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$N = 1$ reductions of $N = 2$ supergravity in the presence of tensor multiplets

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Abstract: We consider consistent truncations of $N = 2$ supergravites in the presence of tensor multiplets (dual to hypermultiplets) as they occur in type-IIB compactifications on Calabi-Yau orientifolds. We analyze in detail the scalar potentials encompassing these reductions when fluxes are turned on and study vacua of the $N = 1$ phases.

Keywords: Superstring Vacua, Supergravity Models.

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Contents

1. Introduction

Massive deformations of extended supergravity play an important role in the description of superstring and M-theory compactifications in the presence of fluxes [1]–[5], either induced by p-forms or by S-S generalized dimensional reduction [6].

In these compactifications to four dimensions one often encounters non-standard supergravities in that some of the scalars have been replaced by antisymmetric tensor fields [7]–[11], which, when fluxes are turned on, may become massive vector fields [11, 12, 13]. The advantage of introducing antisymmetric tensor fields is that one can introduce two kinds of mass-deformations, one of electric and the other of magnetic type. $N = 2 \to N = 1$ reduction of Calabi-Yau compactifications of type-IIA and type-IIB theories [14, 15, 16], corresponds to Calabi-Yau orientifolds [17]–[23], and one encounters in this context such kind on antisymmetric tensor couplings to gravity as coming from NSNS and RR 2-forms or 4-forms. In the present paper we consider, in full detail, such reductions, for the case of different $N = 2 \to N = 1$ truncations which correspond to heterotic string or Calabi-Yau orientifolds with different kind of orientifolding. The paper is organized as follows. In section 1 we describe the $N = 2$ effective supergravity as coming from type-IIB compactifications on a Calabi-Yau 3-fold [24]–[27]. In section 2 to 5 we discuss the different truncations which give different $N = 1$ theories, in the presence of general fluxes. In the remaining sections we discuss the nature of the vacua, the supersymmetric configurations, and the classification of vacua in the case of cubic prepotentials for a given set of electric and magnetic charges.

2. $D = 4$, $N = 2$ supergravity from type-IIB flux compactification

The general $N = 2$ supergravity theory coupled to tensor and vector multiplets has been discussed in references [8, 9, 10]. We recall the field content of the effective theory which, following the notations and conventions of [28, 10], is given by:

- the gravitational multiplet
$$(V^a_\mu,\, \psi_{\mu A},\, \psi^A_\mu,\, A^0_\mu)\,,$$
where $A = 1, 2$ is the SU(2) R-symmetry index of the gravitinos ψ, lower and upper index referring to their left or right chirality respectively, and V^a_μ, A^0_μ are the vierbein and the graviphoton;

- n_V vector multiplets
$$(A^i_\mu,\, \lambda^{Ai},\, \lambda^{\bar{\imath}}_A,\, z^i,\, \bar{z}^{\bar{\imath}})\,,$$
where the chirality convention for upper and lower R-symmetry indices A of the gauginos λ is reversed, and z^i, $i = 1 \dots n_V$ are the complex coordinates of the special Kähler manifold $\mathcal{M}_{SK}$;

- a scalar-tensor multiplet
$$(\zeta_\alpha,\, \zeta^\alpha,\, q^u, B_{I\mu\nu})\,,$$
where $I = 1, \dots, n_T$, label the tensor fields, ζ^α, ζ_α are the (anti)-chiral fermions ("hyperinos") $\alpha = 1, \dots, 2n_H$, transforming in the fundamental of $\mathrm{Sp}(2n_H)$, and q^u are the coordinates of the manifold $\mathcal{M}_T$ associated to the scalar-tensor multiplets, with $u = 1, \dots, 4n_H - n_T$.

If we think of this theory as coming from standard $N = 2$ supergravity, n_H denotes the number of hypermultiplets and n_T the number of quaternionic coordinates which, being axionic, have been dualized into antisymmetric tensors. In the following we shall consider the particular case of a $N = 2$ theory resulting from compactification of type-IIB theory on a Calabi-Yau 3-fold. Therefore the scalar-tensor multiplet will contain just two tensors $B_{1\,\mu\nu}$, $B_{2\,\mu\nu}$, which in the ten dimensional interpretation come from the ten dimensional NSNS and RR 2-forms respectively. Therefore in the following we set $n_T = 2$ so that

$I = 1, 2$. Note that in our conventions the index $I = 1$ for the charges are related to RR fluxes while the index $I = 2$ to NSNS fluxes:

$$
\begin{aligned}
e^1{}_{\mathbf{\Lambda}},\, m^{1\mathbf{\Lambda}} &\leftrightarrow \text{RR fluxes}\,,\\
e^2{}_{\mathbf{\Lambda}},\, m^{2\mathbf{\Lambda}} &\leftrightarrow \text{NS fluxes}\,.
\end{aligned} \tag{2.1}
$$

The lagrangian and transformation laws of the theory have been given in reference [10].

The analysis of the truncation of such theory to $N = 1$ can be done by a careful investigation of the supersymmetry transformation laws, which are given below (up to 3-fermion terms):

$$\delta\psi_{A|\mu} = \nabla_\mu \varepsilon_A - M^{IJ}\tilde{H}_{J\mu}\omega_{I\,A}{}^B\varepsilon_B + \left[iS_{AB}\eta_{\mu\nu} + \epsilon_{AB}T^-_{\mu\nu}\right]\gamma^\nu\varepsilon^B\,, \tag{2.2}$$

$$\delta\lambda^{kA} = i\partial_\mu z^k\gamma^\mu\varepsilon^A + G^{k-}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_B\epsilon^{AB} + W^{kAB}\varepsilon_B\,, \tag{2.3}$$

$$\delta\zeta_\alpha = iP_{uA\alpha}\partial_\mu q^u\gamma^\mu\varepsilon^A - iM^{IJ}\tilde{H}_{J\mu}\mathcal{U}_{IA\alpha}\gamma^\mu\varepsilon^A + N^A_\alpha\varepsilon_A\,, \tag{2.4}$$

$$\delta V^a_\mu = -i\overline{\psi}_{A\mu}\gamma^a\varepsilon^A - i\overline{\psi}^A_\mu\gamma^a\varepsilon_A\,, \tag{2.5}$$

$$
\begin{aligned}
\delta A^{\mathbf{\Lambda}}_\mu &= 2L^{\mathbf{\Lambda}}\overline{\psi}^A_\mu\varepsilon^B\epsilon_{AB} + 2\overline{L}^{\mathbf{\Lambda}}\overline{\psi}_{A\mu}\varepsilon_B\epsilon^{AB} + \\
&\quad + \left(if^{\mathbf{\Lambda}}_k\overline{\lambda}^{kA}\gamma_\mu\varepsilon^B\epsilon_{AB} + i\overline{f}^{\mathbf{\Lambda}}_{\bar{k}}\overline{\lambda}^{\bar{k}}_A\gamma_\mu\varepsilon_B\epsilon^{AB}\right)\,,
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
\delta B_{I\mu\nu} &= -\frac{i}{2}\left(\overline{\varepsilon}_A\gamma_{\mu\nu}\zeta_\alpha\mathcal{U}_I{}^{A\alpha} - \overline{\varepsilon}^A\gamma_{\mu\nu}\zeta^\alpha\mathcal{U}_{IA\alpha}\right) - \\
&\quad -\omega_{IC}{}^A\left(\overline{\varepsilon}_A\gamma_{[\mu}\psi^C_{\nu]} + \overline{\psi}_{[\mu A}\gamma_{\nu]}\varepsilon^C\right)\,,
\end{aligned} \tag{2.7}
$$

$$\delta z^k = \overline{\lambda}^{kA}\varepsilon_A\,, \tag{2.8}$$

$$\delta\overline{z}^{\bar{k}} = \overline{\lambda}^{\bar{k}}_A\varepsilon^A\,, \tag{2.9}$$

$$\delta q^u = P^u{}_{A\alpha}\left(\overline{\zeta}^\alpha\varepsilon^A + \mathbb{C}^{\alpha\beta}\epsilon^{AB}\overline{\zeta}_\beta\varepsilon_B\right)\,. \tag{2.10}$$

Here and in the following we give, for the sake of simplicity, the transformation laws for the left-handed spinor fields only.

Notations are as follows:

- we have collected the $n_V + 1$ vectors into $A^{\mathbf{\Lambda}}_\mu = (A^0_\mu,\, A^i_\mu)$, with $\mathbf{\Lambda} = 0, \ldots, n_V$, and we have defined:[1]

$$\tilde{H}_{I\mu} = \epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}_I\,; \qquad H_{I\,\mu\nu\rho} = \partial_{[\mu}B_{|I\,\nu\rho]}\,; \tag{2.11}$$

- the covariant derivative on the ε parameter is given by:

$$\nabla_\mu\varepsilon_A \equiv \partial_\mu\varepsilon_A - \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\varepsilon_A + \frac{i}{2}Q_\mu\varepsilon_A + \omega_{\mu A}{}^B\varepsilon_B\,, \tag{2.12}$$

$$Q_\mu \equiv \frac{\mathrm{i}}{2}\left(\partial_i K\,\partial_\mu z^i - \partial_{\bar{\imath}}K\,\partial_\mu\overline{z}^{\bar{\imath}}\right)\,, \tag{2.13}$$

$$\omega_A{}^B = \frac{\mathrm{i}}{2}\omega^x\,\sigma^x_A{}^B\,, \tag{2.14}$$

[1] We use boldface indices for the $N = 2$ vector multiplets since we want to reserve the plain capital Greek letters Λ, $\Sigma\ldots$ to label the $N = 1$ vector multiplets.

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where ω^{ab}, $Q(z,\bar{z})$, $\omega_A^{\ B}(q^u)$ denote the Lorentz, U(1)–Kähler and SU(2) 1-form connections, respectively. Here K is the special geometry Kähler potential;

- the transformation laws of the fermions (2.2), (2.3), (2.4) also contain the further structures S_{AB}, W^{kAB}, N_α^A, named "fermion shifts" (or generalized Fayet-Iliopoulos terms) which are related to the presence of electric and magnetic charges ($e^I_{\mathbf{\Lambda}}, m^{I\,\mathbf{\Lambda}}$), and which give rise to a non trivial scalar potential. Their explicit form is:

$$
\begin{aligned}
S_{AB} &= \frac{i}{2}\sigma^x_{AB}\omega^x_I(L^{\mathbf{\Lambda}}e^I_{\mathbf{\Lambda}} - M_{\mathbf{\Lambda}}m^{I\mathbf{\Lambda}})\,, &(2.15)\\
W^{kAB} &= ig^{k\bar{\ell}}\sigma_x^{AB}\omega^x_I(\bar{f}^{\mathbf{\Lambda}}_{\bar{\ell}} - \bar{h}_{\bar{\ell}\mathbf{\Lambda}}m^{I\mathbf{\Lambda}})\,, &(2.16)\\
N^A_\alpha &= 2\mathcal{U}^\alpha_{AI}(\bar{L}^{\mathbf{\Lambda}}e^I_{\mathbf{\Lambda}} - \bar{M}_{\mathbf{\Lambda}}m^{I\mathbf{\Lambda}})\,; &(2.17)
\end{aligned}
$$

- besides $\tilde{H}_{I\mu}$ the transformation laws contain additional I-indexed structures ($I = 1,2$), namely $\mathcal{U}_I{}^{A\alpha}(q^u)$, $\omega_{IA}{}^B(q^u)$ and $M_{IJ}(q^u)$ (M^{IJ} will denote its inverse matrix), which satisfy a number of relations that can be found in ref [8] . We observe that, if one thinks of this theory as coming from the $N=2$ standard supergravity [28], the previous I-indexed quantities can be interpreted as the remnants of the original vielbein $\mathcal{U}^{A\alpha}_{\hat{u}}$, of the SU(2) 1-form connection $\omega_{\hat{u}A}{}^B$ and of the quaternionic metric in the I, J directions, after dualization of the axionic q^I coordinates ($q^{\hat{u}} = (q^u, q^I)$) parametrizing the original quaternionic manifold;

- the quantity $P_{A\alpha} = P_{uA\alpha}(q^u)dq^u$ appearing in equations (2.4), (2.10) is a "rectangular vielbein" [8] related to the metric g_{uv} of $\mathcal{M}_T$ by the relation $P_u{}^{A\alpha}P_{vA\alpha} = g_{uv}$, and $P^{uA\alpha} = g^{uv}P_v{}^{A\alpha}$. It is related to the original vielbein $\mathcal{U}^{A\alpha}_{\hat{u}}$ by:

$$
P_u{}^{A\alpha} = \mathcal{U}^{A\alpha}_u + A^I_u\,\mathcal{U}^{A\alpha}_I\,, \qquad (2.18)
$$

where $A^I_u = M^{IJ}\,h_{Ju}$ and $h_{\hat{u}\hat{v}}$ is the original quaternionic metric. Since the quaternionic vielbein satisfies the reality condition $\mathcal{U}^{A\alpha\star} = \epsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta}$, $\mathbb{C}$ being the $\mathrm{Sp}(2\,n_H)$ invariant metric, an analogous reality condition holds for $P^{A\alpha}$;

- all the other structures appearing in the transformation laws depend on the scalar fields z^i, $z^{\bar{\imath}}$ of the special geometry of the vector multiplets. Here we just recall the fundamental relations obeyed by the symplectic sections of the special manifold:

$$
\begin{aligned}
D_iV &= U_i\,,\\
D_iU_j &= iC_{ijk}g^{k\bar{k}}\bar{U}_{\bar{k}}\,,\\
D_iU_{\bar{\jmath}} &= g_{i\bar{\jmath}}\bar{V}\,,\\
D_i\bar{V} &= 0\,, &(2.19)
\end{aligned}
$$

where i, $j = 1,\ldots,n_V$, D_i is the Kähler and (generally) covariant derivative, and

$$
\begin{aligned}
V &= (L^{\mathbf{\Lambda}}, M_{\mathbf{\Lambda}})\,, \qquad U_i = D_iV = (f^{\mathbf{\Lambda}}_i, h_{\mathbf{\Lambda}i}) \qquad \mathbf{\Lambda} = 0,\ldots,n_V\,; &(2.20)\\
M_{\mathbf{\Lambda}} &= \mathcal{N}_{\mathbf{\Lambda\Sigma}}L^{\mathbf{\Sigma}}\,, \qquad h_{\mathbf{\Lambda}i} = \overline{\mathcal{N}}_{\mathbf{\Lambda\Sigma}}f^{\mathbf{\Lambda}}_i\,, &(2.21)
\end{aligned}
$$

and $\mathcal{N}_{\mathbf{\Lambda\Sigma}}$ is the kinetic vector matrix. Then the "dressed" field-strengths $T^-_{\mu\nu}$ and $G^{k-}_{\mu\nu}$ appearing in the transformation laws of the gravitino and gaugino fields are given by:

$$T^-_{\mu\nu} = 2\mathrm{i}\ Im\mathcal{N}_{\mathbf{\Lambda\Sigma}} L^{\mathbf{\Sigma}} F^{\mathbf{\Lambda}-}_{\mu\nu}\,, \tag{2.22}$$

$$G^{i-}_{\mu\nu} = -g^{i\bar{\jmath}} \bar{f}^{\mathbf{\Gamma}}_{\bar{\jmath}} \mathrm{Im}\mathcal{N}_{\mathbf{\Gamma\Lambda}} F^{\mathbf{\Lambda}-}_{\mu\nu}\,. \tag{2.23}$$

- Finally the scalar potential of the theory can be computed from the shifts (2.15), (2.16), (2.17)and is given by:

$$\begin{aligned}\mathcal{V} = {} & 4\left(\mathcal{M}_{IJ} - \omega^x_I \omega^x_J\right)\left(m^{I\mathbf{\Lambda}}\overline{M}_{\mathbf{\Lambda}} - e^I_{\mathbf{\Lambda}}\overline{L}^{\Lambda}\right)\left(m^{J\mathbf{\Sigma}} M_{\mathbf{\Sigma}} - e^J_{\mathbf{\Sigma}} L^{\mathbf{\Sigma}}\right) + \\ & + \omega^x_I \omega^x_J \left(m^{I\mathbf{\Lambda}}, e^I_{\mathbf{\Lambda}}\right) \mathcal{S} \begin{pmatrix} m^{J\mathbf{\Sigma}} \\ e^J_{\mathbf{\Sigma}} \end{pmatrix}\,,\end{aligned} \tag{2.24}$$

  where the matrix $\mathcal{S}$ is a symplectic matrix given explicitly by:

$$\mathcal{S} = -\frac{1}{2}\begin{pmatrix} I_{\mathbf{\Lambda\Sigma}} + \left(RI^{-1}R\right)_{\mathbf{\Lambda\Sigma}} & -\left(RI^{-1}\right)_{\mathbf{\Lambda}}{}^{\mathbf{\Sigma}} \\ -\left(I^{-1}R\right)^{\mathbf{\Lambda}}{}_{\mathbf{\Sigma}} & I^{-1|\mathbf{\Lambda\Sigma}} \end{pmatrix}. \tag{2.25}$$

  where $R_{\mathbf{\Lambda\,\Sigma}}$ and $I_{\mathbf{\Lambda\,\Sigma}}$ denote $\mathrm{Re}\mathcal{N}_{\mathbf{\Lambda\,\Sigma}}$ and $\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\,\Sigma}}$ respectively . Furthermore the electric and magnetic charges must satisfy the the "generalized tadpole condition":

$$e^I_{\mathbf{\Lambda}} m^{J\mathbf{\Lambda}} - e^J_{\mathbf{\Lambda}} m^{I\mathbf{\Lambda}} = 0\,, \tag{2.26}$$

  as a consequence of the supersymmetry Ward identity of the scalar potential and/or the invariance of the lagrangian under the tensor-gauge transformation:

$$\delta B_{I\,\mu\nu} = \partial_{[\mu}\Lambda_{I\nu]}\,; \qquad \delta A^{\mathbf{\Lambda}}_\mu = -2m^{\mathbf{\Lambda}\,I}\Lambda_{I\mu}\,. \tag{2.27}$$

  $\Lambda_{I\mu}$ being an arbitrary vector.

In the following we shall be concerned with a theory coming from type-IIB compactification on a Calabi-Yau 3-fold. In this case the first term is zero, due to the peculiar properties of the special geometry derived from a cubic prepotential, so that the scalar potential contains only the second term, namely:

$$\mathcal{V} = \omega^x_I \omega^x_J \left(m^{I\mathbf{\Lambda}}, e^I_{\mathbf{\Lambda}}\right) \mathcal{S} \begin{pmatrix} m^{J\mathbf{\Sigma}} \\ e^J_{\mathbf{\Sigma}} \end{pmatrix}\,. \tag{2.28}$$

3. Conditions for a consistent $N = 2 \to N = 1$ truncation

We know that in a special quaternionic manifold $\mathcal{M}_Q$ we can always identify a universal hypermultiplet which is uniquely selected by the isometries of $\mathcal{M}_Q$ [29]. In the dualized theory we are considering, the universal hypermultiplet becomes a double tensor multiplet

$$(B_{1\mu\nu},\, B_{2\mu\nu},\, C_0,\, \varphi)\,,$$

where C_0 is the ten dimensional axion of type-IIB theory, φ is the four dimensional dilaton and $B_{1\mu\nu}$, $B_{2\mu\nu}$ are the the four dimensional 2-forms coming from the NSNS and RR two forms of the type-IIB theory.

The possible truncations to $N = 1$ can be obtained setting to zero a linear combination of the supersymmetry parameters $(\varepsilon_1, \varepsilon_2)$. It is easy to see that there are three essentially different truncations, all the others being equivalent, modulo SU(2) rotations. Two of them will be seen to correspond to $\mathbb{Z}_2$ orientifold projections of type-IIB supergravity on a Calabi-Yau 3-fold, while the third one corresponds to the same compactification of heterotic string.

To understand why we have three different truncations, let us start with the simplest choice, following the guidelines of [14], that is, let us set to zero the parameter ε_2:

$$\varepsilon_2 = \psi_{2\mu} = 0\,. \tag{3.1}$$

Considering the surviving $\psi\varepsilon$ currents in the supersymmetry transformation laws of the tensors (2.7):

$$\delta B_{I\mu\nu} = \frac{i}{2}\omega_I^{(3)}(\overline{\varepsilon}_1\gamma_{[\mu}\psi^1_{\nu]} + \overline{\varepsilon}^1\gamma_{[\mu}\psi_{\nu]1}) + \cdots \tag{3.2}$$

we recognize that in order to truncate one or both of the two tensors B_I, we have to set to zero the corresponding structure $\omega_I^{(3)}$. As we have six ω_I^x, with $I = 1, 2$, $x = 1, 2, 3$, by means of an SU(2) transformation we can always set to zero three of them. A possible choice is the one given in reference [29], that is in our notations:

$$\omega_I^{(1)} = -\frac{1}{2}e^{2\varphi}\begin{pmatrix} 0 \\ \mathrm{Im}\tau \end{pmatrix}; \qquad \omega_I^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad \omega_I^{(3)} = -\frac{1}{2}e^{2\varphi}\begin{pmatrix} 1 \\ \mathrm{Re}\,\tau \end{pmatrix}, \tag{3.3}$$

where $\tau = -C_0 + 4\,i\,e^{-\varphi+\frac{K_Q}{2}}$, K_Q being the Kähler potential of the special Kähler manifold contained in the quaternionic-Kähler manifold $\mathcal{M}_Q$, and $\varphi - \frac{K_Q}{2}$ is the ten-dimensional dilaton.

If we want to consider other possible truncations we have to set to zero different combinations of $(\varepsilon_1, \varepsilon_2)$. For this purpose we can act with a rigid SU(2) transformation on the theory and then set to zero the new ε_2 parameter. There are essentially two more different possibilities which fulfill our requirements, which are obtained by means of a rotation of $\theta = \frac{\pi}{2}$ on the $(x = 1, x = 3)$ and $(x = 2, x = 3)$ planes in $\mathbb{R}^3$ respectively. They correspond to setting to zero ε_2' or ε_2'', namely:

$$\varepsilon_2' = \frac{1}{\sqrt{2}}(-i\varepsilon_1 + \varepsilon_2) = 0\,, \tag{3.4}$$

or

$$\varepsilon_2'' = \frac{1}{\sqrt{2}}(-\varepsilon_1 + \varepsilon_2) = 0\,. \tag{3.5}$$

It will be seen that in these cases we obtain the $N = 1$ theory corresponding to the $O(5)/O(9)$ or $O(3)/O(7)$ orientifold projection of type-IIB theory on a Calabi-Yau 3-fold, respectively. The corresponding values of the rotated ω_I^x are given by:

$O(5)/O(9)$ case:

$$\omega_I^{(1)} = \frac{1}{2}e^{2\varphi}\begin{pmatrix} 1 \\ \mathrm{Re}\,\tau \end{pmatrix}; \qquad \omega_I^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad \omega_I^{(3)} = -\frac{1}{2}e^{2\varphi}\begin{pmatrix} 0 \\ \mathrm{Im}\tau \end{pmatrix}. \tag{3.6}$$

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$O(3)/O(7)$ case:

$$\omega_I^{(1)} = -\frac{1}{2}e^{2\varphi}\begin{pmatrix} 0 \\ \mathrm{Im}\tau \end{pmatrix}; \qquad \omega_I^{(2)} = -\frac{1}{2}e^{2\varphi}\begin{pmatrix} 1 \\ \mathrm{Re}\,\tau \end{pmatrix}; \qquad \omega_I^{(3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.7}$$

Given the correspondence between the choice of the particular supersymmetry parameter to be set to zero and of the values of the corresponding structures ω_I^x, in order to analyze the three cases (3.1), (3.4), (3.5), we will set $\varepsilon_2 = \psi_{2\mu} = 0$ in equations (2.2)–(2.10) and then specify the connections ω_I^x according to the case (3.3), (3.6), (3.7) for an explicit solution of the constraints.

3.1 Truncation of the gravitational multiplet

Let us first consider the gravitino transformation law (2.2) and analyze the consequences of the truncation $\varepsilon_2 = 0$ which do not depend of the three different choices (3.1), (3.4), (3.5). Following the same steps as in [14], setting $\varepsilon_2 = \psi_{2\mu} = 0$ in equation (2.2) gives, for $A = 1$ the supersymmetry transformation law of the $N = 1$ gravitino:

$$\delta\psi_{1\mu} = \nabla_\mu\varepsilon_1 - M^{IJ}\tilde{H}_{J\mu}\omega_1{}^1\varepsilon_1 + iS_{11}\gamma_\mu\varepsilon^1 , \tag{3.8}$$

while for $A = 2$ we obtain the following consistency condition:

$$\delta\psi_{2\mu} = \omega_{\mu 2}{}^1\varepsilon_1 - M^{IJ}\tilde{H}_{J\mu}\omega_{I\,2}{}^1\varepsilon_1 + \left[\mathrm{i}S_{21}\eta_{\mu\nu} + \epsilon_{21}T^-_{\mu\nu}\right]\gamma^\nu\varepsilon^1 = 0 , \tag{3.9}$$

which implies:

$$\omega_{\mu 2}{}^1 = \omega_{u2}{}^1\partial_\mu q^u = 0 , \tag{3.10}$$

$$S_{21} = \frac{i}{2}\sigma^x_{21}\omega^x_I(L^{\mathbf{\Lambda}}e^I_{\mathbf{\Lambda}} - M_{\mathbf{\Lambda}}m^{I\mathbf{\Lambda}}) = 0 , \tag{3.11}$$

$$M^{IJ}\tilde{H}_{J\mu}\omega_{I\,2}{}^1 = 0 , \tag{3.12}$$

$$T^- \equiv 2i\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}}L^{\mathbf{\Lambda}}\mathcal{F}^{-\mathbf{\Sigma}} = 0 . \tag{3.13}$$

The last condition can be solved as in reference [14] since, apart from the fermionic shift related to the scalar potential, the vector multiplet sector is untouched by the dualization in the hypermultiplet sector. A short account of the results given in [14] is reported in the next paragraph.

Conditions (3.12) and (3.11) depend on the choice of one of the three aforementioned cases and will be analyzed separately in the next sections. In this section we concentrate on those conditions which do not depend on the choice of the structure of $\omega_{IA}{}^B$. Condition (3.10) differs from the one in reference [14] because here there appears the SU(2) connection $\omega^x(q^u)$ of the reduced quaternionic manifold [8], instead of the connection $\hat{\omega}^x(q^{\hat{u}})$ of the quaternionic manifold of standard $N = 2$ supergravity [28].[2] In fact, using the expression of the SU(2) curvature $\Omega_A{}^B$ as given in [8], we have:

$$\Omega_1{}^2 \equiv d\omega_2{}^1 + \omega_2{}^1\wedge\omega_1{}^1 + \omega_2{}^2\wedge\omega_2{}^1 + \nabla\left(A^I\wedge\omega_{I\,2}{}^1\right) + P_{2\alpha}\wedge P^{1\alpha} . \tag{3.14}$$

[2] We recall that the tensors of the scalar-tensor multiplet come from the axionic scalars of the $N = 2$ quaternionic manifold which have been dualized implying that the residual $N = 2$ manifold is no more quaternionic.

The consistency condition $\Omega_1{}^2 = 0$ gives:

$$\nabla\left(A^I \wedge \omega_{I\,2}^{\;1}\right) = 0\,, \tag{3.15}$$

$$P_{2\alpha} \wedge P^{1\alpha} = 0\,. \tag{3.16}$$

Since equation (3.15) depends on $\omega_{IA}{}^B$ it will be dealt with later. To analyze the consequences of (3.16), we observe that the holonomy of the scalar manifold $\mathcal{M}_T$ for the $N = 2$ tensor coupled theory is contained in $\mathrm{SU}(2)\times\mathrm{Sp}(2n_H)\otimes\mathrm{SO}(n_T = 2)$ [8]. Performing the truncation from $N = 2$ to $N = 1$ the holonomy must reduce according to:

$$\mathrm{Hol}(\mathcal{M}_Q^{N=2}) \subset \mathrm{SU}(2)\times\mathrm{Sp}(2n_H)\otimes\mathrm{SO}(2) \to \mathrm{Hol}(\mathcal{M}_Q^{N=1}) \subset \mathrm{U}(1)\times\mathrm{SU}(n_H)\,. \tag{3.17}$$

We split therefore the symplectic index α of $P_{A\alpha}$ as follows:

$$\alpha \to (\hat\alpha, \dot\alpha) \in \left(\hat{\mathrm{U}}(1)\times\hat{\mathrm{SU}}(n_H)\right)\times\left(\dot{\mathrm{U}}(1)\times\dot{\mathrm{SU}}(n_H)\right). \tag{3.18}$$

The reality condition on the vielbein $P^{A\alpha}$ becomes:

$$P_{1\hat\alpha} \equiv (P^{1\hat\alpha})^* = \mathbb{C}_{\hat\alpha\dot\beta}P^{2\dot\beta}\,, \tag{3.19}$$

$$P_{2\hat\alpha} \equiv (P^{2\hat\alpha})^* = -\mathbb{C}_{\dot\beta\hat\alpha}P^{2\dot\beta}\,, \tag{3.20}$$

where the symplectic metric has been decomposed according to:

$$\mathbb{C}_{\alpha\beta} = \begin{pmatrix} 0 & \mathbb{C}_{\hat\alpha\dot\beta} \\ \mathbb{C}_{\dot\alpha\hat\beta} & 0 \end{pmatrix}, \tag{3.21}$$

with $\mathbb{C}_{\hat\alpha\dot\beta} = -\mathbb{C}_{\dot\beta\hat\alpha} = \delta_{\hat\alpha\dot\beta}$. Therefore the constraint (3.16) can be rewritten as:

$$\mathbb{C}_{\hat\alpha\dot\beta}P_2^{\hat\alpha}\wedge P^{1\dot\beta} + \mathbb{C}_{\dot\alpha\hat\beta}P_2^{\dot\alpha}\wedge P^{1\hat\beta} = 0\,, \tag{3.22}$$

which can be solved setting, for instance:

$$P_{2\dot\alpha} = 0 \Leftrightarrow P_{1\hat\alpha} = 0\,. \tag{3.23}$$

Equation (3.23), implies further constraints using the results of reference [8] for the covariant derivatives of $P_{A\alpha}$, namely:

$$dP_{2\dot\alpha} + \omega_2{}^1\wedge P_{1\dot\alpha} + \omega_2{}^2\wedge P_{2\dot\alpha} + \Delta_{\dot\alpha}{}^{\dot\beta}\wedge P_{2\dot\beta} + \Delta_{\dot\alpha}{}^{\hat\beta}\wedge P_{2\hat\beta} + F^I\wedge\mathcal{U}_{I2\dot\alpha} = 0\,, \tag{3.24}$$

$$dP_{1\hat\alpha} + \omega_1{}^1\wedge P_{1\hat\alpha} + \omega_1{}^2\wedge P_{2\hat\alpha} + \Delta_{\hat\alpha}{}^{\hat\beta}\wedge P_{1\hat\beta} + \Delta_{\hat\alpha}{}^{\dot\beta}\wedge P_{1\dot\beta} + F^I\wedge\mathcal{U}_{I1\hat\alpha} = 0\,. \tag{3.25}$$

Taking into account equations (3.23), (3.10) we obtain the consistency constraints:

$$F^I\mathcal{U}_{I2\dot\alpha} = F^I\mathcal{U}_{I1\hat\alpha} = 0\,, \tag{3.26}$$

$$\Delta_{\hat\alpha}{}^{\dot\beta} = \Delta_{\dot\alpha}{}^{\hat\beta} = 0\,. \tag{3.27}$$

Furthermore, considering the curvature associated to the vanishing connections (3.27) it is not difficult to see, taking into account the previous constraints, that its vanishing implies:

$$\Omega_{\dot\alpha\dot\beta\hat\gamma\dot\delta} = 0\,, \tag{3.28}$$

where $\Omega_{\alpha\beta\gamma\delta}$ is the completely symmetric tensor entering the expression of the symplectic curvature of the quaternionic manifold $\mathcal{M}_Q$ as well in $\mathcal{M}_T$ [28, 10]. Note that this same constraint was obtained for the truncation $N = 2 \longrightarrow N = 1$ of the standard $N = 2$ supergravity [14].

From the supersymmetry transformation laws of the hypermultiplet scalars, namely:

$$P_{uA\alpha}\delta q^u = \overline{\zeta}_\alpha \varepsilon_A + \mathbb{C}_{\alpha\beta}\overline{\zeta}^\beta \varepsilon^B \,, \tag{3.29}$$

using equation (3.23), we obtain that the truncated spinors of the scalar-tensor multiplet are:

$$\zeta^{\hat{\alpha}} = \zeta_{\hat{\alpha}} = 0 \,, \tag{3.30}$$

and imposing $\delta\zeta^{\hat{\alpha}} = \delta\zeta_{\hat{\alpha}} = 0$, namely:

$$\delta\zeta^{\hat{\alpha}} = iP_u^{1\hat{\alpha}}\partial_\mu q^u \gamma^\mu \varepsilon_1 - iM^{IJ}\tilde{H}_{\mu J}\mathcal{U}_I^{1\hat{\alpha}}\gamma_m \varepsilon_1 + N_1^{\hat{\alpha}}\varepsilon^1 = 0 \,, \tag{3.31}$$

$$\delta\zeta_{\hat{\alpha}} = iP_{u1\hat{\alpha}}\partial_\mu q^u \gamma^\mu \varepsilon^1 - iM^{IJ}\tilde{H}_{\mu J}\mathcal{U}_{I1\hat{\alpha}}\gamma_m \varepsilon^1 + N_{\hat{\alpha}}^1 \varepsilon_1 = 0 \,, \tag{3.32}$$

we obtain the following further conditions:

$$M^{IJ}\tilde{H}_{\mu I}\mathcal{U}_{J|2\hat{\alpha}} = M^{IJ}\tilde{H}_{\mu I}\mathcal{U}_{J|1\hat{\alpha}} = 0 \,, \tag{3.33}$$

$$N_1^{\hat{\alpha}} = N_{\hat{\alpha}}^1 = 0 \,. \tag{3.34}$$

Vice versa, the supersymmetry transformation laws of the retained spinors $\zeta_{\dot{\alpha}}$ (2.4) imply that the vielbein on the reduced manifold must be related to $P_{1\dot{\alpha}}$ (and its complex conjugate $P_{2\hat{\alpha}}$), for which the reduced torsion equation becomes:

$$\begin{gathered} dP_{1\dot{\alpha}} + \frac{i}{2}\omega^{(3)}P_{1\dot{\alpha}} + \Delta_{\dot{\alpha}}^{\ \dot{\beta}}P_{1\dot{\beta}} + F^I\mathcal{U}_{I|1\dot{\alpha}} = 0 \quad , \\ \Updownarrow \\ dP_{2\hat{\alpha}} - \frac{i}{2}\omega^{(3)}P_{2\hat{\alpha}} + \Delta_{\hat{\alpha}}^{\ \hat{\beta}}P_{2\hat{\beta}} + F^I\mathcal{U}_{I|2\hat{\alpha}} = 0 \,. \end{gathered} \tag{3.35}$$

In the sequel we shall derive the precise relation between $P^{1\dot{\alpha}}$ and the vielbein of the $N = 1$ manifold.

3.2 Truncation of the vector multiplets

As far as condition (3.13) is concerned, it can be solved exactly as in reference [14], since the vector multiplet sector is untouched by the dualization of the hypermultiplets. Some differences arise just for the gauge terms and they are discussed in the following. Therefore, in the sequel, we just give a short account of the derivation of the results given in [14].

We recall that the truncation in the vector multiplet sector (including the graviphoton) depends on the way the constraint (3.13) is satisfied. Denoting by $L^{\mathbf{\Lambda}}$ the symplectic section of the special Kähler manifold $\mathcal{M}_{SK}^{N=2}$ of complex dimension n_V of the $N = 2$ standard theory, the most general solution of the constraint is obtained by splitting the index $\mathbf{\Lambda}$ of $L^{\mathbf{\Lambda}}$ as follows:

$$\mathbf{\Lambda} = 0, 1, \ldots, n_V \rightarrow (X = 0, 1, \ldots, n_C, \, \Lambda = 1, \ldots, n'_V) \,, \tag{3.36}$$

with $n_C + n'_V = n_V$. If we set $L^\Lambda = 0$ the remaining $n_C + 1$ sections L^X parametrize a submanifold $\mathcal{M}_V^{N=1}$ of complex dimension n_C of the $N = 1$ scalar manifold. If we further set $\mathcal{F}^X_{\mu\nu} = 0$ we satisfy the constraint (3.13) and only n'_V vectors A^Λ_μ remain in the spectrum. According to the structure of the $N = 1$ multiplets it is easy to see that n_C is the number of the $N = 1$ chiral multiplets and n'_V the number of the $N = 1$ vector multiplets.

Consequently we also split the index $k = 1, \dots, n_V$ which labels the coordinates $(1,\, z^k) = L^{\mathbf{\Lambda}}/L^0$ of the Special Kähler manifold according to $k \to (\dot{k},\, \hat{k})$, where $\dot{k} = 1, \dots, n_C$ refers to the scalars of the chiral multiplets which parametrize the Kähler-Hodge manifold $\mathcal{M}_V^{N=1}$, while $\hat{k} = 1, \dots, n'_V$ labels the scalars which must be truncated out.

According to the previous considerations the $N = 2$ gaugini λ^{kA} therefore decompose as follows:

$$\lambda^{kA} \to (\lambda^{\dot{k}1},\, \lambda^{\dot{k}2},\, \lambda^{\hat{k}1},\, \lambda^{\hat{k}2})\,. \tag{3.37}$$

Defining

$$\lambda^\Lambda_\bullet \equiv -2 f^\Lambda_{\hat{k}}\, \lambda^{\hat{k}2}\,. \tag{3.38}$$

where f^Λ_k is the special geometry object with a world index in the (truncated) directions dz^k, it turns out that $\lambda^\Lambda_\bullet$ is the chiral gaugino of the $N = 1$ vector multiplet such that the associated D-term is given by:

$$D^\Lambda = \mathrm{i}\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\left(P^0_\Sigma + P^3_\Sigma\right). \tag{3.39}$$

The full analysis of reference [14] give furthermore a set of conditions on the special geometry structures that are given by:

$$\mathcal{F}^X_{\mu\nu} = \mathcal{G}_{X\mu\nu} = 0\,, \tag{3.40}$$
$$L^\Lambda = M_\Lambda = f^\Lambda_{\dot{k}} = h_{\dot{k}\Lambda} = 0\,, \tag{3.41}$$
$$f^X_{\hat{k}} = h_{\hat{k}X} = 0\,, \tag{3.42}$$
$$\mathcal{N}_{X\Lambda} = 0\,, \tag{3.43}$$
$$W^{\dot{k}21} = W^{\hat{k}11} = 0\,, \tag{3.44}$$
$$C_{\dot{k}\dot{\ell}\hat{m}} = g_{\dot{k}\hat{\bar{\ell}}} = 0\,, \tag{3.45}$$

where $\mathcal{G}_{X\mu\nu}$ is the dressed field-strength dual to $\mathcal{F}^X_{\mu\nu}$, and $C_{\dot{k}\dot{\ell}\hat{m}}$ and $g_{\dot{k}\hat{\bar{\ell}}}$ are components of the three index tensor and of the Kähler metric of the special geometry with the given particular structure of indices.

4. The heterotic case

Let us consider now the constraints (3.12), (3.15), (3.26), (3.33) for the case (3.1) when the ω^x_I are specified in equation (3.3). Let us first analyze the constraint (3.12). Since ${\omega_{I2}}^1 = \frac{i}{2}\omega^x_I {\sigma^x_2}^1$, using the connections (3.3), the constraint (3.12) implies

$$\tilde{H}^2_\mu = 0\,, \tag{4.1}$$

being $\omega^{(1)}_{I=2}$ the only non-vanishing component of $\omega_{I2}{}^1$. Equation (4.1), explicitly reads:

$$0 = \tilde{H}^2_\mu = M^{21}\tilde{H}_{1\mu} + M^{22}\tilde{H}_{2\mu}\,. \tag{4.2}$$

As shown in the appendix $M^{22} \neq 0$ and $M^{12} \propto \mathrm{Re}\tau$, therefore the solution is given by:

$$\tilde{H}_{2\mu} = 0; \qquad M^{12} = 0 \Leftrightarrow Re\tau = 0\,. \tag{4.3}$$

Since

$$\nabla(A^I\,\omega_I)_2{}^1 = d(A^I\,\omega_{I1}{}^2) + \omega_2{}^1 \wedge A^I\,\omega_{I1}{}^1 + \omega_2{}^2 \wedge A^I\,\omega_{I2}{}^1 = 0\,, \tag{4.4}$$

taking into account eq. (3.10) and the fact that $\omega_{I2}{}^1 \neq 0$ only for $I = 2$, we see that the constraint (3.15) is solved if we set:

$$A^2 = 0\,. \tag{4.5}$$

Using equation (4.1) into equation (3.33) we obtain:

$$\mathcal{U}_{(I=)1|2\dot{\alpha}} = \mathcal{U}_{(I=)1|1\hat{\alpha}} = 0\,. \tag{4.6}$$

Equation (4.6) together with the constraint (4.3) satisfies equation (3.26).

The consistency condition:

$$\begin{aligned}\delta B_{2\mu\nu} = &-\frac{i}{2}(\bar{\varepsilon}_1\gamma_{\mu\nu}\zeta_\alpha\mathcal{U}^{1\alpha}_{(I=)2} - \bar{\varepsilon}^1\gamma_{\mu\nu}\zeta^\alpha\mathcal{U}_{(I=)2|1\alpha} + \\ &+\frac{i}{2}\,\omega^{(3)}_2(\bar{\varepsilon}_1\gamma_{[\mu}\psi^1_{\nu]} + \bar{\varepsilon}^1\gamma_{[\mu}\psi_{\nu]1}) = 0\,,\end{aligned} \tag{4.7}$$

implies again: $\omega^{(3)}_2 \propto \mathrm{Re}\tau = 0$ and furthermore:

$$\zeta_\alpha\mathcal{U}^{1\alpha}_{(I=)2} = \zeta^\alpha\mathcal{U}_{(I=)2|1\alpha} = 0\,, \tag{4.8}$$

which thanks to equation (3.30) gives the following constraints:

$$\mathcal{U}_{(I=)2|1\dot{\alpha}} = \mathcal{U}_{(I=)2|2\hat{\alpha}} = 0\,. \tag{4.9}$$

We now consider the conditions on the fermionic shifts. Let us start with the gravitino shift (3.11), where we take into account condition $\mathrm{Re}\tau = 0$ and equation (3.41). Then we have:

$$S_{12} = \frac{i}{4}e^{2\varphi}(L^X e^1_X - M_X m^{1X}) = 0 \tag{4.10}$$

which implies:

$$e^1_X = m^{1X} = 0\,. \tag{4.11}$$

The conditions on the hyperino shifts (3.34), are satisfied in virtue of condition $\mathrm{Re}\tau = 0$ and equations (4.9), (4.11). Finally the condition from the gaugino shift (3.44) is satisfied if we set:

$$f^\Lambda_{\hat{k}}e^2_\Lambda - h_{\Lambda\hat{k}}m^{2\Lambda} = 0\,, \tag{4.12}$$

which implies that

$$e^2_\Lambda = m^{2\Lambda} = 0\,. \tag{4.13}$$

The manifold $\mathcal{M}_T^{N=1}$ obtained from the reduction of the scalar-tensor multiplet has $2n_H-1$ dimensions and must be the product of a Kähler manifold parametrized by n_H-1 complex coordinates and a one dimensional manifold parametrized by the scalars sitting in the linear multiplet.

In order to identify the vielbein of the Kähler-Hodge manifold and the einbein of the linear multiplet we consider the equation

$$P_{u\alpha[A}\mathcal{U}^{\alpha}_{B]I} = 0\,, \tag{4.14}$$

which is one of the constraints defining the scalar tensor geometry of $\mathcal{M}_T$ [8]. We introduce n_H-1 complex coordinates w^s $(s=1,\dots,n_H-1)$ and one real coordinate $w^0=\bar{w}^0$ and proceed as in reference [14] setting:

$$P_{u1\dot{\alpha}}dq^u = \frac{1}{\sqrt{2}}P_{s\dot{\alpha}}dw^s\,; \qquad P_{u2\hat{\alpha}}dq^u = \frac{1}{\sqrt{2}}P_{\bar{s}\hat{\alpha}}d\bar{w}^{\bar{s}} \qquad (s=0,\dots,n_H-1) \tag{4.15}$$

Thanks to equations (4.15), the $N=2$ relation [8]:

$$P_{uA\alpha}P^{A\alpha}_v = g_{uv}\,, \tag{4.16}$$

reduces to:

$$P_{s\dot{\alpha}}P^{\dot{\alpha}}_{\bar{r}} = g_{s\bar{r}}\,, \tag{4.17}$$

$g_{s\bar{r}}$ being the metric of $\mathcal{M}_V^{N=1}$. In virtue of equations (3.23), (4.15), condition (4.14) reduces to:

$$P_{s\dot{\alpha}}\mathcal{U}^{\dot{\alpha}}_{I2}dw^s = P_{\bar{s}\hat{\alpha}}\mathcal{U}^{\hat{\alpha}}_{I1}d\bar{w}^{\bar{s}},\,; \qquad s=0,\dots,n_H-1\,, \tag{4.18}$$

which implies

$$\begin{aligned} P_{0\dot{\alpha}}\mathcal{U}^{\dot{\alpha}}_{I2} &= P_{0\hat{\alpha}}\mathcal{U}^{\hat{\alpha}}_{I1}\,,\\ P_{s\dot{\alpha}}\mathcal{U}^{\dot{\alpha}}_{I2} &= P_{\bar{s}\hat{\alpha}}\mathcal{U}^{\hat{\alpha}}_{I1} = 0\,; \qquad (s=1,\dots,n_H-1)\,. \end{aligned} \tag{4.19}$$

Taking also $\dot{\alpha}$ running from 0 to n_H-1 we can solve the orthogonality relation (4.19) by setting $\mathcal{U}^{1\dot{\alpha}}_I=0$ except the $\dot{\alpha}=0$ component (see the appendix for an explicit representation of $\mathcal{U}$), by taking $P_{s\dot{\alpha}=0}=0$ for $s=1,\dots,n_H-1$ and requiring $i\,P_{0\dot{\alpha}}\mathcal{U}^{\dot{\alpha}}_{I2}$ to be real. Note that with this position equation (3.35) implies that the Kähler-Hodge manifold $\mathcal{M}_Q^{N=1}$ has a torsionless vielbein $P_{s\dot{\alpha}}$, $\dot{\alpha}=1,\dots,n_H-1$.

According to these considerations, the hyperini $\zeta_{\dot{\alpha}}$ will be also split into one $\zeta=\{\zeta_\bullet,\,\zeta^\bullet\}$ which belongs to the linear multiplet and n_H-1 $\zeta^s=\{\zeta^s_\bullet,\,\zeta^{\bullet s}\}$ belonging to the chiral multiplets.

In summary the reduced $N=1$ theory has a σ-model given by the manifold:

$$\mathcal{M}_V^{(N=1)}\otimes\mathcal{M}_Q^{(N=1)}\otimes\mathbb{R}\,, \tag{4.20}$$

where $\mathcal{M}_V^{(N=1)}$ is the Kähler-Hodge manifold of complex dimension n_C obtained from the reduction of the vector multiplet sector, parametrized by the coordinates $z^{\dot{k}}$, $\mathcal{M}_Q^{(N=1)}$ is again a Kähler-Hodge manifold of complex dimension n_H-1 obtained by the truncation in the scalar-tensor sector and parametrized by the coordinates w^s, and $\mathbb{R}$ is the one dimensional real manifold parametrized by the scalar φ of the residual linear multiplet.

We can derive the supersymmetry transformation laws for $N = 1$ supergravity coupled to one linear multiplet performing the following identifications:

$$\psi_{\bullet\mu} = \psi_{1\mu}\,; \qquad \varepsilon_\bullet = \varepsilon_1\,, \tag{4.21}$$
$$B_{\mu\nu} = B_{1\mu\nu}\,, \tag{4.22}$$
$$P_{\dot\alpha s}\delta w^s = \sqrt{2}P_{u1\dot\alpha}\delta q^u|_{\mathcal{M}^{KH}}\,, \tag{4.23}$$
$$d\varphi = -P_{1\,\dot\alpha=0} \tag{4.24}$$
$$\chi^{\dot k} = \lambda^{\dot k 1}\,; \qquad \lambda^\Lambda_\bullet = -2f^\Lambda_{\hat k}\lambda^{\hat k 2}\,, \tag{4.25}$$
$$\zeta^s = \sqrt{2}P^{\dot\alpha s}\zeta_{\dot\alpha}\,; \qquad \zeta = -\zeta_{\dot\alpha=0}\,; \qquad s = 1,\ldots,n_H - 1\,, \tag{4.26}$$
$$N^s = \sqrt{2}P^{\dot\alpha s}N^1_{\dot\alpha}\,; \qquad N = -N^1_{\dot\alpha=0}\,, \tag{4.27}$$
$$N^{\dot k} = W^{11\dot k}\,; \qquad D^\Lambda = 2if^\Lambda_{\hat k}W^{21\hat k}\,, \tag{4.28}$$
$$L = S_{11}\,, \tag{4.29}$$

where for the $N = 1$ theory we denote left and right-handed spinors with a lower and upper dot $\bullet$ respectively. We these identifications the supersymmetry transformation laws for the $N = 1$ theory are:

$$\delta V^a_\mu = -i\bar\psi_{\bullet\mu}\gamma^a\varepsilon^\bullet + \text{h.c.}\,, \tag{4.30}$$
$$\delta\psi_{\bullet\mu} = \nabla_\mu\varepsilon_\bullet + i\,e^{-2\varphi}\,\tilde H_\mu\varepsilon_\bullet + iL\gamma_\mu\varepsilon^\bullet\,, \tag{4.31}$$
$$\delta A^\Lambda_\mu = \frac{i}{2}\overline{\lambda}^\Lambda_\bullet\gamma_\mu\varepsilon^\bullet + \text{h.c.}\,, \tag{4.32}$$
$$\delta\lambda^\Lambda_\bullet = \mathcal{F}^{(-)\Lambda}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_\bullet + iD^\Lambda\epsilon_\bullet\,, \tag{4.33}$$
$$\delta\chi^{\dot k} = i\partial_\mu z^{\dot k}\gamma^\mu\varepsilon_\bullet + N^{\dot k}\varepsilon_\bullet\,, \tag{4.34}$$
$$\delta\zeta^s = i\partial_\mu w^s\gamma^\mu\varepsilon_\bullet + N^s\varepsilon_\bullet\,, \tag{4.35}$$
$$\delta z^{\dot k} = \overline{\chi}^{\dot k}\varepsilon_\bullet\,, \tag{4.36}$$
$$\delta w^s = \overline{\zeta}^s\varepsilon_\bullet\,, \tag{4.37}$$
$$\delta\varphi = \overline{\zeta}_\bullet\,\epsilon_\bullet + h.c.\,, \tag{4.38}$$
$$\delta B_{\mu\nu} = \frac{1}{4}\,e^{2\varphi}\,\overline{\epsilon}_\bullet\,\gamma_{\mu\nu}\,\zeta_\bullet - \frac{1}{2}\,e^{2\varphi}\,\overline{\epsilon}_\bullet\,\gamma_{[\mu}\psi^\bullet_{\nu]} + \text{h.c.}\,, \tag{4.39}$$
$$\delta\zeta_\bullet = i\partial_\mu\varphi\gamma^\mu\varepsilon^\bullet + 2e^{-2\varphi}\tilde H_\mu\gamma^\mu\varepsilon^\bullet + N\varepsilon_\bullet\,. \tag{4.40}$$

the last three transformation laws referring to the linear multiplet fields

$$\{B_{\mu\nu},\,\zeta_\bullet,\,\zeta^\bullet,\,\varphi\}\,.$$

The term $M^{IJ}\tilde H_{J\mu}$ appearing in equations (2.2), (2.4) has been reduced using the explicit form of M^{IJ} and $\mathcal{U}_{IA\alpha}$ given in the appendix. The covariant derivative in (4.31) is defined as follows:

$$\nabla_\mu\varepsilon_\bullet \equiv \partial_\mu\varepsilon_\bullet - \frac{1}{4}\omega^{ab}\gamma_{ab}\varepsilon_\bullet + \frac{i}{2}\mathbf{Q}_\mu\varepsilon_\bullet \tag{4.41}$$

where

$$\mathbf{Q}_\mu = Q_\mu + \omega^3_\mu\,, \tag{4.42}$$

is the U(1) connection on the $N = 1$ Kähler-Hodge manifold $\mathcal{M}_V^{N=1} \times \mathcal{M}_T^{N=1}$.

The superpotential L, the spin 1/2 fermion shifts and the D-term turn out to be:

$$L = -ie^{\varphi+\frac{K}{2}}\,(L^X e^2_X - M_X m^{2X})\,, \tag{4.43}$$

$$N^{\dot k} = ig^{\dot k\bar{\dot\ell}} e^{\varphi+\frac{K}{2}}\,(\bar f^X_{\bar{\dot\ell}} e^2_X - \bar h_{X\bar{\dot\ell}} m^{2X}) = 2g^{\dot k\bar{\dot\ell}}\nabla_{\bar{\dot\ell}}\overline{L}\,, \tag{4.44}$$

$$N^s = 2\sqrt{2}P^{s\dot\alpha}\mathcal{U}^1_{(I=)2\dot\alpha}(\bar L^X e^2_X - \bar M_X m^{2X}) = 2g^{s\bar s}\nabla_{\bar s}\overline{L}\,, \tag{4.45}$$

$$N = 2\,ie^{\varphi+\frac{K}{2}}\,(\bar L^X e^2_X - \bar M_X m^{2X}) = 2\frac{\partial}{\partial\varphi}\,\overline{L}\,, \tag{4.46}$$

$$D^\Lambda = -\frac{1}{2}e^{2\,\varphi}(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Sigma}(e^1_\Sigma - \mathcal{N}_{\Sigma\Gamma}m^{1\Gamma})\,. \tag{4.47}$$

Let us observe that the electric and magnetic charges entering the superpotential L satisfy the equation (2.26) identically.

The scalar potential can be deduced from the above fermion shifts and reads:

$$\begin{aligned} V = -\frac{1}{8}\,e^{4\,\varphi}\Big[&16\,e^{-2\,\varphi+K_Q}\,(e^2_X - \overline{\mathcal{N}}_{XY}m^{2Y})(\mathrm{Im}\mathcal{N}^{-1})^{YZ}(e^2_Z - \mathcal{N}_{ZW}m^{2W}) + \\ &+ (e^1_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma}m^{1\Sigma})(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Gamma}(e^1_\Gamma - \mathcal{N}_{\Gamma\Delta}m^{1\Delta})\Big]\,. \end{aligned} \tag{4.48}$$

5. The $O5/O9$ case

The reduction corresponding to the case (3.5) is completely analogous to the heterotic case provided we perform in all the equations the substitution $I = 1 \leftrightarrow I = 2$. Thus, for example, equation (3.12), when equations (3.6) are considered, gives the constraints:

$$\tilde H^1_\mu = 0 \;\Leftrightarrow\; B_{1\mu\nu} = 0; \qquad Re\tau = 0 \tag{5.1}$$

replacing the conditions (4.1), (4.3). Proceeding as in previous section we now find that in the $O5/O9$ case we obtain that the equations (4.6), (4.7), (4.8),(4.9), (4.11), (4.12) are valid provided we perform the replacement $I = 1 \leftrightarrow I = 2$. In particular all the considerations of the previous section after (4.12) for the identification of the fields of the $N = 1$ theory remain the same provided we set $B_{\mu\nu} = B_{2\mu\nu}$ and replace in the fermion shifts, (4.43), (4.44), (4.45), (4.46), (4.47), $e^2_X \to e^1_X$, $m^2_X \to m^1_X$, $e^1_\Lambda \to e^2_\Lambda$, $m^1_\Lambda \to m^2_\Lambda$. In particular, the transformation laws of the $N = 1$ theory are the same except for the gravitino and the spinor $\zeta_\bullet$ of the linear multiplet. Indeed the term $M^{IJ}\tilde H_{J\mu}$ appearing in equations (2.2), (2.4) gives a different contribution due to the fact that now we have $\tilde H_{1\mu} = 0$ instead of $\tilde H_{2\mu} = 0$. Using the expression of M^{IJ} in the appendix we now obtain:

$$\delta\psi_{\bullet\mu} = \nabla_\mu\varepsilon_\bullet - 2i\,\frac{1}{\mathcal{I}^{-1|00}}\,e^{-\,\varphi+K_Q/2}\tilde H_\mu\varepsilon_\bullet \tag{5.2}$$

$$\delta\zeta_\bullet = i\partial_\mu\varphi\gamma^\mu\varepsilon^\bullet + \frac{1}{\mathcal{I}^{-1|00}}\tilde H_\mu\gamma^\mu\varepsilon^\bullet + N\varepsilon_\bullet\,. \tag{5.3}$$

As in the Heterotic case the electric and magnetic charges entering the superpotential L satisfy the equation (2.26) identically.

Finally the scalar potential for the $O5/O9$ case is:

$$V = -\frac{1}{8}\, e^{4\varphi}\Big[(e^1_X - \overline{\mathcal{N}}_{XY}m^{1Y})(\mathrm{Im}\mathcal{N}^{-1})^{YZ}(e^1_Z - \mathcal{N}_{ZW}m^{1W}) + \\ + 16\, e^{-2\varphi+K_Q}\,(e^2_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma}m^{2\Sigma})(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Gamma}(e^2_\Gamma - \mathcal{N}_{\Gamma\Delta}m^{2\Delta})\Big]\,. \qquad (5.4)$$

6. The $O3/O7$ case

Let us consider the truncation corresponding to set to zero ε_2 as given in (3.4) and with the connections ω^x_I given in equation (3.7).

We analyze first the constraint (3.12), which, thanks to the expression (3.7), gives the conditions:

$$\tilde{H}^1_\mu = \tilde{H}^2_\mu = 0 \rightarrow \tilde{H}_{1\mu} = \tilde{H}_{2\mu} = 0\,. \qquad (6.1)$$

The same consideration holds for equations (3.15), which is solved setting:

$$F^I = A^I = 0\,, \quad I = 1, 2\,. \qquad (6.2)$$

In virtue of equation (6.2) also the constraint (3.26) is satisfied and thus the constraint (3.23) is consistent. Equations (3.31), (3.32) do not give any constraint on the $\mathcal{U}_I^{A\alpha}$ because of equation (6.1). All the conditions on the $\mathcal{U}_I^{A\alpha}$ come from the supersymmetry transformation law of the tensors:

$$\delta B_{I\mu\nu} = -\frac{i}{2}(\bar{\varepsilon}_1\gamma_{\mu\nu}\zeta_\alpha\mathcal{U}_I^{1\alpha} - \bar{\varepsilon}^1\gamma_{\mu\nu}\zeta^\alpha\mathcal{U}_{I1\alpha} + \frac{i}{2}\omega_I^{(3)}(\bar{\varepsilon}_1\gamma_{[\mu}\psi^1_{\nu]} + \bar{\varepsilon}^1\gamma_{[\mu}\psi_{\nu]1}) = 0\,. \qquad (6.3)$$

Since in this case $\omega_I^{(3)} = 0$ identically, we have to impose:

$$\mathcal{U}_{I1\dot{\alpha}} = \mathcal{U}_{I2\hat{\alpha}} = 0\,. \qquad (6.4)$$

Finally the torsion equation (3.35) for the vielbeins $P_{1\dot{\alpha}}$, taking equations (4.19) and (3.28) into account, becomes:

$$dP_{1\dot{\alpha}} + {\omega_1}^1 P_{1\dot{\alpha}} + {\Delta_{\dot{\alpha}}}^{\dot{\beta}} P_{1\dot{\beta}} = 0\,, \qquad (6.5)$$

ensuring the absence of torsion of the Kähler Hodge manifold.

As far as the fermion shifts are concerned from the gravitino shift we have the condition (3.11):

$$S_{12} = \frac{i}{2}\omega_I^{(3)}(L^X e^I_X - M_X m^{IX}) = 0 \qquad (6.6)$$

which is satisfied since $\omega_I^{(3)} = 0$. Furthermore equation (3.34) is satisfied in virtue of (6.4). Finally the constraint (3.44) imposes:

$$f^\Lambda_{\hat{k}} e_\Lambda - h_{\Lambda\hat{k}} m^\Lambda = 0 \qquad (6.7)$$
$$e_\Lambda = e^1_\Lambda + \tau e^2_\Lambda,\; m^\Lambda = m^{1\Lambda} + \tau m^{2\Lambda}\,. \qquad (6.8)$$

which implies that the Λ-indexed charges must be zero:

$$e^1_\Lambda = e^2_\Lambda = m^{1\Lambda} = m^{2\Lambda} = 0\,. \qquad (6.9)$$

The $N = 1$ theory has in this case a σ-model given by the product of two Kähler-Hodge manifolds

$$\mathcal{M}_V^{(N=1)} \otimes \mathcal{M}_Q^{(N=1)} \tag{6.10}$$

of complex dimension n_C and n_H respectively.

Performing the identifications:

$$\begin{aligned}
\psi_{\bullet\mu} &= \psi_{1\mu} ; \qquad \varepsilon_\bullet = \varepsilon_1 , && (6.11)\\
P_{\dot\alpha s}\delta w^s &= \sqrt{2} P_{u1\dot\alpha}\delta q^u|_{\mathcal{M}^{KH}} , && (6.12)\\
\chi^{\dot k} &= \lambda^{\dot k 1} ; \qquad \lambda^\Lambda_\bullet = -2 f^\Lambda_{\hat k}\lambda^{\hat k 2} , && (6.13)\\
\zeta^s &= \sqrt{2} P^{\dot\alpha s}\zeta_{\dot\alpha} ; \qquad s = 1, \dots, n_H - 1 , && (6.14)\\
N^s &= \sqrt{2} P^{\dot\alpha s} N^1_{\dot\alpha} ; \qquad N = -N^1_{\dot\alpha=0} , && (6.15)\\
N^{\dot k} &= W^{11\dot k} ; \qquad D^\Lambda = 2 i f^\Lambda_{\hat k} W^{21\hat k} , && (6.16)\\
L &= S_{11} , && (6.17)
\end{aligned}$$

the supersymmetry transformation laws of the $N = 1$ theory are given by:

$$\begin{aligned}
\delta V^a_\mu &= -i\bar\psi_{\bullet\mu}\gamma^a\varepsilon^\bullet + \text{h.c.} && (6.18)\\
\delta\psi_{\bullet\mu} &= \nabla_\mu\varepsilon_\bullet + i L\gamma_\mu\varepsilon^\bullet && (6.19)\\
\delta A^\Lambda_\mu &= \frac{i}{2}\overline{\lambda}^\Lambda_\bullet\gamma_\mu\varepsilon^\bullet + \text{h.c.} && (6.20)\\
\delta\lambda^\Lambda_\bullet &= \mathcal{F}^{(-)\Lambda}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_\bullet + i D^\Lambda\epsilon_\bullet && (6.21)\\
\delta\chi^{\dot k} &= i\partial_\mu z^{\dot k}\gamma^\mu\varepsilon_\bullet + N^{\dot k}\varepsilon_\bullet && (6.22)\\
\delta\zeta^s &= i P_{s\dot\alpha}\partial_\mu w^s\gamma^\mu\varepsilon_\bullet + N_{\dot\alpha}\varepsilon_\bullet && (6.23)\\
\delta z^{\dot k} &= \overline{\chi}^{\dot k}\varepsilon_\bullet && (6.24)\\
\delta w^s &= \overline{\zeta}^s\varepsilon_\bullet && (6.25)
\end{aligned}$$

where the fermion shifts are given by:

$$\begin{aligned}
L &= -\frac{1}{4} e^{2\varphi}(L^X e_X - M_X m^X) && (6.26)\\
N^{\dot k} &= -\frac{1}{2} g^{\dot k\bar{\dot\ell}} e^{2\varphi}(f^X_{\bar{\dot\ell}}\bar e_X - h_{X\bar{\dot\ell}}\bar m^X) = 2 g^{\dot k\bar{\dot\ell}}\nabla_{\bar{\dot\ell}}\overline{L} && (6.27)\\
N^s &= 2\sqrt{2} P^{s\dot\alpha}\mathcal{U}_{I1\hat\alpha}(\bar L^X e^I_X - \bar M_X m^{IX}) = 2 g^{s\bar s}\nabla_{\bar s}\overline{L} && (6.28)\\
D^\Lambda &= 0 , && (6.29)
\end{aligned}$$

and we have defined:

$$e_X = e^1_X + \tau e^2_X ; \qquad m^X = m^{1X} + \tau m^{2X} . \tag{6.30}$$

In the present case both the NSNS and RR electric and magnetic charges enter the definition of the superpotential (6.26). We see that the charges $\{e^1_X, e^2_X\}$ and $\{m^{1X}, m^{2X}\}$ are constrained by the tadpole condition:

$$m^{1X}\epsilon^2_X - m^{2X}\epsilon^1_X = m^{1\mathbf{\Lambda}} e^2_{\mathbf{\Lambda}} - m^{2\mathbf{\Lambda}} e^1_{\mathbf{\Lambda}} = 0 . \tag{6.31}$$

For $e \times m = 0$ the scalar potential is given by [3]:

$$V = -\frac{1}{8}\, e^{4\,\varphi}(e_X - \overline{\mathcal{N}}_{XY} m^Y)(\mathrm{Im}\mathcal{N}^{-1})^{XZ}(\overline{e}_Z - \mathcal{N}_{ZW}\overline{m}^W)\,. \tag{6.32}$$

On the other hand, if $e \times m \neq 0$, $N = 2$ supersymmetry is broken but still the theory can have an unbroken $N = 1$ sector. Indeed for $e \times m \neq 0$ the scalar potential is given by [21]:

$$V = -\frac{1}{8}\, e^{4\,\varphi}(e_X - \overline{\mathcal{N}}_{XY} m^Y)(\mathrm{Im}\mathcal{N}^{-1})^{XZ}(\overline{e}_Z - \mathcal{N}_{ZW}\overline{m}^W) + \frac{1}{4}\, e^{4\varphi}\,\mathrm{Im}\tau\, m \times e\,. \tag{6.33}$$

The potential (6.33) can be written in a manifestly $N = 1$ fashion:

$$V = e^{K(z,\,\bar{z})+K(\tau,\,\bar{\tau})+K_D(w,\,\bar{w})}\left[G^{i\bar{\jmath}}\, D_i W\, D_{\bar{\jmath}} W + G^{\tau\bar{\tau}}\, D_\tau W\, D_{\bar{\tau}} W\right] + e^{K_D(w,\,\bar{w})}\, m \times e\,, \tag{6.34}$$

where the superpotential W has the form:

$$W = X^X\, e_X - F_X\, m^X\,, \tag{6.35}$$

which is consistent with the general expression given in [1]. Note that since the first term in (6.34) is separately $N = 1$ supersymmetric, the last term should be supersymmetric as well. In fact it is a F.I. term. A similar term arise from a U(1) gauge field on a $D7$ brane world volume with magnetic fluxes [30]. This term explicitly breaks $N = 2$ supersymmetry. Indeed for $m \times e \neq 0$, the $N = 2$ Ward identity for the scalar potential acquires an additional contribution from the square of the gaugino shifts, which is not proportional to δ_A^B and has the form:

$$\epsilon^{xyz}\, \omega_I^x\, \omega_J^y\, m^{I\mathbf{\Lambda}}\, e_{\mathbf{\Lambda}}^J\, \sigma^z_A{}^B = \frac{1}{4}\, e^{4\varphi}\,\mathrm{Im}\tau\,(m \times e)\,\sigma^3_A{}^B\,. \tag{6.36}$$

From a microscopic point of view, the potential in the form (6.34) does not take into account the contributions due to $O3/O7$ planes. These, as discussed in [21], have the effect of canceling the last term, according to generalized tadpole cancellation condition. The resulting potential will have the form in (6.32) with $m \times e \neq 0$ and will in general have non trivial vacua, as discussed in section 9.

7. Comparison with the orientifold projection

Let us now recover from the previous analysis the results of [18]. For this purpose let us write down the relations between our notations given in the appendix and those of reference [18].

$$\tilde{\xi}_a \rightarrow \rho_a\,; \qquad \tilde{\xi}_0 \rightarrow q^2\,, \tag{7.1}$$

$$\xi^a \rightarrow \ell\, b^a - c^a\,; \qquad \xi^0 = C_0 \rightarrow \ell\,, \tag{7.2}$$

$$\mathrm{Re}(w^a) \rightarrow b^a\,; \qquad \mathrm{Im}(w^a) \rightarrow v^a\,; \qquad a = 1, \ldots, h^{(1,1)}\,, \tag{7.3}$$

where c^a and b^a are the scalars coming from the RR and NSNS two-form respectively, v^a are the scalars coming from the deformations of the Kähler class of the metric, while ρ^a

are the scalars coming from the RR four-form. The scalars $(q^1,\, q^2)$ in this context appear dualized into rank two tensors $(B^1_{\mu\nu},\, B^2_{\mu\nu})$ [8] as they come from the NSNS and RR 2– form respectively.

According to the $\mathbb{Z}_2$ orientifold projection, the quaternionic scalars may appear as the coefficients of the expansion in $H^{(1,1)}_+$ or $H^{(1,1)}_-$ forms. Moreover the real part of the complex dilaton C_0 and the NSNS and RR two forms B_1, B_2 may be even or not under the $\mathbb{Z}_2$ projection. With the previous considerations, we can see, analyzing the truncation of the scalar-tensor multiplet, that the second truncation corresponds to the $O5/O9$ planes case, since $C_0 = B_1 = 0$, while the last corresponds to the $O3/O7$ planes case, since $B_1 = B_2 = 0$. Further analyzing the condition (3.10) using the explicit parametrization of [29] one can also check the consistency of the truncation for the remaining scalars in the hypermultiplet sector.

The first truncation we considered corresponds instead to the orientifold projection of the Heterotic string on a Calabi-Yau 3-fold. Nevertheless from the condition (3.10) we can identify which are the two sets of scalars whose indices must be orthogonal. Furthermore, considering that if the NSNS two forms survive then the scalars b^a may be thought as the coefficients of the $H^{(1,1)}_+$ expansion.

The results are summarized in the following table:

$$\begin{array}{ccc} O5/O9 & O3/O7 & \text{heterotic} \\ b^{\dot a},\ \rho^{\dot a} \in H^{(1,1)}_- & b^{\dot a},\ c^{\dot a} \in H^{(1,1)}_- & c^{\dot a},\ \rho^{\dot a} \in H^{(1,1)}_- \\ c^{\hat a},\ v^{\hat a} \in H^{(1,1)}_+ & v^{\hat a},\ \rho^{\hat a} \in H^{(1,1)}_+ & b^{\hat a},\ v^{\hat a} \in H^{(1,1)}_+ \\ C_0 = 0,\ B_1 = 0 & B_1 = 0,\ B_2 = 0 & C_0 = 0,\ B_2 = 0 \end{array} \tag{7.4}$$

where $\dot a = 1, \ldots, h^{(1,1)}_-$, $\hat a = 1, \ldots, h^{(1,1)}_+$.

As far as the vector multiplets are concerned, before the truncation we had $h^{(2,1)} + 1$ vector multiplets labeled by $\mathbf{\Lambda} = 0, 1, \ldots, h^{(2,1)}$. We split $\mathbf{\Lambda} \to (\Lambda,\, X)$ and retained the vectors $F^\Lambda_{\mu\nu}$ and the symplectic sections $(L^X,\, M_X)$. It is now clear that for the $O5/O9$ case $\Lambda = 1, \ldots, h^{(2,1)}_-$, in order to have $h^{(2,1)}_-$ vector multiplets [18], while $X = 0, 1, \ldots, h^{(2,1)}_+$ such that L^X/L^0 describe the scalars of $h^{(2,1)}_+$ chiral multiplets, while for the $O3/O7$ planes case $\Lambda = 1, \ldots, h^{(2,1)}_+$, labels the $h^{(2,1)}_+$ vector multiplets, while $X = 0, 1, \ldots, h^{(2,1)}_-$ such that L^X/L^0 are the scalars of $h^{(2,1)}_-$ chiral multiplets.

Let us now consider the terms coming from the flux $G = H_2 + \tau H_1$.

For the $O5/O9$ case we have that

$$H_2 \in H^{(3)}_+; \qquad H_1 \in H^{(3)}_- \tag{7.5}$$

therefore consistently we have the following fluxes:

$$(e^1_X,\, m^{1X}), \quad X = 0, 1, \ldots, h^{(2,1)}_+; \qquad (e^2_\Lambda,\, m^{2\Lambda}), \quad \Lambda = 1, \ldots, h^{(2,1)}_-. \tag{7.6}$$

For the $O3/O7$ case we have that

$$H_1 \in H^{(3)}_-; \qquad H_2 \in H^{(3)}_- \tag{7.7}$$

therefore consistently we have the following fluxes:

$$(e^1_X,\, m^{1X}),\quad X=0,1,\dots,h^{(2,1)}_-\,;\qquad (e^2_X,\, m^{2X}),\quad X=0,1,\dots,h^{(2,1)}_-\,. \tag{7.8}$$

As far as the heterotic truncation is concerned, we can rephrase equation (4.1) in the Calabi-Yau language, as the condition:

$$H_1\in H^{(3)}_+;\qquad H_2\in H^{(3)}_- \tag{7.9}$$

therefore consistently with (4.11), (4.12) one obtains the following fluxes:

$$(e^1_\Lambda,\, m^{1\Lambda})\,,\quad \Lambda=1,\dots,h^{(2,1)}_-\,;\qquad (e^2_X,\, m^{2X})\,,\quad X=0,1,\dots,h^{(2,1)}_+ \tag{7.10}$$

which also means that we have $h^{(2,1)}_-$ vector multiplets and $h^{(2,1)}_+$ hypermultiplets. Let us finally observe that equations (6.32), (5.4) coincide respectively with the scalar potentials obtained in reference [18] for the $O3/O7$ and $O5/O9$ planes truncations, and that equation (6.26) gives the superpotential of reference [1].

8. Supersymmetric configurations

In the $N=2$ theory we the following fluxes are present:

$$G^{(0,3)} = e^{-\frac{K}{2}} L^{\mathbf{\Lambda}}(e_{\mathbf{\Lambda}} - \mathcal{N}_{\mathbf{\Lambda\Sigma}} m^{\mathbf{\Sigma}})\,, \tag{8.1}$$
$$G^{(1,2)}_i = e^{-\frac{K}{2}} f^{\mathbf{\Lambda}}_i(e_{\mathbf{\Lambda}} - \overline{\mathcal{N}}_{\mathbf{\Lambda\Sigma}} m^{\mathbf{\Sigma}})\,, \tag{8.2}$$
$$G^{(3,0)} = e^{-\frac{K}{2}} \overline{L}^{\mathbf{\Lambda}}(e_{\mathbf{\Lambda}} - \overline{\mathcal{N}}_{\mathbf{\Lambda\Sigma}} m^{\mathbf{\Sigma}})\,, \tag{8.3}$$
$$G^{(2,1)}_{\overline{k}} = e^{-\frac{K}{2}} \bar{f}^{\mathbf{\Lambda}}_{\overline{k}}(e_{\mathbf{\Lambda}} - \mathcal{N}_{\mathbf{\Lambda\Sigma}} m^{\mathbf{\Sigma}})\,, \tag{8.4}$$

where we recall that $K\equiv K(z,\bar z)$ is the Kähler potential of the complex structure moduli and

$$e_{\mathbf{\Lambda}} = e^1_{\mathbf{\Lambda}} + \tau e_{\mathbf{\Lambda}};\quad m^{\mathbf{\Lambda}} = m^{1\mathbf{\Lambda}} + \tau m^{2\mathbf{\Lambda}}\,, \tag{8.5}$$

and the flux parameters satisfy the tadpole cancellation condition:

$$e^1_{\mathbf{\Lambda}} m^{2\mathbf{\Lambda}} - e^2_{\mathbf{\Lambda}} m^{1\mathbf{\Lambda}} = 0\,. \tag{8.6}$$

The $N=1$ scalar potential can be always written in the following form:

$$\begin{aligned} V &= -\frac{1}{8} e^{4\varphi}(e_{\mathbf{\Lambda}} - \overline{\mathcal{N}}_{\mathbf{\Lambda\Sigma}} m^{\mathbf{\Sigma}})(\mathrm{Im}\mathcal{N}^{-1})^{\mathbf{\Lambda\Gamma}}(\overline{e}_{\mathbf{\Gamma}} - \mathcal{N}_{\mathbf{\Gamma\Delta}}\overline{m}^{\mathbf{\Delta}}) \\ &= \frac{1}{4} e^{4\varphi+K}\left(G^{(3,0)}\overline{G^{(3,0)}} + g^{i\overline{k}} G^{(1,2)}_i \overline{G^{(1,2)}}_{\overline{k}}\right)\,, \end{aligned} \tag{8.7}$$

even in the case in which $e\times m\neq 0$, as discussed at the end of section 6. The Minkowski minimum corresponds to:

$$G^{(3,0)} = G^{(1,2)}_i = 0 \tag{8.8}$$

nevertheless the solutions of (8.8) are not consistent with the constraint (8.6).

If we perform an $N=2\to N=1$ truncation the fluxes, according to equations (3.41), (3.42), (3.43) reduce to:

$$
\begin{aligned}
G^{(0,3)} &= e^{-\frac{K}{2}}L^X(e_X-\mathcal{N}_{XY}m^Y)\,, && (8.9)\\
G^{(1,2)}_{\hat{k}} &= e^{-\frac{K}{2}}f^{\Lambda}_{\hat{k}}(e_\Lambda-\overline{\mathcal{N}}_{\Lambda\Sigma}m^\Sigma)\,, && (8.10)\\
G^{(1,2)}_{\dot{k}} &= e^{-\frac{K}{2}}f^{X}_{\dot{k}}(e_X-\overline{\mathcal{N}}_{XY}m^Y)\,, && (8.11)\\
G^{(3,0)} &= e^{-\frac{K}{2}}\overline{L}^X(e_X-\overline{\mathcal{N}}_{XY}m^Y)\,, && (8.12)\\
G^{(2,1)}_{\bar{\hat{k}}} &= e^{-\frac{K}{2}}\bar{f}^{\Lambda}_{\bar{\hat{k}}}(e_\Lambda-\mathcal{N}_{\Lambda\Sigma}m^\Sigma)\,, && (8.13)\\
G^{(2,1)}_{\bar{\dot{k}}} &= e^{-\frac{K}{2}}\bar{f}^{X}_{\bar{\dot{k}}}(e_X-\mathcal{N}_{XY}m^Y)\,. && (8.14)
\end{aligned}
$$

Consider now the case (3.4) corresponding to the $O3/O7$ truncation. We report here the scalar potential and the superpotential are given by equation (6.32), (6.26).

$$
\begin{aligned}
V &= -\frac{1}{8}e^{4\varphi}(e_X-\overline{\mathcal{N}}_{XY}m^Y)(\mathrm{Im}\mathcal{N}^{-1})^{XZ}(\overline{e}_Z-\mathcal{N}_{ZW}\overline{m}^W)\\
&= \frac{1}{4}e^{4\varphi+K}\left(G^{(3,0)}\overline{G^{(3,0)}}+g^{\dot{\ell}\bar{\dot{k}}}G^{(1,2)}_{\dot{\ell}}\overline{G^{(1,2)}}_{\bar{\dot{k}}}\right)\,, && (8.15)\\
L &= -\frac{1}{4}e^{2\varphi}(L^Xe_X-M_Xm^X)\,. && (8.16)
\end{aligned}
$$

We recall also the condition (6.7) for a consistent truncation:

$$
f^{\Lambda}_{\hat{k}}e_\Lambda-h_{\Lambda\hat{k}}m^\Lambda=f^{\Lambda}_{\hat{k}}(e_\Lambda-\overline{\mathcal{N}}_{\Lambda\Sigma}m^\Sigma)=0\,. \tag{8.17}
$$

Therefore one can observe that the condition for a Minkowski vacuum requires:

$$
G^{(3,0)}=G^{(1,2)}_{\dot{\ell}}=0\,, \tag{8.18}
$$

while the vacuum is supersymmetric if also the gravitino shift vanishes:

$$
L=0\leftrightarrow G^{(0,3)}=0\,. \tag{8.19}
$$

The condition (8.17) for a consistent truncation requires:

$$
G^{(1,2)}_{\hat{\ell}}=0\,. \tag{8.20}
$$

Therefore the theory admits a supersymmetric $N=1$ Minkowski vacuum, just for $(2,1)$ fluxes, according to the previous analysis [31, 17]. Note that the minimum condition of the $N=1$ theory, can not impose any constraint on the component of the $(1,2)$ flux along the truncated scalars $\hat{\ell}$ (8.10). The absence of such a component comes from the constraint (8.20) for a consistent truncation.

In the next section, we will show that conditions (8.18), (8.19), (8.20) do not admit a non trivial solution in the charges if (8.6) holds. The only case in which $e\times m$ can be different from zero after truncation to $N=1$ is the $O3/O7$ case, in which condition (8.6) is indeed relaxed according to the discussion at the end of section 6. In virtue of this in the $O3/O7$ truncation conditions (8.18), (8.19), (8.20) do admit a non-trivial solution.

9. Vacua of IIB on CY_3 orientifolds

Let us now study the vacua of type-IIB string theory compactified on a CY_3 orientifold. We start by recalling some general properties of the scalar potential which hold in all the three truncations considered earlier. To this end we shall write the potential in a general $N = 1$ form which will yield the expressions in eqs. (4.48), (5.4), (6.32) upon performing the corresponding truncation on the fields and charges. The complex 3-form flux across a 3-cycle of the CY_3 can be expanded in a basis of the corresponding cohomology group:

$$\begin{aligned} G_{(3)} &= H_2 + \tau\, H_1 = e_{\mathbf{\Lambda}}\, \beta^{\mathbf{\Lambda}} + m^{\mathbf{\Sigma}}\, \alpha_{\mathbf{\Sigma}}\,, \\ e_{\mathbf{\Lambda}} &= e^1_{\mathbf{\Lambda}} + \tau\, e^2_{\mathbf{\Lambda}}\,; \qquad m^{\mathbf{\Lambda}} = m^{1\mathbf{\Lambda}} + \tau\, m^{2\mathbf{\Lambda}}\,. \end{aligned} \tag{9.1}$$

Let

$$\Omega(z) = \begin{pmatrix} X^{\mathbf{\Lambda}}(z) \\ F_{\mathbf{\Sigma}}(z) \end{pmatrix}, \tag{9.2}$$

be the holomorphic section on the Special Kähler manifold depending only on the complex structure moduli z^i $(i = 1, \ldots, h^{(2,1)})$. According to our previous analysis, the above quantities can be specialized to the three truncations as follows:

- **Heterotic case:** set $\mathrm{Re}(\tau) = 0$, $X^\Lambda = F_\Lambda = 0$, $(\Lambda = 1, \ldots, h^{2,1}_-)$, $e^1_X = m^{1X} = e^2_\Lambda = m^{2,\Lambda} = 0$, $(X = 0, \ldots, h^{2,1}_+)$.
- **O5/O9 case:** set $\mathrm{Re}(\tau) = 0$, $X^\Lambda = F_\Lambda = 0$, $(\Lambda = 1, \ldots, h^{2,1}_-)$, $e^2_X = m^{2X} = e^1_\Lambda = m^{1,\Lambda} = 0$, $(X = 0, \ldots, h^{2,1}_+)$.
- **O3/O7 case:** set $X^\Lambda = F_\Lambda = 0$, $e_\Lambda = m^\Lambda = 0$, $(\Lambda = 1, \ldots, h^{2,1}_+)$.

The general form of the GVW superpotential in the low-energy $\mathcal{N} = 1$ theory [1] is

$$W(\tau, z^i) = e_{\mathbf{\Lambda}}\, X^{\mathbf{\Lambda}} - m^{\mathbf{\Sigma}}\, F_{\mathbf{\Sigma}}\,, \tag{9.3}$$

and the potential has the form:

$$V = e^{K(z,\bar z) + K(\tau,\bar\tau) + K_D(w,\bar w)} \left[G^{i\bar\jmath}\, D_i W\, D_{\bar\jmath} W + G^{\tau\bar\tau}\, D_\tau W\, D_{\bar\tau} W \right], \tag{9.4}$$

where $K(z, \bar z)$, $K(\tau, \bar\tau)$, $K_D(w, \bar w)$ are the contributions to the Kähler potential of the $N = 1$ manifold related to the submanifolds parametrized by the complex structure moduli z^i, the ten dimensional axion/dilaton τ and the Kähler moduli in the ten dimensional Einstein frame w^a. By comparing the expression of the potential (9.4) with the results obtained in the previous sections we find the following identification:

$$\begin{aligned} e^{K(\tau,\bar\tau) + K_D} &= \frac{1}{4}\, e^{4\,\varphi} = \frac{1}{4}\, e^{4\,\phi + 2\,K_Q} = \frac{1}{4}\, e^{\phi + K_D}\,, \\ K(\tau, \bar\tau) &= -\ln[-i\,(\tau - \bar\tau)] + \text{const.}\,; \qquad K_D(w,\, \bar w) = -2\, \ln\left(\frac{1}{3!}\, d_{abc} v^a v^b v^c\right), \end{aligned} \tag{9.5}$$

where $v^a = \text{Im}(w^a)$ are Kähler moduli in the Einstein frame. To understand the identifications in (9.5), recall that from type-II string theory point of view K_Q is related to the volume of the CY_3 expressed in the ten dimensional string frame:

$$K_Q = -\ln\left(\frac{1}{3!}\, d_{abc} v_s^a v_s^b v_s^c\right), \tag{9.6}$$

where v_s^a are Kähler moduli in the string frame. Since the Kähler moduli v in the two frames are related in the following way $v_s^a = v^a\, e^{\frac{\phi}{2}}$, if we define:

$$K_D(w, \bar{w}) = -2\,\ln\left(\frac{1}{3!}\, d_{abc} v^a v^b v^c\right), \tag{9.7}$$

we have

$$2\,K_Q = K_D - 3\,\phi\,. \tag{9.8}$$

The potential V is extremized if $D_\tau W = D_i W = 0$. However if in addition we also require supersymmetry, we have to impose that $0 = D_{w^a} W$. This implies $W = 0$, since W is w^a-independent and thus $D_{w^a} W \propto W$.

We further note that, being $K(\tau, \bar{\tau}) = -\ln(-i(\tau - \bar{\tau})) + \text{const.}$:

$$D_\tau W = \frac{1}{(\bar{\tau} - \tau)}\,\overline{\hat{W}}\,; \qquad \hat{W} = e_\Lambda\,\overline{X}^\Lambda - m^\Sigma\,\overline{F}_\Sigma\,, \tag{9.9}$$

the minimum conditions can then be written in the following form:

$$D_\tau W = D_i W = 0 \Leftrightarrow e_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma} m^\Sigma = 0\,. \tag{9.10}$$

The above equation clearly has solutions only if $m \times e > 0$. This is a consequence of the following relation which holds at the minimum:

$$m \times e = m^{1\Lambda} e^2_\Lambda - m^{2\Lambda} e^1_\Lambda = \frac{1}{\text{Im}\tau}\,\text{Im}(\bar{m} e) = -\frac{1}{\text{Im}(\tau)}\,\bar{m}^T\,\text{Im}\mathcal{N}\,m > 0\,. \tag{9.11}$$

Note that in both the heterotic and the O5/O9 truncations $m \times e = 0$ and thus the potential has no non-trivial vacua. Only in the O3/O7 case we can have $m \times e > 0$. In what follows we shall focus on this latter case and, with an abuse of notation, we shall use the index Λ to label the surviving charges: $\Lambda = 0, \ldots, h^{2,1}_-$.

Supersymmetry further requires $W = 0$, namely:

$$(e_\Lambda - \mathcal{N}_{\Lambda\Sigma} m^\Sigma)\, X^\Lambda = 0\,. \tag{9.12}$$

According to Michelson's analysis [5] the vector of electric/magnetic charges can always be reduced to a form defined by the following non-vanishing entries (Michelson's basis):

$$\begin{aligned} e_0 &= e^1_0 + \tau\, e^2_0\,; \qquad e_1 = e^1_1\,; \qquad m^0 = m^{1\,0} \\ m \times e &= m^{1\,0} e^2_0 > 0\,, \end{aligned} \tag{9.13}$$

which we shall refer to as Michelson's charge basis Q_M. From eqs. (9.10) the minimum conditions read:

$$\overline{\mathcal{N}}_{0,0} = \frac{e_0^2\tau + e_0^1}{m^{10}}\,; \qquad \overline{\mathcal{N}}_{0,1} = \mathcal{N}_{0,1} = \frac{e_1^1}{m^{10}}\,; \qquad \overline{\mathcal{N}}_{0,k} = 0 \;\; k \neq 0,\, 1\,. \tag{9.14}$$

If we also look for supersymmetric vacua should require eq. (9.12), namely

$$(e_0 - \mathcal{N}_{0,0}m^0)\, X^0 + (e_1 - \mathcal{N}_{1,0}m^0)\, X^1 = 0 \;\Rightarrow\; X^0 = 0\,, \tag{9.15}$$

where we have used the reality of $\mathcal{N}_{0,1}$ and the fact that $(e_0 - \mathcal{N}_{0,0}m^0) \neq 0$ (since $(e_0 - \overline{\mathcal{N}}_{0,0}m^0) = 0$ and being the imaginary part of $\mathcal{N}_{0,0}$ always non-vanishing). This minimum, having $X^0 = 0$ cannot be described in the ordinary special coordinate patch in which $X^0 \neq 0$.

In [21] supersymmetric vacua of an STU model (corresponding in the $O3/O7$ case to $h_-^{2,1} = 3$) have been studied in the special coordinate frame, making for the electric/magnetic charges, which are eight complex in general, the following choice:

$$Q_L = \{m^\Lambda,\, e_\Sigma\} = \{-1,\, 0,\, 0,\, \tau,\, -\tau,\, 0,\, 0,\, -1\}\,, \qquad m \times e = 2\,. \tag{9.16}$$

The scalar fields of this model, denoted by s, t, u, in the special coordinate basis are given by:

$$s = \frac{X^1}{X^0}\,; \qquad t = \frac{X^2}{X^0}\,; \qquad u = \frac{X^3}{X^0}\,; \qquad X^0 \neq 0\,. \tag{9.17}$$

We can also define a Michelson's basis Q_M for the STU model in which the non vanishing electric and magnetic charges correspond to $\Lambda = 0,\, 3$. The symplectic bases Q_L and Q_M (in which $m^{10} = 2/e_0^2$ if we require $m \times e = 2$) are related by a symplectic matrix $\mathscr{A}$ given in eq. (B.1) in appendix B:

$$Q_M = \mathscr{A}\, Q_L = \left\{\frac{2}{e_0^2},\, 0,\, 0,\, 0,\, e_0^1 + \tau\, e_0^2,\, 0,\, 0,\, e_1^1\right\}. \tag{9.18}$$

In appendix B, eq. (B.2), the reader may also find the explicit form of the period matrix $\mathcal{N}$ for the STU model in the special coordinate frame. In the special coordinate basis, with the choice of charges Q_L, conditions (9.10) and (9.12) have the following solution [21]:

$$\tau = -u\,; \qquad s = -\frac{1}{t}\,. \tag{9.19}$$

Upon application of $\mathscr{A}$ to Ω we obtain the holomorphic section Ω' in Michelson's basis as function of s, t, u:

$$\Omega' = \mathscr{A}\Omega = \left\{-\frac{1+st}{e_0^2},\, s,\, t,\, \frac{(1-st)(e_0^1 - e_0^2 u)}{e_0^2 e_1^1},\, st(-e_0^1 + e_0^2 u),\, tu,\, su,\, -e_1^1 st\right\}. \tag{9.20}$$

Conditions (9.10) and (9.12) are clearly satisfied by the same values of the moduli (9.19). On this vacuum in the new basis $X'^0 = 0$. It seems that if, in Michelson's basis, we have both electric and magnetic charges in the direction of $X'^0 \neq 0$ (graviphoton) supersymmetry is broken. Therefore in the symplectic basis Ω' we can use special coordinates to describe the supersymmetric vacuum (9.19), in a patch $X'^i \neq 0$ only if $i \neq 0$. In what follows we shall consider the patch $X'^1 \neq 0$. We refer the reader to appendix B, eq. (B.3), for the explicit form of the period matrix in Michelson's basis at $s = -1/t$ and $\tau = -u$. Note that the expression of the components $\mathcal{N}_{0,0}$ and $\mathcal{N}_{0,3}$ in eq. (B.3) are consistent with conditions (9.14), recalling that in this case $m^{10} = 2/e_0^2$.

Let us write the prepotential in Michelson's basis as a function of s, t, u:

$$\mathscr{F} = \frac{1}{2} X'^{\Lambda} F'_{\Lambda} = \frac{t}{s^2}\left(\frac{e_0^1\, t}{e_0^2} + \left(\frac{1}{s} - t\right) u\right). \tag{9.21}$$

We may express the above prepotential in terms of new special coordinates s', t', u' in the patch $X'^1 = 1$:

$$s' = \frac{X'^2}{X'^1}\,; \qquad t' = \frac{X'^0}{X'^1}\,; \qquad u' = \frac{X'^3}{X'^1}\,. \tag{9.22}$$

We refer the reader to eq. (B.4) of appendix B for the explicit form of these coordinates as functions of the old ones s, t, u. The prepotential in these variables is:

$$\mathscr{F} = \frac{e_0^1\, s'}{e_0^2} + \frac{e_0^2\, e_1^1\, t'\, u'}{2} - \frac{e_1^1 \sqrt{-4\, s' + (e_0^2)^2\, t'^2}\, u'}{2}\,. \tag{9.23}$$

One may check that:

$$\begin{aligned} F'_0 &= \partial_{t'} \mathscr{F}\,; \qquad F'_2 = \partial_{s'} \mathscr{F}\,; \qquad F'_3 = \partial_{u'} \mathscr{F}\,, \\ F'_1 &= 2\, \mathscr{F} - t'\, \partial_{t'} \mathscr{F} - s'\, \partial_{s'} \mathscr{F} - u'\, \partial_{u'} \mathscr{F}\,. \end{aligned} \tag{9.24}$$

10. Cubic prepotentials

Let us consider a special Kähler geometry with a generic cubic prepotential:

$$\mathscr{F} = \frac{1}{6}\, \kappa_{ijk}\, z^i\, z^j\, z^k\,. \tag{10.1}$$

Let us denote the real components of z^i as $z^i = x^i + i\, \lambda^i$. The metric has the form:

$$G_{ij} = -\frac{3}{2}\left(\frac{\kappa_{ij}}{\kappa} - \frac{3}{2}\frac{\kappa_i\, \kappa_j}{\kappa^2}\right), \tag{10.2}$$

where

$$\kappa = \kappa_{ijk}\, \lambda^i\, \lambda^j\, \lambda^k\,; \qquad \kappa_i = \kappa_{ijk}\, \lambda^j\, \lambda^k\,; \qquad \kappa_{ij} = \kappa_{ijk}\, \lambda^k\,. \tag{10.3}$$

The real and imaginary components of the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are then computed to be:

$$\begin{aligned} \mathrm{Re}(\mathcal{N}) &= \begin{pmatrix} \frac{1}{3}\, \kappa_{ijk}\, x^i\, x^j\, x^k & -\frac{1}{2}\, \kappa_{ijk}\, x^j\, x^k \\ -\frac{1}{2}\, \kappa_{ijk}\, x^j\, x^k & \kappa_{ijk}\, x^k \end{pmatrix}, \\ \mathrm{Im}(\mathcal{N}) &= \frac{1}{6}\, \kappa \begin{pmatrix} 1 + 4\, G_{ij}\, x^i\, x^j & -4\, G_{ij}\, x^j \\ -4\, G_{ij}\, x^j & 4\, G_{ij} \end{pmatrix}. \end{aligned} \tag{10.4}$$

The positivity domain of the lagrangian requires $\kappa < 0$.

As we have seen for Michelson's basis of charges where $e_0, m^0 \neq 0$, the existence of supersymmetric vacua implies $X^0 = 0$. Consider the case of a cubic prepotential and charges in Michelson's basis, but with no charge along the 0-direction:

$$\begin{aligned} e_{i_0} &= e^2_{i_0}\,\tau + e^1_{i_0}\,; \qquad e_j = e^1_j\,; \quad i_0,\, j \neq 0\,; \quad i_0 \neq j\,, \\ m^{i_0} &= m^{1\,i_0}\,. \end{aligned} \tag{10.5}$$

The minimum conditions (9.14) becomes:

$$\overline{\mathcal{N}}_{i_0 i_0} = \frac{e^1_{i_0} + \tau\, e^2_{i_0}}{m^{1 i_0}}\,; \quad \overline{\mathcal{N}}_{i_0 j} = \frac{e^1_j}{m^{1 i_0}}\,; \quad \overline{\mathcal{N}}_{i_0 k} = 0 \qquad k \neq i_0,\, j\,. \tag{10.6}$$

Using eqs. (10.4) we can write the minimum conditions (10.6) as follows:

$$\text{conditions on } \mathrm{Im}(\mathcal{N}) : \begin{cases} G_{i_0 i_0} = -\frac{3}{2\,\kappa}\,\frac{e^2_{i_0}}{m^{1\,i_0}}\,\tau_2\,, \\ G_{i_0 k} = 0 \;\; k \neq i_0\,, \\ G_{i_0 k}\, x^k = 0\,, \end{cases} \tag{10.7}$$

$$\text{conditions on } \mathrm{Re}(\mathcal{N}) : \begin{cases} \kappa_{i_0 i_0 k}\, x^k = \frac{e^2_{i_0}\,\tau_1 + e^1_{i_0}}{m^{1\,i_0}}\,, \\ \kappa_{i_0 j k}\, x^k = \frac{e^1_j}{m^{1\,i_0}}\,, \\ \kappa_{i_0 k l}\, x^l = 0 \;\; k \neq i_0,\, j\,, \\ \kappa_{i_0 k l}\, x^k\, x^l = 0\,. \end{cases} \tag{10.8}$$

From eqs. (10.7), (10.8) it follows that, if $e_{i_0},\, e_j \neq 0$:

$$x^{i_0} = x_j = 0\,. \tag{10.9}$$

If we further require supersymmetry we need to impose:

$$X^{i_0} = 0\,. \tag{10.10}$$

In the special coordinate basis there are components X^{i_0} which can vanish, their imaginary part should not correspond to Cartan isometries, e.g. brane coordinates.

Let us specialize to cubic prepotentials defining homogeneous spaces. The general form of $\mathscr{F}$ is given in [32]:

$$\mathscr{F} = \frac{1}{2}\,[z^1\,(z^2)^2 - z^1\,(z^\mu)^2 - z^2\,(z^u)^2 + \gamma_{\mu u v}\, z^\mu\, z^u\, z^v]\,, \tag{10.11}$$

where $z^\mu = \{z^3, z^\alpha\}$ is a vector in the fundamental of $\mathrm{SO}(1+q)$, $z^u = \{z^r,\, z^n\}$ ($u = 1, \ldots, 2\,d_s$) transforms in the spinorial representation of $\mathrm{SO}(1+q)$, z^r, z^n being the chiral components with respect to $\mathrm{SO}(q)$, and γ_μ are the generators of the corresponding Clifford algebra. The expression (10.11) can be recast in the following form:

$$\begin{aligned} \mathscr{F} &= stu - \frac{s}{2}\,(z^n)^2 - \frac{u}{2}\,(z^r)^2 - \frac{t}{2}\,(z^\alpha)^2 + \gamma_{\alpha k r}\, z^\alpha\, z^n\, z^r\,, \\ \alpha &= 1, \ldots, q\,; \qquad n = 1, \ldots, d_s\,; \qquad r = 1, \ldots, d_s\,, \end{aligned} \tag{10.12}$$

if we identify $s = z^2 + z^3$, $u = z^2 - z^3$ and $t = 2\,z^1$.

Since s, t, u are moduli whose imaginary parts are related to Cartan isometries, namely $\mathrm{Im}(s) = -e^{\phi_s}$, $\mathrm{Im}(t) = -e^{\phi_t}$, $\mathrm{Im}(u) = -e^{\phi_u}$, they cannot be set to zero. Only the remaining moduli z^α, z^n, z^r can be set to zero, and thus, in the study of supersymmetric vacua, we shall consider three different cases in which the charges e_{i_0}, m^{i_0} are chosen along the directions $X^\alpha = z^\alpha$, $X^n = z^n$, $X^r = z^r$.

Case $i_0 = \bar{\alpha}$. Let us start from the conditions (10.8). The first equation gives

$$\kappa_{\bar{\alpha}\bar{\alpha}t}\, x^t = \frac{1}{m^{2\,\bar{\alpha}}}(e^1_{\bar{\alpha}}\,\tau_1 + e^2_{\bar{\alpha}})\,. \tag{10.13}$$

The remaining conditions depend of the choice of the index j of the additional electric charge e_j. Choosing $j = \beta \neq \bar{\alpha}$ or $j = s, t, u$, conditions (10.8) imply $e_j = 0$. The only cases in which e_j can be non-vanishing correspond to:

$$\begin{aligned} j = \bar{n} \;\Rightarrow\; &\begin{cases} \kappa_{\bar{\alpha}\,n\,r}\, x^r = \frac{e^1_n}{m^{1\,\bar{\alpha}}}\,\delta_{n\bar{n}} \\ \kappa_{\bar{\alpha}\,r\,n}\, x^n = 0 \end{cases}, \\ j = \bar{r} \;\Rightarrow\; &\begin{cases} \kappa_{\bar{\alpha}\,r\,n}\, x^n = \frac{e^1_j}{m^{1\,\bar{\alpha}}}\,\delta_{r\bar{r}} \\ \kappa_{\bar{\alpha}\,n\,r}\, x^r = 0 \end{cases}. \end{aligned} \tag{10.14}$$

The last of conditions (10.8) does not imply any new constraint.

Let us now consider the implications of conditions (10.7). From eq. (10.2) we can write:

$$G_{\bar{\alpha}k} = -\frac{3}{2}\left(\frac{\kappa_{\bar{\alpha}k}}{\kappa} - \frac{3}{2}\,\frac{\kappa_{\bar{\alpha}}\,\kappa_k}{\kappa^2}\right), \tag{10.15}$$

we may distinguish two cases: $\kappa_{\bar{\alpha}} = 0$ and $\kappa_{\bar{\alpha}} \neq 0$. In the former case vanishing of $G_{\bar{\alpha}t} = 0$, which is satisfied if $\kappa_{\bar{\alpha}t} = \kappa_{\bar{\alpha}\bar{\alpha}t}\,\lambda^{\bar{\alpha}} = 0$ which in turn implies $\lambda^{\bar{\alpha}} = 0$. This latter condition, together with $x^{\bar{\alpha}} = 0$ from eqs. (10.7), fixes $X^{\bar{\alpha}} = 0$ and thus *the vacuum is supersymmetric.* The remaining conditions in eqs. (10.7) imply:

$$\kappa_{\bar{\alpha}\bar{\alpha}t}\,\lambda^t = \frac{e^2_{\bar{\alpha}}}{m^{1\,\bar{\alpha}}}\,\tau_2\,, \tag{10.16}$$

$$0 = \kappa_{\bar{\alpha}n} = \kappa_{\bar{\alpha}nr}\,\lambda^r\,, \tag{10.17}$$

$$0 = \kappa_{\bar{\alpha}r} = \kappa_{\bar{\alpha}nr}\,\lambda^n\,. \tag{10.18}$$

Eqs. (10.13), (10.16) imply that the complex scalar t is fixed to the complex value:

$$t = t_0 = \frac{e_{\bar{\alpha}}}{\kappa_{\bar{\alpha}\bar{\alpha}t}\, m^{\bar{\alpha}}}\,. \tag{10.19}$$

The scalars z^β, $\beta \neq \bar{\alpha}$ are moduli in this supersymmetric vacuum.

Relaxing condition $\kappa_{\bar{\alpha}} = 0$ which imply unbroken supersymmetry, we obtain involved non-linear equations to be solved. We shall not discuss here the most general solution of these equations.

Case $i_0 = \bar{n}$. The first of eqs. (10.8) gives

$$\kappa_{\bar{n}\bar{n}s}\, x^s = \frac{1}{m^{2\,\bar{n}}}(e^1_{\bar{n}}\, \tau_1 + e^2_{\bar{n}})\,. \tag{10.20}$$

Let us discuss the remaining conditions for various choices of the electric charge e_j. For $j = n \neq \bar{n}$ or $j = s, t, u$, conditions (10.8) imply $e_j = 0$. The only cases allowing non-vanishing e_j correspond to:

$$\begin{aligned} j = \bar{\alpha} \;\Rightarrow\; &\begin{cases} \kappa_{\bar{n}\,\alpha\,r}\, x^r = \frac{e^1_j}{m^{1\,\bar{n}}}\, \delta_{\alpha\bar{\alpha}} \\ \kappa_{\bar{n}\,r\,\beta}\, x^\beta = 0 \end{cases} , \\ j = \bar{r} \;\Rightarrow\; &\begin{cases} \kappa_{\bar{n}\,r\,\beta}\, x^\beta = \frac{e^1_j}{m^{1\,\bar{n}}}\, \delta_{r\bar{r}} \\ \kappa_{\bar{n}\,\beta\,s}\, x^s = 0 \end{cases} . \end{aligned} \tag{10.21}$$

The last of conditions (10.8) does not imply any new constraint.

As far as conditions (10.7) are concerned, the relevant components of the metric are:

$$G_{\bar{n}i} = -\frac{3}{2}\left(\frac{\kappa_{\bar{n}i}}{\kappa} - \frac{3}{2}\,\frac{\kappa_{\bar{n}}\,\kappa_i}{\kappa^2}\right). \tag{10.22}$$

We start discussing the $\kappa_{\bar{n}} = 0$ case. The vanishing of $G_{\bar{\alpha}s} = 0$, which is satisfied if $\kappa_{\bar{\alpha}s} = 0$ implies $\lambda^{\bar{n}} = 0$. This condition, together with $x^{\bar{n}} = 0$ from eqs. (10.7), fixes $X^{\bar{n}} = 0$ and thus ensures supersymmetry of the vacuum. The remaining conditions in eqs. (10.7) imply:

$$\kappa_{\bar{n}\bar{n}s}\, \lambda^s = \frac{e^2_{\bar{n}}}{m^{1\,\bar{n}}}\, \tau_2\,, \tag{10.23}$$

$$0 = \kappa_{\bar{n}\alpha} = \kappa_{\bar{n}\alpha r}\, \lambda^r\,, \tag{10.24}$$

$$0 = \kappa_{\bar{n}r} = \kappa_{\bar{n}r\alpha}\, \lambda^\alpha\,. \tag{10.25}$$

Eqs. (10.20), (10.23) imply that the complex scalar s is fixed to the complex value:

$$s = s_0 = \frac{e_{\bar{n}}}{\kappa_{\bar{n}\bar{n}s}\, m^{\bar{n}}}\,. \tag{10.26}$$

The scalars z^n, $n \neq \bar{n}$ are moduli in this supersymmetric vacuum.

Also in this case relaxing condition $\kappa_{\bar{n}} = 0$, which imply unbroken supersymmetry, we have to solve involved non-linear conditions. We shall not discuss here the existence of a non-trivial solution.

Case $i_0 = \bar{r}$. This case is analogous to the previous one upon substituting $r \leftrightarrow n$ and $s \leftrightarrow u$.

We now show that in the spacial cases of$L(0, P, \dot{P})$, $L(q, 0)$ manifolds, all vacua are supersymmetric. Consider first the $q = 0$ case defining the $L(0, P, \dot{P})$ manifold. If we take $i_0 = \bar{k}$ from eqs. (10.7) we derive

$$G_{\bar{k}t} = \frac{9}{2}\,\frac{\kappa_{\bar{k}}\,\lambda^s\,\lambda^u}{\kappa^2} = 0 \to \kappa_{\bar{k}} = 0 \;\Rightarrow\; \lambda^{\bar{k}} = 0\,, \tag{10.27}$$

the last condition, together with $x^{\bar{k}} = 0$ ensures supersymmetry of the vacuum.

Similarly if we take $i_0 = \bar{r}$, from $G_{\bar{r}t} = 0$ we derive $\lambda^{\bar{r}} = 0$ and thus that the vacuum is supersymmetric.

The same arguments apply to the $L(q, 0)$.

Sp(6)/U(3) example. The coordinates of the six-dimensional special Kähler manifold are given by the independent entries of a symmetric U(3) complex tensor Z^{ij}, $i, j = 1, 2, 3$. Choosing:

$$z^1 = Z^{11}\,; \qquad z^2 = Z^{22}\,; \qquad z^3 = Z^{33}\,; \qquad z^4 = -Z^{23}\,; \qquad z^5 = -Z^{13}\,; \qquad z^6 = -Z^{12}\,, \tag{10.28}$$

the prepotential $\mathscr{F}$ has the form of the following U(3)-invariant polynomial

$$\mathscr{F} = \frac{1}{6}\,\epsilon_{ijk}\epsilon_{lmn}\,Z^{il}\,Z^{jm}\,Z^{kn} = z^1\,z^2\,z^3 - z^1\,(z^4)^2 - z^2\,(z^5)^2 - z^3\,(z^6)^2 - 2\,z^4\,z^5\,z^6\,. \tag{10.29}$$

Equation (10.29) is consistent with the general form of the cubic polynomial for homogeneous manifolds given in [33], which, for the present $L(1,1)$ case, reads:

$$\kappa(h) = 6[h^1(h^2{+}h^3)(h^2{-}h^3){-}(h^2{-}h^3)(h^5)^2{-}(h^2{+}h^3)(h^6)^2{-}h^1(h^4)^2{-}2\,h^4\,h^5\,h^6]\,. \tag{10.30}$$

In this case we may identify the coordinates s, t, u parametrizing the $[\mathrm{SU}(1,1)/\mathrm{U}(1)]^3$ submanifold, with z^1, z^2, z^3 respectively, and $z^\alpha = z^5$, $z^k = z^4$, $z^r = z^6$. Consider taking e_{i_0}, m^{i_0} along the direction $k = 4$. Conditions (10.7) imply:

$$\begin{aligned}
G_{4s} &= -\frac{18}{\kappa^2}\,(\lambda_4\,\lambda_5 + \lambda_u\,\lambda_6)\,(\lambda_4\,\lambda_6 + \lambda_t\,\lambda_5) = 0\,,\\
G_{4t} &= -\frac{18}{\kappa^2}\,(\lambda_s\,\lambda_u + \lambda_5^2)\,(\lambda_4\,\lambda_s + \lambda_6\,\lambda_5) = 0\,,\\
G_{4u} &= -\frac{18}{\kappa^2}\,(\lambda_s\,\lambda_t + \lambda_6^2)\,(\lambda_4\,\lambda_s + \lambda_6\,\lambda_5) = 0\,,\\
G_{45} &= \frac{18}{\kappa^2}\,(2\,\lambda_1\,\lambda_2\,\lambda_4\,\lambda_5 + \lambda_1\,\lambda_2\,\lambda_3\,\lambda_6 + \lambda_1\,{\lambda_4}^2\,\lambda_6 + \lambda_2\,{\lambda_5}^2\,\lambda_6 - \lambda_3\,{\lambda_6}^3) = 0\,,\\
G_{46} &= \frac{18}{\kappa^2}\,(\lambda_1\,\lambda_2\,\lambda_3\,\lambda_5 + \lambda_1\,{\lambda_4}^2\,\lambda_5 - \lambda_2\,{\lambda_5}^3 + 2\,\lambda_1\,\lambda_3\,\lambda_4\,\lambda_6 + \lambda_3\,\lambda_5\,{\lambda_6}^2) = 0\,,\\
G_{44} &= \frac{18}{\kappa^2}\,({\lambda_1}^2\,\lambda_2\,\lambda_3 + {\lambda_1}^2\,{\lambda_4}^2 - \lambda_1\,\lambda_2\,{\lambda_5}^2 + 2\,\lambda_1\,\lambda_4\,\lambda_5\,\lambda_6 - \lambda_1\,\lambda_3\,{\lambda_6}^2 + 2\,{\lambda_5}^2\,{\lambda_6}^2)\\
&= -\frac{3}{2\,\kappa}\,\frac{e^1_{i_0}}{m^{2 i_0}}\,\tau_2\,.
\end{aligned} \tag{10.31}$$

According to our general analysis condition

$$\kappa_4 = -4\,(\lambda_2\,\lambda_4 + \lambda_5\,\lambda_6) = 0\,, \tag{10.32}$$

characterizes the supersymmetric vacuum which always exists. In this case there are no other solutions.

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A. Quaternionic geometry

In the present appendix we recall our notations. The metric of the original quaternionic manifold is [29]:

$$ds^2 = h_{\hat{u}\hat{v}}\, dq^{\hat{v}} = K_{Q\, a\bar{b}}\, \partial_\mu w^a \partial^\mu \bar{w}^{\bar{b}} + (\partial\varphi)^2 + \frac{e^{4\,\varphi}}{4}\,(\partial a - V \times \partial V)^2 - \frac{e^{2\,\varphi}}{2}\,\partial_\mu V\, \mathscr{M}\, \partial^\mu V\,, \quad \text{(A.1)}$$

where $K_{Q\, a\bar{b}} = \partial_a \partial_{\bar{b}} K_Q$ and we have denoted the scalar fields by $\{\varphi,\, a,\, w^a,\, \xi^\Lambda,\, \tilde{\xi}_\Lambda\}$,[3] V in (A.1) is the symplectic vector defined as:

$$V = \{\xi^\Lambda,\, \tilde{\xi}_\Lambda\}\,, \quad \text{(A.2)}$$

and " $\times$ " denotes the symplectic invariant scalar product:

$$V \times W = V^\Lambda W_\Lambda - V_\Lambda W^\Lambda\,. \quad \text{(A.3)}$$

The matrix $\mathscr{M}$ in (A.1) is negative definite and has the following form:

$$\mathscr{M} = \begin{pmatrix} \mathcal{R}\mathcal{I}^{-1}\mathcal{R} + \mathcal{I} & \mathcal{R}\mathcal{I}^{-1} \\ \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \quad \text{(A.4)}$$

where $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary parts $\mathcal{R} = \mathrm{Re}(\mathcal{M})$, $\mathcal{I} = \mathrm{Im}(\mathcal{M})$ of the "period matrix" $\mathcal{M}_{\Lambda\Sigma}$ associated to the special Kähler submanifold parametrized by z^a.

The scalar fields dual to the NSNS and RR tensors $B_{\mu\nu}$, $C_{\mu\nu}$ are a, $\tilde{\xi}_0$ respectively while ξ^a and $\tilde{\xi}_a$, $a = 1, \ldots, h_{1,1}$, are the remaining RR scalars originating from the 2-form and the 4-form respectively. Finally ξ^0 corresponds to the ten dimensional axion, φ is the four dimensional dilaton and w^a are the Kähler moduli. The metric $M_{IJ} = h_{IJ}$, where the values $I, J = 1, 2$ label the scalars a, $\tilde{\xi}_0$ respectively, and its inverse M^{IJ} have the following form:

$$\begin{aligned} M_{IJ} &= \frac{e^{4\,\varphi}}{4} \begin{pmatrix} 1 & -\xi^0 \\ -\xi_0 & (\xi^0)^2 - 2\, e^{-2\,\varphi}\mathcal{I}^{-1|\,00} \end{pmatrix}, \\ M^{IJ} &= -\frac{2}{\mathcal{I}^{-1|\,00}}\, e^{-2\,\varphi} \begin{pmatrix} (\xi^0)^2 - 2\, e^{-2\,\varphi}\, \mathcal{I}^{-1|\,00} & \xi^0 \\ \xi_0 & 1 \end{pmatrix}. \end{aligned} \quad \text{(A.5)}$$

[3] In the present paper we have chosen to denote the axions deriving from the RR forms by the letter ξ instead of ζ, which is more often used in the literature, in order not to create confusion with the hyperinos.

Note that the expression of M_{IJ} coincides with that of $\omega^x_I \omega^x_J$ given by:

$$\omega^x_I \omega^x_J = \frac{e^{4\varphi}}{4} \begin{pmatrix} 1 & -\xi^0 \\ -\xi_0 & (\xi^0)^2 + 16\, e^{-2\varphi + K_Q} \end{pmatrix}, \tag{A.6}$$

only in the case of cubic quaternionic geometries for which the following relation holds:

$$e^{K_Q} = -\frac{1}{8} \mathcal{I}^{-1|\,00}\,. \tag{A.7}$$

Using the explicit metric (A.1) and eqs. (A.5) we can now compute the quantities A^I_u:

$$\begin{aligned} A^I_u dq^u &= M^{IJ}\, h_{Iu}\, dq^u \qquad (A.8)\\ &= \frac{1}{\mathcal{I}^{-1|00}} \left[\begin{pmatrix} -\mathcal{I}^{-1|00}\, \xi^a + \xi^0\, \mathcal{I}^{-1|0a} \\ \mathcal{I}^{-1|0a} \end{pmatrix} d\tilde{\xi}_a + + \begin{pmatrix} \mathcal{I}^{-1|00}\, \tilde{\xi}_\Lambda + \xi^0\, (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^0 \\ (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^0 \end{pmatrix} d\xi^\Lambda \right]. \end{aligned}$$

If we redefine $a \to a - \xi^\Lambda\, \tilde{\xi}_\Lambda$ the metric will no more depend on $\tilde{\xi}$ and A^I_u will have the form:

$$A^I_u dq^u = \frac{1}{\mathcal{I}^{-1|00}} \left[\begin{pmatrix} -2\,\mathcal{I}^{-1|00}\, \xi^a + 2\, \xi^0\, \mathcal{I}^{-1|0a} \\ \mathcal{I}^{-1|0a} \end{pmatrix} d\tilde{\xi}_a + \begin{pmatrix} 2\, \xi^0\, (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^0 \\ (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^0 \end{pmatrix} d\xi^\Lambda \right]. \tag{A.9}$$

Let us define the following forms:

$$\begin{aligned} v &= \frac{1}{2}\, e^{2\varphi}\, [-2\, e^{-2\varphi}\, d\varphi - i\, (da + \tilde{\xi}^T\, d\xi - \xi^T\, d\tilde{\xi})]\,, \\ u &= i\, e^{\varphi + \frac{K_Q}{2}}\, Z^T\, (\overline{\mathcal{M}}\, d\xi + d\tilde{\xi})\,, \\ E &= i\, e^{\varphi - \frac{K_Q}{2}}\, P\, N^{-1}\, (\overline{\mathcal{M}}\, d\xi + d\tilde{\xi})\,, \\ e &= P\, dZ\,, \end{aligned} \tag{A.10}$$

where $Z^\Lambda = \{1, w^a\}$ and the matrices P and N are defined as follows:

$$P^{\underline{a}}{}_0 = -e_b{}^{\underline{a}} Z^b\,; \qquad P^{\underline{a}}{}_b = e_b{}^{\underline{a}} \qquad (b,\, a = 1, \ldots, h_{2,1})\,, \tag{A.11}$$

$$N_{\Lambda\Sigma} = \frac{1}{2} \mathrm{Re}(\frac{\partial^2 \mathscr{F}_Q}{\partial Z^\Lambda \partial Z^\Sigma})\,. \tag{A.12}$$

$\mathscr{F}_Q$ being the prepotential of the special Kähler manifold embedded in the quaternionic manifold, $e_a{}^{\underline{b}}$ being the corresponding vielbein (the underlined indices are the rigid ones). One can check that in terms of the forms in (A.10), the metric (A.1) has the simple expression:

$$ds^2 = v \otimes \overline{v} + u \otimes \overline{u} + E \otimes \overline{E} + e \otimes \overline{e}\,. \tag{A.13}$$

Let us now give the expression for the vielbein $\mathcal{U}$. In the heterotic case we have:

$$\mathcal{U}^{1\dot{\alpha}} = \begin{pmatrix} v \\ e^{\underline{a}} \end{pmatrix}; \qquad \mathcal{U}^{1\dot{\alpha}} = \begin{pmatrix} u \\ E^{\underline{a}} \end{pmatrix}. \tag{A.14}$$

For the $O5/O9$ case we simply exchange $\mathcal{U}^{1\dot{\alpha}} \leftrightarrow \mathcal{U}^{2\dot{\alpha}}$. In particular we can compute the components of $\mathcal{U}^{A\dot{\alpha}}_I$ where $I = 1$ is the component along da and $I = 2$ along $d\tilde{xi}_0$, a, $\tilde{\xi}_0$

being the scalars dual to $B_{1\mu\nu}$ and $B_{2\mu\nu}$ respectively. We obtain in the heterotic case:

$$\begin{aligned}
\mathcal{U}^{1\dot\alpha}_{I=1} &= -\frac{i}{2}\,e^{2\varphi}\begin{pmatrix}1\\ \mathbf{0}_{(n_H-1)}\end{pmatrix}; \qquad \mathcal{U}^{1\dot\alpha}_{I=2} = \frac{i}{2}\,e^{2\varphi}\begin{pmatrix}\xi^0\\ \mathbf{0}_{(n_H-1)}\end{pmatrix},\\
\mathcal{U}^{2\dot\alpha}_{I=1} &= \begin{pmatrix}0\\ \mathbf{0}_{(n_H-1)}\end{pmatrix}; \qquad \mathcal{U}^{2\dot\alpha}_{I=2} = i\,e^{\varphi+\frac{K_Q}{2}}\begin{pmatrix}1\\ e^{-K_Q}\,P^{\underline{a}}{}_{\Lambda}\,N^{-1|\Lambda 0}\end{pmatrix},
\end{aligned} \tag{A.15}$$

where the first entry of the above vectors corresponds to $\dot\alpha = 0$. We note that in the heterotic case $\xi^0 = 0$ so that $\mathcal{U}^{1\dot\alpha}_{I=2} = \mathcal{U}^{2\dot\alpha}_{I=1} = 0$, consistently with equations (4.6), (4.9). In the $O5/O9$ case, exchanging $\mathcal{U}^{1\dot\alpha} \leftrightarrow \mathcal{U}^{2\dot\alpha}$ we obtain the corresponding conditions.

B. Special Kähler geometry in two different symplectic bases

The matrix $\mathscr{A}$ relating the two relevant symplectic bases Q_M and Q_L is:

$$\mathscr{A} = \begin{pmatrix}
-\frac{1}{e_0^2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{e_0^2}\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\frac{e_0^1}{e_0^2\,e_1^1} & 0 & 0 & -\frac{1}{e_1^1} & -\frac{1}{e_1^1} & 0 & 0 & -\frac{e_0^1}{e_0^2\,e_1^1}\\
0 & 0 & 0 & 0 & -e_0^2 & 0 & 0 & -e_0^1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_1^1
\end{pmatrix}. \tag{B.1}$$

The period matrix in the special coordinate basis is

$$\begin{aligned}
\overline{\mathcal{N}}_{0,0} &= -\frac{s^2\left(\overline{t}u - t\,\overline{u}\right)^2 + \overline{s}^2\left(tu - \overline{t}\overline{u}\right)^2 - 2s\overline{s}\left(t^2\,u\overline{u} + \overline{t}^2\,u\overline{u} + t\,\overline{t}\,\left(u^2 - 4\,u\,\overline{u} + \overline{u}^2\right)\right)}{2\,(s-\overline{s})\,\left(t-\overline{t}\right)\,(u-\overline{u})},\\
\overline{\mathcal{N}}_{0,1} &= \frac{\overline{s}tu - \overline{t}su - \overline{u}ts + s\overline{t}\overline{u}}{2\,(s-\overline{s})},\\
\overline{\mathcal{N}}_{0,2} &= \frac{\overline{s}tu - \overline{t}su + \overline{u}ts - s\overline{t}\overline{u}}{2\,(s-\overline{s})},\\
\overline{\mathcal{N}}_{0,3} &= \frac{\overline{s}tu + \overline{t}su - \overline{u}ts - s\overline{t}\overline{u}}{2\,(s-\overline{s})},\\
\overline{\mathcal{N}}_{1,1} &= -\frac{(t-\overline{t})(u-\overline{u})}{2\,(s-\overline{s})},\\
\overline{\mathcal{N}}_{1,2} &= \frac{u+\overline{u}}{2},\\
\overline{\mathcal{N}}_{1,3} &= \frac{t+\overline{t}}{2},\\
\overline{\mathcal{N}}_{1,1} &= -\frac{(s-\overline{s})(u-\overline{u})}{2\,(t-\overline{t})},\\
\overline{\mathcal{N}}_{2,3} &= \frac{s+\overline{s}}{2},\\
\overline{\mathcal{N}}_{3,3} &= -\frac{(s-\overline{s})(t-\overline{t})}{2\,(u-\overline{u})}.
\end{aligned} \tag{B.2}$$

The period matrix in the new basis Ω' at $s = -1/t$ and $\tau = -u$ reads:

$$
\begin{aligned}
\overline{\mathcal{N}}_{0,0} &= \frac{1}{2}\, e_0^2\, (e_0^1 - u e_0^2)\,, \\
\overline{\mathcal{N}}_{0,1} &= 0\,, \\
\overline{\mathcal{N}}_{0,2} &= 0\,, \\
\overline{\mathcal{N}}_{0,3} &= \frac{1}{2}\, e_0^2\, e_1^1\,, \\
\overline{\mathcal{N}}_{1,1} &= \frac{2\, t^2\, \overline{t}^2\, (u - \overline{u})\, (e_0^1 - e_0^2\, \overline{u})}{e_0^1\, (t - \overline{t})^2 - e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)}\,, \\
\overline{\mathcal{N}}_{1,2} &= \frac{e_0^2\, \overline{u}\, \left(t^2\, u + \overline{t}^2\, u - 2\, t\, \overline{t}\, \overline{u}\right) - e_0^1\, \left(-2\, t\, \overline{t}\, u + t^2\, \overline{u} + \overline{t}^2\, \overline{u}\right)}{-e_0^1\, (t - \overline{t})^2 + e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)}, \\
\overline{\mathcal{N}}_{1,3} &= -\frac{e_0^2\, e_1^1\, t\, \overline{t}\, (t + \overline{t})\, (u - \overline{u})}{-e_0^1\, (t - \overline{t})^2 + e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)}\,, \\
\overline{\mathcal{N}}_{2,2} &= \frac{2\, (u - \overline{u})\, (e_0^1 - e_0^2\, \overline{u})}{e_0^1\, (t - \overline{t})^2 - e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)}\,, \\
\overline{\mathcal{N}}_{2,3} &= \frac{e_0^2\, e_1^1\, (t + \overline{t})\, (u - \overline{u})}{-e_0^1\, (t - \overline{t})^2 + e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)}\,, \\
\overline{\mathcal{N}}_{3,3} &= \frac{-e_0^2\, (e_1^1)^2\, (t - \overline{t})^2}{2\, \left(-e_0^1\, (t - \overline{t})^2 + e_0^2\, \left(t^2\, u + \overline{t}^2\, u + 2\, t\, \overline{t}\, (u - 2\, \overline{u})\right)\right)}\,.
\end{aligned}
\tag{B.3}
$$

In this basis we can define special coordinates referred to the patch in which $X'1 \neq 0$ (we have rescaled Ω' by s):

$$
\begin{aligned}
s' &= \frac{X'^2}{X'^1}\,; \qquad t' = \frac{X'^0}{X'^1}\,; \qquad u' = \frac{X'^3}{X'^1} \\
s &= \frac{-\left(e_0^2\, t'\right) \pm \sqrt{-4\, s' + (e_0^2)^2\, t'^2}}{2\, s'}\,; \qquad t = \frac{-\left(e_0^2\, t'\right) \pm \sqrt{-4\, s' + (e_0^2)^2\, t'^2}}{2}\,; \\
u &= \frac{e_0^1}{e_0^2} \pm \frac{e_1^1\, u'}{\sqrt{-4\, s' + (e_0^2)^2\, t'^2}}\,,
\end{aligned}
\tag{B.4}
$$

we shall use the first solution (with the"+" sign).

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