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# Primes in Almost All Short Intervals II.

DANILO BAZZANELLA

**Sunto.** – *In questo lavoro vengono migliorati i risultati ottenuti in «Primes in Almost All Short Intervals» riguardo la distribuzione dei primi in quasi tutti gli intervalli corti della forma* $[g(n), g(n)+H]$, *con* $g(n)$ *funzione reale appartenente ad una ampia classe di funzioni. Il problema viene trattato mettendo in relazione l'insieme eccezionale per la distribuzione dei primi in intervalli nella forma* $[g(n), g(n)+H]$ *con l'insieme eccezionale per la formula asintotica*

$$\psi(x+H)-\psi(x)\sim H \qquad as\ x\to\infty\,.$$

*I risultati presentati vengono quindi ottenuti grazie allo studio delle proprietà dell'insieme eccezionale per tale formula asintotica..*

## 1. – Introduction.

Throughout this paper $\psi(x)$ will denote Chebyshev's function, $F\approx G$ will mean that $F\ll G$ and $G\ll F$ hold, $F(x)=\infty(G(x))$ will mean that $\lim_{x\to\infty} G(x)/F(x)=0$, $N(\sigma, T)$ denotes the counting function defined as the number of zeros $\varrho=\beta+i\gamma$ of Riemann zeta function which satisfy $\sigma\leqslant\beta\leqslant 1$ and $|\gamma|\leqslant T$ and $N^*(\sigma, T)$ denotes the counting function defined as the number of ordered sets of zeros $\varrho_j=\beta_j+i\gamma_j$ $(1\leqslant j\leqslant 4)$, each counted by $N(\sigma, T)$, for which $|\gamma_1+\gamma_2-\gamma_3-\gamma_4|\leqslant 1$.

It is known that

$$\psi(x+H)-\psi(x)\sim H \qquad \text{as } x\to\infty \tag{1}$$

holds with $x^{7/12-o(1)}\leqslant H\leqslant x$, see Heath-Brown [6], and holds for almost all $x$ with $x^{1/6-o(1)}\leqslant H\leqslant x$, see Zaccagnini [11]. It is also known that, under the assumption of the Riemann Hypothesis (RH), (1) holds with $x^{1/2+\varepsilon}\leqslant H\leqslant x$, and holds for almost all $x$ with $H=\infty(\log^2 x)$, see Selberg [10].

The aim of this paper is to investigate the distribution of primes in intervals of type

$$[g(n), g(n)+H]\,,$$

for $n$ such that $N\leqslant g(n)\leqslant 2N$ and the function $g(x)$ belonging in a wide class of derivable functions.

More precisely we consider the class of derivable functions $g(x)$ such that

$g(x) \approx x^{\alpha}$ and $g'(x) \approx \alpha x^{\alpha-1}$. A function satisfying these requirements will be called of *type* $\alpha$.

The problem that we investigate is how must be large $H$ to have the expected number of primes for almost all intervals of type $[g(n), g(n)+H]$ in $[N, 2N]$, with a fixed function $g(x)$ of *type* $\alpha$.

As in the first paper of the series, see Bazzanella [1], the dimension of $H$ only depends of $\alpha$ and, more precisely, results an increasing function of $\alpha$.

The basic idea of this paper is to connect the exceptional set for the distribution of primes in intervals of type $[g(n), g(n)+H]$ to the exceptional set for the asymptotic formula (1) and use the properties of this set, see Bazzanella and Perelli [2], to obtain the desired results.
Our main unconditional result is the following:

THEOREM 1. – *Let* $g(x)$ *a function of type* $\alpha$ and $\varepsilon > 0$. Then almost all intervals of type $[g(n), g(n)+H]$ in $[N, 2N]$ has the expected number of primes with $H > N^{c(\alpha)+\varepsilon}$ and

$$c(\alpha) = \begin{cases} \dfrac{11\alpha - 10}{16\alpha} & \dfrac{6}{5} \leqslant \alpha \leqslant \dfrac{273}{107} \\ \dfrac{3\alpha - 2}{5\alpha} & \dfrac{273}{107} \leqslant \alpha \leqslant \dfrac{21}{4} \\ \dfrac{47\alpha - 35}{77\alpha} & \dfrac{21}{4} \leqslant \alpha \leqslant \dfrac{210}{29} \\ \dfrac{45\alpha - 12 - \sqrt{144 + 168\alpha + 265\alpha^2}}{48\alpha} & \dfrac{210}{29} \leqslant \alpha \leqslant 23\,. \end{cases}$$

This result mainly depends of the bound for counting functions $N(\sigma, T)$ and $N^*(\sigma, T)$. For this purpose we use the density estimate of Ingham, see ch. 12 of Montgomery [9], Huxley [7] and Jutila [8] for $N(\sigma, T)$ and the density estimate of Heath-Brown [5] for $N^*(\sigma, T)$.

This result is stronger and more explicit than Theorem 1 of [1] for all values of $\alpha$ in considered range. Note that this new technique is not good for large values of $\alpha$. This is due to the fact that we are unable to prove a sufficiently good estimate for $N^*(\sigma, T)$. If we assume the heuristic assumption

$$N^*(\sigma, T) \ll \frac{N(\sigma, T)^4}{T}\,, \tag{2}$$

we can improve Theorem 1 as follows

THEOREM 2. – *Assume* (2), *let* $g(x)$ *a function of type* $\alpha$ *and* $\varepsilon > 0$. *Then almost all intervals of type* $[g(n), g(n)+H]$ *in* $[N, 2N]$ *has the expected number of primes with* $H > N^{c(\alpha)+\varepsilon}$ *and*

$$c(\alpha) = \begin{cases} \dfrac{11\alpha - 10}{16\alpha} & \dfrac{6}{5} \leqslant \alpha \leqslant 2 \\ \dfrac{7\alpha - 5}{12\alpha} & 2 \leqslant \alpha\,. \end{cases}$$

*This conditional result improves Theorem 1 of [1] for all values of* $\alpha$.

*Under RH we obtain in a similar way the following theorem*

THEOREM 3. – *Assume RH and let* $g(x)$ *a function of type* $\alpha$. Then almost all intervals of type $[g(n), g(n)+H]$ in $[N, 2N]$ has the expected number of primes with $H = \infty(N^{\alpha - 1/2\alpha}\log N)$.

This result is stronger than Theorem 2 of [1] for all values of $\alpha$.

We remark that we may obtain results similar to the above theorems slight weakening the hypothesis on the function $g(x)$. Furthermore we remark that in the first two theorems we can replace the positive constant $\varepsilon$ with an appropriate power of $\log N$. We have stated our results in the above form for the sake of simplicity.

## 2. – Preliminary lemmas.

We consider the asymptotic formula

$$\psi(x+H) - \psi(x) \sim H \quad \text{as } x \to \infty\,, \tag{3}$$

with $H = X^{\theta}$.

Let define the set

$$E_{\delta}(X, \theta) = \{X \leqslant x \leqslant 2X\colon\ |\Delta(x, X^{\theta})| \geqslant \delta X^{\theta}\}\,,$$

with

$$\Delta(x, h) = \psi(x+h) - \psi(x) - h\,.$$

It is clear that (3) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_{\delta}(X, \theta) = \emptyset$ for $X \geqslant X_0(\delta)$. Hence for small $\delta > 0$, and for $X \to \infty$ the set $E_{\delta}(X, \theta)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals.

To deal with the problem to estimate the exceptional set for (3) we intro-

duce the functions

$$\mu_\delta(\theta) = \inf\{\xi \geqslant 0\colon\ |E_\delta(X,\theta)| \ll_{\delta,\theta} X^\xi\}$$

and

$$\mu(\theta) = \sup_{\delta>0} \mu_\delta(\theta). \tag{4}$$

The basic lemma needed for the proof of the Theorem 1 is the following unconditional estimate for the exceptional set for the number of primes in short intervals.

LEMMA 1. – *Let* $\mu(\theta)$ *defined by* (4) *then we have*

$$\mu(\theta) \leqslant \begin{cases} \dfrac{11-6\theta}{10} & \dfrac{1}{6} < \theta \leqslant \dfrac{121}{273} \\ \dfrac{3(1-\theta)}{2} & \dfrac{121}{273} \leqslant \theta \leqslant \dfrac{11}{21} \\ \dfrac{47-42\theta}{35} & \dfrac{11}{21} \leqslant \theta \leqslant \dfrac{23}{42} \\ \dfrac{36\theta^2-96\theta+55}{39-36\theta} & \dfrac{23}{42} \leqslant \theta < \dfrac{7}{12}. \end{cases}$$

PROOF. – We first reduce our problem to a similar one, but technically simpler. We begin by observing that if for a given $0<\theta<1$

$$\left|\left\{X \leqslant x \leqslant 2X\colon\ |\Delta(x, X^\theta)| \geqslant \frac{4X^\theta}{\log X}\right\}\right| \ll X^{\alpha+\varepsilon} \tag{5}$$

holds with some $\alpha \geqslant 0$ and for every $\varepsilon > 0$, then clearly $\mu(\theta) \leqslant \alpha$. Further, given any $\varepsilon > 0$, we subdivide $[X, 2X]$ into $\ll X^\varepsilon$ intervals of the type $I_j = [X_j, X_j + Y]$ with $X_j \approx X$ and $Y \ll X^{1-\varepsilon}$. Writing $\xi_j = X^\theta / X_j$ we have

$$\max_{x \in I_j} |X^\theta - \xi_j x| \ll X^{\theta-\varepsilon}$$

uniformly in $j$, and hence

$$\Delta(x, X^\theta) - \Delta(x, \xi_j x) \ll X^{\theta-\varepsilon} \tag{6}$$

uniformly in $j$ and $x \in I_j$.

From (5) and (6) is not difficult to see that if for some $\alpha \geqslant 0$ and any $\varepsilon > 0$

$$\left| \left\{ X \leqslant x \leqslant 2X : \left| \Delta(x, \xi_j x) \right| \geqslant \frac{2X^{\theta}}{\log X} \right\} \right| \ll X^{\alpha + \varepsilon} \tag{7}$$

holds uniformly in $j$, then $\mu(\theta) \leqslant \alpha$. Also, it is clear that in order to prove (7) we may restrict ourselves to the case $\xi_j = \xi = X^{\theta - 1}$, the other cases being completely similar.

In order to prove (7) we use the classical explicit formula, see ch. 17 of Davenport [3], to write

$$\Delta(x, \xi x) = \sum_{|\gamma| \leqslant T} x^{\varrho} c_{\varrho}(\xi) + O\left( \frac{X \log^2 X}{T} \right), \tag{8}$$

uniformly for $X \leqslant x \leqslant 2X$, where $10 \leqslant T \leqslant X$, $\varrho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$,

$$c_{\varrho}(\xi) = \frac{(1 + \xi)^{\varrho} - 1}{\varrho} \quad \text{and} \quad c_{\varrho}(\xi) \ll \min\left( X^{\theta - 1}, \frac{1}{|\gamma|} \right). \tag{9}$$

Choose

$$T = X^{1 - \theta} \log^4 X\,, \tag{10}$$

and use the Ingham-Huxley and Jutila density estimates, which asserts that for every $\varepsilon > 0$ we have respectively

$$N(\sigma, T) \ll \begin{cases} T^{3(1 - \sigma)/(2 - \sigma) + \varepsilon} & \frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4} \\ T^{3(1 - \sigma)/(3\sigma - 1) + \varepsilon} & \frac{3}{4} \leqslant \sigma \leqslant 1 \end{cases} \tag{11}$$

and

$$N(\sigma, T) \ll T^{2(1 - \sigma) + \varepsilon} \quad \text{for} \quad \frac{11}{14} \leqslant \sigma \leqslant 1\,. \tag{12}$$

Let

$$I = [a, b]$$

$$a = \max\left\{ \frac{1}{2}, 3\theta - 1 - \varepsilon \right\}$$

and

$$b = \min\left\{ \frac{11}{14} + \varepsilon,\ \frac{4}{3} - \theta + \varepsilon \right\}.$$

From (9), (10), (11) and (12) by a standard argument we see that

$$\sum_{\substack{|\gamma| \leqslant T \\ \beta \notin I}} x^{\varrho} c_{\varrho}(\xi) \ll X^{\theta - 1} (\log^2 X) \max_{\sigma \notin I} X^{\sigma} N(\sigma, T) \ll \frac{X^{\theta}}{\log^2 X}, \tag{13}$$

uniformly for $X \leqslant x \leqslant 2X$.

Again by standard argument, from (9), (10), (11) and (12) we obtain

$$\int_{X}^{2X} \Big| \sum_{\substack{|\gamma| \leqslant T \\ \beta \in I}} x^{\varrho} c_{\varrho}(\xi) \Big|^2 dx \ll X^{2\theta - 1 + \varepsilon} \max_{\sigma \in I} X^{2\sigma} N(\sigma, T) \ll X^{2\theta + (11 - 6\theta)/10 + \varepsilon},$$

for $\theta \in [1/6,\ 121/273]$ and hence

$$\left| \left\{ X \leqslant x \leqslant 2X:\ \Big| \sum_{\substack{|\gamma| \leqslant T \\ \beta \in I}} x^{\varrho} c_{\varrho}(\xi) \Big| \geqslant \frac{X^{\theta}}{\log X} \right\} \right| \ll X^{(11 - 6\theta)/10 + \varepsilon}.$$

Then we get

$$\mu(\theta) \leqslant \frac{11 - 6\theta}{10} \qquad \text{for} \qquad \frac{1}{6} \leqslant \theta \leqslant \frac{121}{273}.$$

In order to bound $\mu(\theta)$ for the other values of $\theta$ we use Lemma 1 of Heath-Brown [4] to get

$$\int_{X}^{2X} \Big| \sum_{\substack{|\gamma| \leqslant T \\ \beta \in I}} x^{\varrho} c_{\varrho}(\xi) \Big|^4 dx \ll X^{4\theta - 3 + \varepsilon} \max_{\sigma \in I} X^{4\sigma} N^*(\sigma, T). \tag{14}$$

From Theorem 2 of Heath-Brown [5] we have

$$N^*(\sigma, T) \ll \begin{cases} T^{(10 - 11\sigma)/(2 - \sigma) + \varepsilon} & \frac{1}{2} \leqslant \sigma \leqslant \frac{2}{3} \\ T^{(18 - 19\sigma)/(4 - 2\sigma) + \varepsilon} & \frac{2}{3} \leqslant \sigma \leqslant \frac{3}{4} \\ T^{12(1 - \sigma)/(4\sigma - 1) + \varepsilon} & \frac{3}{4} \leqslant \sigma \leqslant 1, \end{cases} \tag{15}$$

Hence from (10), (14) and (15) we obtain

$$\int_X^{2X} \Big| \sum_{\substack{|\gamma| \leqslant T \\ \beta \in I}} x^{\varrho} c_{\varrho}(\xi) \Big|^4 dx \ll \begin{cases} X^{4\theta + 3(1-\theta)/2 + \varepsilon} & \frac{121}{273} \leqslant \theta \leqslant \frac{11}{21} \\ X^{4\theta + (47 - 42\theta)/35 + \varepsilon} & \frac{11}{21} \leqslant \theta \leqslant \frac{23}{42} \\ X^{4\theta + (36\theta^2 - 96\theta + 55)/(39 - 36\theta) + \varepsilon} & \frac{23}{42} \leqslant \theta < \frac{7}{12}. \end{cases}$$

which implies

$$\Big| \Big\{ X \leqslant x \leqslant 2X : \Big| \sum_{\substack{|\gamma| \leqslant T \\ \beta \in I}} x^{\varrho} c_{\varrho}(\xi) \Big| \geqslant \frac{X^{\theta}}{\log X} \Big\} \Big| \ll$$

$$\ll \begin{cases} X^{3(1-\theta)/2 + \varepsilon} & \frac{121}{273} \leqslant \theta \leqslant \frac{11}{21} \\ X^{(47 - 42\theta)/35 + \varepsilon} & \frac{11}{21} \leqslant \theta \leqslant \frac{23}{42} \\ X^{(36\theta^2 - 96\theta + 55)/(39 - 36\theta) + \varepsilon} & \frac{23}{42} \leqslant \theta < \frac{7}{12}. \end{cases}$$

Then Lemma 1 is proved.

Along the same line, using the heuristic assumption (2) instead of the unconditional estimate for $N^*(\sigma, T)$, we can obtain

LEMMA 2. – *Let* $\mu(\theta)$ *defined by* (4) *then we have*

$$\mu(\theta) = \begin{cases} \frac{11 - 6\theta}{10} & \frac{1}{6} < \theta \leqslant \frac{3}{8} \\ \frac{7(1-\theta)}{5} & \frac{3}{8} \leqslant \theta < \frac{7}{12}. \end{cases}$$

## 3. – Proof of the theorems.

As in the proof of theorems of [1] we can immediately obtain the results for large values of $H$. More precisely is not difficult to show that

$$\psi(g(n) + H) - \psi(g(n)) \sim H \qquad \text{for almost all } n$$

for $H \geqslant N^{\alpha - 1/\alpha}$ and $g(x)$ of *type* $\alpha$.

Hence in the following we consider $g(x)$ of *type* $\alpha$ and $H < N^{\alpha - 1/\alpha}$.

To deal with our discrete problem we let $H = N^{\theta}$ and define the set

$$A_{\delta}(N, \theta) = \left\{ N^{1/\alpha} \leqslant n \leqslant (2N)^{1/\alpha} : \left| \psi(g(n) + H) - \psi(g(n)) - H \right| \geqslant \delta H \right\},$$

that contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of type $[g(n), g(n) + H]$ in $[N, 2N]$, and let

$$\eta_{\delta}(\theta) = \inf \left\{ \xi \geqslant 0 : \left| A_{\delta}(N, \theta) \right| \ll_{\delta, \theta} X^{\xi} \right\},$$

$$\eta(\theta) = \sup_{\delta > 0} \eta_{\delta}(\theta).$$

To prove the theorems is sufficiently to show that

$$\eta(\theta) < \frac{1}{\alpha}. \tag{16}$$

The first step of the proof of the theorems is to obtain that

$$\eta(\theta) \leqslant \mu(\theta) - \theta. \tag{17}$$

In order to prove (17) we let $n \in A_{\delta}(N, \theta)$ and consider $g(n) \in [N, 2N]$. From the definition of the set $A_{\delta}(N, \theta)$ we get

$$\left| \psi(g(n) + H) - \psi(g(n)) - H \right| \geqslant \delta H,$$

which implies $g(n) \in E_{\delta}(N, \theta)$. As in $i)$ of Theorem 1 of [2] we can prove that exists an effective constant $c$ such that

$$[g(n), g(n) + cH] \cap [N, 2N] \subset E_{\delta/2}(N, \theta).$$

Let $m \in A_{\delta}(N, \theta)$, $m > n$. In the same way we can consider $g(m) \in [N, 2N]$ such that

$$[g(m), g(m) + cH] \cap [N, 2N] \subset E_{\delta/2}(N, \theta).$$

We observe that

$$g(m) - g(n) \geqslant g(n+1) - g(n) = g'(\xi) \approx \alpha n^{\alpha - 1} \approx N^{\alpha - 1/\alpha} > H,$$

for the appropriate $\xi \in (n, n+1)$.

Choosing the constant $c$ sufficiently small we obtain that for every $m \neq n$ we have

$$[g(n), g(n) + cH] \cap [g(m), g(m) + cH] = \emptyset,$$

which implies

$$|A_\delta(N,\theta)| \leqslant \frac{|E_{\delta/2}(N,\theta)|}{H} \ll \frac{N^{\mu(\theta)+\varepsilon}}{H} \ll N^{\mu(\theta)-\theta+\varepsilon}, \tag{18}$$

for every $\delta > 0$ and $\varepsilon > 0$.

Hence we have

$$\eta_\delta(\theta) \leqslant \mu(\theta) - \theta + \varepsilon \quad \text{for every } \delta > 0 \text{ and } \varepsilon > 0,$$

and then (17) follows.

Theorem 1 and 2 follows by using Lemma 1 and 2 for the estimate of $\mu(\theta)$ and recalling (17) and (16).

To prove Theorem 3 we recall that Selberg [10] proved, under RH, that

$$\int_X^{2X} |\psi(x+H) - \psi(x) - H|^2 dx \ll XH\log^2 X, \tag{19}$$

which implies

$$|E_\delta(X,\theta)| \ll_\delta \frac{X}{H}\log^2 X \quad \text{for all } \delta > 0,$$

and then

$$|A_\delta(N,\theta)| \leqslant \frac{|E_{\delta/2}(N,\theta)|}{H} \ll \frac{N}{H^2}\log^2 N,$$

for all $\delta > 0$, since (18) and (19).

This bound is $o(N^{1/\alpha})$ for $H = \infty(N^{\alpha - 1/2\alpha}\log N)$ and then Theorem 3 follows.

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