The exceptional set for the number of primes in short intervals

THE EXCEPTIONAL SET FOR THE NUMBER OF PRIMES IN SHORT INTERVALS

by

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1. Introduction

Let $\psi(x) = \sum_{n \le x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. A well known conjecture asserts that

$$\psi(x + h(x)) - \psi(x) \sim h(x) \quad \text{as} \quad x \to \infty \tag{1}$$

for every increasing function $h(x)$ satisfying $x^\varepsilon \le h(x) \le x$ with any fixed $\varepsilon > 0$. It is known that (1) holds with $x^{7/12+\varepsilon} \le h(x) \le x$, see Huxley [10], and the wider range $x^{7/12-o(1)} \le h(x) \le x$ has been obtained by Heath-Brown [8] at the cost of a much more difficult proof. It is also known that (1) holds with $x^{1/2+\varepsilon} \le h(x) \le x$ under the assumption of the Riemann Hypothesis (RH). In the opposite direction, Maier [11] showed that (1) does not hold when $h(x) = \log^c x$ with any constant $c \ge 2$.

In this paper we investigate the exceptional set for the asymptotic formula (1). Let X be a large positive number, $\delta > 0$, $|\ \ |$ denote the modulus of a complex number or the Lebesgue measure of a set or the cardinality of a finite set, $h(x)$ be an increasing function such that $x^\varepsilon \le h(x) \le x$ for some $\varepsilon > 0$,

$$\Delta(x, h) = \psi(x + h(x)) - \psi(x) - h(x)$$

and

$$E_\delta(X, h) = \{X \le x \le 2X : |\Delta(x, h)| \ge \delta h(x)\}.$$

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It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, h) = \emptyset$ for $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ and $h(x)$ suitably small with respect to x, the set $E_\delta(X, h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, h) \subset E_{\delta'}(X, h) \quad \text{if} \quad 0 < \delta' < \delta.$$

We will consider increasing functions $h(x)$ of the form $h(x) = x^{\theta+\varepsilon(x)}$, with some $0 < \theta < 1$ and a function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing,

$$\varepsilon(x) = o(1) \quad \text{and} \quad \varepsilon(x + y) = \varepsilon(x) + O(\frac{|y|}{x}).$$

A function satisfying these requirements will be called of *type* θ. It is easy to see that functions like $x^\theta \log^c x$ with $c \in \mathbb{R}$, and similar functions, are of type θ. We are mainly interested in the case $h(x) = x^\theta$, in which case we allow also $\theta = 1$ and write

$$E_\delta(X, h) = E_\delta(X, \theta).$$

Hence, in particular, it is a consequence of the above results that for any $\delta > 0$ and X sufficiently large we have $E_\delta(X, \theta) = \emptyset$ provided $7/12 \leq \theta \leq 1$ and, under RH, provided $1/2 < \theta \leq 1$.

Our first result provides the basic structure of the exceptional set $E_\delta(X, h)$.

Theorem 1. *i)* (inertia property) *Let $0 < \theta < 1$, $h(x)$ be of type θ, X be sufficiently large depending on the function $h(x)$ and $0 < \delta' < \delta$ with $\delta - \delta' \geq exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, h)$ then $E_{\delta'}(X, h)$ contains the interval $[x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_\delta(X, h) \neq \emptyset$ then*

$$|E_{\delta'}(X, h)| \gg_\theta (\delta - \delta')h(X).$$

ii) (decrease property) *Let $0 < \theta' < \theta < 1$, $h(x)$ be of type θ and $h'(x)$ of type θ', X be sufficiently large depending on the functions $h(x)$ and $h'(x)$, and let $0 < \delta' < \delta$ with $\delta - \delta' \geq exp(-\sqrt{\log X})$. Then*

$$\max(|E_{\delta'}(X, h')|, |E_{\delta'}(\frac{3}{2}X, h')|) \gg_{\theta'} (\delta - \delta')|E_\delta(X, h)|.$$

Several deductions can be made from Theorem 1, but prior to that we introduce the functions

$$\mu_\delta(\theta) = \inf\{\xi \geq 0 : |E_\delta(X, \theta)| \ll_{\delta,\theta} X^\xi\}$$

and

$$\mu(\theta) = \sup_{\delta>0} \mu_\delta(\theta),$$

the latter function being well defined since clearly $\mu_\delta(\theta) \leq 1$ for every $\delta > 0$ and $0 < \theta \leq 1$. For convenience we define $\mu_\delta(\theta)$ and $\mu(\theta)$ for

$0 < \theta \le 1$, although these functions are of interest only for $0 < \theta < 7/12$. Clearly

$$\mu_\delta(\theta) \le \mu_{\delta'}(\theta) \quad \text{if} \quad \delta' < \delta$$

and

$$\mu(\theta) = 0 \quad \text{for} \quad 7/12 \le \theta \le 1 \quad \text{and, under RH,} \quad \mu(\theta) = 0 \quad \text{for} \quad 1/2 < \theta \le 1.$$

A first consequence of Theorem 1 is the following

Corollary 1. *i) The function $\mu(\theta)$ is non-increasing.*
ii) $\mu(\theta_0) < \theta_0$ for some $0 < \theta_0 < 1$ if and only if (1) holds with $h(x) = x^\theta$ for every $\theta_0 \le \theta \le 1$. Moreover, in this case $\mu(\theta) = 0$ for every $\theta_0 \le \theta \le 1$.

It follows in particular that if $\mu(\theta)$ were piecewise continuous with jumps of height $< \theta$ at any discontinuity point $\theta \in (0,1)$, then (1) would hold with $h(x) = x^\theta$ for every $0 < \theta \le 1$, and in fact $\mu(\theta) = 0$ in the same range.

The same principle underlying Corollary 1 can be used to infer (1) from suitable mean value estimates. One out of several similar statements in this direction is the following

Corollary 2. *Let $0 < \theta < 1$, $h(x)$ be of type θ, $c > 0$ and $Y = ch(X)$. Assume that for any $0 < c < 1/2$ and X sufficiently large depending on c we have*

$$\int_X^{X+Y} |\Delta(x,h)|^2 dx \le \frac{20}{\theta^2} Y^3. \tag{2}$$

Then (1) holds. The opposite implication holds too.

From Corollary 2 we can deduce the validity of (1) for suitable functions $h(x)$. We deal mainly with conditional results. Although similar statements, here and at later occasions, can be obtained under similar hypotheses such as the Density Hypothesis, we will work out our results only under RH and, in addition, under certain forms of Montgomery's pair correlation conjecture. A form of it, see Goldston-Montgomery [4], states that

$$\int_0^X |\psi(x+H) - \psi(x) - H|^2 dx \sim HX \log \frac{X}{H} \tag{3}$$

uniformly for $X^{1/2-\varepsilon} \le H \le X^{1-\varepsilon}$ for any fixed $\varepsilon > 0$. Moreover, Goldston [3] deduced the validity of a classical conjecture asserting the existence of primes between consecutive squares from a certain stronger form of the following refinement of (3)

$$\int_0^X |\psi(x+H) - \psi(x) - H|^2 dx = HX \log \frac{X}{H} + O(HX), \tag{4}$$

uniformly in the same range as above. We have

Corollary 3. *i) Assume RH. Then (1) holds for any function of type* $1/2$ *of the form* $h(x) = F(x)x^{1/2}\log x$ *with* $F(x) \to \infty$.
ii) Assume RH and (3). Then there exists a function of type $1/2$ *of the form* $h(x) = f(x)(x\log x)^{1/2}$ *with* $f(x) = o(1)$ *for which (1) holds.*
iii) Assume RH and (4). Then (1) holds for any function of type $1/2$ *of the form* $h(x) = F(x)x^{1/2}$ *with* $F(x) \to \infty$*. Moreover, there exists a constant* $c > 0$ *such that the interval* $[x, x + cx^{1/2}]$ *contains a prime for* x *sufficiently large.*

It is not difficult to see that in fact there exist functions $h(x)$ as in *i)* and *iii)* above with $F(x) \to \infty$ arbitrarily slowly. Part *i)* of Corollary 3 should be compared with Cramér's [1] classical result asserting that under RH

$$p_{n+1} - p_n \ll p_n^{1/2}\log p_n,$$

where p_n denotes the n-th prime. Moreover, *ii)* of Corollary 3 should be compared with Heath-Brown - Goldston [9], which contains the proof that under RH and a slightly weaker version of (3)

$$p_{n+1} - p_n = o((p_n\log p_n)^{1/2}).$$

We remark that the above results can be proved by our method too, see the proof of the second part of *iii)* of Corollary 3. Moreover, the latter result is not far from the above quoted conjecture on primes between consecutive squares. We observe that the constant c in *iii)* depends in a simple way on the implicit constant in (4) and on the constant in the Brun-Titchmarsh theorem.
Turning to unconditional results, we only observe that Heath-Brown's [8] result is equivalent to the validity of (2) with some function $h(x)$ of type $7/12$. We remark here that in the conditional treatment of our problem, we in fact do not need to have a "short" mean value estimate of $\Delta(x, h)$, the "long" one being strong enough in this case. Contrary to that, in the unconditional case it is apparently necessary to work with short mean values of $\Delta(x, h)$, see the discussion below.

Mean value estimates can also be used to bound the function $\mu(\theta)$, and hence the size of the exceptional set. A well known consequence of Huxley's [10] density estimate is that (1) holds for almost-all x if $h(x) \geq x^{1/6+\varepsilon}$, and this is essentially the best known result at present. Hence we expect non-trivial bounds for $\mu(\theta)$ in the range $1/6 < \theta < 7/12$. For sake of simplicity we will explicitly work out the bound for $\mu(\theta)$ only in a right neighborhood of $\theta = 1/6$ and in a left neighborhood of $\theta = 7/12$. However, it will be clear from the proof that the same method allows to obtain an explicit bound, strictly decreasing and continuous, in the whole range $1/6 < \theta < 7/12$. The situation is much simpler under RH where, due to Selberg's [14] well known result, we have to consider only the interval $0 < \theta \leq 1/2$. We have the following

Theorem 2. *i) Let $\Delta > 0$ be sufficiently small. Then there exists a constant $c > 0$ such that*

$$\mu(\frac{1}{6} + \Delta) \leq 1 - c\Delta \quad \textit{and} \quad \mu(\frac{7}{12} - \Delta) \leq \frac{5}{8} + \frac{7}{4}\Delta + O(\Delta^2).$$

ii) Assume RH. Then

$$\mu(\theta) \leq 1 - \theta \quad \textit{for} \quad 0 < \theta \leq \frac{1}{2}.$$

For sake of simplicity we will not provide a numerical value to the constant c. Moreover, the density estimates we use in the proof are not necessarily the best known in order to get a good numerical value for c. Our technique for the proof of Theorem 2 is similar to the methods used by Wolke [16] and Heath-Brown [6] for a related problem. In fact, we will use second power moments, and hence estimates for

$$N(\sigma, T) = |\{\varrho = \beta + i\gamma : \zeta(\varrho) = 0 \ , \ \beta \geq \sigma \text{ and } |\gamma| \leq T\}|,$$

when θ is around $1/6$, and fourth power moments, and hence estimates for

$$N^*(\sigma, T) = |\{(\varrho_1, \varrho_2, \varrho_3, \varrho_4) : \varrho_j \text{ is counted by } N(\sigma, T) \text{ and } |\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4| \leq 1\}|,$$

when θ is around $7/12$.

A defect of our method is that we are unable to prove that

$$\lim_{\theta \to 7/12^-} \mu(\theta) \leq \frac{7}{12}, \tag{5}$$

which, according to *i)* of Theorem 1, would indicate that even if the asymptotic formula (1) were to fail just beyond the range where it is presently known to hold, it does so, in some sense, minimally. This also reflects the fact that, for instance, we are unable to reprove Huxley's [10] theorem via long mean values of primes in short intervals. We remark here that (5) can be proved under the "heuristic" assumption

$$N^*(\sigma, T) \ll \frac{N(\sigma, T)^4}{T}, \tag{6}$$

see at the end of section 3.

However, the observation that, for instance, Huxley's theorem is equivalent to suitable short mean value estimates suggests the introduction of the functions

$$\eta_\delta(\theta) = \inf\{\xi \geq 0 : \ b - a \ll_{\delta,\theta} X^\xi \ \text{ for every } \ [a, b] \subset E_\delta(X, \theta)\}$$

and

$$\eta(\theta) = \sup_{\delta>0} \eta_\delta(\theta),$$

where $\delta > 0$ and $0 < \theta \leq 1$. The functions η are a "short intervals" analogue of the functions μ above, and it is easy to prove that (5) holds for $\eta(\theta)$. In fact, our last result is the following

Corollary 4. *For $0 < \theta < 1$ we have*

$$\eta(\theta) \leq \frac{7}{12}.$$

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2. Proof of Theorem 1 and Corollaries 1,2 and 3

We will always assume that x and X are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence.

We first observe that from the definition of function of type θ we have that if $y = O(x^{\alpha+\varepsilon})$ with some $0 < \alpha < 1$, then

$$h(x+y) = h(x) + O(x^{\theta+\alpha-1+\varepsilon}) \tag{7}$$

for every $\varepsilon > 0$. Moreover, $h(2x) \ll h(x)$.

From the Brun-Titchmarsh theorem, see Montgomery-Vaughan [13], we have that

$$\psi(x+y) - \psi(x) \leq \frac{21}{10} y \frac{\log x}{\log y} \tag{8}$$

for $10 \leq y \leq x$. From (8) we easily see that

$$\psi(x+y) - \psi(x) \leq \frac{9}{4\alpha} cY \tag{9}$$

for $X \leq x \leq 3X$ and $0 \leq y \leq cY$, where $0 < \alpha < 1$, $X^{\alpha-\varepsilon} \leq Y \leq X$ and

$$\frac{\alpha}{5} \exp(-\sqrt{\log X}) \leq c \leq 1.$$

We first prove *i*) of Theorem 1. Let h be of type θ, $x_0 \in E_\delta(X,h)$ and $x \in [x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, where c satisfies the above restrictions. We have

$$|\Delta(x,h)| = |\Delta(x_0,h) + \Delta(x,h) - \Delta(x_0,h)| \geq$$

$$|\Delta(x_0,h)| - |\psi(x+h(x)) - \psi(x_0+h(x_0))| - |\psi(x) - \psi(x_0)| - |h(x) - h(x_0)|.$$

But from (7) with $\alpha = \theta$ we get

$$h(x_0) = h(x) + O(X^{2\theta-1+\varepsilon}),$$

hence from (9) with $\alpha = \theta$ we obtain

$$|\Delta(x,h)| \geq \delta h(x) - \frac{9}{2\theta} ch(X) + O(X^{2\theta-1+\varepsilon}) \geq \delta h(x) - \frac{5}{\theta} ch(X) \geq \delta' h(x)$$

by choosing $c = (\delta - \delta')\theta/5$, since h is increasing. Hence $x \in E_{\delta'}(X,h)$ and $i)$ follows.

Now we turn to the proof of $ii)$ of Theorem 1. Let $X \leq \xi \leq 2X$. From (7) with $\alpha = \theta$ we have

$$\int_{\xi}^{\xi+h(\xi)} (\psi(x+h'(x)) - \psi(x))dx = \int_{\xi}^{\xi+h(\xi)} (\psi(x+h'(\xi)) - \psi(x))dx + O(X^{2\theta+\theta'-1+\varepsilon})$$

$$= h'(\xi)(\psi(\xi+h(\xi)) - \psi(\xi)) + O(X^{\max(2\theta', 2\theta+\theta'-1)+\varepsilon})$$

and hence, again by (7) with $\alpha = \theta$,

$$\int_{\xi}^{\xi+h(\xi)} (\psi(x+h'(x)) - \psi(x) - h'(x))dx$$

$$= h'(\xi)(\psi(\xi+h(\xi)) - \psi(\xi) - h(\xi)) + O(X^{\max(2\theta', 2\theta+\theta'-1)+\varepsilon}).$$

Dividing both sides by $h'(x)$ and using once again (7) with $\alpha = \theta$ we get

$$\int_{\xi}^{\xi+h(\xi)} \frac{\Delta(x,h')}{h'(x)} dx = \Delta(\xi,h) + O(X^{\max(\theta', 2\theta-1)+\varepsilon}). \tag{10}$$

Assume now that $E_\delta(X,h) \neq \emptyset$, otherwise $ii)$ is trivial, and let x_1 be the smallest element of $E_\delta(X,h)$, which we may clearly assume to exist. Suppose first that

$$[x_1, x_1 + h(x_1)] \subset [X, 2X]. \tag{11}$$

Then from (10) with $\xi = x_1$ we get

$$\delta h(x_1) \leq |\Delta(x_1,h)| \leq \int_{x_1}^{x_1+h(x_1)} \frac{|\Delta(x,h')|}{|h'(x)|} dx + O(X^{\max(\theta', 2\theta-1)+\varepsilon})$$

and hence, writing

$$A_1 = \{x_1 \leq x \leq x_1 + h(x_1) : |\Delta(x,h')| < \delta' h'(x)\}$$

and

$$B_1 = \{x_1 \leq x \leq x_1 + h(x_1) : |\Delta(x,h')| \geq \delta' h'(x)\},$$

from (9) with $\alpha = \theta'$, $c = 1$ and $Y = h'(x)$ we obtain

$$\delta h(x_1) \leq \delta' |A_1| + \frac{9 - 4\theta'}{4\theta'} |B_1| + O(X^{\max(\theta', 2\theta-1)+\varepsilon}).$$

Therefore

$$|B_1| \gg (\delta - \delta') h(x_1) \tag{12}$$

since $|A_1| \leq h(x_1)$ and $h(x_1) \geq h(X) \gg X^{\theta-\varepsilon}$. Moreover, $B_1 \subset E_{\delta'}(X,h')$.

Let x_2, if it exists, be the smallest element of $E_\delta(X,h) \cap (x_1 + h(x_1), 2X]$ and, in addition, satisfy $[x_2, x_2 + h(x_2)] \subset [X, 2X]$. If such an x_2 does not exist, then $ii)$ clearly follows by (12) , under the assumption (11), since $|E_\delta(X,h)| \ll h(x_2)$ in this case and $h(x_2) \ll h(x_1)$. If x_2 exists, we apply the same argument leading to (12) to the interval $[x_2, x_2 + h(x_2)]$, thus getting a set $B_2 \subset E_{\delta'}(X,h') \cap [x_2, x_2 + h(x_2)]$ with $|B_2| \gg (\delta - \delta')h(x_2)$. We proceed in the same way denoting by x_3, if it exists, the smallest element of $E_\delta(X,h) \cap (x_2 + h(x_2), 2X]$ and, in addition, satisfying $[x_3, x_3 + h(x_3)] \subset [X, 2X]$, and so on until we find an x_k, with $k \geq 1$, but not an x_{k+1} by this procedure. Applying to each interval $[x_j, x_j + h(x_j)]$, $j \leq k$, the argument leading to (12), we obtain k sets $B_1, \ldots, B_k$, with $B_i \cap B_j = \emptyset$ if $i \neq j$, having the property that

$$\bigcup_{j=1}^{k} B_j \subset E_{\delta'}(X,h') \quad \text{and} \quad \sum_{j=1}^{k} |B_j| \gg (\delta - \delta')|E_\delta(X,h)|,$$

and $ii)$ follows, under the assumption (11).

If (11) does not hold, then $|E_\delta(X,h)| \leq h(2X)$ and $[x_1, x_1 + h(x_1)] \subset [\frac{3}{2}X, 3X]$. Hence we apply the first step of the previous argument to obtain that

$$|E_{\delta'}(\frac{3}{2}X, h')| \gg (\delta - \delta')h(x_1),$$

and since $h(x_1) \gg h(2X)$, $ii)$ follows in this case too, thus proving Theorem 1.

The proof of Corollary 1 is very simple. In order to prove $i)$, let $0 < \theta' < \theta < 1$ and choose $h(x) = x^\theta$, $h'(x) = x^{\theta'}$ and $\delta' = \delta/2 \geq exp(-\sqrt{\log X})$ in $ii)$ of Theorem 1. We get

$$\max(|E_{\frac{\delta}{2}}(X,\theta')|, |E_{\frac{\delta}{2}}(\frac{3}{2}X,\theta')|) \gg \delta|E_\delta(X,\theta)|,$$

hence $\mu_{\frac{\delta}{2}}(\theta') \geq \mu_\delta(\theta)$ and so $\mu(\theta') \geq \mu(\theta)$.

To prove $ii)$, let first assume that $\mu(\theta_0) < \theta_0$ for some $0 < \theta_0 < 1$ and observe that from $i)$ we have $\mu(\theta) < \theta$ for every $\theta_0 \leq \theta < 1$. Hence for every $\delta > 0$ we have $\mu_\delta(\theta) < \theta$ in the same range. If (1) fails to hold for $h(x) = x^\theta$ with some $\theta_0 \leq \theta < 1$, then there exists $\delta_0 > 0$ and arbitrarily large values of X such that $E_{\delta_0}(X,\theta) \neq \emptyset$. Hence from $i)$ of Theorem 1 with $h(x) = x^\theta$ and $\delta' = \delta_0/2$ we have, for such values of X, that

$$X^\theta \ll |E_{\frac{\delta_0}{2}}(X,\theta)| \ll X^{\mu_{\frac{\delta_0}{2}}(\theta)+\varepsilon},$$

a contradiction for X sufficiently large and $\varepsilon > 0$ sufficiently small. Hence (1) holds with $h(x) = x^\theta$, $\theta_0 \le \theta < 1$, and $\mu(\theta) = 0$ in the same range.

The opposite implication is trivial since, as we have already observed in the Introduction, the validity of (1) with $h(x) = x^\theta$ implies that $E_\delta(X, \theta) = \emptyset$ for every $\delta > 0$ and X sufficiently large.

In order to prove Corollary 2 we assume that (1) does not hold. Then there exists $\delta_0 > 0$ and a sequence $x_j \to \infty$ with $|\Delta(x_j, h)| \ge \delta_0 h(x_j)$. For x_j sufficiently large, choose $X = x_j$ and $\delta' = \delta_0/2$ in *i*) of Theorem 1. Hence

$$|\Delta(X,h)| \ge \frac{\delta_0}{2} h(x) \ge \frac{\delta_0}{2} h(X) \quad \text{for} \quad X \le x \le X + \frac{\theta\delta_0}{10} h(X).$$

Choosing $Y = \frac{\theta\delta_0}{10} h(X)$, from our assumption we get

$$\frac{\theta\delta_0}{10} h(X) (\frac{\delta_0}{2} h(X))^2 \le \int_X^{X+Y} |\Delta(x,h)|^2 dx \le \frac{20}{\theta^2} (\frac{\theta\delta_0}{10} h(X))^3,$$

a contradiction. The opposite implication is trivial.

To prove *i*) of Corollary 3 we recall that Selberg [14] proved, under RH, that

$$\int_X^{2X} |\psi(x+H) - \psi(x) - H|^2 dx \ll XH \log^2 X \tag{13}$$

for $H \ge 10$. Choosing $h(x)$ as in *i*), $Y = ch(X)$ and $H = h(X)$ we get

$$\int_X^{X+Y} |\psi(x + h(X)) - \psi(x) - h(X)|^2 dx \ll X h(X) \log^2 X. \tag{14}$$

From (7) with $\alpha = \theta = 1/2$ we see that

$$h(x) = h(X) + O(X^\varepsilon) \quad \text{uniformly for} \quad X \le x \le X + Y$$

and hence

$$\int_X^{X+Y} |\Delta(x,h)|^2 dx = \int_X^{X+Y} |\psi(x + h(X)) - \psi(x) - h(X)|^2 dx + O(X^{1/2+\varepsilon}). \tag{15}$$

From (14) and (15) we have

$$\int_X^{X+Y} |\Delta(x,h)|^2 dx \ll X h(X) \log^2 X,$$

and the result follows from Corollary 2.

The proof of *ii*) and of the first part of *iii*) is very similar. We only have to observe that from (3) and (4) by difference we get

$$\int_X^{X+ch(X)} |\psi(x+H) - \psi(x) - H|^2 dx = o(HX \log X)$$

and

$$\int_X^{X+ch(X)} |\psi(x+H) - \psi(x) - H|^2 dx \ll HX$$

respectively, uniformly for $X^{1/2-\varepsilon} \le H \le X^{1-\varepsilon}$. The results follows then arguing as before, by choosing, when proving *ii*), a suitable function $f(x) = o(1)$ such that $h(x) = f(x)(x \log x)^{1/2}$ is of type 1/2. The second part of *iii*) can be proved along similar lines, observing that in this case it is enough to show that $\psi(x + cx^{1/2}) - \psi(x) \ge c' x^{1/2}$ for some constants $c, c' > 0$ and x sufficiently large. Supposing that this is not true, we obtain that for any $c, c' > 0$ there exists a sequence $x_j \to \infty$ such that

$$\psi(x_j + cx_j^{1/2}) - \psi(x_j) < c' x_j^{1/2}$$

and hence, choosing $c' = c/2$, $\delta = 1/2$, $\delta' = 1/4$ and $h(x) = cx^{1/2}$, we obtain

$$[x_j, x_j + \frac{c}{40} x_j^{1/2}] \subset E_{\frac{1}{4}}(x_j, h)$$

by *i*) of Theorem 1. Therefore

$$\int_{x_j}^{x_j + \frac{c}{40} x_j^{1/2}} |\psi(x + cx^{1/2}) - \psi(x) - cx^{1/2}|^2 dx \ge \frac{c^3}{640} x_j^{3/2} \tag{16}$$

for any constant $c > 0$. On the other hand, from (4) by difference we get

$$\int_{x_j}^{x_j + \frac{c}{40} x_j^{1/2}} |\psi(x + cx^{1/2}) - \psi(x) - cx^{1/2}|^2 dx \ll c x_j^{3/2}, \tag{17}$$

and the second part of *iii*) follows from (16) and (17) if c is large enough.

3. Proof of Theorem 2 and Corollary 4

We only give a sketch of the proof, since the arguments involved are fairly standard. We first reduce our problem to a similar one, but technically simpler. We begin by observing that if for a given $0 < \theta < 1$

$$|\{X \le x \le 2X : |\Delta(x, x^\theta)| \ge \frac{4X^\theta}{\log X}\}| \ll X^{\alpha+\varepsilon} \tag{18}$$

holds with some $\alpha \ge 0$ and every $\varepsilon > 0$, then clearly $\mu(\theta) \le \alpha$. Further, given any $\varepsilon > 0$, we subdivide $[X, 2X]$ into $\ll X^\varepsilon$ intervals of the type $I_j = [X_j, X_j + Y]$ with $X \ll X_j \ll X$ and $Y \ll X^{1-\varepsilon}$. Writing $\xi_j = X_j^{\theta-1}$ we have

$$\max_{x \in I_j} |x^\theta - \xi_j x| \ll X^{\theta-\varepsilon}$$

uniformly in j, and hence

$$\Delta(x, x^\theta) - (\psi(x + \xi_j x) - \psi(x) - \xi_j x) \ll X^{\theta-\varepsilon} \tag{19}$$

uniformly in j and $x \in I_j$.
From (18) and (19) is not difficult to see that if for some $\alpha \geq 0$ and any $\varepsilon > 0$

$$|\{X \leq x \leq 2X : |\psi(x+\xi_j x) - \psi(x) - \xi_j x| \geq \frac{2X^\theta}{\log X}\}| \ll X^{\alpha+\varepsilon} \tag{20}$$

holds uniformly in j, then $\mu(\theta) \leq \alpha$. Also, it is clear that in order to prove (20) we may restrict ourselves to the case $\xi_j = \xi = X^{\theta-1}$, the other cases being completely similar.
In order to prove (20) we use the classical explicit formula, see ch. 17 of Davenport [2], to write

$$\psi(x+\xi x) - \psi(x) - \xi x = \sum_{|\gamma| \leq T} x^\varrho c_\varrho(\xi) + O(\frac{X \log^2 X}{T}) = \sum(x) + O(\frac{X \log^2 X}{T}), \tag{21}$$

say, uniformly for $X \leq x \leq 2X$, where $10 \leq T \leq X$, $\varrho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$,

$$c_\varrho(\xi) = \frac{(1+\xi)^\varrho - 1}{\varrho} \qquad \text{and} \qquad c_\varrho(\xi) \ll \min(X^{\theta-1}, \frac{1}{|\gamma|}). \tag{22}$$

We first prove the bound for $\mu(1/6+\Delta)$, and hence we write $\theta = 1/6 + \Delta$. We use Theorem 3 of Halász-Turán [5], which asserts that there exists a constant $c_1 > 0$ such that

$$N(\sigma, T) \ll T^{(1-\sigma)^{3/2} \log^3 \frac{1}{1-\sigma}} \tag{23}$$

for $1 - c_1 \leq \sigma \leq 1$. Choose

$$T = X^{1-\theta} \log^4 X. \tag{24}$$

From (22) - (24) and Vinogradov's zero-free region, see ch. 6 of Titchmarsh [15], by a standard argument we see that there exists a constant $c_2 > 0$ such that

$$\sum_{\substack{|\gamma| \leq T \\ 1-c_2 \leq \beta \leq 1}} x^\varrho c_\varrho(\xi) \ll X^{\theta-1} \log^2 X \max_{1-c_2 \leq \sigma \leq 1} X^\sigma N(\sigma, T) \ll \frac{X^\theta}{\log^2 X} \tag{25}$$

uniformly for $X \leq x \leq 2X$.
We bound the remaining part of $\sum(x)$ in mean square, using the density estimates of Ingham, see ch. 12 of Montgomery [12], and Huxley [10], which imply that

$$N(\sigma, T) \ll T^{\frac{12}{5}(1-\sigma)+\varepsilon} \tag{26}$$

for $1/2 \leq \sigma \leq 1$. Again by a standard argument, from (22), (24) and (26) we obtain

$$\int_X^{2X} |\sum_{\substack{|\gamma|\leq T \\ 0\leq\beta<1-c_2}} x^\varrho c_\varrho(\xi)|^2 dx \ll X^{2\theta-1+\varepsilon} \max_{1/2\leq\sigma\leq 1-c_2} X^{2\sigma} N(\sigma,T) \ll X^{2\theta+1-\frac{12}{5}c_2\Delta+\varepsilon},$$

and hence

$$|\{X \leq x \leq 2X : |\sum_{\substack{|\gamma|\leq T \\ 0\leq\beta<1-c_2}} x^\varrho c_\varrho(\xi)| \geq \frac{X^\theta}{\log^2 X}\}| \ll X^{1-\frac{12}{5}c_2\Delta+\varepsilon}. \tag{27}$$

From (21), (24), (25) and (27) we see that (20) is satisfied with $\alpha = 1 - \frac{12}{5}c_2\Delta$, and the first bound of Theorem 2 is proved with $c = \frac{12}{5}c_2$.

In order to bound $\mu(7/12 - \Delta)$ we proceed along similar lines, using fourth power moments instead of mean square estimates. Here we need the precise version of Ingham's and Huxley's results quoted above, namely

$$N(\sigma,T) \ll \begin{cases} T^{\frac{3(1-\sigma)}{2-\sigma}} \log^k T & \text{if } \dfrac{1}{2} \leq \sigma \leq \dfrac{3}{4} \\ \\ T^{\frac{3(1-\sigma)}{3\sigma-1}} \log^k T & \text{if } \dfrac{3}{4} \leq \sigma \leq 1 \end{cases} \tag{28}$$

where k is an absolute constant.

We write $\theta = 7/12 - \Delta$, with Δ sufficiently small, and $I = [3/4 - 3\Delta, 3/4 + (1+\varepsilon)\Delta]$. From (24), (28) and Vinogradov's zero-free region we see that

$$\sum_{\substack{|\gamma|\leq T \\ 0\leq\beta\leq 1 \\ \beta\notin I}} x^\varrho c_\varrho(\xi) \ll X^{\theta-1}\log^2 X \max_{\substack{1/2\leq\sigma\leq 1 \\ \sigma\notin I}} X^\sigma N(\sigma,T) \ll \frac{X^\theta}{\log^2 X} \tag{29}$$

uniformly for $X \leq x \leq 2X$.

We bound the remaining part of $\sum(x)$ by a fourth power moment estimate. To this end we use Lemma 1 of Heath-Brown [6] to get

$$\int_X^{2X} |\sum_{\substack{|\gamma|\leq T \\ \beta\in I}} x^\varrho c_\varrho(\xi)|^4 dx \ll X^{4\theta-3+\varepsilon} \max_{\sigma\in I} X^{4\sigma} N^*(\sigma,T). \tag{30}$$

From Theorem 2 of Heath-Brown [7] we have

$$N^*(\sigma,T) \ll \begin{cases} T^{\frac{(36-8\sigma)(1-\sigma)}{5}} \log^k T & \text{if } \frac{1}{2} \leq \sigma \leq \frac{3}{4} \\ T^{\frac{12(1-\sigma)}{4\sigma-1}} \log^k T & \text{if } \frac{3}{4} \leq \sigma \leq 1, \end{cases} \tag{31}$$

where k is an absolute constant. Hence from (24), (30) and (31) we obtain

$$\int_X^{2X} \Big| \sum_{\substack{|\gamma|\le T\\ \beta\in I}} x^{\varrho} c_{\varrho}(\xi)\Big|^4 dx \ll X^{4\theta+\frac{5}{8}+\frac{7}{4}\Delta+O(\Delta^2)+\varepsilon}$$

and hence

$$|\{X \le x \le 2X : \Big| \sum_{\substack{|\gamma|\le T\\ \beta\in I}} x^{\varrho} c_{\varrho}(\xi)\Big| \ge \frac{X^{\theta}}{\log^2 X}\}| \ll X^{\frac{5}{8}+\frac{7}{4}\Delta+O(\Delta^2)+\varepsilon}. \tag{32}$$

From (21), (24), (29) and (32) we see that (20) is satisfied in this case with $\alpha = \frac{5}{8} + \frac{7}{4}\Delta + O(\Delta^2)$, and the second bound of Theorem 2 is proved.

The result under RH follows immediately from Selberg's bound (13), and Theorem 2 is proved.

The remark that (5) can be proved under the assumption (6) can be easily checked arguing as before, using only mean square estimates, *i.e.*, by means of (6) and (28) instead of (31).

Finally, we prove Corollary 4. Choose $h(x) = x^{\theta}$ with $\theta > 0$. It is clear that if an interval of type $I = [y, y + Y]$ is contained in $E_{\delta}(X, \theta)$, with $0 < \theta \le \frac{7}{12}$, then $\Delta(x, h)$ has the same sign for all $x \in I$. In fact, $|\Delta(x, h)|$ has jumps of height $\ll \log x$ and $\log x = o(x^{\theta})$. Therefore, the asymptotic formula (1) does not hold for the interval I itself. Hence by [8] we have $Y \ll X^{7/12}$, and Corollary 4 follows.

References

[1] H.Cramér, *Some theorems concerning prime numbers*, Ark. Mat. Astronom. Fys. **15** (1920), 1–32.

[2] H.Davenport, *Multiplicative Number Theory*, second ed., Springer Verlag, 1980.

[3] D.A.Goldston, *Linnik's theorem on Goldbach numbers in short intervals*, Glasgow Math. J. **32** (1990), 285–297.

[4] D.A.Goldston and H.L.Montgomery, *Pair correlation of zeros and primes in short intervals*, Analytic Number Theory and Dioph. Probl., ed. by A.C.Adolphson et al., Birkhäuser Verlag 1987, 183–203.

[5] G.Halász and P.Turán, *On the distribution of roots of Riemann zeta and allied functions, I*, J. Number Theory **1** (1969), 121-137.

[6] D.R.Heath-Brown, *The difference between consecutive primes, II*, J. London Math. Soc. (2) **19** (1979), 207–220.

[7] ———, *Zero density estimates for the Riemann zeta-function and Dirichlet L-functions*, J. London Math. Soc. (2) **19** (1979), 221–232.

[8] ———, *The number of primes in a short interval*, J. reine angew. Math. **389** (1988), 22–63.

[9] D.R.Heath-Brown and D.A.Goldston, *A note on the differences between consecutive primes*, Math. Ann. **266** (1984), 317–320.

[10] M.N.Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164–170.

[11] H.Maier, *Primes in short intervals*, Michigan Math. J. **32** (1985), 221–225.

[12] H.L.Montgomery, *Topics in Multiplicative Number Theory*, Springer L.N. **227**, 1971.

[13] H.L.Montgomery and R.C.Vaughan, *The large sieve* , Mathematika **20** (1973), 119–134.

[14] A.Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), 87–105.

[15] E.C.Titchmarsh, *The Theory of the Riemann Zeta-function*, second ed., Oxford U.P., 1986.

[16] D.Wolke, *Grosse Differenzen aufeinanderfolgender Primzahlen*, Math. Ann. **218** (1975), 269–271.

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