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Original

Availability:
This version is available at: 11583/1397857 since:

Publisher:
Akadémiai Kiadó

Published
DOI:10.1556/SScMath.36.2000.1-2.14

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ON THE ASYMPTOTIC FORMULA FOR GOLDBACH NUMBERS IN SHORT INTERVALS

by

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This is the authors’ post-print version of an article published on

1. INTRODUCTION

Define a Goldbach number (G-number) to be an even number which can be written
as a sum of two primes. In the following we denote by \( N \) a sufficiently large integer
and let \( L = \log N \). Let further
\[
R(k) = \sum_{N < m \leq 2N} \sum_{N < l \leq 2N, m+l=k} \Lambda(l)\Lambda(m)
\]
be the weighted counting function of G-numbers,
\[
\mathcal{S}(k) = \begin{cases} 
2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|k} \left(\frac{p-1}{p-2}\right) & \text{if } k \text{ is even} \\
0 & \text{if } k \text{ is odd}
\end{cases}
\]
be the singular series of Goldbach’s problem and
\[
m(k) = \sum_{N < m \leq 2N} \sum_{N < l \leq 2N, m+l=k} 1.
\]

We recall that a well-known conjecture states that as \( k \to \infty \)
\[
R(k) \sim m(k)\mathcal{S}(k).
\] (1)

In this paper we study the asymptotic formula for the average of \( R(k) \) over short
intervals of type \([n, n + H]\). In the extreme case \( H = 1 \), Chudakov [1], van der Corput [2] and Estermann [4] proved that, as \( N \to \infty \), (1) holds for all \( k \in [1, N] \) but

∗ Research supported by a postdoctoral grant from the University of Genova.

1 This version does not contain journal formatting and may contain minor changes
with respect to the published version. The final publication is available at
http://dx.doi.org/10.1556/SScMath.36.2000.1-2.14. The present version is accessible on PORTO,
the Open Access Repository of Politecnico di Torino (http://porto.polito.it), in com-
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$O(NL^{-A})$ exceptions, for every $A > 0$. Moreover, the same techniques prove, for $H \leq L^D$ and $N \to \infty$, that

$$
\sum_{k \in [n,n+H)} R(k) \sim \sum_{k \in [n,n+H)} m(k) \mathcal{G}(k)
$$

(2)

holds for all $n \in (5N/2, 7N/2]$ but $O(NL^{-A})$ exceptions, for every $A, D > 0$.

We recall that Montgomery-Vaughan [12] improved Chudakov-van der Corput-Estermann’s result proving that there exists a (small) constant $\delta > 0$ such that $|E(N)| \ll N^{1-\delta}$, where $E(N) = E \cap [1, N]$ and $E$ is the exceptional set for Goldbach’s problem. Montgomery-Vaughan’s technique intrinsically does not give any information about the asymptotic formula for $R(k)$.

On the other hand, using the circle method and Ingham-Huxley’s zero density estimate, Perelli [14] proved that (2) holds as $n \to \infty$ uniformly for $H \geq n^{1/6+\varepsilon}$.

Our aim here is to show, using the circle method, that the asymptotic formula (2) holds for almost all $n \in (5N/2, 7N/2]$, uniformly for $L^D \leq H \leq N^{1/6+\varepsilon}$, for all $D > 0$.

Our result is

**Theorem.** Let $D, \varepsilon > 0$ be arbitrary constants and $L^D \leq H \leq N^{1/6+\varepsilon}$. Then, as $N \to \infty$, (2) holds for all $n \in (5N/2, 7N/2]$ but $O(NL^{42+\varepsilon}H^{-2})$ exceptions.

In fact, following the proof of the Theorem, it is easy to see that we have $O(NL^{f(\theta)}H^{-2})$ exceptions, where

$$
H = N^{\theta} \quad \text{and} \quad f(\theta) = \frac{24 - 18\theta}{1 - 3\theta} + \varepsilon.
$$

A direct computation shows that $f(\theta)$ is an increasing function and hence the exponent 42 in the log-factor of the Theorem follows taking $\theta = 1/6 + \varepsilon$.

We observe that our result, for $\theta = 1/6 + \varepsilon$, proves only that the number of exceptions for (2) is $O(N^{2/3-\varepsilon})$ while, from Perelli’s [14] result, we know that there are no exceptions.

We recall that Mikawa, see Lemma 4 of [10], proved a slightly weaker, in the log-factor, result without using the circle method. We finally recall that, under the assumption of the Riemann Hypothesis (RH), (2) holds uniformly for $H \geq \infty(\log^2 n)$, where $f = \infty(g)$ means $g = o(f)$, and that, assuming further the Montgomery pair correlation conjecture, (2) holds uniformly for $H \geq \infty(\log n)$.

**Acknowledgments.** We wish to thank Prof. A. Perelli for some useful discussions.

2. **Outline of the method**

Let

$$
Q = \frac{H}{L^\varepsilon}, \quad T = \frac{N}{Q}L^{2+\varepsilon} \quad \text{and} \quad K_H(n) = \sum_{k \in [n,n+H)} e(-ka),
$$
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where \( e(x) = \exp(2\pi ix) \). Let further \( \beta + i\gamma \) denote the generic non-trivial zero of \( \zeta(s) \),

\[
S(\alpha) = \sum_{N < m \leq 2N} \Lambda(m)e(m\alpha), \quad T(\alpha) = \sum_{N < m \leq 2N} e(m\alpha),
\]

\[
T_\rho(\alpha) = \sum_{N < m \leq 2N} a_\rho(m)e(m\alpha), \quad a_\rho(m) = \int_m^{m+1} t^{\rho-1}dt.
\]

Given an interval \( I = [a, b] \subset [1/2, 1] \) we define

\[
\Sigma_b(\alpha) = \sum_{|\gamma| \leq T} T_\beta(\alpha), \quad \Sigma_g(\alpha) = \sum_{|\gamma| \leq T} T_\rho(\alpha) + \sum_{|\gamma| > T} T_\rho(\alpha) + R(\alpha)
\]

where \( R(\alpha) \) is defined by difference in the approximation

\[
S(\alpha) = T(\alpha) - \Sigma_g(\alpha) - \Sigma_b(\alpha).
\]

Subdivide now \((-\frac{1}{2}, \frac{1}{2})\) into \( O(\log Q) \) subintervals of the following form

\[
A_0 = (-\frac{1}{Q}, \frac{1}{Q}), \quad A_j = (-\frac{1}{2^j}, -\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}}, \frac{1}{2^j})
\]

for \( j \in [1, K] \), where \( K = \lceil \log Q / \log 2 \rceil \). Hence we have

\[
\sum_{k \in [n, n+H)} R(k) = \int_{-1/2}^{1/2} S(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/4}^{1/4} S(\alpha)^2 K_H(\alpha) d\alpha
\]

\[
+ \sum_{j=1}^K \int_{A_j} S(\alpha)^2 K_H(\alpha) d\alpha = \Sigma_1 + \Sigma_2,
\]

say. We will prove that

\[
\Sigma_1 = \sum_{k \in [n, n+H)} m(k)G(k) + \int_{-1/4}^{1/4} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha + O(HN),
\]

\[
\sum_{\frac{3}{4}N < n \leq \frac{5}{4}N} |\int_{-1/4}^{1/4} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha|^2 \ll N^3 L(\theta),
\]

and

\[
\Sigma_2 = O(HN).
\]

We will need also that

\[
\sum_{k \in [n, n+H)} m(k)G(k) \gg HN
\]

which can be obtained immediately using \( G(2k) \gg 1 \). Since \( \epsilon > 0 \) is arbitrarily small, our Theorem follows at once from (4)-(8).
3. Preliminary Lemmas

In the following we will need two auxiliary lemmas.

**Lemma 1.** Let \( N(\sigma, T) \) be the number of zeros \( \rho = \beta + i\gamma \) of the Riemann zeta-function such that \( |\gamma| \leq T \) and \( \beta \geq \sigma \), and let \( I \subset [1/2, 1] \) be an interval. Then

\[
\int_N^{2N} \left| \sum_{|\gamma| \leq T, \beta \in I} x^\rho \left(1 + \frac{Q/x}{\rho} - 1 \right) \right|^2 \, dx \ll Q^2 L^4 \max_{\sigma \in I} N^{2\sigma - 1} N(\sigma, N/Q).
\]

The proof of Lemma 1 is standard. It can be obtained using, e.g., Saffari-Vaughan’s [15] technique and hence we omit it.

**Lemma 2.** We have, for \( |\gamma| \ll N \) and \( N \) sufficiently large, that

\[
T_\rho(\alpha) \ll N^3 |\gamma|^{-1/2}.
\]

**Proof.** We follow the line of Perelli [13] and hence we give only a brief sketch of the proof. Since

\[
a_\rho(m) = \int_m^{m+1} t^{\rho-1} \, dt = \frac{m^\rho}{\rho} \left((1 + \frac{1}{m})^\rho - 1\right),
\]

and, for \( P \) sufficiently large but fixed,

\[
(1 + \frac{1}{m})^\rho - 1 = \sum_{j=1}^{P} \frac{\rho(\rho - 1) \cdots (\rho - j + 1)}{j!} \left(\frac{1}{m}\right)^j + O(N^{-11}),
\]

we can write

\[
T_\rho(\alpha) = T_{\rho,1}(\alpha) + \sum_{j=2}^{P} \left(\frac{\rho - 1)(\rho - 2) \cdots (\rho - j + 1)}{j!}\right) T_{\rho,j}(\alpha) + O(N^{3-10}),
\]

(9)

where

\[
T_{\rho,j}(\alpha) = \sum_{N < m \leq 2N} m^{\rho-j} e(m\alpha).
\]

From Abel’s inequality we have

\[
|T_{\rho,j}(\alpha)| \ll N^{\beta-j} \max_{N \leq y \leq 2N} \| \sum_{N \leq m \leq y} e^{2\pi i f_\rho(\alpha)} \|,
\]

where \( f_\rho(\alpha) = \frac{\alpha}{2\pi} \log n + \alpha n \). We can assume that the maximum is attained at \( Y = 2N \), and so, using van der Corput’s second derivative method, see Theorem 2.2 of Graham-Kolesnik [5], we get

\[
T_{\rho,j}(\alpha) \ll N^{\beta-j+1} |\gamma|^{-1/2}.
\]

(10)

Lemma 2 now follows inserting (10) in (9).
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4. Estimation of $\Sigma_2$

Letting $S(\alpha) = T(\alpha) + R_1(\alpha)$, where $R_1(\alpha)$ is defined by difference, and using

$$K_H(\alpha) \ll \min(H, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}],$$

we have

$$\Sigma_2 \ll \sum_{j=1}^{K} \left( \int_{A_j} |T(\alpha)|^2 |K_H(\alpha)|d\alpha + \int_{A_j} |R_1(\alpha)|^2 |K_H(\alpha)|d\alpha \right)$$

$$\ll \sum_{j=1}^{K} 2^j \left( \int_{A_j} |T(\alpha)|^2 d\alpha + \int_{A_j} |R_1(\alpha)|^2 d\alpha \right) = \Sigma_{2,1} + \Sigma_{2,2},$$

say. Using

$$T(\alpha) \ll \min(N, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}],$$

we obtain

$$\Sigma_{2,1} \ll \sum_{j=1}^{K} 4^j \ll 4^K \ll Q^2 = o(HN).$$

(14)

By Gallagher’s lemma, see, e.g., Lemma 1.9 of Montgomery [11], and the Brun-Titchmarsh theorem we get

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^j \int_{-2^{-j-1}/Q}^{2^{-j}} \left( \sum_{N < m \leq 2N} (\Lambda(m) - 1)e(m\alpha) \right)^2 d\alpha \ll \sum_{j=1}^{K} 2^{-j}(J(N, 2^j) + L^2 2^{3j}),$$

(15)

where $J(N, h)$ is the Selberg integral. Inserting the estimate $J(N, h) \ll h^2 N + hNL$ for all $h \geq 1$, see the Lemma in Languasco [7], in (15) we have

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^{-j}(2^{3j}L^2 + 2^{2j}N + 2^j NL) \ll L^2 Q^2 + NQ + NL \log Q = o(HN).$$

(16)

Hence, inserting (14) and (16) in (12), we finally have that (7) holds.

5. Estimation of $\Sigma_1$

Inserting the identity

$$S(\alpha)^2 = (2S(\alpha)T(\alpha) - T(\alpha)^2) - \Sigma_g(\alpha)^2 - 2T(\alpha)\Sigma_g(\alpha) + 2S(\alpha)\Sigma_g(\alpha) + \Sigma_b(\alpha)^2$$

into the definition of $\Sigma_1$, we obtain

$$\Sigma_1 = \Sigma_{1,1} - \Sigma_{1,2} - \Sigma_{1,3} + \Sigma_{1,4} + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha)d\alpha,$$

(17)

where

$$\Sigma_{1,1} = \int_{-1/Q}^{1/Q} (2S(\alpha)T(\alpha) - T(\alpha)^2)K_H(\alpha)d\alpha,$$
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\[ \Sigma_{1,2} = \int_{-1/Q}^{1/Q} \Sigma_g(\alpha)^2 K_H(\alpha) d\alpha, \]
\[ \Sigma_{1,3} = \int_{-1/Q}^{1/Q} 2T(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha \]
and
\[ \Sigma_{1,4} = \int_{-1/Q}^{1/Q} 2S(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha. \]

In this section we will prove
\[ \Sigma_{1,1} = \sum_{k \in [n, n+H)} m(k) \mathcal{G}(k) + o(HN) \tag{18} \]
and
\[ \Sigma_{1,2} = o(HN), \tag{19} \]
while the estimation of the mean-square of \( \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \) will be performed in the next section.

Assuming that (19) holds, the contribution of \( \Sigma_{1,3} \) and \( \Sigma_{1,4} \) can be estimated using the Cauchy-Schwarz inequality and
\[ \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha \ll N, \tag{20} \]
which can be proved using the same argument in the proof of Corollary 3 of Languasco-Perelli [9]. We obtain
\[ \Sigma_{1,3} = o(HN) \quad \text{and} \quad \Sigma_{1,4} = o(HN). \tag{21} \]

Hence, by (17)-(19) and (21), we have that (5) holds.

Now we proceed to evaluate \( \Sigma_{1,1} \) and \( \Sigma_{1,2} \).

**Contribution of \( \Sigma_{1,1} \)**

Squaring out we obtain
\[ \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha = \sum_{k \in [n, n+H)} m(k) \]
and hence, using (11) and (13), we get
\[ \int_{-1/Q}^{1/Q} T(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha + O(Q^2) = \sum_{k \in [n, n+H)} m(k) + o(HN). \tag{22} \]

Using the Prime Number Theorem, the Cauchy-Schwarz inequality and arguing analogously, we can write
\[ \int_{-1/Q}^{1/Q} S(\alpha) T(\alpha) K_H(\alpha) d\alpha = \sum_{k \in [n, n+H)} m'(k) + o(HN), \tag{23} \]
where
\[ m'(k) = \sum_{N < m \leq 2N} \Lambda(m) \sum_{N < h \leq 2N, m+h=k} 1. \]

Again by the Prime Number Theorem, we get
\[ \sum_{k \in [n,n+H)} m(k) = \sum_{k \in [n,n+H)} m'(k) + o(HN) \quad (24) \]
and hence, by (22)-(24), we have
\[ \Sigma_{1,1} = \sum_{k \in [n,n+H)} m(k) + o(HN). \quad (25) \]

Using the Theorem of Languasco [8] and by partial summation, it is easy to prove
\[ \sum_{k \in [n,n+H)} m(k) = \sum_{k \in [n,n+H)} m(k)\mathcal{G}(k) + o(HN) \quad \text{for} \quad H \geq L^{2/3+\varepsilon}. \quad (26) \]

Now (18) follows from (25) and (26).

**Contribution of \( \Sigma_{1,2} \)**

Since
\[ \Sigma_{\beta}(\alpha)^2 \ll \int_{|\gamma| \leq T} |T_{\beta}(\alpha)|^2 |K_H(\alpha)|d\alpha + \int_{|\gamma| > T} |T_{\beta}(\alpha)|^2 |K_H(\alpha)|d\alpha \]
\[ \quad \text{and} \quad A_3 = \int_{-1/Q}^{1/Q} |R(\alpha)|^2 |K_H(\alpha)|d\alpha. \]

Using (11) and Gallagher’s lemma, we obtain
\[ A_1 \ll \frac{H}{Q^2} \left( \int_N^{2N} |\sum_{x<m<x+Q} \sum_{|\gamma| \leq T} a_{\nu}(m)|^2 dx + \int_N^{N-Q} |\sum_{N<m<x+Q} \sum_{|\gamma| \leq T} a_{\nu}(m)|^2 dx \right. 
\[ + \left. \int_{2N-Q}^{2N} |\sum_{x<m \leq 2N} \sum_{|\gamma| \leq T} a_{\nu}(m)|^2 dx \right) = A_{1,1} + A_{1,2} + A_{1,3}, \quad (28) \]
say. Interchanging summation and integration in $A_{1,1}$, we get

$$A_{1,1} \ll \frac{H}{Q^2} \int_N^{2N} \left| \sum_{\beta \in I} t^{\rho-1} dt \right|^2 \, dx \quad \text{(29)}$$

To bound the contribution of the integral on $[x, [x] + 1]$ in (29), we argue as follows. Interchanging summation and integration, we get

$$\int_N^{2N} \left| \sum_{\beta \in I} t^{\rho-1} dt \right|^2 \, dx \ll \sum_{N < n \leq 2N} \sum_{|\gamma| \leq T} \frac{(n+1)/x}{\rho} - 1 \ll 2 \, dx.$$ 

and then, using $\frac{(n+1)/x}{\rho} - 1 \ll \min\left(\frac{1}{N}, \frac{1}{|\gamma|}\right)$, we have

$$\int_N^{2N} \left| \sum_{\beta \in I} t^{\rho-1} dt \right|^2 \, dx \ll L^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, N). \quad \text{(30)}$$

To estimate the integral on $[[x + Q], x + Q]$ in (29) we proceed analogously and hence we get

$$\int_N^{2N} \left| \sum_{\beta \in I} t^{\rho-1} dt \right|^2 \, dx \ll L^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, N). \quad \text{(31)}$$

Now we treat the integral on $[x, x + Q]$ in (29). Proceeding as above we obtain

$$\int_N^{2N} \left| \sum_{\beta \in I} t^{\rho-1} dt \right|^2 \, dx \ll \int_N^{2N} \left| \sum_{\beta \in I} x^{\rho} \frac{1}{\rho} - 1 \right|^2 \, dx \quad \text{(32)}$$

$$\ll Q^2 L^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, N),$$

where the last inequality follows by Lemma 1.

Choosing, in the definition of the interval $I$,

$$a = \frac{1 + 3\theta}{2} - l \log L \quad \text{and} \quad b = \frac{5 - 3\theta}{6} + k \log L,$$

where $l > 27(1 - \theta)/211 - 3\theta$ and $k$ is a sufficiently large constant, we have, using Ingham-Huxley’s density estimate, see, e.g., Ivić [6], and (29)-(33), that

$$A_{1,1} \ll HL^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, N) = o(HN). \quad \text{(34)}$$
Interchanging summation and integration in $A_{1,2}$, we get

$$A_{1,2} \ll \frac{H}{Q^2} \int_{N-Q}^{N} \sum_{|\gamma| \leq T} x^\rho c_{\rho,Q} |\gamma|^2 dx,$$

where $c_{\rho,Q} = \left(\frac{(\frac{x+Q}{x})^\rho - (\frac{N}{x})^\rho}{\rho}\right)$. Splitting the summation according to $|\gamma| \leq N/Q$ and $N/Q \leq |\gamma| \leq T$ and using $c_{\rho,Q} \ll \min\left(\frac{Q}{N}, \frac{1}{|\gamma|}\right)$, we obtain

$$A_{1,2} \ll \frac{H}{Q^2} \left(\frac{Q^2}{N^2} \int_{N-Q}^{N} |\sum_{|\gamma| \leq N/Q} x^\rho|^2 dx + \int_{N-Q}^{N} |\sum_{N/Q \leq |\gamma| \leq T} x^\rho|^2 dx\right)$$

$$\ll HQL^4 \max_{\sigma \notin I} N^{2\sigma-2} N\left(\frac{N}{Q}\right)^2.$$

Using Ingham-Huxley’s density estimate, we see that the maximum is attained at $\sigma = 1/2$ and hence we can write

$$A_{1,2} \ll HQL^4 N^{-1}\left(\frac{N}{Q}\right)^2 L^2 = \frac{HN L^6}{Q} = o(HN).$$

(A35)

$A_{1,3}$ can be bounded following the lines of the estimation of $A_{1,2}$. We have

$$A_{1,3} = o(HN).$$

(A36)

Inserting (34) and (35)-(36) in (28) we obtain

$$A_{1} = o(HN).$$

(A37)

Now we proceed to estimate $A_{2}$. By (11) we get

$$A_{2} \ll H \int_{-\frac{1}{Q}}^{\frac{1}{Q}} \sum_{N \leq m \leq 2N} \sum_{|\gamma| > T} a_{\rho}(m) e(\rho \alpha) |\gamma|^2 d\alpha.$$

(A38)

Using (38), Gallagher’s lemma and the explicit formula for $\psi(x)$, see equations (9)-(10) in ch. 17 of Davenport [3], we have

$$A_{2} \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \frac{N^2 L^4}{T^2} dx \ll \frac{HN^3}{Q^2 T^2} L^4 = o(HN).$$

(A39)

To bound $A_{3}$ we use (11), Gallagher’s lemma and the explicit formula for $\psi(x)$, see equation (1) in ch. 17 of Davenport [3]. Hence

$$A_{3} \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \sum_{x < m \leq x+Q \atop N \leq m \leq 2N} (\Lambda(m) - 1 + \sum_{\rho} a_{\rho}(m))^2 dx$$

$$\ll \frac{H}{Q^2} \int_{N-Q}^{2N} L^4 dx \ll \frac{HN L^4}{Q^2} = o(HN).$$

(A40)

Now (19) follows inserting (37) and (39)-(40) in (27).
6. Mean-square estimate of $\Sigma_b(\alpha)^2$

Squaring out and using the definition of $\Sigma_b(\alpha)$, we get

$$\sum_{\frac{3}{2}N < n \leq \frac{5}{2}N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2$$

$$= \sum_{\frac{3}{2}N < n \leq \frac{5}{2}N} \int_{-1/Q}^{1/Q} \left( \sum_{|\gamma| \leq T} T_\rho(\alpha) K_H(\alpha) \right) d\alpha \int_{-1/Q}^{1/Q} \left( \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right) \overline{K_H}(\delta) d\delta$$

$$\ll \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right|^2 \sum_{\frac{3}{2}N < n \leq \frac{5}{2}N} K_H(\alpha) \overline{K_H}(\delta) |d\delta d\alpha| = \Sigma_3,$$

say. Since $K_H(\alpha) = \frac{\sin \pi H \alpha}{\sin \pi \alpha} e\left(\frac{1-H}{2} \alpha \right) e(-n\alpha)$, we have

$$\Sigma_3 \ll H^2 \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right|^2 K_N(\alpha - \delta) d\delta d\alpha \right)$$

where $K_N(t) = \sum_{\frac{3}{2}N < n \leq \frac{5}{2}N} e(-nt) \ll \min(N, \frac{1}{|t|})$.

Using the latest estimate and (42), we obtain

$$\Sigma_3 \ll H^2 N \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{(-\frac{1}{4}, \frac{1}{4}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right|^2 d\delta d\alpha \right)$$

$$+ H^2 \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{(-\frac{1}{4}, \frac{1}{4}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right|^2 \frac{1}{|\alpha - \delta|} d\delta d\alpha \right)$$

$$= \Sigma_{3,1} + \Sigma_{3,2},$$

say. Using (3) and arguing as in section 6, we get

$$\int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 d\alpha \ll \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha + O(N) \ll N,$$  \hspace{1cm} (44)

where the latest inequality follows from (20).

Now, inserting (44) in $\Sigma_{3,1}$, we have

$$\Sigma_{3,1} \ll H^2 N^2 \left( \max_{\alpha \in (-1/Q, 1/Q)} \int_{(-\frac{1}{4}, \frac{1}{4}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{|\gamma'| \leq T} T_{\rho'}(\delta) \right|^2 d\delta \right)$$

$$\ll H^2 N \left( \max_{\delta \in (-1/Q, 1/Q)} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \right).$$  \hspace{1cm} (45)
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To bound $\Sigma_{3,2}$, we argue as for $\Sigma_{3,1}$ and we can prove that the bound in (45) holds, with an extra $L$ factor, for $\Sigma_{3,2}$ too. Finally, by (41), (43), (45) and the above remark, we obtain

$$\sum_{3/2 < n \leq 7/2} \left| \int_{-1/Q}^{1/Q} \left( \sum_{\beta \in I} T_{\beta}(\alpha)^2 K_H(\alpha) \right)^2 d\alpha \right| \leq H^2 NL \left( \max_{|\gamma| \leq T} \left| \sum_{\beta \in I} T_{\beta}(\delta) \right|^2 \right).$$

Using Lemma 2 and a standard argument to bound sums over zeros of $\zeta(s)$, we have

$$\sum_{|\gamma| \leq T} T_{\gamma}(\delta) \ll L^2 \left( \max_{\sigma \leq \sigma_{7/9}} N^\sigma \max_{|t| \leq T} |t|^{-1/2} + \max_{\sigma \geq \sigma_{7/9}} N(\sigma, t) |t|^{-1/2} \right)$$

$$\ll L^2 \left( \max_{\sigma \leq \sigma_{7/9}} N^\sigma N(\sigma, T) T^{-1/2} + \max_{\sigma \geq \sigma_{7/9}} N^\sigma \right).$$

By Ingham-Huxley’s density estimate, we have that the first maximum is attained at $\sigma = a$ and the second at $\sigma = b$. Hence, by (46) and (47), we see that (6) holds.

REFERENCES


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