Nontrivial solutions of variational inequalities. The degenerate case

Nontrivial solutions of variational inequalities. The degenerate case

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Author's version
Published in: *Topological Methods in Nonlinear Analysis* 18 2 (2001), 303-319

Abstract

We consider a class of asymptotically linear variational inequalities. We show the existence of a nontrivial solution under assumptions which allow the problem to be degenerate at the origin.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 with $g(0) = 0$ and linear growth at infinity. The existence of nontrivial solutions u to the semilinear elliptic problem

$$\begin{cases} \Delta u + g(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

was first studied by Amann and Zehnder in [1] by means of Conley index. The main result was then refind by Chang, Lazer and Solimini [3, 13], using Morse theory, and Saccon [15], again by means of Conley index. The key assumptions are that there exists

$$g'(\infty) := \lim_{|s|\to\infty} \frac{g(s)}{s}$$

and that the quadratic forms

$$\begin{aligned} Q_0(u) &= \int_\Omega \left(|Du|^2 - g'(0)u^2\right) dx\,, \\ Q_\infty(u) &= \int_\Omega \left(|Du|^2 - g'(\infty)u^2\right) dx \end{aligned}$$

have different index in $H_0^1(\Omega)$.

*Supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40% – 1999).
1991 *Mathematics Subject Classification.* Primary: 35J85, Secondary: 58E05.
Key words and phrases: Variational inequalities, nonsmooth critical point theory, Morse theory.

More recently, the result has been also extended to variational inequalities by Saccon [16] and to quasilinear equations by Corvellec, Degiovanni, and Lancelotti [6, 12]. In the first case, one also considers a closed convex subset K of $H_0^1(\Omega)$ with $0 \in K$ and looks for nontrivial solutions $u \in K$ of the variational inequality

$$\int_\Omega [DuD(v-u) - g(u)(v-u)]\, dx \geq 0 \qquad \forall v \in K\,. \tag{1.1}$$

It is interesting to remark that the constraint K can induce the existence of nontrivial solutions also when $g(s) = \lambda s$ with $\lambda \in \mathbb{R}$. However, if for instance

$$K = \left\{u \in H_0^1(\Omega) :\ \varphi_1 \leq u \leq \varphi_2\right\}$$

with $\varphi_1 < 0 < \varphi_2$, the assumptions considered in [16] require the quadratic form Q_0 to be nondegenerate at the origin, a restriction which is not needed for semilinear equations (see [13]).

Our purpose is to prove the existence of nontrivial solutions to (1.1) without assuming such a nondegeneracy at 0. While the approach of [16] was based on Conley index, we find it more convenient to use Morse theory. More precisely, since the precence of the constraint K makes the problem nonsmooth, we take advantage of the extension of Morse theory to continuous functionals developed in [5].

Our main result is theorem 2.2, where we prove the existence of a nontrivial solution to (1.1) in the degenerate case, even if the family of constraints K considered is not so wide as in [16] (see assumption (2.1)). Since our approach is different, we also treat in theorem 2.4 the nondegenerate case already considered in [16].

As in [13], the first step in the proof is to find a saddle point u of the functional $f : K \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx - \int_\Omega G(u)\, dx\,, \qquad G(s) = \int_0^s g(t)\, dt\,,$$

with a suitable information about its critical groups. This is done by an adaptation of Rabinowitz saddle theorem (see theorem 4.2). Then the main point is to obtain estimates about the critical groups of f at the origin. Since 0 is possibly degenerate, we adapt to our nonsmooth setting some ideas of the generalized Morse lemma (see [4, 9, 14]). After that, it is possible to show that $u \neq 0$, obtaining the existence of a nontrivial solution.

The author wishes to thank Marco Degiovanni for helpful discussions.

2 Statement of the main results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, $\varphi_1 : \Omega \to [-\infty, 0]$ and $\varphi_2 : \Omega \to [0, +\infty]$ be two functions such that φ_1 is quasi-upper semicontinuous and φ_2 is quasi-lower semicontinuous. We consider the convex set

$$K = \left\{u \in H_0^1(\Omega) :\ \ \varphi_1(x) \leq \tilde{u}(x) \leq \varphi_2(x) \text{ for q.e. } x \text{ in } \Omega\right\}\,,$$

where $\tilde{u}$ is a quasi-continuous representative of u. We also consider $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that:

(*a*) the function $\{s \longmapsto g(x,s)\}$ is of class C^1 for a.e. $x \in \Omega$ and the function $\{x \longmapsto g(x,s)\}$ is measurable for every $s \in \mathbb{R}$;

(*b*) $g(x,0) = 0$ for a.e. $x \in \Omega$;

(*c*) there exists $b \in L^{\frac{n}{2}}(\Omega)$ such that for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$

$$|D_s g(x,s)| \leq b(x);$$

(*d*) for a. e. $x \in \Omega$ there exists

$$D_s g(x,\infty) := \lim_{|s|\to\infty} \frac{g(x,s)}{s}.$$

Let us consider the following subsets of Ω:

$$\begin{aligned} F_1^0 &= \{x \in \Omega : \varphi_1(x) = 0\}, \\ F_1^\infty &= \{x \in \Omega : \varphi_1(x) = -\infty\}, \\ F_2^0 &= \{x \in \Omega : \varphi_2(x) = 0\}, \\ F_2^\infty &= \{x \in \Omega : \varphi_2(x) = +\infty\}. \end{aligned}$$

Moreover, let us consider the following closed linear subspaces of $H_0^1(\Omega)$:

$$\begin{aligned} H_0 &= \{u \in H_0^1(\Omega) : \widetilde{u} = 0 \text{ for q.e. } x \text{ in } F_1^0 \cap F_2^0\}, \\ H_\infty &= \{u \in H_0^1(\Omega) : \widetilde{u} = 0 \text{ for q.e. } x \text{ outside } F_1^\infty \cup F_2^\infty\}, \\ H_0' &= \{u \in H_0^1(\Omega) : \widetilde{u} = 0 \text{ for q.e. } x \text{ in } F_1^0 \cup F_2^0\}, \\ H_\infty' &= \{u \in H_0^1(\Omega) : \widetilde{u} = 0 \text{ for q.e. } x \text{ outside } F_1^\infty \cap F_2^\infty\}. \end{aligned}$$

Finally, let us denote by $\left(\lambda_k^{(0)}\right)$, $\left(\mu_k^{(0)}\right)$ the eigenvalues of the linear operator $-\Delta - D_s g(x,0)$ respectively in H_0 and H_0', and by $\left(\lambda_k^{(\infty)}\right)$, $\left(\mu_k^{(\infty)}\right)$ the eigenvalues of the linear operator $-\Delta - D_s g(x,\infty)$ respectively in H_∞ and H_∞' (Δ is the Laplace operator and eigenvalues are repeated according to multiplicity).

Remark 2.1 *Since $H_0' \subseteq H_0$ and $H_\infty' \subseteq H_\infty$, we have that*

$$\forall k \in \mathbb{N}: \quad \lambda_k^{(0)} \leq \mu_k^{(0)}, \quad \lambda_k^{(\infty)} \leq \mu_k^{(\infty)}.$$

Theorem 2.2 *Assume that*

$$(\varphi_1(x) \neq 0 \text{ and } \varphi_2(x) \neq 0) \implies (\varphi_1(x) = -\infty \text{ and } \varphi_2(x) = +\infty) \qquad \text{q.e. in } \Omega. \tag{2.1}$$

and that $\mu_k^{(\infty)} < 0 < \lambda_{k+1}^{(\infty)}$ for some k. Moreover, suppose there exists $h \neq k$ such that either

$$h < k \quad \textit{and} \quad \mu_h^{(0)} \leq 0 < \lambda_{h+1}^{(0)}$$

or

$$h > k \quad \text{and} \quad \mu_h^{(0)} < 0\,.$$

Then there exists a nontrivial solution u of the semilinear variational inequality

$$\begin{cases} u \in K\,, \\ \displaystyle\int_\Omega DuD(v-u)\,dx - \int_\Omega g(x,u)(v-u)\,dx \;\geq\; 0 \quad \forall v \in K\,. \end{cases} \tag{2.2}$$

Remark 2.3 *Assumption* (2.1) *is satisfied, for instance, if K has the form*

$$K = \left\{ u \in H_0^1(\Omega) : \; \tilde{u}(x) \geq 0 \text{ for q.e. } x \text{ in } E_1 \text{ and } \tilde{u}(x) \leq 0 \text{ for q.e. } x \text{ in } E_2 \right\}\,,$$

where E_1, E_2 are two subsets of Ω.

The novelty of theorem 2.2 is that we allow the cases $h < k$ with $\mu_h^{(0)} = 0 < \lambda_{h+1}^{(0)}$ and $h > k$ with $\mu_h^{(0)} < 0 = \lambda_{h+1}^{(0)}$, which were excluded in [16].

The next result has been proved also in [16].

Theorem 2.4 *Assume that there exist $h \neq k$ such that*

$$\mu_k^{(\infty)} < 0 < \lambda_{k+1}^{(\infty)}\,,$$

$$\mu_h^{(0)} < 0 < \lambda_{h+1}^{(0)}\,.$$

Then there exists a nontrivial solution u of the semilinear variational inequality

$$\begin{cases} u \in K\,, \\ \displaystyle\int_\Omega DuD(v-u)\,dx - \int_\Omega g(x,u)(v-u)\,dx \;\geq\; 0 \quad \forall v \in K\,. \end{cases}$$

3 Background in nonsmooth critical point theory

In this section we recall from [5, 7, 8] some basic facts that will be needed in the following. Let X denote a metric space endowed with the metric d and $f : X \to \mathbb{R}$ a continuous function. Moreover, let $\mathrm{B}_r(u)$ be the open ball of radius $r > 0$ centered at $u \in X$. For every $c \in \mathbb{R}$ let us set

$$f^c = \{u \in X : \; f(u) \leq c\}\,.$$

Definition 3.1 *For every $u \in X$ let us denote by $|\mathrm{d}\, f|\,(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : \mathrm{B}_\delta(u) \times [0, \delta] \to X$ with*

$$\forall v \in \mathrm{B}_\delta(u)\;,\; \forall t \in [0,\delta] \quad : \qquad d(\mathcal{H}(v,t), v) \;\leq\; t\,,$$

$$\forall v \in \mathrm{B}_\delta(u)\;,\; \forall t \in [0,\delta] \quad : \qquad f(\mathcal{H}(v,t)) \;\leq\; f(v) - \sigma t\,.$$

The extended real number $|\mathrm{d}\, f|\,(u)$ is called the weak slope *of f at u.*

It is easily seen that the function $|\mathrm{d}\, f| : X \to [0, +\infty]$ is lower semicontinuous. Moreover, if X is an open subset of a normed space and f a function of class C^1, it turns out that $|\mathrm{d}\, f|\,(u) = \|f'(u)\|$ for every $u \in X$.

Let us point out that the above notion has been independently introduced also in [11], while a variant can be found in [10].

Definition 3.2 *An element $u \in X$ is said to be* a critical point *of f, if $|\mathrm{d}\, f|\,(u) = 0$. A real number c is said to be* a critical value *of f, if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be* a regular value *of f.*

Definition 3.3 *Let c be a real number. The function f is said to satisfy* the Palais - Smale condition at level c *($(PS)_c$ for short), if every sequence (u_h) in X with $|\mathrm{d}\, f|\,(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence (u_{h_k}) converging in X (any cluster point of (u_h) is a critical point of f by the lower semicontinuity of $|\mathrm{d}\, f|$).*

Definition 3.4 *Let $\mathbb{K}$ be a field. For $u \in X$ and $c = f(u)$ set $C_q(f; u) = H^q(f^c, f^c \setminus \{u\})$, where $H^q(A, B)$ denotes the $q-$th cohomology group of the pair (A, B), with coefficients in $\mathbb{K}$ (here we consider the Alexander-Spanier cohomology [17]). The vector space $C_q(f; u)$ is called* the $q-$th critical group *of f at u.*

Because of the excision property, for every neighbourhood U of u we have

$$C_q(f; u) \approx H^q(f^c \cap U, (f^c \cap U) \setminus \{u\})\,.$$

Therefore $C_q(f; u)$ depends only on the behaviour of f near u.

Theorem 3.5 *Let X be a Banach space which splits into a direct sum $X = X^- \oplus X^+$ with $\dim X^- = m < +\infty$ and X^+ closed. Let K be a closed subset of X and $f : K \to \mathbb{R}$ a continuous function. Assume there exist $a, b \in \mathbb{R}$ with $a < b$ and $r > 0$ such that*

$$X^- \cap \overline{\mathrm{B}_r\,(0)} \subseteq K\,,$$

$$\max_{X^- \cap \partial\, \mathrm{B}_r(0)} f < a < \inf_{K \cap X^+} f \quad \textit{and} \quad \max_{X^- \cap \overline{\mathrm{B}_r(0)}} f < b\,.$$

Suppose also that f satisfies the $(PS)_c$ condition for any $c \in [a, b]$.

Then f admits a critical value in $[a, b]$; more precisely, either f admits infinitely many critical points in $f^{-1}([a, b])$, or there exists a critical point u of f in $f^{-1}([a, b])$ such that $C_m(f, 0) \neq \{0\}$.

Proof. Consider the homomorphisms, induced by inclusion maps,

$$H^m(X, X \setminus X^+) \longrightarrow H^m(f^b, f^a) \longrightarrow H^m(X^- \cap \overline{\mathrm{B}_r\,(0)}, X^- \cap \partial\, \mathrm{B}_r\,(0))\,.$$

Since the inclusion map $(X^- \cap \overline{\mathrm{B}_r\,(0)}, X^- \cap \partial\, \mathrm{B}_r\,(0)) \to (X, X \setminus X^+)$ induces an isomorphism in cohomology, the homomorphism

$$H^m(f^b, f^a) \longrightarrow H^m(X^- \cap \overline{\mathrm{B}_r\,(0)}, X^- \cap \partial\, \mathrm{B}_r\,(0))$$

is surjective. On the other hand, it is well-known that $H^m(X^- \cap \overline{\mathrm{B}_r\,(0)}, X^- \cap \partial\, \mathrm{B}_r\,(0)) \neq \{0\}$. It follows that $H^m(f^b, f^a) \neq \{0\}$.

From [5, Theorem 4.4] the assertion follows. ∎

4 The saddle point

In this section let us consider K and g as in sect. 2. Let $f_1 : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$f_1(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx\,,$$

where $G(x,s) = \displaystyle\int_0^s g(x,t)\,dt$, and let $f : K \to \mathbb{R}$ be the restriction of f_1 to K.

Let also $Q_\infty : H_0^1(\Omega) \to \mathbb{R}$ be the quadratic form defined by

$$Q_\infty(u) = \int_\Omega |Du|^2\,dx - \int_\Omega D_s g(x,\infty)u^2\,dx\,.$$

In the following, $\|\cdot\|_{1,2}$ and $\|\cdot\|_{-1,2}$ will denote the standard norms in $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Proposition 4.1 *The following facts hold:*

(*a*) *K is a convex closed subset of $H_0^1(\Omega)$ containing 0;*

(*b*) *the functional f_1 is of class C^2 with $f_1'(0) = 0$;*

(*c*) *for every $u \in K$ there exists $\eta \in H^{-1}(\Omega)$ such that $\|\eta\|_{-1,2} \le |\mathrm{d}\,f|\,(u)$ and*

$$\int_\Omega DuD(v-u)\,dx - \int_\Omega g(x,u)(v-u)\,dx \ge \langle \eta, v-u\rangle \quad \forall v \in K\,.$$

Proof. Assertions (*a*) and (*b*) are well-known. Assertion (*c*) follows from [8, Theorem (2.11) and Proposition (2.10)]. ∎

Theorem 4.2 *Let us assume that there exists $k \in \mathbb{N}$ such that*

$$\mu_k^{(\infty)} < 0 < \lambda_{k+1}^{(\infty)}\,.$$

Then, if f has only a finite number of critical points, there exists a critical point u of f such that $C_k(f,u) \ne \{0\}$.

Proof. Let X^- be a maximal subspace of H'_∞ where Q_∞ is negative definite. Since $\mu_k^{(\infty)} < 0 < \lambda_{k+1}^{(\infty)} \le \mu_{k+1}^{(\infty)}$, we have $\dim X^- = k$. Let us set

$$\widehat{X}^+ = \left\{ u \in H_\infty : \int_\Omega DuDv\,dx - \int_\Omega D_s g(x,\infty)uv\,dx \;=\; 0 \quad \forall v \in X^- \right\}\,,$$

so that $H_\infty = X^- \oplus \widehat{X}^+$. Moreover, Q_∞ is positive definite on $\widehat{X}^+$. In fact, consider for a contradiction $u \in \widehat{X}^+$, $u \ne 0$, such that $Q_\infty(u) \le 0$. It follows that Q_∞ is negative semidefinite on $(X^- \oplus \mathrm{span}\,(u)) \subseteq H_\infty$ with $\dim\,(X^- \oplus \mathrm{span}\,(u)) = k+1$: a contradiction, because $\lambda_{k+1}^{(\infty)} > 0$.

Now we have the decomposition $H_0^1(\Omega) = X^- \oplus X^+$, where $X^+ = \left(\widehat{X}^+ \oplus H_\infty^\perp\right)$ and $H_\infty^\perp$ is the orthogonal of H_∞ in $H_0^1(\Omega)$ with respect to the standard scalar product.

We want to apply theorem 3.5 to the functional $f : K \to \mathbb{R}$. First of all we have that f is bounded from below on $K \cap X^+$. In fact, by contradiction, let us consider a sequence (u_h) in $K \cap X^+$ such that $f(u_h) \to -\infty$. Since f is bounded on bounded subsets, we have that $\|u_h\|_{1,2} \to +\infty$. Let $u_h = \rho_h w_h$, with $\rho_h = \|u_h\|_{1,2}$ and $\|w_h\|_{1,2} = 1$. Up to a subsequence, (w_h) is weakly convergent to some $w \in X^+$. Since

$$\forall t > 0 : \quad t\varphi_1 \le w_h \le t\varphi_2 \quad \text{q.e. in } \Omega$$

eventually as $h \to \infty$, we also have $w \in K_\infty := \bigcap_{t>0}(tK)$. On the other hand $K_\infty \subseteq H_\infty$, so that $w \in \widehat{X}^+$. Since Q_∞ is positive definite on $\widehat{X}^+$, $\|w_h\|_{1,2} = 1$ and

$$\lim_h \int_\Omega \frac{G(x, \rho_h w_h)}{\rho_h^2}\, dx = \int_\Omega D_s g(x, \infty) w^2\, dx\,,$$

it follows

$$\liminf_h \left[\frac{1}{2}\int_\Omega |Dw_h|^2\, dx - \int_\Omega \frac{G(x, \rho_h w_h)}{\rho_h^2}\, dx\right] > 0\,.$$

In particular, we have

$$\lim_h f(u_h) = \lim_h \rho_h^2 \left[\frac{1}{2}\int_\Omega |Dw_h|^2\, dx - \int_\Omega \frac{G(x, \rho_h w_h)}{\rho_h^2}\, dx\right] = +\infty\,,$$

whence a contradiction.

In a similar (and simpler) way, one can show that

$$\lim_{\substack{\|u\|_{1,2}\to\infty \\ u\in X^-}} f_1(u) = -\infty\,,$$

so that there exists $r > 0$ such that

$$\max_{u\in X^- \cap \partial \mathrm{B}_r(0)} f_1 < \inf_{u\in K\cap X^+} f_1\,.$$

Since $X^- \subseteq H'_\infty \subseteq K$, we trivially have $X^- \cap \overline{\mathrm{B}_r(0)} \subseteq K$.

Now let us prove that f satisfies $(PS)_c$ for every $c \in \mathbb{R}$. Let (u_h) be a sequence in K such that $f(u_h) \to c$ and $|\mathrm{d}\, f|\,(u_h) \to 0$. First of all, let us prove that (u_h) is bounded. By contradiction, let $\|u_h\|_{1,2} \to +\infty$. By proposition 4.1, there exists a sequence (η_h) in $H^{-1}(\Omega)$ with $\eta_h \to 0$ and

$$\int_\Omega Du_h D(v - u_h)\, dx - \int_\Omega g(x, u_h)(v - u_h)\, dx \ge \langle \eta_h, v - u_h\rangle \quad \forall v \in K\,. \tag{4.1}$$

Let $u_h = \rho_h w_h$, with $\rho_h = \|u_h\|_{1,2}$ and $\|w_h\|_{1,2} = 1$. As in the previous step, up to a subsequence (w_h) is weakly convergent to some $w \in K_\infty \subseteq H_\infty$. Moreover, we have that

$$\int_\Omega Dw_h D(v - w_h)\, dx - \int_\Omega \frac{g(x, \rho_h w_h)}{\rho_h}(v - w_h)\, dx \ \ge \langle \frac{\eta_h}{\rho_h}, v - w_h\rangle \quad \forall v \in K_\infty\,. \tag{4.2}$$

Going to the limit as $h \to \infty$, we get

$$\int_\Omega DwD(v-w)\,dx - \int_\Omega D_s g(x,\infty)w(v-w)\,dx \ \geq 0 \quad \forall v \in K_\infty\,. \tag{4.3}$$

On the other hand, choosing $v = 0$ in (4.2) we obtain

$$\int_\Omega |Dw_h|^2\,dx \leq \int_\Omega \frac{g(x,\rho_h w_h)}{\rho_h} w_h\,dx + \langle \frac{\eta_h}{\rho_h}, w_h\rangle\,,$$

whence $w \neq 0$.

Let $w = w_- + w_+$ with $w_- \in X^-$ and $w_+ \in \widehat{X}^+$. Since $X^- \subseteq H'_\infty \subseteq K_\infty$, we may choose $v = 2w_- = (w_- - w_+) + w$ in (4.3), obtaining

$$\int_\Omega D(w_- + w_+)D(w_- - w_+)\,dx - \int_\Omega D_s g(x,\infty)(w_- + w_+)(w_- - w_+)\,dx \ \geq 0\,,$$

hence

$$\int_\Omega |Dw_-|^2\,dx - \int_\Omega D_s g(x,\infty)(w_-)^2\,dx \ \geq \int_\Omega |Dw_+|^2\,dx - \int_\Omega D_s g(x,\infty)(w_+)^2\,dx\,.$$

Since Q_∞ is negative definite on X^- and positive definite on $\widehat{X}^+$, we have that $w = 0$ and a contradiction follows.

Being bounded, (u_h) is weakly convergent, up to a subsequence, to some $u \in K$. If we choose $v = u$ in (4.1), we obtain

$$\int_\Omega |Du_h|^2\,dx \leq \int_\Omega \left[Du_h Du - g(x,u_h)(u-u_h)\right]\,dx \ - \langle \eta_h, u - u_h\rangle\,.$$

It follows

$$\limsup_h \int_\Omega |Du_h|^2\,dx \leq \int_\Omega |Du|^2\,dx\,,$$

so that (u_h) is strongly convergent to u.

Since $\dim X^- = k$, by theorem 3.5 there exists a critical point u of f such that $C_k(f,u) \neq \{0\}$. ■

5 Critical groups for q large enough

In this section we consider a reflexive Banach space X, a convex closed subset K of X with $0 \in K$ and a continuous function $f : K \to \mathbb{R}$. Let us assume that X splits into a direct sum $X = V \oplus W$, with $\dim V = m < +\infty$ and W closed, and denote by P_V and P_W the associated projections. Moreover, let us suppose that:

- (i) for every sequence (u_h) in K weakly convergent to u with $\lim_h f(u_h) = f(u)$, one has that (u_h) is strongly convergent to u;
- (ii) for every $u \in K$, the function f is strictly convex on $K \cap (u + W)$;

(*iii*) there exists a continuous function $\psi : V \to \mathbb{R}$ such that $\{u \longmapsto f(u) + \psi(P_V u)\}$ is convex on K;

(*iv*) $V \subseteq \overline{\bigcup_{t>0} (tK)}$;

(*v*) $f(w) \geq f(0)$ for every $w \in W \cap K$.

Theorem 5.1 *We have $C_q(f, 0) = \{0\}$ for every $q \geq m + 1$.*

Theorem 5.2 *Under the previous assumptions, let us suppose that there exists $\delta > 0$ such that $V \cap \mathrm{B}_\delta(0) \subseteq K$ and $f(v) \leq f(0)$ for every $v \in V \cap \mathrm{B}_\delta(0)$.*

Then

$$C_q(f, 0) \approx \begin{cases} \{0\} & \text{if } q \neq m\,, \\ \mathbb{K} & \text{if } q = m\,. \end{cases}$$

The section will be devoted to the proof of these results.

Theorem 5.3 *Assume that K is also bounded.*

Then, for every $v \in P_V(K)$, the function $\{w \longmapsto f(v + w)\}$ has one and only one minimum point in $(K - v) \cap W$.

Moreover, if we denote by $\Phi(v)$ such a minimum point, then the following properties hold:

(*a*) $0 \in \mathrm{int}_V(P_V(K))$ *and the map* $\Phi : \mathrm{int}_V(P_V(K)) \to W$ *is continuous with* $\Phi(0) = 0$*;*

(*b*) *the function* $\varphi : \mathrm{int}_V(P_V(K)) \to \mathbb{R}$ *defined by* $\varphi(v) = f(v + \Phi(v))$ *is continuous;*

(*c*) $C_q(\varphi, 0) \approx C_q(f, 0)$ *for every* q.

Proof. Suppose, for a contradiction, that $0 \notin \mathrm{int}_V(P_V(K))$. Since $\dim V < +\infty$, there exists $\eta \in V^* \setminus \{0\}$ such that $\langle \eta, v \rangle \leq 0$ for any $v \in P_V(K)$. It follows $\langle \eta, P_V u \rangle \leq 0$ for any $u \in K$, hence for any $u \in \overline{\bigcup_{t>0} (tK)}$. From assumption (*iv*) we deduce that $\langle \eta, v \rangle \leq 0$ for any $v \in V$, which is clearly impossible.

By assumption (*ii*), for every $v \in P_V(K)$, the function $\{w \longmapsto f(v + w)\}$ has one and only one minimum point $\Phi(v)$ in $(K - v) \cap W$. From assumption (*v*) it follows that $\Phi(0) = 0$.

Let us define a function $\hat{f} : K \to \mathbb{R}$ by

$$\hat{f}(u) = f(u) + \psi(P_V u)\,.$$

By assumption (*iii*), $\hat{f}$ is convex and continuous. Moreover, for every $v \in P_V(K)$, we have that $\Phi(v)$ is also the unique minimum point of the function $\{w \longmapsto \hat{f}(v + w)\}$ in $(K - v) \cap W$. Define $\hat{\varphi} : P_V(K) \to \mathbb{R}$ by $\hat{\varphi}(v) = \hat{f}(v + \Phi(v)) = \varphi(v) + \psi(v)$.

We claim that $\hat{\varphi}$ is convex and lower semicontinuous. Actually, let $v_0, v_1 \in P_V(K)$ and let $t \in [0,1]$. Since $\hat{f}$ is convex, we have

$$\begin{aligned}
\hat{\varphi}((1-t)v_0 + tv_1) &= \hat{f}\left((1-t)v_0 + tv_1 + \Phi((1-t)v_0 + tv_1)\right) \leq \\
&\leq \hat{f}\left((1-t)v_0 + tv_1 + (1-t)\Phi(v_0) + t\Phi(v_1)\right) \leq \\
&\leq (1-t)\hat{f}\left(v_0 + \Phi(v_0)\right) + t\hat{f}\left(v_1 + \Phi(v_1)\right) = \\
&= (1-t)\hat{\varphi}(v_0) + t\hat{\varphi}(v_1)\,.
\end{aligned}$$

Now, let (v_h) be a sequence in $P_V(K)$ converging to v. Up to a subsequence, $(\Phi(v_h))$ is weakly convergent to some $w \in W$ with $v + w \in K$. It follows

$$\hat{\varphi}(v) = \hat{f}(v + \Phi(v)) \leq \hat{f}(v + w) \leq \liminf_h \hat{f}(v_h + \Phi(v_h)) = \liminf_h \hat{\varphi}(v_h)\,.$$

Being convex and lower semicontinuous, $\hat{\varphi}$ is continuous on $\mathrm{int}_V\left(P_V(K)\right)$. Therefore, if (v_h) is convergent to v in $\mathrm{int}_V\left(P_V(K)\right)$, we have that $(\Phi(v_h))$ is weakly convergent to $\Phi(v)$. From assumption (i) it follows that $(\Phi(v_h))$ is strongly convergent to $\Phi(v)$. At the end, also φ is continuous on $\mathrm{int}_V\left(P_V(K)\right)$.

Finally, let us prove property (c). Without loss of generality, we may assume that $f(0) = \varphi(0) = 0$. If we set

$$U = \mathrm{int}_V\left(P_V(K)\right) + W\,,$$

$$M = \left\{v + \Phi(v) : \, v \in \mathrm{int}_V\left(P_V(K)\right)\right\}\,,$$

then $\{v \longmapsto v + \Phi(v)\}$ is a homeomorphism of $\mathrm{int}_V\left(P_V(K)\right)$ onto M. Since $\Phi(0) = 0$, the pair $(\varphi^0, \varphi^0 \setminus \{0\})$ is homeomorphic to the pair $\left(\left(f_{|M}\right)^0, \left(f_{|M}\right)^0 \setminus \{0\}\right)$. In particular,

$$C_q(\varphi, 0) = H^q\left(\varphi^0, \varphi^0 \setminus \{0\}\right) \approx H^q\left(\left(f_{|M}\right)^0, \left(f_{|M}\right)^0 \setminus \{0\}\right)\,.$$

On the other hand, since $\{w \mapsto f(v+w)\}$ is convex, the map $\eta : (f^0 \cap U, (f^0 \cap U) \setminus \{0\}) \times [0,1] \to (f^0 \cap U, (f^0 \cap U) \setminus \{0\})$ defined by

$$\eta(u,t) = P_V u + (1-t)P_W u + t\Phi(P_V u)\,,$$

is a strong deformation retraction of $(f^0 \cap U, (f^0 \cap U) \setminus \{0\})$ in $\left(\left(f_{|M}\right)^0, \left(f_{|M}\right)^0 \setminus \{0\}\right)$. In particular,

$$H^q\left(\left(f_{|M}\right)^0, \left(f_{|M}\right)^0 \setminus \{0\}\right) \approx H^q\left(f^0 \cap U, (f^0 \cap U) \setminus \{0\}\right) \approx C_q(f, 0)\,.$$

and assertion (c) follows. ■

Now we may prove the main results of this section.

Proof of theorem 5.1. By substituting K with $K \cap \overline{\mathrm{B}_1(0)}$, we may assume that K is also bounded. Let $\varphi : \mathrm{int}_V\left(P_V(K)\right) \to \mathbb{R}$ be as in theorem 5.3. We know that

$$\forall q : \quad C_q(f, 0) \approx C_q(\varphi, 0)\,.$$

Since $\varphi^0 \setminus \{0\} \subseteq \varphi^0 \subseteq V$ with $\dim V = m$, it follows that $C_q(\varphi, 0) = \{0\}$ whenever $q \geq m+1$. ■

Proof of theorem 5.2. Again, we may assume K to be bounded. Let $\Phi : \mathrm{int}_V\,(P_V(K)) \to W$ and $\varphi : \mathrm{int}_V\,(P_V(K)) \to \mathbb{R}$ be as in theorem 5.3. For every $v \in K \cap \mathrm{B}_\delta\,(0)$, we have

$$\varphi(v) = f(v + \Phi(v)) \leq f(v) \leq f(0) = \varphi(0)\,.$$

Therefore

$$\begin{aligned} C_q(f,0) &\approx C_q(\varphi,0) \approx H^q\left(\varphi^0 \cap \mathrm{B}_\delta\,(0)\,, (\varphi^0 \cap \mathrm{B}_\delta\,(0)) \setminus \{0\}\right) = \\ &= H^q\left(V \cap \mathrm{B}_\delta\,(0)\,, (V \cap \mathrm{B}_\delta\,(0)) \setminus \{0\}\right) \end{aligned}$$

and the assertion follows. ■

6 Critical groups for q small enough

In this section we consider a Banach space X, a convex closed subset K of X with $0 \in K$ and a continuous function $f : K \to \mathbb{R}$. Let us assume that X splits into a direct sum $X = V \oplus W$, with $\dim V = m < +\infty$ and W closed. Moreover, let us suppose that:

(*i*) there exists $\delta > 0$ such that

$$(V \cap \mathrm{B}_\delta\,(0)) + (K \cap W \cap \mathrm{B}_\delta\,(0)) \subseteq K\,;$$

(*ii*) for every $w \in K \cap W \cap \mathrm{B}_\delta\,(0)$, the function $\{v \mapsto f(v+w)\}$ is strictly concave on $V \cap \mathrm{B}_\delta\,(0)$.

Theorem 6.1 *We have $C_q(f,0) = \{0\}$ for every $q \leq m-1$.*

The section will be devoted to the proof of this result.

Lemma 6.2 *Let S be a symmetric subset of V and C be a convex subset of W such that $0 \in C$ and $S + (K \cap C) \subseteq K$.*

Then $S + (K \cap C) = K \cap (S + C)$. In particular, it is

$$(V \cap \mathrm{B}_\delta\,(0)) + (K \cap W \cap \mathrm{B}_\delta\,(0)) = K \cap \left[(V \cap \mathrm{B}_\delta\,(0)) + (W \cap \mathrm{B}_\delta\,(0))\right].$$

Proof. Let $v + w \in K$ with $v \in S$ and $w \in C$. If $\hat{w} \in K \cap C$, we have $-v + \hat{w} \in K$, hence $(w+\hat{w})/2 \in K \cap C$. Starting from $0 \in K \cap C$, we find by induction that $(1-2^{-k})w \in K \cap C$ for any $k \in \mathbb{N}$. It follows that $w \in K$, whence $v + w \in S + (K \cap C)$. The opposite inclusion is obvious. ■

Lemma 6.3 *Let $V = \operatorname{span}(e) \oplus Z$ with $e \neq 0$ and assume that $f(v) \leq f(0)$ for every $v \in V \cap \mathrm{B}_\delta(0)$.*

Then there exist $r > 0$ and $\rho \in]0, r]$ such that:

(a) *for every $u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)})$ and every $t \in [-r, r]$ we have $te + u \in K$;*

(b) *for every $u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)})$ the function $\{t \mapsto f(te + u)\}$ has one and only one maximum point $\vartheta(u)$ on $[-r, r]$ with $|\vartheta(u)| < r$;*

(c) *the function $\vartheta : (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \to \mathbb{R}$ is continuous with $\vartheta(0) = 0$;*

(d) *the function $\varphi : (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \to \mathbb{R}$ defined by $\varphi(u) = f(\vartheta(u)e + u)$ is continuous;*

(e) *we have*

$$\forall z \in Z \cap \mathrm{B}_\rho(0) : \quad \varphi(z) \leq \varphi(0)$$

and for every $w \in K \cap W \cap \mathrm{B}_\rho(0)$ the function $\{z \mapsto \varphi(z + w)\}$ is strictly concave on $Z \cap \mathrm{B}_\rho(0)$.

Proof. Let $r \in]0, \delta[$ be such that $\|te + z\| < \delta$ whenever $|t| \leq r$ and $z \in Z \cap \overline{\mathrm{B}_r(0)}$. From assumption (i), it follows that

$$\forall u \in (Z \cap \overline{\mathrm{B}_r(0)}) + (K \cap W \cap \overline{\mathrm{B}_r(0)}), \forall t \in [-r, r] : \quad te + u \in K .$$

By assumption (ii) we have that $f(-re) < f(0)$ and $f(re) < f(0)$. Therefore, there exists $\rho \in]0, r]$ such that

$$\forall u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) : \quad f(-re + u) < f(u), \quad f(re + u) < f(u).$$

Then assertions (a) and (b) easily follow. Moreover, since $f(te) \leq f(0)$ for every $t \in [-r, r]$, we have $\vartheta(0) = 0$.

Now, let (u_h) be a sequence in $(Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)})$ converging to u. Up to a subsequence, $(\vartheta(u_h))$ is convergent to some $t \in [-r, r]$. On the other hand $f(\vartheta(u_h)e + u_h) \geq f(\vartheta(u)e + u_h)$. Since f is continuous it follows that $f(te + u) \geq f(\vartheta(u)e + u)$, whence $t = \vartheta(u)$. Therefore ϑ is continuous. Of course, φ also is continuous.

For every $z \in Z \cap \mathrm{B}_\rho(0)$ and $t \in [-r, r]$ we have $f(te + z) \leq f(0)$, whence $\varphi(z) \leq \varphi(0)$. Finally, let $w \in K \cap W \cap \mathrm{B}_\rho(0)$, $z_0, z_1 \in Z \cap \mathrm{B}_\rho(0)$ with $z_0 \neq z_1$ and let $t \in]0, 1[$. From assumtion (ii) it follows that

$$\begin{aligned}
\varphi((1-t)z_0 + tz_1 + w) &= f(\vartheta((1-t)z_0 + tz_1 + w)e + (1-t)z_0 + tz_1 + w) \geq \\
&\geq f([(1-t)\vartheta(z_0 + w) + t\vartheta(z_1 + w)]e + (1-t)z_0 + tz_1 + w) > \\
&> (1-t)f(\vartheta(z_0 + w)e + z_0 + w) + tf(\vartheta(z_1 + w)e + z_1 + w) = \\
&= (1-t)\varphi(z_0 + w) + t\varphi(z_1 + w).
\end{aligned}$$

Therefore the function $\{z \mapsto \varphi(z + w)\}$ is strictly concave on $Z \cap \mathrm{B}_\rho(0)$. ∎

Let us set

$$\begin{aligned}
E^+ &= \left\{ te + u : \, u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \, , \, \vartheta(u) \le t \le r \right\} , \\
E^- &= \left\{ te + u : \, u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \, , \, -r \le t \le \vartheta(u) \right\} , \\
E &= E^+ \cap E^- = \left\{ \vartheta(u)e + u : \, u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \right\} , \\
U_{r,\rho} &= E^+ \cup E^- = \left\{ te + u : \, u \in (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \, , \, -r \le t \le r \right\} .
\end{aligned}$$

Lemma 6.4 *Under the assumptions of the previous lemma, we have $C_q(f,0) \approx C_{q-1}(\varphi,0)$ for any q.*

Proof. Without loss of generality, we may assume that $f(0) = 0$. From lemma 6.2 and (a) of lemma 6.3, it follows that $U_{r,\rho}$ is a neighbourhood of 0 in K.

Now, let $\tau : E^+ \to [0, +\infty[$ be a continuous function such that $\tau(u)e + u \in E^+$ for any $u \in E^+$ and $\tau(0) > 0$. Let us define $\mathcal{H} : (f^0 \cap E^+) \times [0,1] \to f^0 \cap E^+$ by

$$\mathcal{H}(u,s) = u + s\tau(u)e \, .$$

Then $\mathcal{H}$ is continuous and takes actually its values in $f^0 \cap E^+$ by assumption (ii). Moreover, we have

$$\forall u \in f^0 \cap E^+ : \quad \mathcal{H}(u,0) = u \, , \quad \mathcal{H}(u,1) \neq 0 \, ,$$
$$\forall u \in (f^0 \cap E^+) \setminus \{0\} \, , \forall s \in [0,1] : \quad \mathcal{H}(u,s) \neq 0.$$

It follows

$$\forall q : \quad H^q \left(f^0 \cap E^+, (f^0 \cap E^+) \setminus \{0\} \right) = \{0\} \, .$$

In a similar way, we find that

$$\forall q : \quad H^q \left(f^0 \cap E^-, (f^0 \cap E^-) \setminus \{0\} \right) = \{0\} \, .$$

Since E^+ and E^- are closed in $U_{r,\rho}$ and we are considering Alexander-Spanier cohomology in a metric space, we have the Mayer-Vietoris exact sequence

$$\begin{aligned}
&\to \; H^{q-1} \left(f^0 \cap E^-, (f^0 \cap E^-) \setminus \{0\} \right) \oplus H^{q-1} \left(f^0 \cap E^+, (f^0 \cap E^+) \setminus \{0\} \right) \to \\
&\to \; H^{q-1} \left(f^0 \cap E, (f^0 \cap E) \setminus \{0\} \right) \to H^q \left(f^0 \cap U_{r,\rho}, (f^0 \cap U_{r,\rho}) \setminus \{0\} \right) \to \\
&\to \; H^q \left(f^0 \cap E^-, (f^0 \cap E^-) \setminus \{0\} \right) \oplus H^q \left(f^0 \cap E^+, (f^0 \cap E^+) \setminus \{0\} \right) .
\end{aligned}$$

It follows that

$$H^q \left(f^0 \cap U_{r,\rho}, (f^0 \cap U_{r,\rho}) \setminus \{0\} \right) \approx H^{q-1} \left(f^0 \cap E, (f^0 \cap E) \setminus \{0\} \right) ,$$

hence $C_q(f,0) \approx C_{q-1}(f_{|E},0)$. On the other hand, $\Phi(u) = \vartheta(u)e + u$ is a homeomorphism of $(Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)})$ onto E with $\Phi(0) = 0$. It follows that $C_{q-1}(\varphi,0) \approx C_{q-1}(f_{|E},0)$. ∎

Proof of theorem 6.1. If there exists $v_0 \in V \cap \mathrm{B}_\delta(0)$ such that $f(v_0) > f(0)$, then $|\mathrm{d}\, f|\,(0) \neq 0$. Actually, by assumption (ii) we may assume that $\|v_0\| < \delta/2$. If

$$0 < \sigma < \frac{f(v_0) - f(0)}{\|v_0\|}\,,$$

by lemma 6.2 there exists $\delta' > 0$ such that, for every $u \in K \cap \mathrm{B}_{\delta'}(0)$, one has that $\{u + sv_0 : -\delta' \leq s \leq 1\} \subseteq K$, f is concave on $\{u + sv_0 : -\delta' \leq s \leq 1\}$ and

$$\frac{f(u + v_0) - f(u)}{\|v_0\|} \geq \sigma\,.$$

Let $\mathcal{H} : (K \cap \mathrm{B}_{\delta'}(0)) \times [0, \delta'] \to K$ be defined by

$$\mathcal{H}(u,t) = u - t\frac{v_0}{\|v_0\|}\,.$$

Then $\mathcal{H}$ is continuous and $\|\mathcal{H}(u,t) - u\| = t$. Moreover, for every $u \in K \cap \mathrm{B}_{\delta'}(0)$ and $t \in [0, \delta']$, we have

$$u = \frac{t}{t + \|v_0\|}\,(u + v_0) + \frac{\|v_0\|}{t + \|v_0\|}\,\mathcal{H}(u,t)\,,$$

hence

$$f(u) \geq \frac{t}{t + \|v_0\|}\,f(u + v_0) + \frac{\|v_0\|}{t + \|v_0\|}\,f(\mathcal{H}(u,t))\,,$$

which is equivalent to

$$f(\mathcal{H}(u,t)) \leq f(u) - \frac{t}{\|v_0\|}\,(f(u + v_0) - f(u))\,.$$

It follows $f(\mathcal{H}(u,t)) \leq f(u) - \sigma t$, whence $|\mathrm{d}\, f|\,(0) \geq \sigma > 0$. By [5, Proposition (3.4)] we deduce that $C_q(f, 0) = \{0\}$ for every q.

Therefore, we may assume that $f(v) \leq f(0)$ for every $v \in V \cap \mathrm{B}_\delta(0)$. Let us argue by induction on $m = \dim V$.

If $m = 0$, i.e. $V = \{0\}$, there is nothing to prove. Now let $m \geq 1$ and assume the assertion is true for $m - 1$. Let $\varphi : (Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)}) \to \mathbb{R}$ be as in lemma 6.4. We know that

$$\forall q : \quad C_q(f, 0) \approx C_{q-1}(\varphi, 0)\,.$$

On the other hand, by lemma 6.3 φ satisfies the same assumptions of f, with V substituted by Z and K by $(Z \cap \overline{\mathrm{B}_\rho(0)}) + (K \cap W \cap \overline{\mathrm{B}_\rho(0)})$. Since $\dim Z = m - 1$, by the inductive assumption we have that

$$\forall q \leq m - 2 : \quad C_q(\varphi, 0) = \{0\}$$

and the assertion follows. ∎

7 Proof of the main results

This section is devoted to the proof of Theorems 2.2 and 2.4.

Let K, g, f_1 and f be as in sect. 4. Let also $Q_0 : H_0^1(\Omega) \to \mathbb{R}$ be the quadratic form defined by

$$Q_0(u) = \int_\Omega |Du|^2\,dx - \int_\Omega D_s g(x,0)u^2\,dx\,.$$

By proposition 4.1, each critical point of f is a solution of (2.2). Therefore, without loss of generality, we may assume that f has only a finite number of critical points. By theorem 4.2 there exists a critical point $\overline{u}$ of f such that $C_k(f,\overline{u}) \neq \{0\}$. Therefore, it is sufficient to show that $C_k(f,0) = \{0\}$.

Proof of theorem 2.2. Suppose first that there exists $h < k$ such that $\mu_h^{(0)} \leq 0 < \lambda_{h+1}^{(0)}$. Let V be a maximal subspace of H_0' where Q_0 is negative semidefinite and let $\widehat{W}$ be a maximal closed subspace of H_0 where Q_0 is positive definite. Since $\mu_h^{(0)} \leq 0 < \lambda_{h+1}^{(0)}$, we have $\dim V = \operatorname{codim}_{H_0} \widehat{W} = h$. Let $H_0 = \widehat{V} \oplus \widehat{W}$ and let $P_{\widehat{V}}$, $P_{\widehat{W}}$ be the projections associated with the decomposition. We clearly have $V \cap \widehat{W} = \{0\}$. Therefore $P_{\widehat{V}} : V \to \widehat{V}$ is injective, hence bijective. For any $u \in H_0$, let $u = \hat{v} + \hat{w}$ with $\hat{v} \in \widehat{V}$ and $\hat{w} \in \widehat{W}$. Let also $v \in V$ with $P_{\widehat{V}} v = \hat{v}$. Then we have

$$u = P_{\widehat{V}} v + \hat{w} = v + \big(\hat{w} - P_{\widehat{W}} v\big) \in V + \widehat{W}\,.$$

Therefore $H_0 = V \oplus \widehat{W}$.

Consequently, we have the decomposition $H_0^1(\Omega) = V \oplus W$, where $W = \left(\widehat{W} \oplus H_0^\perp\right)$ and $H_0^\perp$ is the orthogonal of H_0 in $H_0^1(\Omega)$ with respect to the standard scalar product. Let P_V be the associated projection on V.

We want to apply theorem 5.1. Assumption (i) is clearly satisfied. Since f_1 is of class C^2, $f_1'(0) = 0$ and $f_1''(0)$ is positive definite on $\widehat{W}$, there exist $\omega, \delta > 0$ such that

$$\begin{aligned}
&\forall w \in \widehat{W} \cap \overline{\mathrm{B}_\delta(0)} : \quad f_1(w) \geq f_1(0)\,,\\
&\forall u \in \overline{\mathrm{B}_\delta(0)} : \quad \text{the function } f_1 \text{ is strictly convex on } \overline{\mathrm{B}_\delta(0)} \cap (u + \widehat{W})\,,\\
&\big\{u \mapsto f_1(u) + \omega\|P_V u\|^2\big\} \text{ is convex on } H_0 \cap \overline{\mathrm{B}_\delta(0)}\,.
\end{aligned}$$

Since we want to estimate the critical groups of f at 0, we may substitute K with $K \cap \overline{\mathrm{B}_\delta(0)}$. As $K \subseteq H_0$, it follows that assumptions (ii), (iii) and (v) are satisfied. Finally, according to [16], we have

$$\overline{\bigcup_{t>0}(tK)} = \big\{u \in H_0^1(\Omega) : \ u(x) \geq 0 \text{ q.e. in } F_1^0 \text{ and } u(x) \leq 0 \text{ q.e. in } F_2^0\big\}\,.$$

It follows $V \subseteq H_0' \subseteq \overline{\bigcup_{t>0}(tK)}$. By theorem 5.1 we conclude that $C_k(f,0) = \{0\}$.

Now, suppose that there exists $h > k$ such that $\mu_h^{(0)} < 0$. Arguing as in the proof of theorem 4.2, we find a decomposition of the form $H_0^1(\Omega) = V \oplus W$, where W is closed and

V is a subspace of H_0' with $\dim V = h$ such that $Q_0 = f_1''(0)$ is negative definite on V. By assumption (2.1) we have $H_0' \subseteq K$, hence $V + K = K$. Moreover, there exists $\delta > 0$ such that, for every $w \in \mathrm{B}_\delta(0)$, the function $\{v \mapsto f_1(v+w)\}$ is strictly concave on $V \cap \mathrm{B}_\delta(0)$. By theorem 6.1 we conclude that $C_k(f,0) = \{0\}$. ■

Proof of theorem 2.4. Arguing as in the proof of theorem 4.2, we find a decomposition of the form $H_0^1(\Omega) = \widetilde{V} \oplus \widehat{W} \oplus H_0^\perp$, where $\widehat{W}$ is closed in H_0, $\widetilde{V}$ is a subspace of H_0' with $\dim \widetilde{V} = h$, $Q_0 = f_1''(0)$ is negative definite on $\widetilde{V}$ and positive definite on $\widehat{W}$. Set also $W = \widehat{W} \oplus H_0^\perp$. It is readily seen that $H_0' \subseteq \overline{\bigcup_{t>0}(t(K \cap (-K)))}$. Let $\{z_1, \dots, z_h\}$ be a basis in $\widetilde{V}$. Given $\varepsilon > 0$, there exist $t_j > 0$ and $v_j \in t_j(K \cap (-K))$ with $\|v_j - z_j\| < \varepsilon$. Let V be the linear subspace spanned by $\{v_1, \dots, v_h\}$. If ε is sufficiently small, we have $\dim V = h$, $H_0^1(\Omega) = V \oplus W$ and $f_1''(0)$ is negative definite also on V. Moreover, we have $[-t_j^{-1}, t_j^{-1}]v_j \subseteq K$, hence

$$\left[-\frac{1}{ht_1}, \frac{1}{ht_1}\right] v_1 + \cdots + \left[-\frac{1}{ht_h}, \frac{1}{ht_h}\right] v_h \subseteq K\,. \tag{7.1}$$

As in the proof of theorem 2.2, we see that assumptions $(i) - (v)$ of sect. 5 are satisfied. Since $f_1''(0)$ is negative definite on V, by (7.1) we find $\delta > 0$ such that $V \cap \mathrm{B}_\delta(0) \subseteq K$ and $f_1(v) \le f_1(0)$ for any $v \in V \cap \mathrm{B}_\delta(0)$. From theorem 5.2 we conclude that $C_k(f,0) = \{0\}$. ■

Remark 7.1 *Let us point out that assumption* (2.1) *is actually needed only to treat the case $h > k$ with $\mu_h^{(0)} < 0$, while it is not used in the case $h < k$ with $\mu_h^{(0)} \le 0 < \lambda_{h+1}^{(0)}$.*

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