The Normal holonomy group of Kahler submanifolds

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The normal holonomy group of Kähler submanifolds

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Abstract

We study the (restricted) holonomy group $\mathrm{Hol}(\nabla^\perp)$ of the normal connection $\nabla^\perp$ (shortly, normal holonomy group) of a Kähler submanifold of a complex space form. We prove that if the normal holonomy group acts irreducibly on the normal space then it is linear isomorphic to the holonomy group of an irreducible Hermitian symmetric space. In particular, it is a compact group and the complex structure J belongs to its Lie algebra.

We prove that the normal holonomy group acts irreducibly if the submanifold is full (i.e. it is not contained in a totally geodesic proper Kähler submanifold) and the second fundamental form at some point has no kernel. For example, a Kähler -Einstein submanifold of $\mathbb{C}P^n$ has this property.

We define a new invariant μ of a Kähler submanifold of a complex space form. For non-full submanifolds, the invariant μ measures the deviation of J from belonging to the normal holonomy algebra. For a Kähler -Einstein submanifold, the invariant μ is a rational function of the Einstein constant. By using the invariant μ, we prove that the normal holonomy group of a not necessary full Kähler -Einstein submanifold of $\mathbb{C}P^n$ is compact and give a list of possible holonomy groups.

The approach is based on a definition of the holonomy algebra $\mathrm{hol}(P)$ of arbitrary curvature tensor field P on a vector bundle with a connection and on a De Rham type decomposition theorem for $\mathrm{hol}(P)$.

1 Introduction.

A theorem by Nomizu-Ozeki states that any connected linear Lie group $H \subset GL_n(\mathbb{R})$ is the holonomy group of a linear connection. On the other hand, the list of (restricted) holonomy groups H of irreducible Riemannian manifolds is very short. It was discovered by M. Berger that besides isotropy groups of irreducible Riemannian symmetric spaces it contains only the following linear groups: $SU_n, Sp_n, Spin_7$ and G_2 [Be].

G/K	K	V
$Gr_p(\mathbb{C}^{p+q}) := SU(p+q)/S(U(p) \times U(q))$	$S(U(p) \times U(q))$	$\mathbb{C}^p \otimes \mathbb{C}^q$
$SO(2n)/U(n)$	$U(n)$	$\Lambda^2(\mathbb{C}^n)$
$Gr_2(\mathbb{R}^{n+2}) := SO(n+2)/SO(2) \times SO(n)$	$SO(2) \times SO(n)$	$\mathbb{R}^2 \otimes \mathbb{R}^n$
$Sp(n)/U(n)$	$U(n)$	$S^2\mathbb{C}^n$
$E_6/T^1 \cdot Spin_{10}$	$T^1 \cdot Spin_{10}$	$\mathbb{C}^{16}$
$E_7/T^1 \cdot E_6$	$T^1 \cdot E_6$	$\mathbb{C}^{27}$

Table 1:
Isotropy representations $K \hookrightarrow SO(V)$ of compact irreducible Hermitian symmetric spaces G/K.

An investigation of the (restricted) holonomy group $\mathrm{Hol}(\nabla^\perp)$ of the normal connection $\nabla^\perp$ of a submanifold $M \subset \mathbb{R}^n$ (or shortly, the normal holonomy group) was initiated by C. Olmos [Ol1]. He proved that the list of normal holonomy groups is even more restrictive: The normal holonomy group $\mathrm{Hol}(\nabla^\perp)$ of a submanifold $M \subset \mathbb{R}^n$ is isomorphic (as a linear group) to the isotropy representation of a Riemannian symmetric space. The same result holds for a submanifold of a space of constant curvature. The classification of Olmos turned out to be very useful for geometry of submanifolds of the spaces of constant curvature. Thus, it is natural to study the normal holonomy group of submanifolds of other rank ones symmetric spaces.

The aim of this paper is to describe the normal holonomy group $\mathrm{Hol}(\nabla^\perp)$ of a complex submanifold M of the complex space forms $\mathbb{S}_c^n$ with constant holomorphic sectional curvature c. Since our considerations are local, without loss of generality we may assume that $\mathbb{S}_c^n$ is the complex projective space $\mathbb{C}P^n$ with the standard metric with $c = 4$, or $\mathbb{S}_c^n = \mathbb{C}^n$ with the standard flat metric with $c = 0$ or the complex hyperbolic space $\mathbb{S}_c^n = \mathbb{C}H^n$ with the standard metric with $c = -4$.

Recall that a submanifold M of $\mathbb{S}_c^n$ is called *full* if it is not contained in a proper complex totally geodesic submanifold of $\mathbb{S}_c^n$. Our main results can be stated as follows:

Theorem 1.1 *Let $\mathbb{S}_c^n = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ be the complex space form of holomorphic constant sectional curvature c and $M \subset \mathbb{S}_c^n$ a Kähler submanifold. If the normal holonomy group $\mathrm{Hol}_p(\nabla^\perp) \subset SO(N_p(M))$ of M at a point $p \in M$ acts irreducibly on the normal space $N_p(M)$ then $\mathrm{Hol}_p(\nabla^\perp)$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1. In particular, this is true if $M \subset \mathbb{C}^n$ is a locally irreducible Kähler submanifold of $\mathbb{C}^n$.*

We will denote by α the second fundamental form of a submanifold $M \subset \mathbb{S}_c^n$ and by $\nu(p) := \dim_{\mathbb{R}}(Ker(\alpha)) = \dim_{\mathbb{R}}(\{X \in T_pM : \alpha(X, \cdot) = 0\})$ the index of relative nullity (see [KNII, Note 16.]).

Theorem 1.2 *Let M be a full Kähler submanifold of $\mathbb{S}_c^n$ such that the index of relative nullity $\nu(p) = 0$ at some point $p \in M$. If the normal holonomy group $\mathrm{Hol}_p(\nabla^{\perp})$ at $p \in M$ does not act irreducibly on the normal space $N_p(M)$, then $c = 0$ and M is extrinsically a product, i.e. there exist a decomposition $\mathbb{S}_0^n = \mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ such that locally $M = M_1 \times M_2 = M \cap \mathbb{C}^{n_1} \times M \cap \mathbb{C}^{n_2}$.*

Note that if a complete Kähler submanifold $M \subset \mathbb{C}P^n$ of the complex projective space $\mathbb{C}P^n$ has index of relative nullity $\nu(p) > 0$ at any point then by the result of K. Abe and M.A. Magid [AbMa, Thm. 3] M is a totally geodesic submanifold. So we get the following global result.

Theorem 1.3 *Let $M \subset \mathbb{C}P^n$ be a full and complete Kähler submanifold. Then the normal holonomy group $\mathrm{Hol}_p(\nabla^{\perp})$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

Corollary 1.4 *Let M be a full Kähler submanifold of the complex space form $\mathbb{S}_c^n$ such that the index of relative nullity $\nu(p) = 0$ at some point $p \in M$. Then the complex structure J_q of the normal space $N_q(M)$ at any point $q \in M$ belongs to the normal holonomy algebra $\mathrm{hol}_q(\nabla^{\perp})$ and to the normal holonomy group $\mathrm{Hol}_q(\nabla^{\perp})$.*

For a full curve and a full surface we get the following results without assumption about the index of relative nullity ν.

Theorem 1.5 *Let $M^2 \subset \mathbb{S}_c^n$ be a full Kähler surface of $\mathbb{S}_c^n$. Then the normal holonomy group $\mathrm{Hol}_p(\nabla^{\perp})$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

Theorem 1.6 *Let $M^1 \subset \mathbb{S}_c^n$ be a full Kähler curve of $\mathbb{S}_c^n$. Then the normal holonomy group $\mathrm{Hol}_p(\nabla^{\perp})$ is linearly isomorphic to $U(n-1)$.*

Nomizu and Smyth [NoSm, Thm.2, pag. 505] proved that a complex hypersurface M^n in $\mathbb{C}H^{n+1}$ has the (tangent) holonomy group $U(n)$. In the following theorem we describe the (tangent) holonomy group for a Kähler submanifold of arbitrary codimension in a complex space form $\mathbb{C}^{n+p}, \mathbb{C}H^{n+p}$.

Theorem 1.7 *Let $M^n \subset \mathbb{C}^{n+p}, \mathbb{C}H^{n+p}$ be a full Kähler submanifold of a complex space form $\mathbb{S}_c^n$ of non positive curvature $c \leq 0$. Then,*

i) If $M^n \subset \mathbb{C}H^{n+p}$ then the restricted holonomy group of M is $U(n)$.

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ii) *If* $M^n \subset \mathbb{C}^{n+p}$ *then locally* M *splits into an Euclidean factor and irreducible ones as* $M = \mathbb{C}^r \times M_1^{n_1} \times \cdots \times M_k^{n_k} \subset \mathbb{C}^{n+p} = \mathbb{C}^r \times \mathbb{C}^{n_1+p_1} \times \cdots \times \mathbb{C}^{n_k+p_k}$. *The restricted holonomy group of each irreducible factor* $M_j^{n_j}$ *is* $U(n_j)$ *($j = 1, \cdots, k$). Moreover, the normal holonomy group of the factor* $M_j^{n_j} \subset \mathbb{C}^{n_j+p_j}$ *is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

If the submanifold it is non-full then the complex structure J can not belong to the normal holonomy group. More precisely, we prove the following theorem.

Theorem 1.8 *Let* M^n *be Kähler submanifold of the complex space form* $\mathbb{S}_c^{n+2} = \mathbb{C}P^{n+2}, \mathbb{C}^{n+2}, \mathbb{C}H^{n+2}$ *such that the index of relative nullity* $\nu(p) = 0$ *at some point* $p \in M$. *Then,*

    (i) if $c = -4$ *then* $J_q \in \mathrm{hol}_q(\nabla^\perp)$ *for* $q \in M$ *and* M *is full;*

    (ii) if $c = 0$ *then* $J_q \in \mathrm{hol}_q(\nabla^\perp)$ *for* $q \in M$ *if and only if* M *is full.*

    (iii) if $c = 4$ *then either* $J_q \in \mathrm{hol}_q(\nabla^\perp)$ *for* $q \in M$ *or* $J_q \notin \mathrm{hol}_q(M)$ *and* M *is congruent to an open subset of the (non-full) complex quadric* $Q^n := \{[z_0 : \cdots : z_{n+2}] \in \mathbb{C}P^{n+2} \mid z_0^2 + \cdots + z_{n+1}^2 = 0, \, z_{n+2} = 0\}$.

In case that M is non-full, we get the following results.

Theorem 1.9 *Let* $M^m \subset \mathbb{S}_c^{\overline{m}} \subset \mathbb{S}_c^n$ *($c \neq 0$) be a non-full and non-totally geodesic Kähler submanifold of* $\mathbb{S}_c^n$ *which is full as a submanifold of* $\mathbb{S}_c^{\overline{m}}$ $m < \overline{m} < n$. *Let us decompose the normal space* $N(M)$ *of* $M \subset \mathbb{S}_c^n$ *into an orthogonal sum of* $\nabla^\perp$*-parallel subbundles*

$$N_q(M) = \overline{N}_q(M) \oplus N_q(\mathbb{S}_c^{\overline{m}})$$

where $\overline{N}_q(M) := T_q\mathbb{S}_c^{\overline{m}} \cap N_q(M)$.

Assume that the index of relative nullity $\nu(p) = 0$ *at some point* $p \in M$. *Then, the Lie algebra* $\mathrm{hol}_q(\nabla^\perp)$ *of the normal holonomy group satisfies:*

    (i) $\mathrm{hol}_q(\nabla^\perp)|_{N_q(\overline{M})} = \mathbb{R}J_q$.

    (ii) $\mathrm{hol}_q(\nabla^\perp)|_{\overline{N}_q(M)}$ *is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

Using this theorem we can define an invariant μ for Kähler submanifolds $M \subset \mathbb{S}_c^n$ with index of relative nullity $\nu(p) = 0$ at some point $p \in M$.

Definition 1.10 *Let* $M^m \subset \mathbb{S}_c^{\overline{m}} \subset \mathbb{S}_c^n$ *($c \neq 0$) be a non-full and non-totally geodesic Kähler submanifold of* $\mathbb{S}_c^n$ *which is full as a submanifold of* $\mathbb{S}_c^{\overline{m}}$. *Assume that the index of relative nullity* $\nu(p) = 0$ *at some point* $p \in M$. *We define* $\mu \in \mathbb{R}$ *as the minimal in absolute value number such that* $\mathrm{diag}(J, \mu J) \in \mathrm{hol}_p(\nabla^\perp) \subset \mathbb{R}J|_{N_p(\overline{M})} \oplus SO(|_{\overline{N}_p(M)})$.

By Corollary 1.4 and Theorem 1.9 such μ exists and is unique. We will show that μ can also be defined for full submanifolds $M \subset \mathbb{S}^n_c$ with index of relative nullity $\nu(p) = 0$ at some point $p \in M$.

Since normal holonomy groups at different points are conjugated, μ does not depend on $p \in M$. Moreover, we get the following result.

Theorem 1.11 *The invariant μ defined above depends only on the Kähler metric of M.*

Notice that if $M \subset \mathbb{S}^n_c$ is a full submanifold then we can regard M as a submanifold of a bigger $\mathbb{S}^{n'}_c$ i.e. $M \subset \mathbb{S}^n_c \subset \mathbb{S}^{n'}_c$. Moreover, the index of relative nullity $\nu(p)$ of M considered as a submanifold of $\mathbb{S}^n_c$ and $\mathbb{S}^{n'}_c$ is the same. So, by the above theorem we can define μ for all Kähler submanifolds of $\mathbb{S}^n_c$ with index of relative nullity $\nu(p) = 0$ at some point $p \in M$.

The main properties of the invariant μ are summarized in the following theorem.

Theorem 1.12 *Let $M^m \subset \mathbb{S}^{\overline{m}}_c \subset \mathbb{S}^n_c$ be a non-full Kähler submanifold of $\mathbb{S}^n_c$ ($c \neq 0$) which is full as a submanifold of $\mathbb{S}^{\overline{m}}_c$. Assume that the index of relative nullity $\nu(p) = 0$ at some point $p \in M$. Then,*

(a) *$\mu = 0$ if and only if $\dim(\text{center}(\text{Hol}_p(\nabla^\perp))) = 2$. Moreover, in this case $\text{Hol}_p(\nabla^\perp) = S^1 \cdot H$ (direct product modulo a finite central subgroup) where H is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

So, $\text{Hol}_p(\nabla^\perp)$ is compact and $J_p \in \text{hol}_p(\nabla^\perp)$, $J_p \in \text{Hol}_p(\nabla^\perp)$.

(b) *$\mu \neq 0$ if and only if $\dim(\text{center}(\text{Hol}_p(\nabla^\perp))) = 1$. Moreover, in this case we have:*

    (i) *$\text{Hol}_p(\nabla^\perp)$ is a compact group if and only if $\mu \in \mathbb{Q}$,*

    (ii) *$J_p \in \text{hol}_p(\nabla^\perp)$, $J_p \in \text{Hol}_p(\nabla^\perp)$ if and only if $\mu = 1$.*

It is known that any Kähler-Einstein submanifold of $\mathbb{C}^n$ and $\mathbb{C}H^n$ is a totally geodesic submanifold [Um]. On the other hand, there exist non-totally geodesic Kähler-Einstein submanifolds of $\mathbb{C}P^n$ [NT] but the full classification is not known. Recall also that by a Theorem of Chern [Ch] a Kähler-Einstein hypersurface of $\mathbb{C}P^n$ must be either totally geodesic or congruent to an open subset of the complex quadric $Q^{n-1} := \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_0^2 + \cdots + z_n^2 = 0\}$.

Since the index of relative nullity ν of a non-totally geodesic Kähler-Einstein submanifold of $\mathbb{C}P^n$ never vanishes we obtain the following results.

Theorem 1.13 *Let M be a full and non-totally geodesic Kähler-Einstein submanifold of $\mathbb{C}P^n$. Then, the normal holonomy group $\text{Hol}_p(\nabla^\perp)$ at $p \in M$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.*

Theorem 1.14 *Let* $(M^m, g) \subset \mathbb{C}P^n$ *be a non-full and non-totally geodesic Kähler-Einstein submanifold with Ricci tensor* $Ric_M = k \cdot g$. *Let* $\mathbb{C}P^{\overline{m}} \subset \mathbb{C}P^n$ *be the totally geodesic Kähler submanifold of* $\mathbb{C}P^n$ *such that* M *is full in* $\mathbb{C}P^{\overline{m}}$. *Then,*

$$\mu = \frac{\overline{m} - m}{\overline{m} + 1 - \frac{k}{2}}$$

where μ *is the invariant of Definition 1.10.*

By combining the above theorems with results of D. Hullin [Hu],[Hu1] we obtain the following result.

Corollary 1.15 *Let* $(M^m, g) \subset \mathbb{C}P^n$ *be a Kähler-Einstein submanifold with Ricci tensor* $Ric_M = k \cdot g$. *Then, the normal holonomy group* $\mathrm{Hol}_p(\nabla^\perp)$ *is compact and* $\dim(\mathrm{center}(\mathrm{Hol}_p(\nabla^\perp))) = 1$. *If* M *is non-full and non-totally geodesic then*

$$\mu \leq \frac{\overline{m} - m}{\overline{m} - m + 1},$$

and, $J_p \notin \mathrm{hol}_p(\nabla^\perp)$. *Moreover, if* M *is compact then*

$$\frac{\overline{m} - m}{\overline{m} + 1} < \mu.$$

We obtain the following generalization of a theorem of [BS].

Theorem 1.16 *Let* M^m *be a Kähler submanifold of the space form* $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ *with* $c = 4, 0, -4$ *respectively. If* $\dim(\mathrm{hol}(\nabla^\perp)) = 1$ *then one of the following holds:*

  (i) $c \neq 0$ *and* M *is a complex hypersurface,*

 (ii) $c \neq 0$ *and* M *is a totally geodesic submanifold,*

(iii) $c = 0$ *and* M *is a complex hypersurface in a complex affine subspace of* $\mathbb{C}^n$,

 (iv) $c = 4$ *and* M^m *is congruent to an open subset of a non-full complex quadric* $Q^m := \{[z_0 : \cdots : z_{m+1}] \in \mathbb{C}P^{m+1} \mid z_0^2 + \cdots + z_{m+1}^2 = 0\} \subset \mathbb{C}P^n$, $m + 1 < n$.

We obtain the following characterization of the complex quadric.

Theorem 1.17 *Let* M^m *be a Kähler submanifold of the space form* $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ ($n > m > 1$) *with* $c = 4, 0, -4$ *and* A^ξ *its shape operator. Then, the following conditions are equivalents:*

  (i) $(A^\xi)^2 = f\|\xi\|^2 Id$, *where* f *is a positive function on* M,

 (ii) $c = 4$ *and* M *is an open subset of the complex quadric* $Q^m := \{[z_0 : \cdots : z_{m+1}] \in \mathbb{C}P^{m+1} \mid z_0^2 + \cdots + z_{m+1}^2 = 0\}$. *In particular* $f = (\frac{4}{c})^2$ *is a constant.*

We remark that the above theorem is false for $m = 1$ (i.e. for holomorphic curves). But it remains true for holomorphic curves under the additional assumption that $f = const$.

2 Preliminaries.

2.1 Kähler submanifolds of Kähler manifolds.

Let $(\widetilde{M}, g = \langle,\rangle, J)$ be a Kähler manifold with metric g and complex structure J and $\widetilde{\nabla}$, $\tilde{R}$ its connection and curvature tensor, respectively. Recall that any complex submanifold M of $\widetilde{M}$ is a Kähler manifold with respect to the induced metric $g = \tilde{g}|_M$ and complex structure $J|_M$. We will call a complex submanifold M of $\widetilde{M}$ with the induced structure (g, J) the Kähler submanifold of $\widetilde{M}$. We denote by ∇ the Levi-Civita connection of g and by R its curvature tensor.

We denote by $\nabla^\perp$ the normal connection in the normal bundle $N(M) = TM^\perp$ (defined by the orthogonal projection of $\widetilde{\nabla}$ on $N(M)$) and by $R^\perp$ its curvature tensor. The shape operator A^ξ in the direction of a normal vector field $\xi \in \Gamma(N(M))$ and $N(M)$-valued second fundamental form α are defined by the following Gauss-Weingarten formulas:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) \tag{1}$$

$$\widetilde{\nabla}_X \xi = -A^\xi X + \nabla^\perp_X \xi \tag{2}$$

where X, Y are tangent vector fields of M.

Recall also that

$$\langle \alpha(X, Y), \xi \rangle = \langle A^\xi X, Y \rangle. \tag{3}$$

and that the shape operator anticommutes with J:

$$A^\xi J = -J A^\xi = -A^{J\xi}. \tag{4}$$

This implies that $trace(A^\xi) = 0$. Hence, any Kähler submanifold M is a minimal submanifold of $\widetilde{M}$.

We will denote by $\mathbb{S}^n_c$ ($n = \dim_{\mathbb{C}}(\mathbb{S}^n_c)$) the complex space form of constant holomorphic sectional curvature c. We will assume that $\mathbb{S}^n_c$ is simply connected and complete. Then, it is homothetic to $\mathbb{C}P^n$, $\mathbb{C}^n$ or $\mathbb{C}H^n$ equipped with the standard metric with holomorphic curvature $c = 4, 0$ or -4.

Definition 2.1 *A Kähler submanifold $M \subset \widetilde{M}$ is called* full *if M is not contained in a proper Kähler totally geodesic submanifold of $\widetilde{M}$.*

Recall that in terms of homogeneous coordinates of $\mathbb{S}^n_c$, a complex totally geodesic submanifold $T \subset \mathbb{S}^n_c$ is defined by linear equations (See [Mon] for the case $\mathbb{S}^n_c = \mathbb{C}H^n$). Since Kähler submanifold of $\mathbb{S}^n_c$ are analytic we get the following result.

Proposition 2.2 *Let M be a Kähler submanifold of $\mathbb{S}^n_c$ and $U \subset M$ an open subset. If U is contained in a complete complex totally geodesic submanifold T of $\mathbb{S}^n_c$ then M is also contained in T.*

Recall that the first normal space $N^1(M)$ is defined by $N^1(M) := span\{\alpha(X,Y)\}$.

Theorem 2.3 *[Ce], [ChO1] Let M be a Kähler submanifold of $\mathbb{S}_c^n$. If there exists a complex $\nabla^\perp$-parallel subbundle $V \neq 0$ of the normal bundle $N(M)$ such that $V \perp N^1(M)$, then M is non-full.*

Definition 2.4 *We say that $M \subset \widetilde{M}$ is extrinsically reducible at $p \in M$ if there exist a (complex) parallel distribution H, locally defined around p, such that $\alpha(H, H^\perp) = 0$.*

Theorem 2.5 *(Complex Moore's Lemma, [Moo],[DS]) Let M be a Kähler submanifold of the space $\mathbb{C}^n$. Then M is extrinsically reducible at $p \in M$ if and only if there exist a decomposition $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ such that locally near p $M = M_1 \times M_2 = (M \cap \mathbb{C}^{n_1}) \times (M \cap \mathbb{C}^{n_2})$.*

For a Riemannian manifold M the *index of nullity* at $p \in M$ is the dimension $r(p)$ of the subspace

$$T_0(p) : \{X_p \in T_p M : R(X_p, \cdot) = 0\}$$

where R is the curvature tensor of M.

We will prove that the index of nullity of Kähler submanifold of $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is zero on an open and dense subset.

Definition 2.6 *[KNII, Note 16.] Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ and α its second fundamental form. The* nullity *space $\mathcal{N}_p$ of α at a point $p \in M$ is defined as $\mathcal{N}_p := \{X \in T_p M : \alpha(X, \cdot) = 0\} = \bigcap_{\xi \in N_p(M)} ker(A^\xi)$. The index of relative nullity at $p \in M$ $\nu(p)$ is the dimension $\nu(p) := \dim_{\mathbb{R}}(\mathcal{N}_p)$ of the nullity space at $p \in M$.*

If M is a Kähler submanifold of a Kähler manifold $\widetilde{M}$ then the equation (4) shows that the nullity space $\mathcal{N}_p$ is a complex subspace of $T_p M$.

Proposition 2.7 *Let M be a Kähler submanifold of a Kähler manifold $\widetilde{M}$. Let V_1, V_2 be two complex subspaces of $N_p(M)$ such that $[A^{\xi_1}, A^{\xi_2}] = 0$ for $\xi_1 \in V_1$ and $\xi_2 \in V_2$. Then, $A^{\xi_1} A^{\xi_2} = A^{\xi_2} A^{\xi_1} = 0$. In particular, if $V_1 = V_2$ then $A^\xi = 0$ for $\xi \in V_1 = V_2$.*

Proof. Since V_2 is complex, for $\xi_1 \in V_1, \xi_2 \in V_2$ we get $[A^{\xi_1}, A^{\xi_2}] = A^{\xi_1} A^{\xi_2} - A^{\xi_2} A^{\xi_1} = 0$, $[A^{\xi_1}, A^{J\xi_2}] = [A^{\xi_1}, JA^{\xi_2}] = -JA^{\xi_1} A^{\xi_2} - JA^{\xi_2} A^{\xi_1} = 0$. Hence, $A^{\xi_1} A^{\xi_2} = 0$. If $V_1 = V_2$ we get $(A^\xi)^2 = 0$ which implies $A^\xi = 0$. $\square$

2.2 Gauss-Codazzi equations.

Let $M \subset \widetilde{M}$ be a submanifold of a Riemannian manifold and $\tilde{\nabla}, \tilde{R}$ (resp. ∇, R) the Levi-Civita connection of $\widetilde{M}$ and its curvature tensor (resp. the Levi-Civita connection of M and its curvature tensor).

Let X, Y be tangent vector fields of M. We decompose the curvature operator $\tilde{R}_{X,Y} \in gl(T_p\widetilde{M}) = gl(T_pM \oplus N_p(M))$ into tangential part $\tilde{R}_{X,Y}^{\top\top} \in gl(T_pM)$, normal part $\tilde{R}_{X,Y}^{\perp\perp} \in gl(N_p(M))$ and mixed parts $\tilde{R}_{X,Y}^{\perp\top} \in Hom(N_p(M), T_p(M))$ and $\tilde{R}_{X,Y}^{\top\perp} \in Hom(T_p(M), N_p(M))$ such that

$$\tilde{R}_{X,Y} = \tilde{R}_{X,Y}^{\top\top} + \tilde{R}_{X,Y}^{\perp\top} + \tilde{R}_{X,Y}^{\top\perp} + \tilde{R}_{X,Y}^{\perp\perp}.$$

We have the following Gauss-Codazzi equations:

$$\tilde{R}_{X,Y}^{\top\top} = R_{X,Y} - \sum_i A^{\xi_i}X \wedge A^{\xi_i}Y \tag{5}$$

$$\langle \tilde{R}_{X,Y}^{\perp\perp}\xi, \eta \rangle = \langle R_{X,Y}^{\perp}\xi, \eta \rangle - \langle [A^{\xi}, A^{\eta}]X, Y \rangle \tag{6}$$

$$\tilde{R}_{X,Y}^{\top\perp}\xi = (\nabla_X A^{\xi})(Y) - (\nabla_Y A^{\xi})(X) \tag{7}$$

$$\tilde{R}_{X,Y}^{\perp\top}Z = (\nabla_X \alpha)(Y, Z) - (\nabla_Y \alpha)(X, Z) \tag{8}$$

where ξ, η are normal vector fields, ξ_i is an orthonormal basis of $N(M)$ and $(\nabla_X A^{\xi})(Y) := \nabla_X(A^{\xi}(Y)) - A^{\nabla_X^{\perp}\xi}(Y) - A^{\xi}(\nabla_X Y)$ and $(\nabla_X \alpha)(Y, Z) := \nabla_X^{\perp}\alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$. Here we identify skew-symmetric endomorphisms with bivectors as: $(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$.

In terms of the second fundamental form α, the equation (5) can be written as [KNII, p.23]:

$$\langle \tilde{R}_{X,Y}^{\top\top}Z, W \rangle = \langle R_{X,Y}Z, W \rangle - (\langle \alpha(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle). \tag{9}$$

The following result is a generalization of Proposition 4 of [Sm].

Proposition 2.8 *Let M be a Kähler submanifold of a Kähler manifold $\widetilde{M}$. Then the Ricci tensors Ric_M, $Ric_{\widetilde{M}}$ of M and $\widetilde{M}$ and the curvature tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$ are related by:*

$$Ric_{\widetilde{M}}(X, JY) = Ric_M(X, JY) - \frac{1}{2}trace(JR_{X,Y}^{\perp}) = Ric_M(X, JY) - \frac{1}{2}\sum_i \langle JR_{X,Y}^{\perp}\xi_i, \xi_i \rangle$$

where ξ_i is an orthonormal basis of $N(M)$.

Proof. Recall that $Ric_{\widetilde{M}}(X, JY) = -\frac{1}{2}\sum_i \langle J\tilde{R}_{X,Y}e_i, e_i\rangle$ where $\tilde{R}$ is the curvature tensor of $\widetilde{M}$ and e_i is an orthonormal basis of $T\widetilde{M}$ [KNII, p. 149]. We can assume that $e_1, \ldots, e_n$ $(n = \dim_{\mathbb{R}}(M))$ are tangent to the submanifold M and $e_{n+1} = \xi_1, \ldots, e_m = \xi_{m-n}$ $(m = \dim_{\mathbb{R}}(\widetilde{M}))$ are normal to M. So,

$$2Ric_{\widetilde{M}}(X, JY) = \sum_i \langle \tilde{R}^{\top\top}_{X,Y}e_i, Je_i\rangle + \sum_i \langle \tilde{R}^{\perp\perp}_{X,Y}\xi_i, J\xi_i\rangle.$$

By using Gauss-Codazzi equations (5) and (6) we obtain

$$2Ric_{\widetilde{M}}(X, JY) = \sum_i \langle R_{X,Y}e_i, Je_i\rangle - \sum_{i,j}\langle (A^{\xi_j}X \wedge A^{\xi_j}Y)e_i, Je_i\rangle +$$

$$+ \sum_i \langle R^{\perp}_{X,Y}\xi_i, J\xi_i\rangle - \langle [A^{\xi_i}, A^{J\xi_i}]X, Y\rangle.$$

Note that

$$\sum_i \langle (A^{\xi_j}X \wedge A^{\xi_j}Y)e_i, Je_i\rangle = \sum_i \langle e_i, A^{\xi_j}Y\rangle\langle A^{\xi_j}X, Je_i\rangle - \langle e_i, A^{\xi_j}X\rangle\langle A^{\xi_j}Y, Je_i\rangle =$$

$$= \sum_i \langle e_i, A^{\xi_j}X\rangle\langle A^{J\xi_j}Y, e_i\rangle - \langle e_i, A^{\xi_j}Y\rangle\langle A^{J\xi_j}X, e_i\rangle = \langle [A^{J\xi_j}, A^{\xi_j}]X, Y\rangle$$

that is

$$-\sum_{i,j}\langle (A^{\xi_j}X \wedge A^{\xi_j}Y)e_i, Je_i\rangle = \sum_j \langle [A^{\xi_j}, A^{J\xi_j}]X, Y\rangle.$$

So we obtain

$$2Ric_{\widetilde{M}}(X, JY) = \sum_i \langle R_{X,Y}e_i, Je_i\rangle + \sum_i \langle R^{\perp}_{X,Y}\xi_i, J\xi_i\rangle = 2Ric_M(X, JY) - trace(JR^{\perp}_{X,Y}).\,\square$$

Remark 2.9 *The above formula can be rewritten as:*

$$\langle \tilde{R}_{X,Y}, J\rangle = \langle R_{X,Y}, J\rangle + \langle R^{\perp}_{X,Y}, J\rangle$$

where the scalar product is the natural extension of $\langle,\rangle$ to tensors. Also, we can write

$$\tilde{\rho}_{|TM} = \rho + \rho(R^{\perp})$$

where $\tilde{\rho} := Ric_{\widetilde{M}}(\cdot, J\cdot)$ (resp. ρ) is the Ricci form of $\widetilde{M}$ (resp. M) and $\rho(R^{\perp})(X, Y) := \frac{1}{2}\langle R^{\perp}_{X,Y}, J\rangle$ is the Ricci form of the normal connection.

As a corollary we get the following generalization of the Proposition 5 from [Sm].

Corollary 2.10 *Let M be a compact Kähler submanifold of a Kähler manifold $\widetilde{M}$. If M and $\widetilde{M}$ are both Einstein manifolds and their Ricci forms ρ and $\tilde{\rho}$ are not equal, then the first Chern class $c_1(N(M))$ of the normal bundle is not trivial.*

Proof. Indeed, under assumption of corollary, the Chern form of the normal bundle $\frac{1}{2\pi}\rho(\mathbb{R}^{\perp}) = \frac{1}{2\pi}(\tilde{\rho}|_{TM} - \rho) = c\,\omega_M$ is proportional to the Kähler form $\omega_M := \langle\cdot, J\cdot\rangle$ of M with a constant coefficient $c \neq 0$. Hence, it is not cohomological to zero. $\square$

Let now $\mathbb{S}_c^n = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ be the complex space form of constant holomorphic sectional curvature $c = -4, 0, 4$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{S}_c^n$ and $R_{X,Y}^{\mathbb{S}_c}Z = \tilde{\nabla}_X\tilde{\nabla}_YZ - \tilde{\nabla}_Y\tilde{\nabla}_XZ - \tilde{\nabla}_{[X,Y]}Z$ its curvature tensor. Recall that $R^{\mathbb{S}_c}$ is given by

$$R_{X,Y}^{\mathbb{S}_c}Z = \frac{c}{4}((X \wedge Y)Z + (JX \wedge JY)Z - 2\langle JX, Y\rangle JZ). \tag{10}$$

Let M^m $(m = \dim_{\mathbb{C}}(M^m))$ be a Kähler submanifold of $\mathbb{S}_c^n$. Then, we can rewrite Gauss-Codazzi equations (7), (6) as

$$-\frac{c}{2}\langle JX, Y\rangle\langle J\xi, \eta\rangle = \langle R_{X,Y}^{\perp}\xi, \eta\rangle - \langle[A^{\xi}, A^{\eta}]X, Y\rangle, \tag{11}$$

$$(\nabla_X A^{\xi})(Y) = (\nabla_Y A^{\xi})(X). \tag{12}$$

From Gauss-Codazzi equation (9) we obtain the following useful equations for the holomorphic bisectional curvature of M and for the Ricci tensor Ric_M of M.

$$\frac{c}{2}(\langle X, Y\rangle^2 + \langle X, JY\rangle^2 + \|X\|^2\|Y\|^2) = \langle R_{X,JX}JY, Y\rangle + 2\|\alpha(X, Y)\|^2, \tag{13}$$

$$Ric_M(X, Y) = c\frac{m+1}{2}\langle X, Y\rangle - \sum_i\langle A^{\xi_i}(X), A^{\xi_i}(Y)\rangle. \tag{14}$$

Theorem 2.11 *Let M be a Kähler submanifold of $\mathbb{S}_c^n$. If $c < 0$, then M is locally irreducible (i.e. the local holonomy group of M acts irreducibly on T_pM). If $c = 0$ and M is locally reducible then M is locally extrinsically reducible, i.e. there exist a decomposition $\mathbb{S}_c^n = \mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ such that locally $M = M_1 \times M_2 = (M \cap \mathbb{C}^{n_1}) \times (M \cap \mathbb{C}^{n_2})$. If $c > 0$, then M is extrinsically irreducible at any point $p \in M$.*

Proof. Assume that M is locally reducible, i.e. $M = M_1 \times M_2$, where M_1, M_2 are Kähler manifolds. Then equation (13) for $X \in T_p(M_1)$, $Y \in T_pM_2$ gives a contradiction if $c < 0$. If $c = 0$ then equation (13) shows that

$$\alpha(X, Y) = 0.$$

Then the extrinsic splitting follows from Theorem 2.5. If $c > 0$ and M is extrinsically reducible, i.e. $TM = H \oplus H^{\perp}$ is the sum of two parallel distributions with $\alpha(H, H^{\perp}) = 0$, then equation (13) for $X \in H$, $Y \in H^{\perp}$ gives a contradiction. $\square$

We denote by $Sym(V)$ the space of symmetric endomorphisms of $(V, \langle,\rangle)$. We need the following proposition.

Proposition 2.12 *Let M be a Riemannian manifold and $\pi : E \to M$ a vector bundle endowed with a connection ∇' and with a ∇'-parallel Riemannian metric $\langle , \rangle'$. Let $T : E \to Sym(TM)$ be smooth a map such that $T^\xi \in Sym(T_pM)$ for $\xi \in E_p = \pi^{-1}(p)$. Assume that T satisfies the following Codazzi type equation:*

$$(\nabla_X T^\xi)(Y) = (\nabla_Y T^\xi)(X), \qquad \xi \in \Gamma E \text{ and } X,Y \in \Gamma TM$$

where $(\nabla_X T^\xi)(Y) := \nabla_X T^\xi Y - T^{\nabla'_X \xi} Y - T^\xi \nabla_X Y$.

If the dimension of the kernel $\mathcal{D}_p := \bigcap_{\xi_p \in E_p} ker(T^{\xi_p})$ does not depend on $p \in M$ then the kernel distribution $\mathcal{D} : p \in M \to \mathcal{D}_p$ is autoparallel i.e. $\nabla_X Y \in \Gamma \mathcal{D}$ for $X,Y \in \Gamma \mathcal{D}$.

Proof. Observe that $\langle (\nabla_X T^\xi) Y, Z \rangle$ is symmetric in X,Y,Z. Let $X,Y \in \Gamma \mathcal{D}$ and $Z \in \Gamma TM$ be tangent vector fields and ξ a section of E. Then, $-\langle T^\xi \nabla_X Y, Z \rangle = \langle (\nabla_X T^\xi) Y, Z \rangle - \langle \nabla_X T^\xi Y, Z \rangle + \langle T^{\nabla'_X \xi} Y, Z \rangle = \langle (\nabla_X T^\xi) Y, Z \rangle = \langle (\nabla_Z T^\xi) Y, X \rangle = \langle \nabla_Z T^\xi Y, X \rangle - \langle T^{\nabla'_Z \xi} Y, X \rangle - \langle T^\xi \nabla_Z Y, X \rangle = -\langle T^\xi \nabla_Z Y, X \rangle = -\langle \nabla_Z Y, T^\xi X \rangle = 0$. So, $T^\xi (\nabla_X Y) = 0$ and since ξ is arbitrary we get that $\nabla_X Y \in \Gamma \mathcal{D}$. $\square$

Applying this proposition to the shape operator $A : N(M) \to Sym(TM)$ we get the following basic proposition concerning the nullity distribution.

Proposition 2.13 *Let $M \subset \widetilde{M}$ be a curvature invariant submanifold of a Riemannian manifold $\widetilde{M}$ i.e. $\tilde{R}^{\top \perp} = 0$. Let V be a $\nabla^\perp$-parallel subbundle of the normal bundle $N(M)$. Define $\mathcal{N}_p = \bigcap_{\xi_p \in V} ker(A^{\xi_p})$. Assume that the dimension $\dim(\mathcal{N}_p)$ of the space $\mathcal{N}_p$ does not depend on $p \in M$. Then the distribution $\mathcal{N}$ is autoparallel, i.e. $\nabla_X Y \in \Gamma \mathcal{N}$ for $X,Y \in \Gamma \mathcal{N}$. In other words, the distribution $\mathcal{N}$ is integrable and its leaves are totally geodesic submanifolds of M. Moreover, if $V = N(M)$ then the leaves of $\mathcal{N}$ are totally geodesic submanifolds of $\widetilde{M}$.*

Proof. The first part is a direct consequence of Proposition 2.12. Assume now that $V = N(M)$. If X,Y are tangent vector fields to a leave L of $\mathcal{N}$ then $\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y)$ where $\nabla_X Y \in \Gamma TL$ by the first part. Then equation (3) shows that $\alpha(X,Y) = 0$. Hence, the second fundamental form of the leave as submanifold of $\widetilde{M}$ vanish identically. $\square$

Let us recall the following remarkable theorem by S.S. Chern.

Theorem 2.14 *[Ch] Let M be a Kähler-Einstein hypersurface of the complex space form $\mathbb{S}_c^n = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$. Then, if $c \leq 0$ then M is totally geodesic. If $c = 4$ then either M is totally geodesic or M is congruent to an open subset of the complex quadric $Q^{n-1} := \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_0^2 + \cdots + z_n^2 = 0\}$.*

We also recall the following result of Chen and Ogiue. For completeness we give a simple proof of this theorem without the use of the above theorem by S.S. Chern.

Theorem 2.15 *[ChO1] Let M be a Kähler submanifold of $\mathbb{S}^n_c$. If M admits a parallel unit normal vector field then $c = 0$ and M is non-full.*

Proof. Assume that there exists a $\nabla^\perp$-parallel normal section $\xi \neq 0 \in \Gamma(N(M))$ and that $c \neq 0$. By putting $\eta = J\xi$ in Gauss-Codazzi equation (11) and by using equation (4) we get $(A^\xi)^2 = -\frac{c}{4}\|\xi\|^2$. This shows that $c < 0$ and that the shape operator A^ξ has only two constant eigenvalues $\lambda, -\lambda$. Let $V = ker(A^\xi - \lambda Id)$ be the eigendistribution of A^ξ. Thus, $V = ker(T^\xi)$ where $T^\xi := A^\xi - \lambda Id$. Observe that T^ξ satisfies Codazzi equation due to Gauss-Codazzi equation (12). Then, Proposition 2.12 implies that V is autoparallel. Similarly we prove that $JV = ker(A^\xi + \lambda Id) = V^\perp$ is autoparallel. Thus, $TM = V \oplus JV$ where both V and $JV = V^\perp$ are autoparallel. Then V and $JV = V^\perp$ are parallel distributions and M is locally reducible. Then, by Theorem 2.11 we obtain $c \geq 0$. This shows that $c = 0$ and then $A^\xi = 0$. Finally, M is non-full since ξ is the restriction to $N(M)$ of a parallel field of $\mathbb{S}^n_c = \mathbb{C}^n$, i.e. $D_X\xi = 0$ for $X \in TM$, where D is the Levi-Civita connection of $\mathbb{C}^n$. $\square$

Corollary 2.16 *The normal holonomy group of a Kähler submanifold $M \subset \mathbb{S}^n_c$ has no fixed point if $c \neq 0$. In particular, the normal holonomy group is not trivial.*

We will need the following theorem by Barros and Santos. For completeness we give a simple proof of this theorem.

Theorem 2.17 *[BS] Let $M \subset \mathbb{S}^n_c$ be a Kähler submanifold. If $\mathrm{hol}_p(\nabla^\perp) = \mathbb{R}J$ then either M is a hypersurface or M is totally geodesic.*

Proof. Note that $\mathrm{hol}_p(\nabla^\perp) = \mathbb{R}J$ if and only if $R^\perp_{X,Y} = \langle HX, Y\rangle J$ where H is a skew-symmetric endomorphism of TM. Gauss-Codazzi equation (11) implies that $[A^\xi, A^\eta] = \langle J\xi, \eta\rangle(H + \frac{c}{2}J)$. Assume that M is not an hypersurface and let ξ, η two unit normals such that the complex lines $\mathbb{C}\xi$ and $\mathbb{C}\eta$ are orthogonal. Then, $[A^\xi, A^\eta] = 0$ and by Proposition 2.7 $A^\xi A^\eta = 0$. Then, $-4(H + \frac{c}{2}J)^2 = 2J(A^\xi)^2 2J(A^\eta)^2 = 0$. Since $H + \frac{c}{2}J$ is skew-symmetric we obtain that $H + \frac{c}{2}J \equiv 0$. Then $[A^\xi, A^\eta] = 0$ for any $\xi, \eta \in N(M)$ and by Proposition 2.7 we obtain $A^\xi \equiv 0$ for any $\xi \in N(M)$. Hence, M is a totally geodesic submanifold. $\square$

We will need the following theorem in which the equivalence $(i) \Leftrightarrow (ii)$ was proved by [Sa].

Theorem 2.18 *Let $M \subset \mathbb{S}^n_c$ be a Kähler submanifold. The following facts are equivalents.*

(i) M has parallel second fundamental form i.e. $\nabla\alpha = 0$;

(ii) The curvature tensor $R^\perp$ of the normal connection $\nabla^\perp$ is parallel i.e. $\nabla R^\perp = 0$;

(iii) The square of the second fundamental form $(A^\xi)^2$ is parallel i.e. $\nabla(A^\xi)^2 = 0$.

Proof. $(ii) \Leftrightarrow (iii)$ follows from Gauss-Codazzi equation (11) by putting $\eta = J\xi$ and a standard polarization argument. $\square$

We will need the following proposition.

Proposition 2.19 *Let* M *be a Kähler submanifold of* $\mathbb{S}_c^n$ *. Assume that the normal bundle* $N(M)$ *splits as* $N(M) = V \oplus V^\perp$ *where* $V \neq 0$ *is a complex* $\nabla^\perp$ *-parallel subbundle of* $N(M)$ *.* $\mathcal{D}_p = \bigcap_{\xi_p \in V} ker(A^{\xi_p})$ *and* $\mathcal{D}'_p = \bigcap_{\xi_p \in V^\perp} ker(A^{\xi_p})$ *. Then*

(i) $A^{\xi_1} A^{\xi_2} = A^{\xi_2} A^{\xi_1} = 0$ *for any* $\xi_1 \in V$ *and* $\xi_2 \in V^\perp$ *,*

(ii) $T_p M = \mathcal{D}'_p + \mathcal{D}_p$ *where the sum is direct if and only if the second fundamental form has no kernel.*

(iii) *If the dimension of* $\mathcal{D}_p$ *(resp.* $\mathcal{D}'_p$ *) does not depend of* $p \in M$ *, the distribution* $\mathcal{D} : p \to \mathcal{D}_p$ *(resp.* $\mathcal{D}' : p \to \mathcal{D}'_p$ *) is autoparallel i.e.* $\nabla_X Y \in \Gamma\mathcal{D}$ *for* $X, Y \in \Gamma\mathcal{D}$ *(resp.* $\nabla_X Y \in \Gamma\mathcal{D}'$ *for* $X, Y \in \Gamma\mathcal{D}'$ *).*

Proof. Since V is $\nabla^\perp$-parallel we have $\langle R^\perp_{X,Y} \xi_1, \xi_2 \rangle = 0$ for $\xi_1 \in V$ and $\xi_2 \in V^\perp$. Since V is complex, Gauss-Codazzi equation (11) shows that $[A^{\xi_1}, A^{\xi_2}] = 0$. Now statement (i) follows from Proposition 2.7. The statement (ii) follows from the formula $T_p M = \sum_{\xi_p \in V} Image(A^{\xi_p}) \oplus \mathcal{D}_p$ and (i). The last statement is a consequence of Proposition 2.13. $\square$

The following theorem is a corollary of a Theorem by Calabi [C].

Theorem 2.20 *Let* M *be a Kähler submanifold of a complex space form* $\mathbb{S}_c^n$ *. If* $c \neq 0$ *then the index of nullity vanish on an open and dense subset of* M *.*

Proof. Since the index of nullity $r(p)$ is the dimension of the space of solutions of the linear system $R(X_p, \cdot) = 0$, we have that $r(q) \leq r(p)$ for $q \in M$ near $p \in M$. Assume that there exists an open subset G such that $r|_G \geq 1$. Then there exist an open subset $G_0 \subset G$ such that $r = r_0 \geq 1$ is constant on G_0. By Proposition 2 in [KNII, Note 16] the distribution $T_0 : p \to T_0(p)$ on G_0 is integrable with totally geodesic leaves. Note that T_0 is a complex distribution. Let L be a leave of the distribution T_0. Gauss-Codazzi equation (5) implies that the curvature tensor of L vanish. So, L is a flat Kähler submanifold of $\mathbb{S}_c^n$ $c \neq 0$. This contradict a Theorem by Calabi [C] which states that a flat Kähler manifold has no complex isometric imbedding in $\mathbb{S}_c^n$, $c \neq 0$. $\square$

3   The holonomy algebra associated with a tensor field.

Let $\pi : E \to M$ be a vector bundle over a manifold M endowed with a connection ∇ and a ∇-parallel Riemannian metric. We denote by $R^\nabla \in \Gamma(End(E) \otimes \Lambda^2 T^* M)$ the curvature form of the connection ∇.

Let $P \in \Gamma(E^k_l)$ be a section of a tensor bundle $\pi^k_l : E^k_l = E^{\otimes k} \otimes E^{*\otimes l} \to M$ for $k, l > 0$. We associate with P and a point $p \in M$ a linear Lie algebra $\mathrm{hol}_p(P) \subset gl(E_p)$ which we call the holonomy algebra of the tensor P at p. For any curve $\gamma : [0,1] \to M$ with $\gamma(1) = p$ we denote by $\tau_\gamma : (E^k_l)_{\gamma(0)} \to (E^k_l)_p$ the parallel transport along γ.

Definition 3.1 *Let $P \in \Gamma(E^k_l)$ be a section of a tensor bundle $\pi^k_l : E^k_l = E^{\otimes k} \otimes E^{*\otimes l} \to M$ for $k, l > 0$. The holonomy algebra $\mathrm{hol}_p(P)$ of P at a point $p \in M$ is defined as the Lie subalgebra $\mathrm{hol}_p(P) \subset gl(E_p)$ generated by all endomorphisms of the form*

$$\tau_\gamma P_q(\xi_1, \ldots, \xi_{l-1}, \eta_1, \ldots, \eta_{k-1}) \qquad q \in M, \xi_1, \ldots, \xi_{l-1} \in E_p, \eta_1, \ldots, \eta_{k-1} \in E^*_p$$

where γ is a curve joint a point $q \in M$ with the point $p \in M$ and

$A(\xi_1, \ldots, \xi_{l-1}, \eta_1, \ldots, \eta_{k-1})$ denotes the contraction of a tensor $A \in (E^k_l)_p$ with vectors ξ_i and covectors η_j. The linear group $\mathrm{Hol}_p(P)$ generated by the Lie algebra $\mathrm{hol}_p(P)$ is called the holonomy group of P at p.

Remark 3.2 *Ambrose-Singer Theorem states that the holonomy algebra $\mathrm{hol}_p(\nabla)$ of the connection ∇ at a point p is the same as the holonomy algebra of the curvature tensor R^∇ :*

$$\mathrm{hol}_p(\nabla) = \mathrm{hol}_p(R^\nabla).$$

Proposition 3.3 *The holonomy algebras of a tensor P at different points $p, q \in M$ are conjugated: $\mathrm{hol}_p(P) = \tau_c^{-1}\mathrm{hol}_q(P)\tau_c$, where c is a curve jointed p and q.*

Proof. Let c be any C^1 piecewise smooth curve from p to another point $q \in M$. Let $\tau_\gamma P$ be parallel transport of P, where γ is a curve ending at p. Then, $\tau_c \circ \tau_\gamma P = \tau_{c.\gamma}P \in \mathrm{hol}_q(P)$. Thus, $\tau_c \mathrm{hol}_p(P)\tau_c^{-1} \subset \mathrm{hol}_q(P)$. So, by interchanging p and q we obtain $\tau_c^{-1} \mathrm{hol}_q(P)\tau_c \subset \mathrm{hol}_p(P)$, which implies $\mathrm{hol}_p(P) = \tau_c^{-1}\mathrm{hol}_q(P)\tau_c$. $\square$

Corollary 3.4 *Let P be a tensor field and let $V_0 \subset E$ be the maximal subbundle of E such that $\mathrm{hol}_p(P)$ acts trivially on $(V_0)_p$ for $p \in M$. Then, V_0 is a parallel subbundle of E.*

Remark 3.5 *If M is a Riemannian manifold and P is an $\mathfrak{so}(E)$-valued 2-form then $\mathrm{hol}_p(P) \subset \mathfrak{so}(E_p)$.*

We denote by

$$\mathcal{R}(E_p) = \{P \in Sym(\Lambda^2 E_p) : \beta(P) = 0\}$$

the space of Riemannian curvature tensors of the Euclidean space E_p, i.e. the space of symmetric endomorphisms of the exterior square $\Lambda^2 E_p$, which satisfy the Bianchi identity

$$\beta(P)(\xi_1, \xi_2, \xi_3, \xi_4) = \langle P(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4 \rangle + \langle P(\xi_3 \wedge \xi_1)\xi_2 \wedge \xi_4 \rangle + \langle P(\xi_2 \wedge \xi_3), \xi_1 \wedge \xi_4 \rangle = 0$$

A section $P \in \Gamma(\mathcal{R}(E))$ of the bundle $\mathcal{R}(E)$ of curvature tensors is called a curvature tensor field on E. We identify P with a $\mathfrak{so}$-value 2-form by using the formula: $\langle P(\xi_1 \wedge \xi_2)\xi_3, \xi_4 \rangle \overset{\sim}{=} \langle P(\xi_1 \wedge \xi_2), \xi_4 \wedge \xi_3 \rangle$, $\xi_i \in E_p$.

3.1 De Rham decomposition.

We prove the following De Rham type theorem:

Proposition 3.6 *Let $\pi : E \to M$ be a vector bundle with a connection ∇ and ∇-invariant Riemannian metric $g^E = \langle , \rangle$. Let $P \in \Gamma(\mathcal{R}(E))$ be a non zero curvature tensor field, $\mathrm{hol}_p(P) \subset \mathfrak{so}(E_p)$ its holonomy algebra at a point $p \in M$ and*

$$E_p = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \qquad \dim(V_i) > 1 \text{ for } i > 0$$

the decomposition of $\mathrm{hol}_p(P)$-module E_p into direct sum of trivial module V_0 and irreducible modules V_i, $i = 1, \ldots, k$. Then

$$\mathrm{hol}_p(P) = h_1 \oplus \cdots \oplus h_k \qquad (\text{direct sum of ideals})$$

such that the ideal h_i $i = 1, \ldots, k$ acts irreducibly on V_i and acts trivially on the other modules V_j. Moreover, the restriction $h_i|_{V_i}$ is either isomorphic to the isotropy representation of an irreducible Riemannian symmetric space or to one of the linear Lie algebras $\mathfrak{su}_n, \mathfrak{sp}_n, \mathfrak{spin}_7$ or $\mathfrak{g}_2$.

Corollary 3.7 *Under the above assumptions, $\mathrm{Hol}_p(P) \subset SO(E_p)$ is a closed (hence, compact) subgroup of $SO(E_p)$.*

Proof of Proposition 3.6. For $\xi \in E_p$ we denote by ξ^i its projection to V_i. We need the following lemma of Simons.

Lemma 3.8 *[Si, pp. 217-218] Let $R = \tau_\gamma P_q$ be the parallel transport of the curvature tensor $P_q \in \mathcal{R}(E_q)$ to $p \in M$ along a curve γ joining $q \in M$ with p. Then for any $\xi, \eta \in E_p$*

(i) $R(\xi^i, \eta^j) = 0$ if $i \neq j$;

(ii) $R(\xi^i, \eta^i)V_j = \{0\}$ if $i \neq j$;

Proof of lemma. If $i \neq j$ and $u, v \in E_p$ then

$$\langle R(\xi^i, \eta^j)u, v \rangle = \langle R(u, v)\xi^i, \eta^j \rangle = 0$$

because $R \in \mathrm{hol}_p(\mathcal{P})$ and $\mathrm{hol}_p(P)$ leaves V_i invariant. This prove (i). Using Bianchi identity for R, we get

$$R(\xi^i, \eta^i)u^j = R(u^j, \xi^i)\eta^i - R(\eta^i, u^j)\xi^i = 0$$

by part (i), which proves (ii). $\square$

Now we continue the proof of Proposition. The first statement of Proposition follows from the above lemma:

$$\mathrm{hol}_p(P) = h_1 \oplus \cdots \oplus h_k$$

where h_i is generated by all the curvature tensors $(\tau_\gamma P)(\xi^i, \eta^i), \xi^i, \eta^i \in V_i$. The last assertion follows from the following classical theorem by M. Berger. $\square$

Theorem 3.9 *(M. Berger, see [Be]) Let $\mathfrak{g} \subset \mathfrak{so}(V), V = \mathbb{R}^n$ be an irreducible subalgebra of the orthogonal algebra $\mathfrak{so}(V)$ i.e. its action on V is irreducible. Assume that there exists a non zero curvature tensor which is a $\mathfrak{g}$-valued 2-form.*

Then, either $\mathfrak{g}$ is the holonomy algebra of an irreducible Riemannian symmetric space or $\mathfrak{g} = \mathfrak{su}_n, \mathfrak{sp}_n, \mathfrak{spin}_7$ or $\mathfrak{g}_2$. In the last case all $\mathfrak{g}$-valued curvature tensors has zero Ricci tensor.

4 The normal curvature tensor associated to a Kähler submanifold of $\mathbb{S}_c^n$.

In this section we will associate with a Kähler submanifold M of a complex space form $\mathbb{S}_c^n$ a curvature tensor field P in the normal bundle $N(M)$ of M and prove that (under some conditions) the holonomy algebra $\mathrm{hol}_p(\nabla^\perp)$ of the normal connection $\nabla^\perp$ coincides with the holonomy algebra of P:

$$\mathrm{hol}_p(\nabla^\perp) = \mathrm{hol}_p(P).$$

We keep the notation of section 2.1, in particular A^ξ denote the shape operator of $M \subset \mathbb{S}_c^n$ in direction of a normal vector $\xi \in N(M)$.

We define a linear map $\bar{R}^\perp : \Lambda^2(TM) \to \Lambda^2(N(M))$ by

$$\langle \bar{R}^\perp(X \wedge Y), \xi \wedge \eta \rangle := \langle [A^\xi, A^\eta]X, Y \rangle, \text{ where } X, Y \in T_pM, \xi, \eta \in N_p(M)$$

and denote by $\bar{R}^{\perp*} : \Lambda^2(N(M)) \to \Lambda^2(TM)$ the conjugated linear map, given by

$$\bar{R}^{\perp*}(\xi \wedge \eta) = [A^\xi, A^\eta] \in \mathfrak{so}(T_pM) = \Lambda^2(T_pM).$$

Then the second Gauss-Codazzi equation (11) of a Kähler submanifold of $\mathbb{S}_c^n$ can be rewritten as

$$R^\perp_{X,Y} = \bar{R}^\perp(X \wedge Y) - \frac{c}{2}\langle JX, Y \rangle J. \tag{15}$$

Definition 4.1 *The tensor field $P := \bar{R}^\perp \circ \bar{R}^{\perp*}$ defined by the formula*

$$\langle P(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4 \rangle = \langle [A^{\xi_1}, A^{\xi_2}], [A^{\xi_3}, A^{\xi_4}] \rangle = -tr([A^{\xi_1}, A^{\xi_2}], [A^{\xi_3}, A^{\xi_4}])$$

is called the normal curvature tensor of the Kähler submanifold M.

We identify the normal curvature tensor P with a $\mathfrak{so}(N(M))$-valued 2-form using the formula $\langle P(\xi_1, \xi_2)\xi_4, \xi_3 \rangle = \langle P(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4 \rangle$.

Lemma 4.2 *The tensor field P is a field of Kähler curvature tensor, i.e $P \in \mathcal{R}(N(M))$ and $[P(\xi_1, \xi_2), J] = 0$. Moreover, it has non-positive sectional curvature and $P \equiv 0$ if and only if M is a totally geodesic submanifold of $\mathbb{S}_c^n$.*

Proof. The crucial fact that the tensor P satisfied Bianchi identity was remarked by C. Olmos [Ol1]. It follows from the identity

$$\langle P(\xi_1, \xi_2)\xi_3, \xi_4\rangle = -\langle P(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4\rangle = tr([A^{\xi_1}, A^{\xi_2}], [A^{\xi_3}, A^{\xi_4}]) =$$

$$= tr(A^{\xi_2}[A^{\xi_3}, A^{\xi_4}]A^{\xi_1}) - tr((A^{\xi_1}[A^{\xi_3}, A^{\xi_4}]A^{\xi_2})^t) = 2tr(A^{\xi_2}[A^{\xi_3}, A^{\xi_4}]A^{\xi_1}) =$$

$$= 2tr(A^{\xi_1}A^{\xi_2}A^{\xi_3}A^{\xi_4}) - 2tr(A^{\xi_3}A^{\xi_1}A^{\xi_2}A^{\xi_4}).$$

Using the cyclic sum we get $\beta(P) = 0$. The claim that P commute with J follows from the identity (4) as follows:

$$\langle P(\xi_1, \xi_2)J\xi_3, \xi_4\rangle = \langle P(\xi_1 \wedge \xi_2), \xi_4 \wedge J\xi_3\rangle = \langle [A^{\xi_1}, A^{\xi_2}], [A^{\xi_4}, A^{J\xi_3}]\rangle =$$

$$\langle [A^{\xi_1}, A^{\xi_2}], -J(A^{\xi_4}A^{\xi_3} + A^{\xi_3}A^{\xi_4})\rangle = -\langle [A^{\xi_1}, A^{\xi_2}], [A^{J\xi_4}, A^{\xi_3}]\rangle =$$

$$-\langle P(\xi_1 \wedge \xi_2), J\xi_4 \wedge \xi_3\rangle = -\langle P(\xi_1, \xi_2)\xi_3, J\xi_4\rangle = \langle JP(\xi_1, \xi_2)\xi_3, \xi_4\rangle.$$

So, $P(\xi_1, \xi_2)J\xi_3 = JP(\xi_1, \xi_2)\xi_3$. To prove the last statement note that the sectional curvature $\langle P(\xi, \eta)\eta, \xi\rangle = -\langle P(\xi \wedge \eta), \xi \wedge \eta\rangle = -\|[A^\xi, A^\eta]\|^2$. Thus, the sectional curvature of P vanish identically if and only if all the shapes operators commute. Now the last statement follows from Proposition 2.7. $\square$

We denote by $\bar{R}^\perp_{X,Y}$ a skew-symmetric endomorphism of $N(M)$ defined by $\langle \bar{R}^\perp_{X,Y}\xi, \eta\rangle = \langle \bar{R}^\perp(X \wedge Y), \xi \wedge \eta\rangle$.

The following lemma link the normal holonomy algebra $\mathrm{hol}_p(\nabla^\perp)$ with the holonomy algebra $\mathrm{hol}_p(P)$. We denote by $\mathrm{hol}_p(\bar{R}^\perp) \subset \mathfrak{so}(N_p(M))$ the holonomy algebra of the $\mathfrak{so}(N(M))$-valued 2-form $\bar{R}^\perp$.

Proposition 4.3 *Let M^m be a Kähler submanifold of $\mathbb{S}^n_c$ and $P \in Sym(\Lambda^2(N(M)))$ the normal curvature tensor of M. Then*

(i) $P(\Lambda^2 N(M)) = \bar{R}^\perp(\Lambda^2 TM)$;

(ii) $\mathrm{hol}_p(P) = \mathrm{hol}_p(\bar{R}^\perp)$;

(iii) $\mathrm{hol}_p(\nabla^\perp) \equiv \mathrm{hol}_p(P) \ (mod(\mathbb{R}J))$, *i.e.* $\mathrm{hol}_p(\nabla^\perp) + \mathbb{R}J = \mathrm{hol}_p(P) + \mathbb{R}J$.

Proof. Part (i) follows from the identity $ker(\bar{R}^\perp) = Image(\bar{R}^{\perp*})^\perp$. Statement (ii) follows from the fact that the subspace of skew-symmetric endomorphism of $N_q(M)$ spanned by P and $\bar{R}^\perp$ at each point q of M are the same by part (i). To prove the last statement, note that equation (15) implies that

$$\mathrm{hol}_p(\bar{R}^\perp) \equiv \mathrm{hol}_p(R^\perp) \ (mod(\mathbb{R}J)).$$

Now, (iii) follows from (ii). $\square$

Proposition 4.4 *Let M be a complex totally geodesic submanifold of $\mathbb{S}^n_c$ $c \neq 0$. Then, the normal holonomy algebra $\mathrm{hol}(\nabla^\perp)_p$ at $p \in M$ is $\mathrm{hol}(\nabla^\perp)_p = \mathbb{R}J_p$.*

Proof. If M is totally geodesic then $\bar{R}^\perp \equiv 0$. Then, equation (15) implies that $R^\perp_{X,Y} = -\frac{c}{2}\langle JX, Y\rangle J$. So, by the Ambrose-Singer Theorem we have $\text{hol}(\nabla^\perp)_p = \mathbb{R}J$. $\square$

The following lemma is a generalization of Lemma 2.1 in [DS].

Lemma 4.5 *Let* $M^m \subset \mathbb{S}^n_c$ *be a Kähler submanifold* $(m < n)$. *Then* M *is full if and only if the holonomy algebra* $\text{hol}(P)$ *has no invariant vector, i.e.* $\text{hol}(P)\xi = 0$ *implies* $\xi = 0$.

Proof. Let $V_0 = ker(\text{hol}(P))$ be the set of fixed points of $\text{hol}(P)$. By Corollary 3.4 V_0 is a parallel subbundle of $N(M)$. Recall that the sectional curvature $\langle P(\xi, J\xi)J\xi, \xi\rangle = -\langle P(\xi \wedge J\xi), \xi \wedge J\xi\rangle = -\|[A^\xi, A^{J\xi}]\|^2 = -\|2(A^\xi)^2\|^2$. Then $V_0 \subset N^1(M)^\perp$. So, if M is full then by Theorem 2.3 V_0 must be trivial. Conversely, let $M \subset \overline{M}$ be a non-full submanifold i.e. $\overline{M}$ is complex totally geodesic. Then it is not hard to check that $N(\overline{M}) \subset V_0$. $\square$

Theorem 4.6 *Let* $M^m \subset \mathbb{S}^n_c$ *be a Kähler submanifold* $(m < n)$ *and* P *its normal curvature tensor. Then*

(i) $\text{hol}_p(P)$ *is isomorphic to the holonomy algebra of a Hermitian symmetric space with flat factor* $V_0 = Ker(\text{hol}(P))$;

(ii) M *is full if and only if* $J_p \in \text{hol}_p(P)$;

(iii) *If* M *is full then* $\text{hol}_p(\nabla^\perp) \subset \text{hol}_p(P)$ *and* $\dim(\text{hol}_p(P)) - \dim(\text{hol}_p(\nabla^\perp)) \leq 1$. *Moreover,* $\text{hol}_p(\nabla^\perp)$ *is an ideal of* $\text{hol}_p(P)$.

Proof. (i) By Proposition 3.6, it is sufficient to check that a h_i-valued curvature tensor can not be Ricci flat for $i > 0$. By part (iii) of Lemma 4.2 P has non-positive sectional curvature, so its restriction to V_i is Ricci flat if and only if its restriction $P|_{V_i}$ is zero and then $i = 0$. Part (ii) is a direct consequence of the above lemma and part (i). For (iii) note that part (ii) and part (iii) of Proposition 4.3 we have that $\text{hol}_p(\nabla^\perp) \subset \text{hol}_p(P)$ and the codimension is one. So, $\text{hol}_p(P) = \text{hol}_p(\nabla^\perp) + \mathbb{R}J_p$ and since J is parallel we have that $\text{hol}_p(\nabla^\perp)$ is an ideal of $\text{hol}_p(P)$. $\square$

4.1 Proof of the main theorems.

Now we can prove our principal results.

Proof of Theorem 1.1. If $c = 0$ the result follows from Theorem 3.1 in [Ol1] and Lemma 1 in [DS]. Let us assume that $c \neq 0$. If the normal curvature tensor P vanishes then Lemma 4.2 and Proposition 4.4 implies that $\text{hol}_p(\nabla^\perp) = \mathbb{R}J$. Since $\text{Hol}_p(\nabla^\perp)$ acts irreducible on $N_p(M)$ we conclude that M is a complex totally geodesic hypersurface of $\mathbb{S}^n_c$, hence $\text{Hol}_p(\nabla^\perp)$ is isomorphic to the isotropy group of the Hermitian symmetric space $\mathbb{C}P^1$.

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Assume now that $P \neq 0$. So, we can apply Proposition 3.6 and its corollaries with $E = N_p(M)$ to decompose $\text{hol}_p(P) = h_1 \oplus \cdots \oplus h_k$ and $N_p(M) = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ into $\text{Hol}_p(P)$-invariant subspaces. Since $\text{Hol}_p(\nabla^\perp)$ acts irreducibly, Corollary 3.4 implies that $V_0 = \{0\}$. Then part (iii) in Theorem 4.6 implies that $\text{Hol}_p(P)$ acts irreducibly on $N_p(M)$, hence $N_p(M) = V_1$. Part (i) of Theorem 4.6 shows that $\text{hol}_p(P)$ is isomorphic to the holonomy algebra of an irreducible Hermitian symmetric space i.e. one of the groups in Table 1.

Assume that $\text{hol}_p(\nabla^\perp) \subsetneqq \text{hol}_p(P)$. Then part (iii) of Theorem 4.6 shows that $\text{hol}_p(P) - \dim(\text{hol}_p(\nabla^\perp)) = 1$ and $\text{hol}_p(\nabla^\perp)$ is an ideal of $\text{hol}_p(P)$. Then either $\text{hol}_p(\nabla^\perp) = \mathbb{R}J$ or $\text{hol}_p(\nabla^\perp) \subset su(N_p)$. It remains to study these two cases.

Case I. $\text{hol}_p(\nabla^\perp) = \mathbb{R}J$. By Theorem 2.17 M must be either a hypersurface or a totally geodesic submanifold. Since M is full, M is an hypersurface and then $\text{hol}_p(P) = \text{hol}_p(\nabla^\perp)$ which is a contradiction with the hypothesis $\text{hol}_p(\nabla^\perp) \neq \text{hol}_p(P)$.

Case II. $\text{hol}_p(\nabla^\perp) \subset su(N_p)$. This means that the Ricci form $\rho(R^\perp)$ of the normal bundle vanish identically. So by Theorem 2.8 we have that $Ric_{\mathbb{S}^n_c}|_M = Ric_M = c\frac{n+1}{2}\langle,\rangle$. Thus M is Kähler-Einstein submanifold of $\mathbb{S}^n_c$. If $c \geq 0$ then we obtain a contradiction with equation (14). If $c < 0$ a theorem of Umehara [Um] states that M is totally geodesic, hence non-full. This complete the proof of Theorem 1.1. $\square$

We need the following proposition.

Proposition 4.7 *Let $G \subset U(n) \subset SO(2n)$ be a subgroup which acts $\mathbb{C}$-irreducible on $\mathbb{C}^n = \mathbb{R}^{2n}$. Then, either G acts $\mathbb{R}$-irreducible or there exits a G-invariant orthogonal $\mathbb{R}$-splitting $\mathbb{R}^{2n} = V \oplus iV$.*

Proof. Assume that $V \subset \mathbb{R}^{2n}$ is a G-invariant subspace. Then, the subspaces $V + iV$ and $V \cap iV$ are complex and G-invariant. Since G acts $\mathbb{C}$-irreducible we obtain that $\mathbb{R}^{2n} = V \oplus iV$ (a direct sum of real vector spaces). In order to show that $V \perp iV$ we define a new inner product b on $\mathbb{C}^n = V \oplus iV$ such that $V \perp iV$, $b|_V = \langle,\rangle|_V$ and $b|_{JV} = \langle,\rangle|_{JV}$, where $\langle,\rangle$ denote the standard inner product on $\mathbb{C}^n$. Note that b is G-invariant and J invariant. Then $b(X,Y) = \langle BX, Y\rangle$ where B is a symmetric operator which commutes with J and G. Now Schur's Lemma implies that $B = mId$ and $b = m\langle,\rangle$ where $m \in \mathbb{R}^+$. Hence V and JV are perpendicular with respect to the standard inner product $\langle,\rangle$. $\square$

Proof of Theorem 1.2. It is standard to show that the index of relative nullity ν is zero in a small ball B_p at $p \in M$. Note that $\text{Hol}_p(\nabla^\perp) \subset U(N_pM) \subset SO(N_p)$. Suppose that there exists a nontrivial complex $\text{Hol}_p(\nabla^\perp)$-invariant subspace $V_p \subset N_p(M)$. Then the normal bundle on B_p splits as $N(M) = V \oplus V^\perp$ where V is a complex $\nabla^\perp$-parallel subbundle of the normal bundle $N(M)$. Define $D_q := \bigcap_{\xi \in V} ker(A^\xi)$ and $D'_q := \bigcap_{\xi \in V^\perp} ker(A^\xi)$ for $q \in B_p$. Proposition 2.19 shows that $TM = D \oplus D'$ (a orthogonal sum). Since $\dim(D_q) \leq \dim(D_p)$ (resp. $\dim(D'_q) \leq$

$\dim(D'_p))$ for $q \in B_p$, the dimension of D_p (resp. D'_p) is constant on B_p. Then by (iii) of Proposition 2.19 D and D' are autoparallel, hence parallel. Since M is full, Theorem 2.3 implies that D is not trivial, i.e. $D \neq \{0\}, TM$. So, M splits locally around p as $M = M_1 \times M_2$ where M_1 (resp. M_2) is the leaf through p of D (resp. of $D^\perp$). Then, by putting $X \in TM_1$ and $Y \in TM_2$ in equation (13) we obtain that $c = 0$. Now Theorem 2.11 shows that M is extrinsically a product at $p \in M$.

Assume now that $\mathrm{Hol}_p(\nabla^\perp)$ acts $\mathbb{C}$-irreducible. By Proposition 4.7 if $\mathrm{Hol}_p(\nabla^\perp)$ does not act irreducible on $N_p(M)$ there exist a orthogonal splitting $N_p(M) = V_p \oplus JV_p$. Then the normal bundle can be written as $N(M) = V \oplus JV$ on B_p. Define $D_p := \bigcap_{\xi \in V} ker(A^\xi)$ and $D'_p := \bigcap_{\xi \in V^\perp} ker(A^\xi)$. Since $JV = V^\perp$ we get that $D^\perp = D' = JD$. Then it is not difficult to check by using equations (3) and (4) that the second fundamental form α of M vanish on B_p. Then Proposition 2.2 implies that M is non-full and the theorem is proved. □

It is well known that in Hermitian symmetric spaces the complex structure J belongs to the holonomy algebra. So, Corollary 1.4 is a consequence of the above theorems.

Proof of Theorem 1.5. Let M^2 be a full complex surface of $\mathbb{S}^n_c$. If the holonomy group $\mathrm{Hol}_p(\nabla^\perp)$ acts irreducible on the normal space $N_p(M)$ then the result follows from Theorem 1.1. Assume now that $\mathrm{Hol}_p(\nabla^\perp)$ acts reducible on the normal space. Then $N_p(M) = V \oplus V^\perp$ where $V, V^\perp$ are $\nabla^\perp$-parallel subbundles.

Suppose now that V is a complex subbundle. Then the distributions D and D' defined as in the proof of Theorem 1.2 are complex distributions on M such that $TM = D + D'$, hence $\dim_\mathbb{C}(D) \leq 2$. Since we assume M full we get that $1 = \dim_\mathbb{C}(D) = \dim_\mathbb{C}(D')$. In fact, if $D = TM$ (resp. $D = \{0\}$) then $V \perp N^1(M)$ (resp. $V^\perp \perp N^1(M)$) and M is non-full by Theorem 1.9. Then we obtain the same contradiction as in Theorem 1.2.

Suppose now that $\mathrm{Hol}_p(\nabla^\perp)$ acts $\mathbb{C}$-irreducible but not irreducible. By using Proposition 4.7 we can assume that $JV = V^\perp$. Then a similar argument as in proof of Theorem 1.2 shows that the second fundamental form α of M vanish. Since we assume M full we get a contradiction.

Thus, $\mathrm{Hol}_p(\nabla^\perp)$ acts irreducibly on the normal space and the theorem is proved. □

Proof of Theorem 1.6. Let M^1 be a full complex curve of $\mathbb{S}^n_c$. Note that the second fundamental form has no kernel. So, by Theorem 1.2 and Theorem 1.1 the holonomy group $\mathrm{Hol}_p(\nabla^\perp)$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, i.e. one of the groups in Table 1.Since only the group $U(n-1)$ from Table 1 acts transitively on the unit sphere it is sufficient to prove that $\mathrm{Hol}_p(\nabla^\perp) = \mathrm{Hol}_p(P)$ acts transitively on the unit sphere of $N_p(M)$. If $\mathrm{Hol}_p(P)$ does not acts transitively on the unit sphere then by a theorem of J. Simons [Si, Thm. 4] the holonomy system $[N_p(M), P, \mathrm{Hol}_p(P)]$ is

symmetric. Hence, the holonomy algebra $\mathrm{hol}_p(P)$ is spanned by the endomorphism $P(\xi, \eta)$. This implies that the first normal space $N_p^1(M)$ is equal to the normal space $N_p(M)$. In fact, otherwise there exist a $\xi \in N_p(M)$ such that $A^\xi = 0$ and this implies that $P\xi = 0$ i.e. $\mathrm{hol}_p(P)\xi = 0$, which contradict irreducibility. So, $\dim_\mathbb{C}(N_p(M)) = \dim_\mathbb{C}(N_p^1(M)) \leq 1$. It is clear that also in this case $\mathrm{Hol}_p(P)$ acts transitively on the unit sphere. $\square$

4.2   Holonomy group of Kähler submanifolds of $\mathbb{C}^n$ and $\mathbb{C}H^n$.

In this section we give the proof of Theorem 1.7. The main part of the proof of statement $ii)$ was proved in [DS1],[DS].

Proof of Theorem 1.7. $i)$ Theorem 2.11 implies that a Kähler submanifold $M^n \subset \mathbb{C}H^{n+p}$ is locally irreducible. The classification of irreducible restricted holonomy groups by M. Berger shows that either $\mathrm{Hol}(\nabla)$ is $U(n), SU(n), Sp(n)$ or M is a locally irreducible Hermitian symmetric space [Be]. So if $\mathrm{Hol}(\nabla) \neq U(n)$ then M is an Einstein manifold. Then Umehara's Theorem [Um] implies that M is a totally geodesic submanifold which contradict that M is full. Thus, we get $\mathrm{Hol}(\nabla) = U_n$ which proves $i)$.

To prove $ii)$, we remark that by Theorem 2.11 there exists a local decomposition $M = \mathbb{C}^r \times M_1^{n_1} \times \cdots \times M_k^{n_k} \subset \mathbb{C}^{n+p} = \mathbb{C}^r \times \mathbb{C}^{n_1+p_1} \times \cdots \times \mathbb{C}^{n_k+p_k}$. The same argument as in part $i)$ shows that the holonomy group of each factor $M_k^{n_k}$ is U_{n_k}. Finally, the last claim follows from [DS]. $\square$

4.3   Non-full Kähler submanifolds.

Proof of Theorem 1.9. The first statement follows from Gauss-Codazzi equation (11). The last statement is an immediate consequence of Theorem 1.1. $\square$

Proof of Theorem 1.11. By a theorem of Calabi [C] a Kähler submanifold M of $\mathbb{S}_c^n$ is rigid, i.e. any isometric and holomorphic immersion $\tilde{f}$ of M in $\mathbb{S}_c^n$ is congruent to the original one. So, it is sufficient to prove that μ depends only on the submanifold $M \subset \mathbb{S}_c^{\overline{m}}$. Suppose that $M \subset \mathbb{S}_c^{\overline{m}} \subset \mathbb{S}_c^n \subset \mathbb{S}_c^{n'}$. Let $\mathrm{diag}(J, \mu' J)$ be the element of $\mathrm{hol}_p(\nabla^\perp)$ with respect to the immersion $M \subset \mathbb{S}_c^{\overline{m}} \subset \mathbb{S}_c^{n'}$. Note that the normal bundle of the totally gedesic submanifold $\mathbb{S}_c^{\overline{m}} \subset \mathbb{S}_c^n$ along M is invariant with respect to the normal connection of the submanifold $M \subset \mathbb{S}_c^{n'}$. This shows that $\mu = \mu'$, which completes the proof. $\square$

Proof of Theorem 1.12. Assume that $\mu = 0$. Then Theorem 1.1 and the fact that a one dimensional representation of a semisimple Lie algebra is trivial imply that

$$\mathrm{hol}_p(\nabla^\perp) = \mathbb{R}J \oplus \mathrm{diag}(\mathfrak{k}, 0)$$

where $\mathfrak{k}$ is one of the holonomy algebra of the holonomy groups K in Table 1. Now statement (a) is clear, since the center of $\mathfrak{k}$ is $1-$dimensional.

Assume now that $\mu \neq 0$. Then there exists only one endomorphism of the form $\mathrm{diag}(J, \mu J)$ in $\mathrm{hol}_p(\nabla^\perp)$. Thus,

$$\mathrm{hol}_p(\nabla^\perp) = \mathbb{R}\mathrm{diag}(J, \mu J) \oplus \mathrm{diag}(\mathfrak{k}', 0)$$

where $\mathfrak{k}'$ is the semisimple part of the holonomy algebra $\mathfrak{k} = \mathbb{R}J \oplus \mathfrak{k}'$ of one of the groups K in Table 1. So, it is clear that $\dim(\mathrm{center}(\mathrm{hol}_p(\nabla^\perp))) = 1$ and that $\mu \in \mathbb{Q}$ if and only if $\mathrm{Hol}_p(\nabla^\perp)$ is compact. Note that $J_p \in \mathrm{hol}_p(\nabla^\perp)$ if and only if $\mu = 1$. This completes the proof. $\square$

4.4 The complex quadric. Proof of Theorems 1.8, 1.17 and 1.16.

Proof of Theorem 1.8. It is clear that if $M \subset \mathbb{C}^n$ is non-full then $J \notin \mathrm{Hol}(\nabla^\perp)$. Conversely, it was proved in [DS] that if $M \subset \mathbb{C}^n$ is full then $\mathrm{Hol}(\nabla^\perp)$ is isomorphic to the isotropy group of a Hermitian symmetric space without flat factor. So, $J \in \mathrm{Hol}(\nabla^\perp)$ and this show (ii).

Assume now that $c \neq 0$. Then, if $M \subset \mathbb{S}_c^{n+2}$ is full, statements (i) and (iii) follows from Corollary 1.4. So, we can assume that M is contained in a proper totally geodesic hypersurface $\overline{M} := \mathbb{S}_c^{n+1}$. Let $V_1 = N\overline{M}$ be the normal bundle of $\overline{M}$. Then, the normal bundle $N(M)$ decompose into $\nabla^\perp$-parallel complex bundles, as $N(M) = V_1 \oplus V_2$ where $V_2 := N(M) \cap T\overline{M}$. Let $X, Y, Z \in \Gamma(TM)$ be vector fields on M. Recall that $(\nabla_Z R_{X,Y}^\perp)_p \in \mathrm{hol}_p(\nabla^\perp)$. From Gauss-Codazzi equation (11) we obtain that:

$$R_{X,Y}^\perp \xi_1 = -\frac{c}{2}\langle JX, Y\rangle J\xi_1$$

$$R_{X,Y}^\perp \xi_2 = (-\frac{c}{2}\langle JX, Y\rangle - \langle 2J(A^{\xi_2})^2 X, Y\rangle)J\xi_2$$

where $\xi_1 \in \Gamma V_1$ and $\xi_2 \in \Gamma V_2$, and A^ξ is the shape operator of M. Here we used the condition $\dim_{\mathbb{C}}(V_1) = \dim_{\mathbb{C}}(V_2) = 1$. Thus, $(\nabla_Z R_{X,Y}^\perp)_p \xi_1 = 0$ and $(\nabla_Z R_{X,Y}^\perp)_p \xi_2 = \langle -2J(\nabla_Z(A^{\xi_2})^2)X, Y\rangle_p J_p \xi_2$. Then, $J_p \notin \mathrm{hol}_p(\nabla^\perp)$ if and only if $\nabla_{Z_p}(A^\xi)^2 = 0$. By Theorem 2.18 we have $J_p \notin \mathrm{hol}_p(\nabla^\perp)$ if and only if M is an hypersurface of $\overline{M}$ with parallel second fundamental form. Hence, by [Kon, Remark 3, p.1148] we obtain that M is Einstein. Then, Theorem 2.14 implies that M is either totally geodesic or the complex quadric $Q^n \subset \overline{M}$. Thus, if $c < 0$ then M is a totally geodesic submanifold which shows (i). If $c > 0$ and $J_p \notin \mathrm{Hol}_p(\nabla^\perp)$ then Proposition 4.4 shows that M is not a totally geodesic submanifold, hence M is an open subset of the complex quadric.

Finally, assume that M is an open subset of the complex quadric. The shape operator A^ξ of M satisfies $(A^\xi)^2 = Id$ [Re]. So, the above equalities for the curvature tensor $R^\perp$ can be written as :

$$R_{X,Y}^\perp \xi_1 = -2\langle JX, Y\rangle J\xi_1$$

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$$R^{\perp}_{X,Y}\xi_2 = -4\langle JX, Y\rangle J\xi_2.$$

Thus, $\mathrm{hol}_p(\nabla^{\perp})$ is 1-dimensional algebra spanned by the skew-symmetric endomorphism $\mathrm{diag}(J, 2J)$. This shows that $J_p \notin \mathrm{hol}_p(\nabla^{\perp})$. $\square$

Proof of Theorem 1.17. It is known that the shape operator of the complex quadric satisfies $(A^{\xi})^2 = Id$ [Re]. We prove $(i) \Rightarrow (ii)$. Note that M is full in $\mathbb{S}^n_c$. In fact, if M is contained in a complex totally geodesic hypersurface $\mathbb{S}^{n-1}_c$ of $\mathbb{S}^n_c$ then Gauss-Codazzi equation (11) shows that $[A^{\xi}, A^{\eta}] = 0$ for $\xi \in N(\mathbb{S}^{n-1}_c)$ and $\eta \in N(M) \cap T\mathbb{S}^{n-1}_c$. Then Proposition 2.7 implies that $A^{\xi} A^{\eta} = 0$ which contradict the fact that A^{ξ} is invertible. Now, Gauss-Codazzi equation (11) implies that the curvature tensor $R^{\perp}$ satisfies $\langle R^{\perp}_{X,Y}\xi, J\xi\rangle = \langle JX, Y\rangle\|\xi\|^2(-2f - \frac{c}{2})$. By polarization, we get $\langle R^{\perp}_{X,Y}\xi, J\eta\rangle = 0$ for $\eta \perp \xi$. Thus, $R^{\perp}_{X,Y} = \langle JX, Y\rangle\|\xi\|^2(-2f - \frac{c}{2})J$ which implies that the normal holonomy algebra $\mathrm{hol}_p(\nabla^{\perp})$ is spanned by J. Theorem 2.17 shows that M is a hypersurface. Finally, Gauss-Codazzi equation (9) implies that M is Einstein. Thus, f is constant and the result follows from Theorem 2.14 . $\square$

Proof of Theorem 1.16. We consider two cases according with the index of relative nullity $\nu = inf_{p\in M}\nu(p)$ of M. Let G_{ν} be an open subset of M such that $\dim(\mathcal{N}_p)|_{G_{\nu}} = \nu$.

Case I. $\nu > 0$. So, we choose a unit vector field $X \in \Gamma\mathcal{N}$ such that $JX \in \Gamma\mathcal{N}$. Then Gauss-Codazzi equation (11) gives

$$R^{\perp}_{X,JX} = -\frac{c}{2}J.$$

Thus, $\mathrm{hol}(\nabla^{\perp}) = \mathbb{R}J$ and Theorem 2.17 implies (i) or (ii).

Case II. $\nu = 0$. If $M \subset \mathbb{S}^n_c$ is full then Corollary 1.4 shows that $\mathrm{hol}(\nabla^{\perp}) = \mathbb{R}J$. Then Theorem 2.17 implies (i) or (ii). If $M \subset \mathbb{S}^n_c$ is non-full and $c = 0$ we can reduce codimension (see [DS]) and Corollary 1.4 and Theorem 2.17 implies (iii).

Assume now that $c \neq 0$ and that $M \subset \mathbb{S}^n_c$ is non-full and non-totally geodesic. Then by Theorem 1.9 we have a decomposition $N_p(M) = \overline{N}_p(M) \oplus N_p(\overline{M})$, where $\overline{M}$ is a proper totally geodesic submanifold such that M is full in $\overline{M}$. By the same theorem there exist a constant μ such that $\mathrm{diag}(J, \mu J) \in \mathrm{hol}(\nabla^{\perp})$. Then we can write

$$R^{\perp}_{X,Y} = \langle HX, Y\rangle\mathrm{diag}(J, \mu J)$$

where H is a skew-symmetric endomorphism of TM. Since $c \neq 0$, Gauss-Codazzi equation (11) implies that $\mu \neq 0$. By Gauss-Codazzi equation (11) and identity $A^{\xi} = 0$ for $\xi \in N_p(\overline{M})$ we get $H = -\frac{c}{2\mu}J$. Then for $\xi \in \overline{N}_p(M)$ Gauss-Codazzi equation (11) gives $-2J(A^{\xi})^2 = H + \frac{c}{2}J$. Thus, $(A^{\xi})^2 = cte \cdot Id$. Since $\overline{M} = \mathbb{S}^{n'}_c$ is a totally geodesic submanifold of $\mathbb{S}^n_c$, the shape operator $\overline{A}^{\xi}$ of M as submanifold of $\overline{M}$ is the restriction of the shape operator A^{ξ} of $M \subset \mathbb{S}^n_c$. Now the result follows from Theorem 1.17. $\square$

4.5 Kähler-Einstein submanifolds. Proof of Theorems 1.13, 1.14 and Corollary 1.15.

Proof of Theorem 1.13. Let M be a full Kähler-Einstein submanifold of $\mathbb{C}P^n$. Equation (14) shows that $\sum_i (A^{\xi_i})^2 = kId$ with $k > 0$ (Note that $k \neq 0$ because M is full). Thus, $\mathcal{N} = \{0\}$ and the result follows from Theorem 1.2 and Theorem 1.1. $\square$

Proof of Theorem 1.14. Proposition 2.8 and Remark 2.9 implies

$$(\frac{c}{2}(n+1) - k)\omega(X,Y) = \frac{\langle R^\perp_{X,Y}, J\rangle}{2}$$

where ω is the Kähler form of M. So, by taking successive covariant derivatives of both sides of the above equation we obtain that

$$\langle (\nabla^i R^\perp)_{X,Y,Z_1,\ldots,Z_i}, J\rangle = 0 \qquad X, Y, Z_i \in TM,\, i = 1, 2, \cdots . \qquad (*)$$

Since the normal connection of a Kähler submanifold is analytic, the holonomy algebra $\mathrm{hol}_p(\nabla^\perp)$ is spanned by operators $\langle (\nabla^i R^\perp)_{X,Y,Z_1,\ldots,Z_i}$, $X, Y, Z_i \in TM$ [KNI, p. 101]. There exists real numbers $r, s \in \mathbb{R}$ and tangent vectors X, Y such that

$$r R^\perp_{X,Y} + sT = \mathrm{diag}(J, \mu J)$$

where $T \in span\{\nabla R^\perp, \nabla^2 R^\perp, \ldots\}$.
Then,

$$r\,\mathrm{diag}(R^\perp_{X,Y}|_{T\overline{M}},\, \frac{c}{2}\omega(X,Y)J) + s\,\mathrm{diag}(T, 0) = \mathrm{diag}(J, \mu J)$$

So, $r\frac{c}{2}\omega(X,Y) = \mu$ and since $T \perp J$ (by (*)), we have

$$r(\frac{c}{2}(n+1) - k)\omega(X,Y) = r\frac{\langle R^\perp_{X,Y}, J\rangle}{2} = (\overline{m} - m) + \mu(n - \overline{m})$$

Hence,

$$(\frac{c}{2}(n+1) - k)\frac{2}{c}\mu = (\overline{m} - m) + \mu(n - \overline{m})$$

Now putting $c = 4$ and expressing μ as a function of k, we complete the proof of the theorem. $\square$

Proof of Corollary 1.15. If M is full, then the compactness of $\mathrm{Hol}_p(\nabla^\perp)$ follows from Theorem 1.13. If M is non-full, it follows from Theorem 1.14, Theorem 1.12 and the following remarkable result by D. Hullin. The first inequality is consequence of equation (14). $\square$

Theorem 4.8 *[Hu],[Hu1] The Einstein constant k of a Kähler-Einstein submanifold M of $\mathbb{C}P^n$ is rational. Moreover, if M is compact then $k > 0$.*

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