}_{SM}, \quad K_0, \quad \mathcal{J}=N_a+N_c+K_3 \ , \tag{21}$$

only two, $\mathcal{H}_{SM}$, and $\mathcal{J}$, survive, in that K_0 has been explicitly used to eliminate K_3. In other words, we are left with three non-Hermitian equations with only two constants of motion. This number of conserved quantities is not sufficient to guarantee integrability. Nonintegrability is directly related to the property that the dynamical algebra of $\mathcal{H}_{SM}$ is ∞ dimensional.

In view of the above discussion, and keeping in mind the purpose of describing a global dynamical behavior of the system characterized by chaotic features, we propose, first of all, to adopt in our model the su(1,1) alternative. Moreover, we notice that, since the dynamics expressed by the simplified single-phonon equations (19) has itself great inherent complexity yet allowing us some analytic discussion, whereas the complete form (17) gives no hope of being tractable other than numerically, we propose to model our system resorting simply to the one-phonon picture. Physically, such choice is motivated by two considerations. On the one hand, if the electric field E is close to $E_* \equiv \chi B(v_1-v_s)/c$, the selection rules allow indeed only processes which can be interpreted within a single-phonon scheme. On the other hand, for different values of E, one can easily devise the following procedure. It is a plausible assumption to set $\Gamma_2^{(j)}=\rho\Gamma_1^{(j)}$, $\forall j$, with some mode-independent phenomenological parameter ρ; moreover, since the physically relevant phonon modes are the acoustic modes, replacing $\Omega_j \to \langle \Omega_j \rangle \equiv \Omega$ (also independent of j) amounts to a simple hypothesis of isotropy of the speed of sound tensor. Upon adopting these assumptions, one can then define the average effective phonon

$$c \equiv \frac{1}{\Gamma_1}\sum_j \Gamma_1^{(j)} c_j \ , \tag{22}$$

with

$$\Gamma_1 \equiv \left[\sum_j (\Gamma_1^{(j)})^2\right]^{1/2} . \tag{23}$$

Obviously, $\Gamma_2=\rho\Gamma_1$. In terms of these new variables, and within the approximation scheme discussed above, Eqs. (17) manifestly identify with (19).

In the following sections we shall therefore discuss in detail the spectrum, dynamics, and physical properties of the system, in its model representation given by the Hamiltonian $\mathcal{H}_{SM}$ (20).

III. SYSTEM DYNAMICS: THE INTEGRABLE CASE $\Gamma_2=0$

A. The quantum picture

The quantum dynamics described by Eq. (19) is far from trivial because of the nonlinear terms. The latter obviously disappear if $\Gamma_1=0$ (even when Γ_2 remains $\neq 0$, i.e., ρ is infinitely large), in which case the system is trivially equivalent to three harmonic oscillators, one of which K_-, of frequency ϵ, is decoupled from the others, whereas a and c give rise to normal modes with hybridized frequencies

$$\nu_\pm = \tfrac{1}{2}\{\Omega+\omega_c \pm [(\Omega-\omega_c)^2+4\Gamma_2^2]^{1/2}\} \ .$$

There is another limiting case in which one of the modes is decoupled $[a=a(0)\exp(-i\omega_c t)]$, namely, $\rho=0$ (implying $\Gamma_2=0$), that has much more interest in that it preserves the nonlinearity of the system. In fact, the remaining equations of motion in this case read

$$\begin{aligned} i\dot{c} &= \Omega c + \Gamma_1 K_- \ , \\ i\dot{K}_- &= \epsilon K_- + 2\Gamma_1 c K_3 \ , \\ i\dot{K}_3 &= \Gamma_1 (cK_+ - K_- c^\dagger) \ . \end{aligned} \tag{24}$$

The dynamical equations (24) still exhibit two constants of motion (besides K_0, which once more allows us to get rid of K_3): $\mathcal{J}_* = N_c + K_3$, and the Hamiltonian

$$\mathcal{H}_* = \hbar[\Omega N_c + \epsilon K_3 + \Gamma_1(c^\dagger K_- + K_+ c)] + h_a \ , \tag{25}$$

where

$$h_a = \hbar[\omega_c(N_a+\tfrac{1}{2})+\tfrac{1}{2}\Omega] - v_1^2/2m \ ,$$

N_a denoting the Landau-level number. The (quantum) dynamical system described by (24) inherits from the classical case some of the features of integrability. Indeed, not only has it, as already mentioned, the right number of constants of motion, but it can be formally integrated by means of the exponential of the adjoint action of $it\mathcal{H}_*/\hbar$ on the variables. The explicit evaluation of such an action is extremely difficult, in that the dynamical algebra induced by $\mathcal{H}_*$ in terms of the variables c, K_-, their Hermitian conjugates, and K_3 is complex. In order to clarify this point with more detail, we perform therefore a symplectic reduction of the system, by introducing the following new bosonic variables:

$$Q \equiv c e^{-i\chi x_1}, \quad Q^\dagger \equiv e^{i\chi x_1} c^\dagger \ . \tag{26}$$

It is worth noticing that $Q^\dagger Q = N_c$. It is straightforward to check that the $Q, Q^\dagger$ satisfy the commutation relations

$$\begin{aligned} &[Q,Q^\dagger]=1, \quad [Q,\mathcal{J}_*]=0 \ , \\ &[Q,N_c]=Q, \quad [Q^\dagger,N_c]=-Q^\dagger \ , \\ &[Q,K_3]=-Q, \quad [Q^\dagger,K_3]=Q^\dagger \ . \end{aligned}$$

In terms of these new variables, the equations of motion (24) are reduced to those of a single nonlinear oscillator:

$$\begin{aligned} i\dot{Q} &= -\delta Q - \Gamma_1 Q^2 + \Gamma_1(\alpha - 2N_c) \ , \\ -i\dot{Q}^\dagger &= -\delta Q^\dagger - \Gamma_1 Q^{\dagger 2} + \Gamma_1(\bar{\alpha} - 2N_c) \ , \end{aligned} \tag{27}$$

corresponding to the Hamiltonian

$$\mathcal{H}_* = \hbar[\epsilon \mathcal{J}_* - \delta N_c + \Gamma_1(\bar{\alpha} - N_c)Q + \Gamma_1 Q^\dagger(\alpha - N_c)] + h_a \ , \tag{28}$$

where $\delta = \epsilon - \Omega$ and $\alpha = \mathcal{J}_* + l$. Recall that $K_3 = \mathcal{J}_* - N_c$.

The equations of motion (27) can be formally solved, upon denoting for simplicity by Q the initial condition for the operator Q at $t=0$, as

$$Q(t)=\exp\left[\text{ad}\; it\frac{\mathcal{H}_*}{\hbar}\right](Q)$$
$$\equiv\exp\left[it\frac{\mathcal{H}_*}{\hbar}\right]Q\exp\left[-it\frac{\mathcal{H}_*}{\hbar}\right]. \quad (29)$$

Indeed, (29) can be evaluated at any given order in t because the corresponding (infinite-dimensional) dynamical algebra $\mathcal{A}_{\text{dyn}}$ is isomorphic with the algebra (enveloping algebra of the Weil-Heisenberg algebra generated by Q and $Q^\dagger$) generated by the set of operators

$$W_{ab}\equiv Q^{\dagger a}N_c^b,\quad W_{de}^\dagger\equiv N_c^dQ^e, \quad (30)$$

whose commutation relations

$$[W_{a,b},W_{c,d}]=\sum_{z=\min(b,d)}^{b+d}f_{ab,cd}^{(z)}W_{a+c,z},$$
$$[W_{a,b},W_{c,d}^\dagger]=\sum_{j=0}^{b+d}\sum_{i=0}^{c}\tilde{f}_{ab,cd}^{(i,j)}W_{a-c,i+j}$$

are known (the structure constants f and $\tilde{f}$ are explicitly given in the Appendix).

In terms of the operators W the Hamiltonian reads

$$\mathcal{H}_*=\hbar\left[\epsilon\mathcal{J}_*-\frac{\delta}{2}(W_{0,1}+W_{0,1}^\dagger)\right.$$
$$\left.+\Gamma_1(\alpha W_{1,0}+\bar{\alpha}W_{1,0}^\dagger-W_{1,1}-W_{1,1}^\dagger)\right]+h_a. \quad (31)$$

The possibility of evaluating to any given order the multiple commutators in (29),

$$[\mathcal{H}_*,[\mathcal{H}_*,\ldots,[\mathcal{H}_*,Q]\cdots]]\in\mathcal{A}_{\text{dyn}},$$

can be thought of as analogous to the classical case of integrability in which, after reduction to quadratures of the solutions of the equations of motion, the integrals can be evaluated to any given order in t by some recursive algorithm. If $\mathcal{A}_{\text{dyn}}$ were finite dimensional, we should have a situation similar to the case when the integrals can be solved in closed form.

B. The semiclassical picture

In view of the difficulties inherent in the quantum case pointed out in the previous section, we focus our attention now on the semiclassical version of the dynamical problem (27) and (28). In order to derive such a version, we observe first that the Weil-Heisenberg algebra generated by $Q,Q^\dagger$ leads, in the customary way,[9] to the coherent states

$$|z\rangle=e^{(-1/2)|z|^2}e^{zQ^\dagger}|0\rangle,\quad z\in\mathbb{C}, \quad (32)$$

with the property that $Q|z\rangle=z|z\rangle$, where $|0\rangle$ denotes the vacuum ($Q|0\rangle=0$). It is known that the quantum dynamics of the system is mapped on the classical Hamiltonian flow on the coherent-state manifold (here $\mathbb{C}$) defined by the Hamiltonian function[10]

$$\mathcal{H}_*(z,\bar{z})\equiv\langle z|\mathcal{H}_*|z\rangle$$
$$=\hbar[\epsilon\mathcal{J}_*-\delta|z|^2+\Gamma_1(\bar{\alpha}-|z|^2)z$$
$$+\Gamma_1(\alpha-|z|^2)\bar{z}]+h_a, \quad (33)$$

which is manifestly quite similar to (28) except for the fact that it is written as if the noncommutativity of Q (and $Q^\dagger$) with N_c had been neglected. The classical equations of motion corresponding to (33) are identical with (27) provided one replaces in the latter Q with z, $Q^\dagger$ with $\bar{z}$, and N_c with $|z|^2$. For example, $y\equiv|z|^2$ is a solution to the equation, in the rescaled time $\tau\equiv\Gamma_1t$,

$$\frac{d^2y}{d\tau^2}=6y^2-4(\mathcal{J}_*+W_1)y+2[\mathcal{J}_*(\mathcal{J}_*-1)-W_2], \quad (34)$$

where

$$W_1\equiv\mathcal{J}_*+\tfrac{1}{2}-\left[\frac{\delta}{2\Gamma_1}\right]^2 \quad (35)$$

and

$$W_2\equiv K_0+\frac{\delta}{2\Gamma_1^2}\left[\Omega\mathcal{J}_*-\frac{\mathcal{H}_*-h_a}{\hbar}\right]. \quad (36)$$

From Eq. (34), recalling the simple relation between N_c and K_3, one immediately obtains the corresponding semiclassical equation for K_3:

$$\frac{d^2K_3}{d\tau^2}=-6K_3^2+4W_1K_3+2W_2. \quad (37)$$

This equation can be thought of as the equation of motion in one dimension for a particle of mass 1 with potential energy

$$U(K_3)=2K_3^3-2W_1K_3^2-2W_2K_3+U_0,$$

where U_0 is a constant that we select, in view of the discussion to follow, equal to $\frac{4}{27}W_1^3+\frac{2}{3}W_1W_2$. The cubic potential U suggests that, depending on the initial conditions, one can have both confined solutions, oscillating within the potential well, and solutions escaping to $-\infty$. A straightforward analysis shows that positivity of the control parameter $g_2=\frac{4}{3}(W_1^2+3W_2)$ guarantees the existence of the minimum of U. In turn $g_2>0$ implies, resorting to the explicit expressions (35) and (36) for W_1 and W_2, an upper bound for $s^2=-K_0-\frac{1}{4}$, which—as we shall show later—provides a constraint on the range of possible initial conditions. It is worth noticing that we have a separatrix between the two regimes corresponding to the initial condition $K_3(0)=\frac{1}{3}W_1$, $K_3'(0)=0$, for which the system moves from the starting point to the maximum of U in an infinitely long time (the prime denotes derivative with respect to τ).

This equation is formally identical with the quantum one (in which both K_3 and $\mathcal{H}_*$ should be considered as noncommutating operators). The dynamics of the system in normal conditions can be expected to be related to the class of oscillating confined solutions. However, the existence of the hyperbolic points at the extremes of the separatrix (which will be discussed in detail in Sec. IV) hints at the possibility that the onset of chaotic motion may take place there. We argue that, at least in the framework of the simplified model adopted, which—we recall—leads to a cylindrical configuration space, the diverging solutions play no physical role.

Equation (37) has the solution, in terms of the Weierstrass $\wp$ function,

$$K_3(\tau)=\tfrac{1}{3}W_1-\wp(\tau+\tau_0;g_2,g_3) , \tag{38}$$

where the parameters τ_0 and g_3 allow us to fix the initial conditions. Since the equation for $\wp$,

$$\frac{d^2\wp}{d\tau^2}=6\wp^2-\tfrac{1}{2}g_2 , \tag{39}$$

is equivalent to[11,12]

$$\left[\frac{d\wp}{d\tau}\right]^2=4\wp^3-g_2\wp-g_3 , \tag{40}$$

one easily checks that g_3 is given by

$$g_3=[K_3'(0)]^2-\tfrac{4}{27}[W_1-3K_3(0)]^3+\tfrac{1}{3}g_2[W_1-3K_3(0)] , \tag{41}$$

whereas τ_0, which fixes the origin of time, can in principle be obtained from $K_3(0)$ by resorting to (38). We point out that the motivation for focusing our attention on the variable K_3 was simply the feature that one can reconduct its dynamics to a well-known equation [see (39)]. On the other hand, once the time dependence of K_3 is known, inserting it in the second of Eqs. (24) and reducing the system of the two remaining equations of the first order thus obtained to two uncoupled equations of the second order, the latter also have a form which is known. For instance, the resulting equation for the variable $b=c\exp\{-(i/2)(\omega_c+\epsilon)t\}$ turns out to be a Lamé equation[13]

$$\frac{d^2b}{dt^2}+[\lambda_1\wp(\Gamma_1 t)+\lambda_2]b=0 , \tag{42}$$

where

$$\lambda_1\equiv-2\Gamma_1^2,\quad \lambda_2\equiv\tfrac{1}{4}(\omega_c-\epsilon)^2+\tfrac{2}{3}W_1\Gamma_1^2 . \tag{43}$$

Equation (38), together with g_2 as given above and (41), provide the solution of the equation of motion (37). In view of the complexity of the dependence of $\wp$ on g_2 and g_3, it is convenient to express the same solution in a different form. First, recalling the relation between the Weierstrass function and the Jacobi elliptic functions,[12] we write

$$K_3(\tau)=\tfrac{1}{3}W_1+\tfrac{1}{3}\gamma^2(1+\kappa^2)-\gamma^2\kappa^2\mathrm{sn}^2(\gamma\tau+\phi_0;\kappa) , \tag{44}$$

where sn denotes the Jacobi elliptic "sine," $\phi_0=\gamma\tau_0$, γ and κ related to the parameters g_2 and g_3 by

$$g_2=\tfrac{4}{3}\gamma^4(1-\kappa^2+\kappa^4) ,$$
$$g_3=\tfrac{4}{27}\gamma^6(1+\kappa^2)(2-\kappa^2)(1-2\kappa^2) . \tag{45}$$

Upon introducing the amplitude $\mathcal{A}\equiv\frac{1}{2}\kappa^2\gamma^2$, Eq. (44) becomes

$$K_3(\tau)=\tfrac{1}{3}W_1+\tfrac{2}{3}\mathcal{A}^2(1+\kappa^{-2})-2\mathcal{A}\,\mathrm{sn}^2\left[\frac{\sqrt{2\mathcal{A}}}{\kappa}\tau+\phi_0;\kappa\right] , \tag{46}$$

where now

$$\kappa^2=2[1+\sqrt{3}(g_2/4\mathcal{A}^2-1)^{1/2}]^{-1} .$$

The manifold of initial conditions whereby $g_2>0$—which guarantees, as already mentioned, stability of the solutions for K_3—is determined by

$$4\{K_3(0)+\tfrac{1}{4}[|c(0)|^2+\tfrac{1}{2}(\lambda^2+1)]\}^2+\tfrac{3}{4}[|c(0)|^2+\tfrac{1}{2}][|c(0)|^2+\tfrac{1}{2}-\lambda^2]>3|K_+(0)+\tfrac{1}{2}\lambda\bar{c}(0)|^2 ,$$

where $\lambda\equiv\delta/\Gamma_1$. The last condition describes, for fixed $c(0)$, in the space ($\sim\mathbb{R}^3$) spanned by $K_3(0)$, $K_+(0)$, and $K_-(0)$, the domain external to the two-sheeted hyperboloid defined by setting = instead of > in the above inequality.

In conclusion we signal two interesting limiting cases: (i) $\kappa=1$, i.e., $\mathcal{A}=\frac{1}{4}\sqrt{3g_2}$, where

$$K_3(\tau)=\tfrac{1}{3}W_1+\tfrac{4}{3}\mathcal{A}-\mathcal{A}\tanh^2(\sqrt{2\mathcal{A}}\tau+\phi_0) , \tag{47}$$

which describes the relaxationlike separatrix dynamics, and (ii) $\kappa\to0$, which gives, with vanishing amplitude $\mathcal{A}\to0$, the harmonic solution at the bottom of the potential well,

$$K_3(\tau)\approx\tfrac{1}{3}W_1+\tfrac{1}{6}\sqrt{3g_2}-2\mathcal{A}\sin^2[(\tfrac{3}{4}g_2)^{1/4}\tau+\phi_0] . \tag{48}$$

IV. SEMICLASSICAL DISCUSSION OF THE CASE $\Gamma_2\neq0$

In the previous Sec. III B we were able to show that, in the semiclassical approach, the special case $\Gamma_2=0$ of the equations of motion (19) is integrable. In this section we intend to examine the phase portrait of the system when the interaction Γ_2 is switched on. It will be shown that the associated Hamiltonian flow has several critical points, whose number and position are controlled by the parameters which specify the constants of motion [$\mathcal{J},\mathcal{H}_{SM}$, and K_0 (or s); (21)] as well as by the detunings between the various bosonic modes ($\lambda\equiv\delta/\Gamma_1$, $\Lambda\equiv\Delta/\Gamma_1$, where $\Delta=\epsilon-\omega_c$), and the ratio $\xi\equiv\Gamma_2/\Gamma_1$. Our analysis is aimed to show that there exist both elliptic and hyperbolic critical points. The very existence of the latter suggests, in view of the quadratic nonlinearity of the equations of motion, the possibility of the onset of chaos. Moreover, variation of the control parameters induces bifurcations of the unstable critical points, allowing the switching on of known mechanisms of onset of chaos through the intersection of different branches of separatrixes.[14]

The simplest scheme to perform the stability analysis is that provided, together with the variables $Q, Q^\dagger$, by

$$A \equiv a e^{-i\chi x_1}, \quad A^\dagger \equiv e^{i\chi x_1} a^\dagger , \tag{49}$$

in terms of which $N_a \equiv A^\dagger A = a^\dagger a$. $A, A^\dagger$ satisfy the commutation relations

$$[A, A^\dagger] = 1, \quad [A, \mathcal{J}] = 0 ,$$
$$[A, N_a] = A, \quad [A^\dagger, N_a] = -A^\dagger ,$$

and obviously commute with $Q, Q^\dagger$, so that also $[Q, \mathcal{J}] = 0$.

The Hamiltonian, reformulated in these new variables, reads

$$\begin{aligned}\mathcal{H}_{\rm SM} = \hbar\{ & \epsilon\mathcal{J} - \delta N_c - \Delta N_a \\ & + \Gamma_1[(\bar\sigma - N_a - N_c)Q + Q^\dagger(\sigma - N_a - N_c)] \\ & + \Gamma_2(AQ^\dagger + QA^\dagger)\} + h_a ,\end{aligned} \tag{50}$$

where $\sigma \equiv \mathcal{J} + l$. The corresponding semiclassical equations of motion can be obtained by the natural extension of (32) and (33) to the two-mode (Q and A) case:

$$\begin{aligned} i\dot z &= -\delta z - \Gamma_1 z^2 + \Gamma_1(\sigma - |\zeta|^2 - 2|z|^2) + \Gamma_2 \zeta , \\ -i\dot{\bar z} &= -\delta \bar z - \Gamma_1 \bar z^2 + \Gamma_1(\bar\sigma - |\zeta|^2 - 2|z|^2) + \Gamma_2 \bar\zeta , \\ i\dot\zeta &= -\Delta\zeta + \Gamma_2 z - \Gamma_1 \zeta(z + \bar z) , \\ -i\dot{\bar\zeta} &= -\Delta\zeta + \Gamma_2 \bar z - \Gamma_1 (z + \bar z)\bar\zeta , \end{aligned} \tag{51}$$

where $\zeta \in \mathbb{C}$ is the label of the A-mode coherent state. On the other hand, Eqs. (51) can be derived, by the canonical variational method, imposing stationarity of the action

$$S = \int_{t_0}^{t} d\tau \langle \eta | \left[i\hbar \frac{\partial}{\partial t} - \mathcal{H}_{\rm SM} \right] | \eta \rangle . \tag{52}$$

Here $|\eta\rangle$ is the composite coherent state for a, c, and K_- (namely, the point η represents a state in the six-dimensional semiclassical phase space). Symplectic reduction follows, via the constant of the motion $\mathcal{J}$, to the four-dimensional manifold (z, ζ).

Within the above description of the relevant dynamical degrees of freedom, assuming that—consistently with the semiclassical approximation scheme—averages can be identified with expectation values in the restricted states $|\mathcal{J}; \eta\rangle$, the experimentally observed quantity j_L (proportional to V_1 if $\rho_L \neq 0$ is assumed to be constant) is $\propto \langle \dot x_1 \rangle$:

$$\frac{V_1}{L_1 \rho_L} = j_L = \frac{e n_e}{m} \langle p_1 \rangle \equiv e n_e \langle \dot x_1 \rangle , \tag{53}$$

where n_e is the macroscopic electron surface density. $\langle \dot x_1 \rangle$, in turn, is readily seen to be, up to the unitary transformations (5), $\propto \mathrm{Re}\langle A e^{i\chi x_1} \rangle$. Explicit calculation of the latter shows that it is an analytic function of z, ζ, and β, where β is the label of the su(1,1) coherent states of p_1 and x_1.

Stability analysis follows the customary scheme: one first locates the set of fixed points, solutions [$P_\kappa \equiv (z_\kappa, \zeta_\kappa)$, where κ labels the different solutions] of the equations one obtains by setting the right-hand sides of (51) equal to zero; one then linearizes Eqs. (51) in the neighborhood of such points, and finally—by the Routh-Hurwitz[15] criterion applied to the dynamical matrix thus obtained—one discusses the stability features of the fixed points. Such a scheme, in spite of the relative simplicity of its statement, is in practice a complex problem in that both the search for the fixed points and the stability criterion (which in the present case refers to a matrix of rank 4) imply a proliferation of cases. The equations for P_κ define the submanifold (discrete subset) of the configuration space $\Sigma \sim \mathbb{C} \times \mathbb{C}$ for whose points $\dot z = 0$ and $\dot\zeta = 0$. Such a submanifold is the intersection of manifolds $\mathcal{M}$, diffeomorphic to the complex plane $\mathbb{C}$, defined by

$$\zeta = \frac{\xi z}{\Lambda + z + \bar z} , \tag{54}$$

and $\mathcal{K}$, obtained by setting the right-hand side of the first of Eqs. (51) equal to zero. The P_k's are embedded in both $\mathcal{M}$ and $\mathcal{K}$ and their projections over the complex plane z are given (one should recall that $\sigma \equiv \mathcal{J} - \frac{1}{2} + is = \eta + is \in \mathbb{C}$) by the equation

$$\begin{aligned}\frac{1}{|\Lambda + z + \bar z|^2} \{ & -\xi^2 z(\Lambda + z) \\ & + |\Lambda + z + \bar z|^2 [2|z|^2 + z^2 + \lambda z - \sigma]\} = 0 ,\end{aligned} \tag{55}$$

obtained by inserting (54) in the equation for $\mathcal{K}$. Upon setting $x = \mathrm{Re} z$, $y = \mathrm{Im} z$, Eq. (55) is equivalent to the sys-

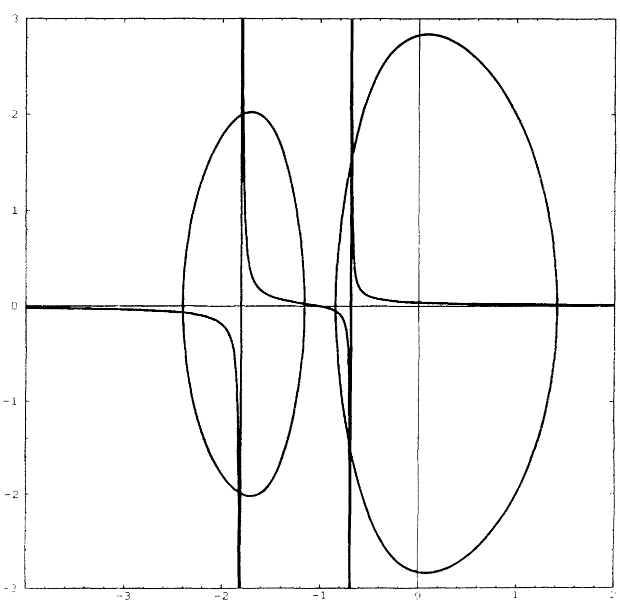

FIG. 1. Projection over the complex z plane of the manifold $\mathcal{K} \cap \mathcal{M}$, diffeomorphic with the system phase space in the neighborhood of the fixed points $\{P_\kappa\}$: generic case, s small, $\kappa = 1, \ldots, 8$.

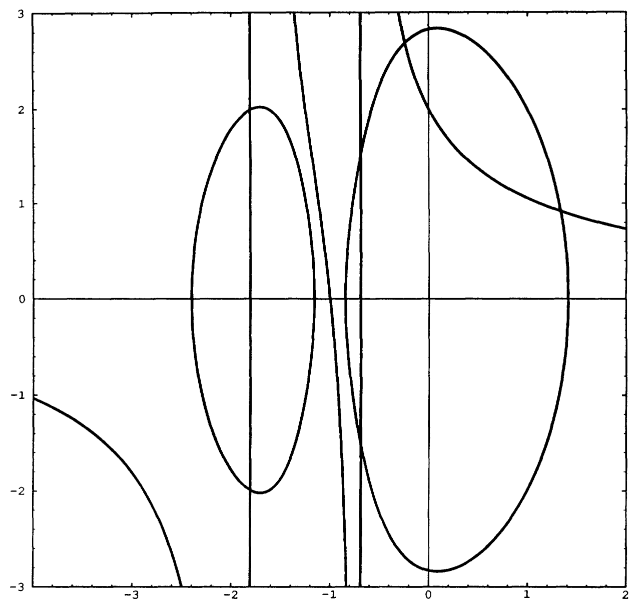

FIG. 2. Same as Fig. 1, with s larger, such that $\kappa=1,\ldots,2$.

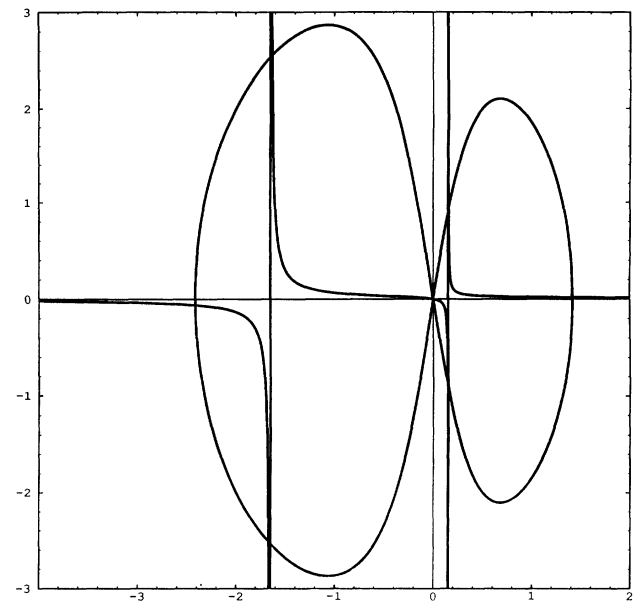

FIG. 3. Same as Figs. 1 and 2, resonating case $\kappa=1,\ldots,6$.

tem (describing the intersection of two curves—one rational, the other irrational two branched—in the plane z)

$$C_1 : y=\frac{s(\Lambda+2x)}{(\Lambda+2x)(\lambda+2x)-\xi^2}\ , \tag{56}$$

$$C_2 : y=\pm\sqrt{\mathcal{N}(x)/\mathcal{D}(x)}\ , \tag{57}$$

where

$$\mathcal{N}(x)=(\Lambda+2x)^2[\eta-x(\lambda+3x)]+\xi^2 x(\Lambda+x)\ ,$$

$$\mathcal{D}(x)=\xi^2+(\Lambda+2x)^2\ .$$

Figures 1, 2, and 3 illustrate the three generic situations one encounters in this analysis. For s small enough, one finds that C_2, consisting of two disjoint lobes, is intersected eight times by C_1, which is characterized by two vertical asymptotes at $x_{\pm}=\frac{1}{4}(-\lambda\pm\sqrt{\lambda^2+4\xi^2})$ (Fig. 1).

Upon increasing s one finds a sequence of bifurcations at which pairs of intersections successively disappear (one after the other: an elliptic fixed point disappears and two hyperbolic points merge into a single one), and one moves from the generic case of eight to six, four, two, and zero solutions. Figure 2 shows a situation in which six solutions were lost.

An example of some interest from the point of view of physics is the resonating case, in which $\Lambda=0$ corresponding to $\omega_c=\epsilon$. In that case, the two lobes of C_1 coalesce into a unique self-intersecting curve, exhibiting a double point at the origin. Figure 3 shows the same example as Fig. 1, with $\delta=0$. It should be noted that the degeneracy of points P_1 and P_5 (which have moved to the origin) is only apparent in the z-plane projection: indeed, the two corresponding fixed points are separated in Σ (and, of course, in $\mathcal{M}$) in that their ζ coordinates are, respectively, $\zeta_{1,5}=\frac{1}{2}(\xi\mp\sqrt{\xi^2+4\eta})\in\mathbb{R}$.

Linearization around the fixed points P_κ in the form

$$z=z_\kappa+\delta z_\kappa,\quad \zeta=\zeta_\kappa+\delta\zeta_\kappa\ , \tag{58}$$

[one should recall that, due to (54), $\zeta_\kappa=\xi z_\kappa/(\Lambda+2\,\mathrm{Re}z_\kappa)$] gives, as customary, the local equations of motion as

$$\frac{d}{dt}|\delta\mathbf{x}_\kappa\rangle=\mathbf{M}_\kappa|\delta\mathbf{x}_\kappa\rangle\ , \tag{59}$$

where $|\delta\mathbf{x}_\kappa\rangle$ is a column vector of components $\delta z_\kappa,\delta\bar{z}_\kappa,\delta\zeta_\kappa,\delta\bar{\zeta}_\kappa$, and the dynamical matrix

$$\mathbf{M}_\kappa=\Gamma_1\begin{bmatrix}-(\lambda+4\,\mathrm{Re}z_\kappa) & -2z_\kappa & (\xi-\bar{\zeta}_\kappa) & -\zeta_\kappa\\ 2\bar{z}_\kappa & \lambda+4\,\mathrm{Re}z_\kappa & \bar{\zeta}_\kappa & -(\xi-\zeta_\kappa)\\ (\xi-\zeta_\kappa) & -\zeta_\kappa & -(\Lambda+2\,\mathrm{Re}z_\kappa) & 0\\ \bar{\zeta}_\kappa & -(\xi-\zeta_\kappa) & 0 & \Lambda+2\,\mathrm{Re}z_\kappa\end{bmatrix}\ . \tag{60}$$

The eigenvalues $\{\mu_\kappa^{(\alpha)},\ \alpha=1,\ldots,4\}$ of $\mathbf{M}_\kappa$ are solutions of the secular equation

$$\det(\mathbf{M}-\mu_\kappa\mathbf{I})=\sum_{j=0}^{4}c_j\mu_\kappa^{4-j}\ . \tag{61}$$

One can easily check that $c_0=1$ and $c_1=c_3=0$, whereas

$$c_2=(\lambda+4\,\mathrm{Re}z_\kappa)^2+(\Lambda+2\,\mathrm{Re}z_\kappa)^2+\frac{2\Lambda\xi^2}{\Lambda+2\,\mathrm{Re}z_\kappa}-4|z_\kappa|^2\ , \tag{62}$$

$$c_4=(\lambda+4\,\mathrm{Re}z_\kappa)^2(\Lambda+2\,\mathrm{Re}z_\kappa)^2-4|z_\kappa|^2\left[(\Lambda+2\,\mathrm{Re}z_\kappa)^2+2\xi^2(\Lambda+2\,\mathrm{Re}z_\kappa)+\xi^2\frac{\lambda+2\,\mathrm{Re}z_\kappa}{\Lambda+2\,\mathrm{Re}z_\kappa}\right]$$
$$-2\xi^2\Lambda(\lambda+4\,\mathrm{Re}z_\kappa)+\frac{\Lambda^2\xi^2}{(\lambda+2\,\mathrm{Re}z_\kappa)^2}\ . \tag{63}$$

The Routh-Hurwitz criterion states that the fixed point P_κ is respectively stable or unstable depending on whether both $c_2\geq 0$ and $c_4\geq 0$, or at least one of them is <0. Figure 4 shows the projection on the complex z plane of the phase portrait in $\mathcal{M}$ of the dynamical system (51), in the case when it is symmetric with respect to the Rez axis ($s=0$). Such a phase portrait clearly shows that the points P_{2n}, $n=1,\ldots,4$, are hyperbolic. Among the remaining fixed points, P_3 and P_5 are maxima, whereas P_1 and P_7 are minima of the manifold defined by $\mathcal{H}_{\mathrm{SM}}=\mathrm{const}$.

In view of the invariant manifold theorem,[14] in some sufficiently small neighborhood $\mathcal{U}$ of the hyperbolic fixed points P_{2n}, $n=1,\ldots,4$, there exist local stable and unstable manifolds $\mathcal{N}^s(P_{2n})$ and $\mathcal{N}^u(P_{2n})$ for the linearized dynamical system (59). Since the branches of the separatrix have two distinct endpoints which are both hyperbolic, such a property has deep global repercussions on the complexity of the dynamics, in that it gives rise to homoclinic tangles (see Fig. 5), which are responsible for the occurrence of "chaotic orbits." This can be straightforwardly checked by noticing how the extended dynamics in $\mathcal{U}$ has, at least, both quadratic and cubic nonlinearities.

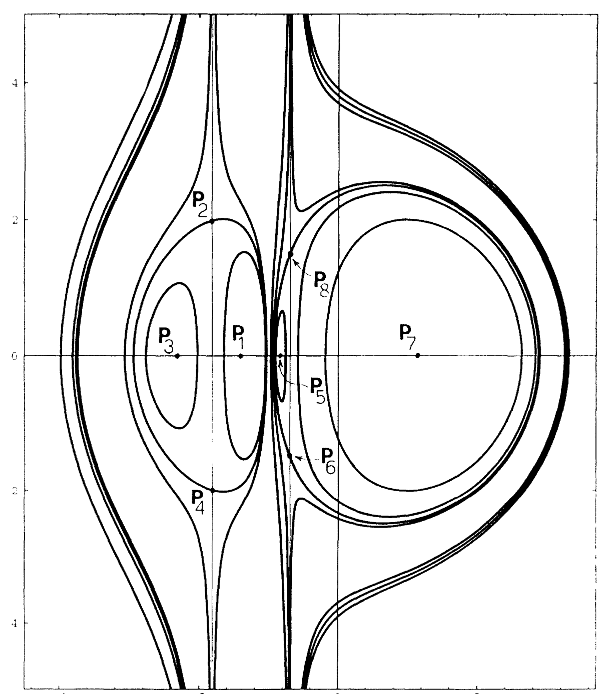

FIG. 4. Phase-portrait ($\sim\mathcal{M}$) projection, for $s=0$, showing that points P_{2n}, $n=1,\ldots,4$, are hyperbolic, and points P_{2n+1}, $n=0,\ldots,3$, are elliptic.

From the physical point of view, since we already pointed out that the experimentally observed quantity V_1 is proportional to an analytic function of z and ζ, we argue that—at least at the microscopic level characteristic of our description—chaotic behavior of the latter variables should imply (possibly intermittent) chaos of the former.

V. CONCLUSIONS

In this paper, taking inspiration from a set of experiments[2,1] confirming the appearance of an undoubtedly random behavior of the longitudinal voltage V_1 vs time in samples for QHE measurements at high current, and for magnetic fields lying at the edges of a Hall plateau (see Refs. 1 and 2), we argued that coupling between the Landau motion of the electrons and the phonons, either thermal or possibly generated by the sample elastic deformations induced by the Hall voltage V_2, might lead to such chaotic dynamics. We have shown how even a very simple modeling of the phenomenon resorting to only a single "average" phonon and to a coupling mechanism based on a suitable nonlinear realization of the emerging dynamical algebra su(1,1) (referred to in the mathematical literature as the continuous principal series[6]) does indeed provide a semiclassical picture of the electron and phonon dynamics whose phase portrait and Hamiltonian guarantee the existence of chaotic orbits.

It is, in fact, interesting that the dynamical system to which one reconducts the equations of motion in the frame of the proposed model turns out to be integrable in

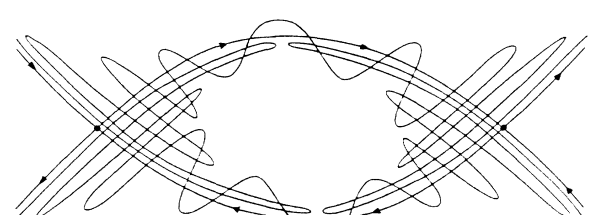

FIG. 5. The homoclinic tangle which arises in proximity to the hyperbolic points.

the limit when one of the electron-phonon couplings (more precisely, that between the electron Landau oscillating mode and the phonons) is switched off. This implies, on the one hand, that use of the KAM theorem could furnish meaningful information on the range of couplings for which one can—at least in the nonresonant cases—save the system's invariant tori, and still reconduct the solutions to the equations of motion to quadratures. On the other hand, however, the integrable case itself has highly structured solutions, given in terms of elliptic functions, whose behavior bifurcates between confined and unbounded (unphysical) orbits, depending on the control parameter provided just by the label s of the dynamical algebra representation. It is worth pointing out as well the formal but also physical analogy of the separatrix dynamics that one gets in the integrable case [see Eq. (47)] with the equations for a laser.[16] Finally, the integrable case has an intrinsic interest in that it not only provides an example of a physically originated integrable dynamical system, possibly relevant for application to a wealth of problems in quantum optics, but also could lead—through a more conventional approach—to a systematic perturbative analysis of the nonintegrable case.

We stress once more that the equations of motion on which we focused our attention in the paper are related to dynamical variables connected with the relevant quantum numbers [$K_3=p_1/\hbar\chi$ with n_1, proportional to p_1; $N_a=a^\dagger a=A^\dagger A$ with n_2, being $A=a\exp(-i\chi x_1)$]. Such quantum numbers, n_1 and n_2, are just those specifying the Hall spectrum. According to the QUILLS scheme discussed in the Introduction, the dynamical transitions allowed in the model consist in changes of both the Landau quantum number n_2, and the level degeneracy number n_1. Our dynamical analysis, on the other hand, shows how the representative of the system's quantum state, in the space $\mathcal{S}=\{(n_1,n_2)\}$, can be thought of as chaotically resonating—in a way that, of course, we know, in general, only in the semiclassical picture—among a finite set of points (depending on B_c and the effective phonon mode adopted) in $\mathcal{S}$. The important feature of the approach presented is that, in order to have chaotic response, not only should phonons be present and interact with electrons, but they should be coupled with the proper electronic Landau mode, whereas the coupling with the transverse mode does not play a role in this effect.

In this paper no thorough numerical analysis of the system dynamics is reported, in that the main aim was just to demonstrate that the effect of the proposed interactions could indeed give rise to the expected chaotic motion. We expect that more realistic models could be originated from that discussed here, taking into account features that have been neglected here, in particular, the existence of many phonon modes. Indeed, one can argue that increasing the number of phonon modes should increase the number of points in $\mathcal{S}$ accessible from any given initial condition, and therefore increase the number of quantum states among which the system intermittently or chaotically resonates. In the one-phonon picture the semiclassical phase portrait exhibits—as we showed in Sec. IV—only two minima, and it is in the neighborhood of these two points that the system orbits oscillate, assuming a random structure if they are close enough to the separatrixes, and thus producing a two-basin strange attractor. It is plausible, and we conjecture, that the consideration of several independent phonon modes would generate a (strange) attractor whose much richer structure presents many basins, which might explain the multilevel noisy form reported in Ref. 2.

ACKNOWLEDGMENTS

The authors gratefully acknowledge stimulating discussions with G. Rinaudo, G. Marullo-Reedtz, and coworkers who originated their interest in the subject.

APPENDIX

The structure constants appearing in the Weyl-Heisenberg enveloping algebra commutators

$$[W_{a,b},W_{c,d}]=\sum_{z=\min(b,d)}^{b+d} f^{(z)}_{ab,cd}W_{a+c,z}\ ,$$

$$[W_{a,b},W^\dagger_{c,d}]=\begin{cases}\displaystyle\sum_{j=0}^{b+d}\sum_{i=0}^{c}\tilde{f}^{(i,j)}_{ab,cd}W_{a-c,i+j} & \text{if } a>c\\ \displaystyle\sum_{j=0}^{b+d}\sum_{i=0}^{a}\tilde{f}^{(i,j)}_{cd,ab}W^\dagger_{c-a,i+j} & \text{if } a\le c\ ,\end{cases}$$

are

$$f^{(z)}_{ab,cd}=\left[\binom{b}{z-d}\Theta(z-d)-\binom{d}{z-b}\Theta(z-b)\right]c^{b+d-z}\ ,$$

$$\tilde{f}^{(i,j)}_{ab,cd}=\binom{b+d}{j}S_c^{(i)}(-c)^{b+d-j}$$
$$-\Theta(j-b)\binom{d}{j-b}\mathcal{S}^{(i)}_{c,a}(a-c)^{b+d-j}\ ,$$

where $S_c^{(i)}$ is the Stirling number of the first kind (see Ref. 11), $\Theta(x)$ is the Heaviside function [here we assume $\Theta(0)=1$] and $\mathcal{S}^{(i)}_{c,a}$ is defined as

$$\mathcal{S}^{(i)}_{c,a}=\sum_{m=i}^{c}S_c^{(m)}\binom{m}{i}a^{m-i}\ .$$

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