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Diffusive coupling as a quadratic constraint: Synchronization and stability in heterogeneous networks ☆

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ABSTRACT

This paper studies synchronization of heterogeneous multi-agent networks from the perspective of absolute stability theory. We consider networks of nonlinear agents coupled via simple diffusive protocols and ask: how much can the classical passivity assumption on the agents be relaxed while retaining a verifiable, fully distributed synchronization criterion? We show that passivity can be replaced by the strictly weaker property of *input feedforward passivity* (IFP) – the ability to be rendered passive by a special parallel feedforward compensator. The key idea is a quadratic constraint satisfied by the diffusive coupling protocol. This relaxation becomes especially pronounced in discrete time, where even the first-order integrator fails to be passive yet is IFP. We develop synchronization criteria for networks of heterogeneous IFP agents in both continuous and discrete time within a unified dissipativity framework, and further show that the presence of sufficient damping in a subset of agents ensures global output stability of the coupled network. The results are applied to cooperative adaptive cruise control for vehicle platoons.

1. Introduction

The classical formulation of absolute stability theory, inseparably associated with the name of V.A. Yakubovich, concerns the stability of a Lur'e-type system: a feedback interconnection of a known linear time-invariant (LTI) block and a nonlinear element whose structure is unknown but assumed to satisfy certain constraints (e.g., sector or slope inequalities). The objective is to obtain tractable conditions ensuring a prescribed stability property (usually, global asymptotic stability of an equilibrium) for all nonlinearities satisfying the specified constraints; the term "absolute" underscores that the property holds uniformly across all such nonlinearities.

Broadly speaking, absolute stability theory can be divided into two major parts. The *operator approach* to absolute stability, rooted in the Popov method of "integral indices" [1,2], treats dynamical systems as input–output operators on function spaces. This approach allows one to treat many classes of dynamical systems (e.g., differential, difference, and integro-differential equations) and general integral quadratic constraints in a uniform way, yielding elegant frequency-domain stability conditions [3–5]. This approach, however, provides very limited information about the transient behavior of solutions, and becomes quite complicated in the case of partial stability, where, e.g., only certain outputs of the system can be estimated. The *Lyapunov approach*, pioneered by Lur'e and Postnikov [6,7], establishes absolute stability via special classes of Lyapunov functions and requires an explicit state-space representation of the system and inherently limits the class of admissible quadratic constraints. At the same time, the availability of a Lyapunov function enables the derivation of explicit convergence rates and transient performance estimates [8,9]. Furthermore, modern semidefinite programming solvers allow for the efficient computation of stability conditions expressed as linear matrix inequalities (LMIs) [10], in contrast to frequency-domain criteria whose direct validation, especially in the MIMO case, can be nontrivial. Moreover, Lyapunov methods permit the replacement of the linear time-invariant subsystem, standard in classical absolute stability theory, by a more general nonlinear block that is *dissipative* in the sense of Willems [11] with respect to an appropriate quadratic supply rate. The quadratic constraint on the nonlinearity ensures that the interconnection remains dissipative, while the storage function serves as a Lyapunov certificate. Dissipativity properties often arise naturally from the physical structure of the system: Euler–Lagrange and port-Hamiltonian models are inherently passive with respect to power-conjugate variables (generalized forces as inputs and generalized velocities as outputs); the associated energy function serves as a natural storage function [12,13].

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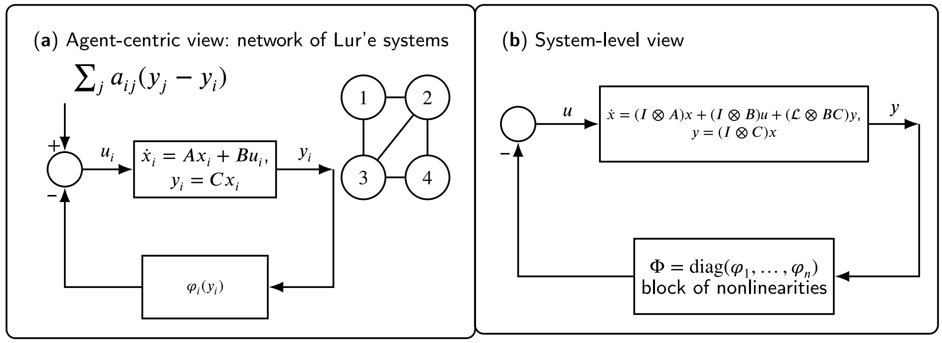

Fig. 1. A network of Lur'e-type agents (a) recast as a single Lur'e-type system (b): a linear network in feedback with local static nonlinearities. We illustrate a special case of output diffusive coupling; $\mathcal{L}$ denotes the Laplacian matrix.

Absolute stability for dynamical networks

Dynamical networks, where uncertain and time-varying interconnections align naturally with the classical interpretation of "nonlinearity" as an uncertain feedback element, have benefited surprisingly little from absolute stability theory. Methods rooted in absolute stability theory, treating activation functions (such as ReLU, sigmoid, and tanh) as sector-bounded or slope-restricted nonlinearities, have been developed for stability and contraction analysis of some neural network architectures [14–16]. The main model considered in the multi-agent context, however, is a network of diffusively coupled[1] Lur'e-type systems with identical linear parts. Equivalently, such a network may be interpreted as a feedback interconnection between a linear network – comprising the agents' linear subsystems and the coupling topology – and local nonlinear feedback elements at the nodes (Fig. 1). This representation lends itself to direct application of the multivariate Popov criterion [19], circle criterion [20] and more general conditions based on the integral quadratic constraint (IQC) framework [21]. The Kronecker-product structure of the system matrices allows efficient validation of the frequency-domain conditions. More commonly, however, analysis and design employs Lyapunov functions whose quadratic part is obtained by summing identical quadratic forms $x_i^\top P x_i$ over all agents, where $P > 0$ is a common positive definite matrix found from a Kalman–Yakubovich–Popov-type LMI or an algebraic Riccati equation [22–24]. Many works require not only identical linear parts but also identical nonlinearities, as is the case, for instance, when incremental slope restrictions are imposed rather than sector inequalities [25,26] – while works allowing full or partial heterogeneity remain scarce [27,28].

Another approach, which motivates the current work, treats the interconnections themselves as the "nonlinearity". The coupling among agents can indeed be nonlinear – as in the seminal Kuramoto model [29] – but even linear interconnections are often uncertain: one may only know that the graph remains connected. In many applications, the graph is time-varying owing to, e.g., packet dropouts, link failures, or changing agent neighborhoods in mobile networks [30]. As shown in [31–33], symmetric sector-bounded couplings satisfy a quadratic constraint that encodes only minimal information about the interconnection graph. Exploiting this quadratic constraint, analogues of the circle [31] and Popov [33] criteria are derived, ensuring synchronization of networks of LTI agents subject to nonlinear diffusive couplings. These results were largely motivated by earlier work [34], exploiting a simple quadratic constraint on diffusive couplings to establish a synchronization criterion for passive agents. The passivity-based synchronization criterion, further developed in [13,35] is important for a number of reasons. First, it is fully distributed: no information about the network beyond connectivity is required – neither its size nor the eigenvalues of the Laplacian matrix. Second, it requires only simple diffusive couplings among the agents, without dynamic controllers on edges or state observers. Third, and most importantly, passivity-based criteria are among the few that can accommodate *fully heterogeneous* agents that can even have different dimensions of the state vector.

Heterogeneous node dynamics

Stability and synchronization of networks with heterogeneous node dynamics remain challenging and incompletely resolved, and passivity-based conditions [13,35–37] are among the few practically verifiable criteria available to date. General stability criteria for heterogeneous networks, including input-to-state stability results, have been obtained via small-gain theory [38,39]; however, these methods do not extend easily to partial stability and synchronization.

Controlled synchronization of heterogeneous autonomous agents is typically achieved via an intermediate control layer. Each agent is endowed with a virtual copy of a prescribed dynamical system, referred to as the local reference model, and a model-matching controller that enforces tracking of this reference model. Since the reference models are identical, their synchronization — which constitutes the distributed layer of the control architecture – can be achieved using standard tools, including reduction to the first-order consensus dynamics [40], Laplacian-based modal decomposition [18,41,42], contraction analysis [43], and incremental dissipativity [44]. The systematic design of the "upper" layer control, including the choice of a reference model and agent-specific model-matching controllers, is, however, largely confined to linear dynamical systems, since it relies on the solvability of the regulator equations [45–49], whereas results for heterogeneous nonlinear agents remain comparatively limited [27,50,51]. Frequency-domain methods for synchronization control design are likewise largely confined to linear systems [52,53].

Contribution of this work

Surveying the landscape of existing methods for network stability and synchronization, one may notice a clear divide. On the one hand, passivity-based designs that inherit the core ideas of absolute stability theory accommodate arbitrarily heterogeneous passive agents under simple diffusive couplings. On the other hand, the designs prevailing in the multi-agent systems literature – including those based on absolute stability methods – are either confined to identical agents or reduce the heterogeneous problem to one involving identical agents.

While bridging this gap entirely remains a long-term goal, the present work takes an important step in this direction. We show that the passivity condition can be substantially relaxed, and that a simple diffusive coupling protocol can synchronize a broad class of *non-passive* heterogeneous agents. It suffices that each agent becomes passive upon addition of a parallel feedforward compensator – a property referred to

[1] The term, formally defined below, originates from models of interconnected living cells [17]. It has a physical interpretation: the coupling between neighboring agents represents a flow of mass, charge, heat, or energy, with rate proportional to the difference between their outputs [18].

as (input) *feedforward passivity*. Many important dynamical systems possess this property, and it has been widely used in adaptive control [54]; recently it has also been employed in synchronization of certain classes of homogeneous networks [55,56]. Unlike passive agents, agents that are feedforward passive can be synchronized by sufficiently *weak* diffusive coupling. We further show that in the presence of sufficiently weak damping in some of the agents, a diffusively coupled network becomes output stable. The difference between feedforward and usual passivity becomes dramatic for discrete-time synchronization criteria, since even the simplest discrete-time dynamics, such as the first-order integrator, are not passive yet feedforward passive. We apply our results to the design of cooperative adaptive cruise control (CACC) for vehicle platoons.

Some results of this work were reported, without proofs, in the conference paper [57]. Here, we not only provide complete proofs, but also treat the continuous-time and discrete-time cases (absent from [57]) in a unified framework.

2. Preliminaries and problem setup

In this section, we introduce basic concepts from graph theory and define input-feedforward passivity (IFP).

Graphs and connectivity. A weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ consists of a node set $\mathcal{V} = \{1, \dots, N\}$, an arc set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a non-negative adjacency matrix $A = (a_{ij})$, where $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. Since N and the node indices are fixed, graphs and adjacency matrices are in one-to-one correspondence via $A \mapsto \mathcal{G}[A]$; we always assume $a_{ii} = 0$. The weighted in- and out-degrees of node i are $d_i^+ \stackrel{\Delta}{=} \sum_j a_{ij}$ and $d_i^- \stackrel{\Delta}{=} \sum_j a_{ji}$.

A *walk* from v to v' is a sequence $v = v_{i_0}, v_{i_1}, \dots, v_{i_s} = v'$ with $(v_{i_{k-1}}, v_{i_k}) \in \mathcal{E}$ for each k. A graph is *strongly connected* if a walk exists between every ordered pair of distinct nodes, and *quasi-strongly connected* (equivalently, has a *directed spanning tree*) if some node can reach all others by walks. For undirected and weight-balanced graphs both conditions reduce to the usual notion of *connectedness*.

The *Laplacian* matrix of $\mathcal{G}[A]$ is defined as $L[A] \stackrel{\Delta}{=} D^+[A] - A$, where $D^+[A] \stackrel{\Delta}{=} \mathrm{diag}(d_1^+, \dots, d_N^+)$ is the out-degree matrix. By construction, $L[A]\mathbf{1} = 0$, i.e., $\mathbf{1} \stackrel{\Delta}{=} (1, \dots, 1)^\top$ is a right eigenvector of $L[A]$ corresponding to the eigenvalue 0. If $\mathcal{G}[A]$ is quasi-strongly connected, then $\ker L[A] = \mathrm{span}\,\mathbf{1}$ and there exists a unique left eigenvector $p \in \mathbb{R}^N$ such that $p^\top L[A] = 0$ and $\sum_{i=1}^N p_i = 1$. Since $(-L[A])$ is a Metzler matrix, the Perron–Frobenius theorem [18] guarantees that p is nonnegative; p is strictly positive if and only if the graph is strongly connected.

Dissipativity and its special cases. In this subsection, we recall the concept of dissipativity for the system

$$\sigma x(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{Σ}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ are the state, input, and output, and σ denotes either the derivative $\dot{x}(t)$ (continuous time, $t \geq 0$) or the forward shift $x(t) \mapsto x(t+1)$ (discrete time, $t = 0, 1, \dots$). We use the shorthand $\int_{t_1}^{t_2} (\cdot)\, dt$ and $\sum_{t_1 \leq t < t_2} (\cdot)$ for the continuous- and discrete-time accumulated supply, respectively.

Definition 1 (*[11]*). System (Σ) is *dissipative* with storage function $V : \mathbb{R}^n \to [0, \infty)$ and supply rate $s : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ if, for all $0 \leq t_1 < t_2$ and all solutions defined on $[t_1, t_2]$,

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} s(y, u)\, dt \quad \left(\text{resp.} \sum_{t_1 \leq t < t_2} s(y, u)\right). \tag{1}$$

The most studied case of dissipativity is *passivity*, i.e., dissipativity with supply rate $s(y, u) = y^\top u$. While natural for many mechanical systems and electric circuits, these conditions fail to hold for a wide range of practically important systems. We therefore consider a much broader class, called *input feedforward passive* (IFP) [58].

Definition 2. Let $\dim y = \dim u = m$. System (Σ) is IFP with index $R \in \mathbb{R}^{m \times m}$, denoted IFP($R$), if it is dissipative with $s(y, u) = y^\top u + u^\top R u$. When $R = rI$, $r \geq 0$, we will simply write IFP(r) for brevity.

Equivalently, system (Σ) is IFP(R) if and only if it is passive with respect to the modified output $\tilde{y} = y + Ru$, i.e., it is rendered passive by introducing a *parallel feedforward compensator*, called a *shunt* in the Russian control literature [54]. Since the antisymmetric part of R does not affect the supply rate $u^\top R u$, we may assume without loss of generality that $R = R^\top$. Typically $R \geq 0$; the case $R = 0$ reduces to conventional passivity.

Example 1. The discrete-time integrator $y(t+1) = y(t) + \tau u(t)$ is not passive: sufficiently strong negative feedback $u(t) = -k y(t)$ (with $k\tau > 1$) not only violates (1) for any nonnegative V, but also destabilizes the system. Choosing the storage function $V(y) = \frac{1}{2\tau} y^2$, it can be checked that this system is IFP($\tau/2$), because

$$V(y(t+1)) - V(y(t)) = y(t)u(t) + \frac{\tau}{2} u(t)^2.$$

For general LTI systems, IFP can be tested via the Positive Real Lemma [11,58,59] as shown by the following.

Example 2. Consider the continuous-time SISO system[2]

$$\rho\left(\tfrac{d}{dt}\right) \tfrac{d}{dt} \zeta(t) = u(t) \in \mathbb{R}, \quad y(t) = \eta\left(\tfrac{d}{dt}\right) \zeta(t), \qquad \rho(\lambda) \stackrel{\Delta}{=} \textstyle\sum_{k=0}^r \rho_k \lambda^k, \quad \eta(\lambda) \stackrel{\Delta}{=} \textstyle\sum_{k=0}^q \eta_k \lambda^k. \tag{2}$$

which has a single unstable pole at zero, with transfer function $W(s) = \eta(s)/(s\rho(s))$.

Lemma 1. *Suppose $\rho(\lambda)$ is Hurwitz, $\rho_0 > 0$ and $\eta_0 \geq 0$ (the residue at the zero pole is nonnegative). Then system (2) is IFP(α^*), where the passivity index is $\alpha^* \stackrel{\Delta}{=} -\inf_{\omega \in \mathbb{R} \setminus \{0\}} Re\, W(\imath\omega)$.*

Proof. Notice first that $Re\, W(\imath\omega)$ is bounded below and thus $\alpha^* < \infty$. Indeed, as $\omega \to \infty$, $W(\imath\omega) \to 0$. Near $\omega = 0$, one has $W(\lambda) = (\eta_0/\rho_0)\lambda^{-1} + O(1)$, so $Re\, W(\imath\omega) = O(1)$ as $\omega \to 0$ (since the residue η_0/ρ_0 is real and $Re\,[(\imath\omega)^{-1}] = 0$). Hence $\alpha^* < \infty$. As noted in Section 2, the IFP(α) condition is equivalent to passivity of system (2) with respect to the modified output $\tilde{y} = y + \alpha^* u$, with transfer function $\tilde{W}(\lambda) = W(\lambda) + \alpha^*$. By definition of α^*, one has $Re\, \tilde{W}(\imath\omega) \geq 0$ for all $\omega \neq 0$. Moreover, $\lim_{\lambda \to 0} \lambda \tilde{W}(\lambda) = \eta_0/\rho_0 \geq 0$, and all nonzero poles of $\tilde{W}$ are stable since $\rho(\lambda)$ is Hurwitz. Hence $\tilde{W}$ is positive real, and the result follows from the Positive Real Lemma [11]. ■

Many examples of nonlinear IFP systems can be constructed by noticing that the IFP property is preserved under output feedback as illustrated by the following lemma.

Lemma 2. *Consider system (Σ) with the feedback law $u(t) = v(t) - \varphi(y(t))$, where $v(t)$ is a new input and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity. If system (Σ) is IFP(R), where $R \geq 0$ and $y^\top \varphi(y) \geq (1+\varepsilon)\varphi(y)^\top R \varphi(y)$ for all $y \in \mathbb{R}^m$ for some $\varepsilon > 0$, then the closed-loop system is IFP$\left(\left(1 + \varepsilon^{-1}\right) R\right)$ with respect to the input v and output y.*

Proof. Denoting for brevity $\varphi = \varphi(y)$, weighted Young's inequality ensures that $2|v^\top R \varphi| \leq \varepsilon^{-1} v^\top R v + \varepsilon \varphi^\top R \varphi$. The statement is now immediate from the definition of IFP and the inequalities

$$\begin{aligned} y^\top u + u^\top R u = y^\top v - y^\top \varphi + (v - \varphi)^\top R (v - \varphi) &\leq y^\top v + (1 + \varepsilon^{-1}) v^\top R v \\ &\quad - y^\top \varphi + (1 + \varepsilon)\varphi^\top R \varphi \\ &\leq y^\top v + (1 + \varepsilon^{-1}) v^\top R v \quad ■. \end{aligned}$$

[2] To apply the definition of IFP, assume that we are dealing with a minimal (observable and controllable) state-space realization of this model.

Diffusive couplings. a quadratic constraint. Following [17,36], we introduce the concept of a *diffusively coupled* network. Consider $N \geq 2$ heterogeneous agents of the form (Σ), indexed $i = 1, \dots, N$:

$$\sigma x_i(t) = f_i(x_i(t), u_i(t)), \;\; y_i(t) = h_i(x_i(t)), \quad i = 1, \dots, N, \tag{3}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^m$ are the state, input, and output of the ith agent. All agents share the same time domain (all continuous-time or all discrete-time).

Definition 3. The systems (3) interconnected via

$$u_i(t) = K \sum_{j=1}^{N} a_{ij}(y_j(t) - y_i(t)), \quad i = 1, \dots, N, \tag{4}$$

with $a_{ij} \geq 0$ constitute a *diffusively coupled* network with *coupling matrix* $A = (a_{ij})$ and matrix gain $K \in \mathbb{R}^{m\times m}$.

Typically $K = I$ or, more generally, $K = kI$ for some scalar $k > 0$; the dependence of synchronization on a single scalar gain is a common formulation in the physics literature [60]. More generally, a diagonal matrix $K = \mathrm{diag}(k_1, \dots, k_m)$ allows independent weighting across output channels, which is useful when the output components have different physical units or scales. In the synchronization criteria below, we admit an arbitrary symmetric positive definite matrix gain $K \succ 0$. As discussed below, the matrix gain K must be sufficiently small to ensure synchronization.

The following lemma, generalizing the result from [13], shows that the diffusive coupling protocol imposes a *quadratic constraint*. This observation places the synchronization problem within the framework of absolute stability theory and the quadratic constraint methodology developed by V.A. Yakubovich: the role of the sector-bounded nonlinearity is played by the diffusive coupling, and the standard LMI conditions on the linear block are replaced by IFP conditions on the (possibly nonlinear) agents.

Lemma 3. *Let the graph $\mathcal{G}[A]$ be strongly connected and p be the strictly positive Perron–Frobenius eigenvector for the matrix $L[A]$ such that $p^\top L[A] = 0$ and $p^\top \mathbf{1} = 1$. Choose constants $\alpha_i \in \left[0, \frac{1}{2d_i^+[A]}\right)$ for all $i = 1, \dots, N$. Then $\varepsilon = \varepsilon(\alpha, A) > 0$ exists such that all vectors $(u_1, \dots, u_N, y_1, \dots, y_N)$ satisfying (4) obey the quadratic constraint*

$$\sum_{i=1}^{N} p_i \left(y_i^\top u_i + \alpha_i u_i^\top K^{-1} u_i\right) \leq -\varepsilon \sum_{i,j=1}^{N} a_{ij}(y_j - y_i)^\top K (y_j - y_i). \tag{5}$$

Proof. Consider first the case where $K = I$. As shown in [13, Theorem 8.5], the Eqs. (4) in this case imply

$$\sum_{i=1}^{N} p_i y_i^\top u_i = -\frac{1}{2} \sum_{i,j=1}^{N} p_i a_{ij} |y_j - y_i|^2. \tag{6}$$

On the other hand, using the Cauchy–Schwarz inequality, for all $i = 1, \dots, N$ one has

$$d_i^+[A] \sum_{j=1}^{N} a_{ij} |y_j - y_i|^2 = \sum_{k=1}^{N} a_{ik} \sum_{j=1}^{N} a_{ij} |y_j - y_i|^2 \geq \left| \sum_{j=1}^{N} a_{ij}^{1/2} a_{ij}^{1/2} (y_j - y_i) \right|^2 = |u_i|^2.$$

Combining these inequalities, we obtain

$$\sum_{i=1}^{N} p_i y_i^\top u_i + \sum_{i=1}^{N} p_i \alpha_i |u_i|^2 \leq \sum_{i,j=1}^{N} \left(p_i \alpha_i d_i^+[A] - \frac{1}{2} p_i\right) a_{ij} |y_j - y_i|^2 \leq -\varepsilon \sum_{i,j=1}^{N} a_{ij} |y_j - y_i|^2,$$

where ε can be chosen as $\varepsilon \triangleq \min_i p_i(1/2 - \alpha_i d_i^+[A])$. This proves (5) with $K = I$. The general case is obtained by replacing $u_i \mapsto K^{-1/2} u_i$, $y_i \mapsto K^{1/2} y_i$ for all i. ■

Although computing p_i and ε in Lemma 3 may be computationally expensive for large-scale graphs, these coefficients do not appear explicitly in the synchronization criteria derived below.

Output synchronization and stability. In this paper, we study distributed protocols that synchronize the outputs y_j asymptotically or in L_2-norm.[3] Specifically, we seek conditions under which the coupling protocol (4) achieves output synchronization or L_2-synchronization of the agents (3). We also consider output stability in the presence of damping terms. To this end, we introduce the following notions.

Definition 4. Solutions $(x_j, u_j, y_j)_{j=1}^N$ of (3), defined for $t \geq 0$, are called:

- (asymptotically) *output synchronized* if $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all i, j;
- *output L_2-synchronized* if $\int_0^\infty |y_i(t) - y_j(t)|^2 \, dt < \infty$ for all i, j;
- (asymptotically) *output stable* if $|y_i(t)| \to 0$ as $t \to \infty$ for all i;
- *output L_2-stable* if $\int_0^\infty |y_i(t)|^2 \, dt < \infty$ for all i.

The discrete-time analogues are defined identically, with $\int_0^\infty (\cdot)\, dt$ replaced by $\sum_{t=0}^\infty (\cdot)$.

3. Main results

To simplify the formulation of continuous-time synchronization criteria, we impose some regularity conditions.

Assumption 1 (*Regularity*). The functions $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^m$ are defined globally. In the discrete-time case, we assume that h_i is locally bounded[4] (i.e., maps bounded sets to bounded sets). In the continuous-time case, f_i is continuous and h_i is C^1 for each $i = 1, \dots, N$.

Henceforth we assume, unless otherwise stated, that agents are IFP (whereas their passivity indices may differ).

Assumption 2 (*The IFP Property*). For each $i = 1, \dots, N$, the ith agent in (3) is IFP(R_i) with a known passivity index $R_i = R_i^\top \succ 0$ and a storage function $V_i(x_i) \geq 0$.

In some statements, we will impose an additional assumption on the storage functions.

Assumption 3 (*Radial Unboundedness*). The storage functions V_i are radially unbounded: $\lim_{|x_j| \to \infty} V_j(x_j) = \infty$.

3.1. Synchronization criterion

We now formulate the first main result of this paper, establishing synchronization in a network of systems (3), interconnected via a protocol (4) with a symmetric gain matrix $K = K^\top \succ 0$.

Theorem 1. *Suppose Assumptions 1 and 2 hold, $G[A]$ is strongly connected, and*

$$2d_i^+[A]\, K \prec R_i^{-1}, \qquad i = 1, \dots, N, \tag{7}$$

for some symmetric $K = K^\top \succ 0$. Then:

[3] In practice, the distinction between asymptotic and L_2-synchronization is minor. In the discrete-time case, L_2-synchronization implies asymptotic synchronization, but the converse does not hold. In the continuous-time case, neither condition implies the other. The implication from L_2- to asymptotic synchronization or stability is usually established via Barbalat's lemma (cf. [58, Lemma 8.2]).

[4] In other words, if the state vectors remain bounded, then the outputs remain bounded.

1. *In the discrete-time case, every solution of (3), (4) is output synchronized and output L_2-synchronized.*
2. *In the continuous-time case, every solution defined for all $t \geq 0$ is output L_2-synchronized.*
3. *If, in addition, Assumption 3 holds, then all variables (x_i, u_i, y_i) are bounded. In the continuous-time case this implies that solutions are defined for all $t \geq 0$ (hence also output L_2-synchronized) and are output synchronized.*

Proof. In view of (7), for each $i = 1, \dots, N$ we may choose α_i such that

$$\alpha_i K^{-1} > R_i, \quad 0 < 2d_i^+[A]\alpha_i < 1, \quad \text{whence}$$
$$y_i^\top u_i + u_i^\top R_i u_i \leq y_i^\top u_i + \alpha_i u_i^\top K^{-1} u_i. \tag{8}$$

We begin with the continuous-time case, which is more involved. Define the stack vector $X = [x_1^\top, \dots, x_N^\top]^\top$ and the aggregate storage function $V(X) \triangleq \sum_{i=1}^N p_i V_i(x_i)$. The IFP property together with (8) and Lemma 3 gives

$$\begin{aligned} V(X(T)) - V(X(0)) &\leq \sum_{i=1}^N \int_0^T p_i \left(y_i^\top u_i + u_i^\top R_i u_i\right) dt \\ &\overset{(5),\,(8)}{\leq} -\varepsilon \sum_{i,j=1}^N \int_0^T a_{ij} (y_j - y_i)^\top K (y_j - y_i)\, dt \leq 0. \end{aligned} \tag{9}$$

Since $V \geq 0$ and $K > 0$, for each pair of agents with $a_{ij} > 0$ one has

$$\int_0^T |y_j - y_i|^2\, dt \leq \frac{V(X(0))}{a_{ij}\lambda_{\min}(K)} \quad \forall T.$$

Hence, if the solution is defined for all $t \geq 0$, then $y_j - y_i \in L_2[0,\infty)$ for every pair of adjacent nodes ($a_{ij} > 0$), as follows by taking $T \to \infty$ in (9). Since $\mathcal{G}[A]$ is strongly connected, every pair of agents is connected by a walk, and $y_i - y_j \in L_2[0,\infty)$ for all i, j follows by the triangle inequality. This proves statement 2.

To prove statement 3, note that radial unboundedness of each $V_i(x_i)$, together with $p_i > 0$ for all i, implies that $V(X)$ is also radially unbounded. Since $V(X(T)) \leq V(X(0))$ for all $T \geq 0$, the state vectors $x_i(t)$ are bounded uniformly in t. Since h_i is continuous, the outputs $y_i(t)$ are bounded; boundedness of $u_i(t)$ follows from (4). In particular, the solution exists for all $t \geq 0$. Since f_i is continuous, $\dot{x}_i(t)$ is bounded, and since h_i is C^1, $\dot{y}_i(t) = h_i'(x_i)\dot{x}_i(t)$ is also bounded. The differences $|y_i(t) - y_j(t)|^2$ are thus uniformly continuous, and output synchronization follows from Barbalat's lemma.

The discrete-time proof is identical, with integrals replaced by sums. The only difference concerns statement 1: asymptotic output synchronization follows immediately from L_2-synchronization, since $\sum_{t=0}^\infty |y_i(t) - y_j(t)|^2 < \infty$ implies $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$. In statement 3, the outputs are bounded in view of local boundedness of h_i. ■

Discussion and remarks

Note that when $R_i = 0$ (the passive case), condition (7) is satisfied for any $K > 0$ and any coupling matrix A, and Theorem 1 reduces to [13, Theorem 8.3]. A well-known special case is consensus of single integrators ($\dot{x}_j = u_j$, $y_j = x_j$) under diffusive coupling [61]. As noted earlier, discrete-time systems are generally not passive, which underscores the importance of the IFP framework in the discrete-time setting.

The small gain condition (7) can be fulfilled in two ways. On the one hand, one can choose the matrix gain K small, e.g., $K = kI$ with $k > 0$ sufficiently small. This approach is common in the physics literature, where synchronization is controlled via a single scalar coupling parameter [60]. On the other hand, one can fix K (e.g., use the standard diffusive coupling $K = I$) and choose the edge weights so that the weighted degrees $d_i^+[A]$ are sufficiently small. The small-gain condition (7) *cannot*, in general, be omitted, as the following example demonstrates.

Example 3. For any $\beta, \gamma > 0$, the system

$$\dddot{y}_j(t) + \beta \ddot{y}_j(t) + \gamma \dot{y}_j(t) = u_j(t) \in \mathbb{R}, \quad t \geq 0, \tag{10}$$

has transfer function $W(s) = 1/(s(s^2 + \beta s + \gamma))$ and is IFP(α^*), where

$$\alpha^* = -\inf_{\omega \neq 0} Re\, W(\imath\omega) = \sup_{\omega \neq 0} \frac{\beta}{\beta^2\omega^2 + (\gamma - \omega^2)^2} = \begin{cases} \frac{4}{\beta(4\gamma - \beta^2)} & \text{if } \gamma > \beta^2/2, \\ \frac{\beta}{\gamma^2} & \text{if } \gamma \leq \beta^2/2. \end{cases}$$

Applying protocol (4) with $K = 1$ and all-to-all coupling $a_{ij} = a > 0$ for all $i \neq j$ to a group of identical agents (10), the only nonzero eigenvalue of the Laplacian matrix is $\lambda = a(N-1)$. The output synchronization is thus achieved if and only if the feedback $u_i = -a(N-1)y_i$ stabilizes the agent's dynamics [42], that is, the polynomial $s^3 + \beta s^2 + \gamma s + a(N-1)$ is Hurwitz. By the Routh–Hurwitz criterion, this means that $a(N-1) < \beta\gamma$. In particular, as the network size N is growing, the coupling strength must decay as $O(N^{-1})$.

It is interesting to compare this necessary and sufficient condition with Theorem 1. Being applied in the case where $\gamma > \beta^2/2$, $R_i = \alpha^*$, $K = 1$ and $d_i^+ = (N-1)a$, the condition (7) results in

$$a(N-1) < \frac{\beta(4\gamma - \beta^2)}{8} = \frac{\beta\gamma}{2} - \frac{\beta^3}{8} < \beta\gamma.$$

This conservatism is not surprising: Theorem 1 guarantees synchronization under *any* strongly connected graph satisfying (7), so some loss of sharpness on specific graphs is to be expected. Moreover, Theorem 1 accommodates heterogeneous agents, whereas the Laplacian modal decomposition [42] is applicable only to identical systems.

An interesting phenomenon is that, for heterogeneous agents, diffusive coupling does not always lead to synchronization when the graph has a directed spanning tree but is not strongly connected,[5] as the example demonstrates.

Example 4. Consider a pair ($N = 2$) of harmonic oscillators $\ddot{\xi}_1 + \omega_1^2\xi_1 = u_1$ and $\ddot{\xi}_2 + \omega_2^2\xi_2 = u_2$ of different frequencies $\omega_1 \neq \omega_2$. These two systems are passive with respect to the inputs u_i and the outputs $y_i = \dot{\xi}_i$. Consider the master–slave coupling protocol $u_1 = a(\dot{\xi}_2 - \dot{\xi}_1)$, $u_2 = 0$, corresponding to a graph with the single arc $2 \to 1$ (i.e., $a_{12} = a > 0$, $a_{21} = 0$). It can be shown that the system has a family of solutions $\xi_1(t) = Re[W(\imath\omega_2)ce^{\imath\omega_2 t}]$, $\xi_2 = Re[ce^{\imath\omega_2 t}]$, where $c \in \mathbb{C}$ and $W(s) \triangleq as/(s^2 + as + \omega_1^2)$, with corresponding outputs $y_i = \dot{\xi}_i$ obtained by differentiation. Since $|W(\imath\omega_2)| < 1$, the outputs are harmonics with the same frequency but different amplitudes $|W(\imath\omega_2)\omega_2 c|$ and $|\omega_2 c|$.

3.2. Stability in presence of damping terms

We now turn to a related problem: stabilization of a diffusively coupled network via *pinning control* [62], where local damping feedback is applied to a subset of agents (the "pinned" nodes) with the goal of driving all outputs to zero. In many applications, diffusive couplings do not arise from a designed control algorithm but from the natural physical interconnection between subsystems – for instance, diffusion of chemical substances across cell membranes or current flow between nodes in an electrical network. In such settings, the engineer has little or no freedom to modify the coupling, and the goal is not to achieve synchronization, but rather to stabilize the outputs at a prescribed value. The linear damping terms, added to the inputs of a subset of agents, provide actuation that achieves this goal: they anchor the network's output level to zero, while the diffusive coupling propagates this effect throughout the network.

[5] Under additional assumptions, synchronization over quasi-strongly connected graphs can be established by using some *dynamic* controller, e.g., via internal model control [51].

We thus consider the algorithm

$$u_i(t) = -b_i y_i(t) + K \sum_{j=1}^{N} a_{ij}(y_j(t) - y_i(t)), \quad i = 1, \dots, N, \tag{11}$$

which extends (4) by adding damping terms $-b_i y_i$ at the pinned nodes ($b_i > 0$), while $b_i = 0$ at the remaining nodes. In practice, one is often interested in using one or a few pinned nodes – for instance, in vehicle platoons or robot flocks, the velocity can be determined by the leading vehicle, while the remaining agents track this reference velocity through diffusive interactions with their neighbors. For simplicity, we assume throughout that $G[A]$ is strongly connected.

The output stability criterion is based on the following modification of Lemma 2.

Lemma 4. *Suppose system (Σ) is IFP(rI) with some $r \geq 0$ and storage function V. Let $b \in \left[0, \frac{1}{2r}\right)$ and define*

$$\hat{r} \overset{\Delta}{=} \frac{r}{1-2rb}, \qquad \delta \overset{\Delta}{=} \frac{b(1-rb)}{1-2rb} \geq 0, \qquad \hat{V}(x) \overset{\Delta}{=} \frac{1}{1-2rb} V(x).$$

Then (Σ) is IFP$(\hat{r})$ with respect to the same output y, the modified input $\hat{u} \overset{\Delta}{=} u + by$ and storage function $\hat{V}$. Moreover,

$$\hat{V}(x(T)) - \hat{V}(x(0)) \leq \int_0^T \left(y^\top \hat{u} + \hat{r}|\hat{u}|^2 - \delta|y|^2\right) dt \quad \forall T \geq 0 \tag{12}$$

with $\int_0^T (\cdot)\, dt$ is replaced by $\sum_{t=0}^{T-1}(\cdot)$ in the discrete-time case.

Proof. The result follows from the definition of IFP by noting that $\frac{1}{1-2rb}\left(y^\top u + r|u|^2\right) = y^\top \hat{u} + \hat{r}|\hat{u}|^2 - \delta|y|^2$. ■

The additional term $-\delta_i|y_i|^2$ in (12) provides the dissipation needed to establish output stability in place of synchronization. We formulate the corresponding modification of Theorem 1.

Theorem 2. *Suppose Assumptions 1 and 2 hold with $R_i = r_i I$, $G[A]$ is strongly connected, $b_i \geq 0$ for all i with $b_i > 0$ for at least one i, and*

$$2b_i I + 2d_i^+[A]\, K \prec R_i^{-1}, \qquad i = 1, \dots, N, \tag{13}$$

for some symmetric $K = K^\top \succ 0$. Then:

1. *In the discrete-time case, every solution of (3), (11) is output stable and output L_2-stable.*
2. *In the continuous-time case, every solution defined for all $t \geq 0$ is output L_2-stable.*
3. *If, in addition, Assumption 3 holds, then all variables (x_i, u_i, y_i) are bounded, every solution is defined for all $t \geq 0$, and all solutions are output L_2-stable and output stable.*

Proof. The proof follows that of Theorem 1. Condition (13) implies $b_i \in \left[0, \frac{1}{2r_i}\right)$, so introducing the modified inputs $\hat{u}_i \overset{\Delta}{=} u_i + b_i y_i$ and applying Lemma 4 replaces IFP$(r_i I)$ by IFP$(\hat{r}_i I)$ with $\hat{r}_i \overset{\Delta}{=} r_i/(1 - 2r_i b_i)$ and storage function $\hat{V}_i$. By construction, the modified inputs $\hat{u}_i$ satisfy (4), and condition (13) coincides with (7) upon replacing R_i by $\hat{r}_i I$; the proof thus reduces to that of Theorem 1 with updated passivity indices. Introducing the aggregate storage function $\hat{V}(X) = \sum_i p_i \hat{V}_i(x_i)$, the inequality (9) is replaced by

$$\hat{V}(X(T)) - \hat{V}(X(0)) \leq -\int_0^T \left[\sum_{i,j=1}^{N} \varepsilon a_{ij}(y_j - y_i)^\top K(y_j - y_i) + p_i \delta_i |y_i|^2\right] dt,$$

where the expression in brackets is a positive definite quadratic form in $(y_1, \dots, y_N)$, since $\mathcal{G}[A]$ is strongly connected and $p_i \delta_i > 0$ for at least one index i. This entails statement 2 (by passing to the limit as $T \to \infty$). The remainder of the proof follows that of Theorem 1, with synchronization replaced by stability throughout. ■

The proof of Theorem 2 in fact allows one to obtain explicit estimates of the L_2 norms of the outputs in terms of the initial state $X(0)$ and the system parameters; such estimates are standard in absolute stability theory [4].

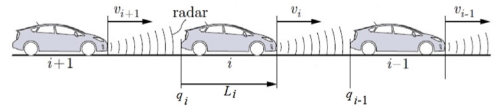

Fig. 2. Platoon of vehicles. Notation used in the text.

4. Synchronization in vehicle platooning

Cooperative adaptive cruise control (CACC) refers to a class of distributed algorithms enabling vehicles in a platoon to maintain prescribed inter-vehicle spacings and a common speed via vehicle-to-vehicle communication. The stability of CACC algorithms, naturally cast as an output synchronization and stability problem, is well understood for identical vehicle models: Laplacian modal decomposition [63] reduces the analysis to scalar subsystems indexed by the Laplacian eigenvalues and applies to a broad class of interconnection topologies. For heterogeneous platoons, no comparably general theory is available: stability depends on the individual vehicle models, the information topology, and the adopted spacing and speed policies. For predecessor–follower topologies, the stronger condition of *string stability* is typically imposed: the $\mathcal{H}_\infty$ gain from each vehicle's tracking error to its successor's does not exceed one. This is a conservative yet easily verifiable condition guaranteeing that disturbances do not amplify as they propagate downstream [64]. Bidirectional platoons, in which each vehicle reacts to both its predecessor and successor, are more challenging to analyze. Their disturbance attenuation capability often exceeds that of unidirectional architectures [65], whereas the stability margin can deteriorate as the platoon size increases, leading to slower convergence [66,67].

As an application of Theorem 2, we examine the stability of a heterogeneous vehicle platoon under a bidirectional-leader topology, in which each vehicle measures the relative positions of its predecessor and successor, and the leader's speed is broadcast to all vehicles. Following [66], we consider a linear control algorithm with a constant-spacing policy, here extended to vehicles with heterogeneous powertrain dynamics.

Mathematically, we consider a platoon consisting of a leading vehicle (indexed 0) and N follower vehicles indexed $i = 1, \dots, N$ (Fig. 2). The leader maintains a constant speed v_0, broadcast to all followers. Vehicles 1 through $N-1$ measure the distances to both their predecessors and successors, while the rear vehicle N measures only the distance to its predecessor. Denoting the rear bumper position of vehicle i by $q_i \in \mathbb{R}$ (Fig. 2), the goal of the CACC algorithm is to maintain the prescribed bumper-to-bumper spacings s_i and converge to the leader's speed:

$$q_{i-1}(t) - q_i(t) \xrightarrow[t\to\infty]{} s_i, \qquad v_i(t) = \dot{q}_i(t) \xrightarrow[t\to\infty]{} v_0. \tag{14}$$

As is standard in CACC modeling [63,64,68], each follower vehicle obeys the first-order powertrain lag model

$$\tau_i \dot{a}_i + a_i = a_{i,des}(t), \quad a_i = \ddot{q}_i, \tag{15}$$

where $a_{i,des}$ is the desired acceleration and $\tau_i > 0$ is a time constant characterizing the vehicle's powertrain.

The follower vehicles apply the control law [66]

$$\begin{aligned} a_{i,des}(t) &= \mu_i(v_0 - v_i(t)) + \eta_i(q_{i-1}(t) - q_i(t) - s_i) \\ &\quad + \nu_i(q_{i+1}(t) - q_i(t) + s_{i+1}), \quad i = 1, \dots, N-1, \\ a_{N,des}(t) &= \mu_N(v_0 - v_N(t)) + \eta_N(q_{N-1}(t) - q_N(t) - s_N), \end{aligned} \tag{16}$$

where $\mu_i, \eta_i, \nu_i > 0$ are the velocity, predecessor-spacing, and successor-spacing gains, respectively. The second equation in (16) reflects the fact that the rear vehicle N interacts only with its predecessor.

Defining the inputs $u_i(t) \overset{\Delta}{=} a_{i,des}(t) + \mu_i(v_i(t) - v_0)$ and outputs $y_i(t) \overset{\Delta}{=} q_i(t) + \sum_{k=1}^{i} s_k - q_0(t)$, the closed-loop system (15), (16) reduces to a network of agents

$$\tau_i \dddot{y}_i + \ddot{y}_i + \mu_i \dot{y}_i = u_i, \quad i = 1, \dots, N, \tag{17}$$

interconnected by the following algorithm:

$$\begin{aligned} u_1(t) &= -\eta_1 y_1(t) + \nu_1(y_2(t) - y_1(t)),\\ u_i(t) &= \eta_i(y_{i-1}(t) - y_i(t)) + \nu_i(y_{i+1}(t) - y_i(t)),\ 1 < i < N,\\ u_N(t) &= \eta_N(y_{N-1}(t) - y_N(t)). \end{aligned} \tag{18}$$

This is a special case of the protocol (11) with $K = 1$ (scalar outputs), $b_1 = \eta_1$, $b_i = 0$ for $i > 1$ and weighted degrees

$$d_1^+[A] = \nu_1, \quad d_N^+[A] = \eta_N, \quad d_i^+[A] = \eta_i + \nu_i\ \forall 1 < i < N. \tag{19}$$

The graph $G[A]$ is a bidirectional chain and is therefore strongly connected. Note that only the first node is pinned ($b_1 > 0$): while the leader's speed υ_0 is broadcast to all vehicles as a constant reference, only vehicle 1 measures the leader's position $q_0(t)$ and can enforce the spacing constraint $q_0(t) - q_1(t) \to s_1$.

We now state the convergence criterion for the CACC algorithm.

Theorem 3. *Suppose that for all $i = 1, \dots, N$ the inequalities hold: $\mu_i, \eta_i, \nu_i > 0$, $2\mu_i\tau_i \le 1$, and*

$$\mu_i^2 > 2\varpi_i, \quad \text{where} \quad \varpi_i \overset{\Delta}{=} b_i + d_i^+[A] = \begin{cases} \eta_i + \nu_i, & 1 \le i < N;\\ \eta_N, & i = N. \end{cases} \quad \forall i \tag{20}$$

Then the algorithm (16) ensures that (14) holds for every solution.

Proof. By Lemma 1, agent (17) is IFP(r_i) with

$$\begin{aligned} r_i &= -\inf_{\omega\neq 0} Re\, \frac{1}{\tau_i(\iota\omega)^3 + (\iota\omega)^2 + \mu_i(\iota\omega)}\\ &= \sup_{\omega\neq 0} \frac{1}{\mu_i^2 + (1 - 2\tau_i\mu_i)\omega^2 + \tau_i^2\omega^4} = \frac{1}{\mu_i^2}. \end{aligned}$$

where the last equality uses $1 - 2\tau_i\mu_i \ge 0$. Since (17) is a linear system, every solution is defined for all $t \ge 0$. In accordance with Theorem 2, the condition (20) is equivalent to

$$r_i^{-1} = \mu_i^2 > 2(b_i + d_i^+[A]) \quad \forall i = 1, \dots, N, \tag{21}$$

thus implying output L_2-stability. Since $u_i \in L_2[0, \infty)$ and $\dot{y}_i$ is the output of a stable linear system driven by u_i, Eq. (17) implies $\dot{y}_i \in L_2[0, \infty)$, hence $y_i^\top \dot{y}_i \in L_1[0, \infty)$. Therefore, the limit exists

$$\lim_{T\to\infty} |y_i(T)|^2 = |y_i(0)|^2 + 2\int_0^\infty y_i^\top \dot{y}_i\, dt < \infty.$$

Since $y_i \in L_2[0, \infty)$, we conclude $y_i(T) \to 0$ as $T \to \infty$ for each $i = 1, \dots, N$, and thus also $u_i(T) \to 0$ in view of (18). This proves the achievement of the CACC goal (14). ■

We note that Theorem 3 provides a sufficient condition for convergence, in contrast to the necessary and sufficient conditions available for homogeneous platoons [63,67]. At the same time, Theorem 3 accommodates heterogeneous vehicles in which not only the control gains but also the powertrain time constants τ_i may differ, and guarantees convergence for any finite N, unlike PDE-based results which are asymptotically valid as $N \to \infty$ [66,68]. We note that [66] considers double-integrator vehicle dynamics (corresponding to $\tau_i \to 0$) and focuses primarily on the stability margin as N grows, showing that "mistuning" – asymmetric coupling with $\eta_i \neq \nu_i$ – dramatically improves the margin compared to the symmetric case; this effect is further quantified in [67] for more general control architectures.

5. Conclusions and future work

In this paper, we have developed simple distributed protocols for output synchronization and stabilization of heterogeneous non-passive agents satisfying an input feedforward passivity (IFP) condition. The key insight is that diffusive coupling imposes a quadratic constraint on the agents' inputs and outputs – an observation that places the synchronization problem naturally within the framework of absolute stability theory, developed by Vladimir Andreevich Yakubovich. We have applied the results to cooperative adaptive cruise control (CACC) for heterogeneous vehicle platoons. Several directions for future work are worth pursuing. First, the proposed quadratic-constraint approach is not restricted to finite-dimensional dynamics: certain infinite-dimensional agents, such as delayed integrators [57], can also be treated within the same weak IFP framework. The results also extend to nonlinearly coupled networks, where the couplings satisfy anti-symmetry and sector conditions [13,33,69]. Second, we conjecture that Theorem 2 extends to the more general quasi-strongly connected case, where every node is either pinned or can reach a pinned node via a directed path, as in containment control and certain opinion dynamics models [70]. Third, the robustness of the synchronization criteria against measurement noise and communication delays is a subject of ongoing research. Fourth, a natural direction for future research is the extension of Lemma 2 to more general Lur'e-type models, where a sector-bounded nonlinearity in the feedback loop preserves the IFP property of the stable LTI part. This would enable the extension of Theorem 3 to nonlinear vehicle dynamics arising when the inner-loop linearization is imperfect.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] V. Rasvan, Popov theories and qualitative behavior of dynamic and control systems, Eur. J. Control 8 (2002) 190–199.
[2] V. Yakubovich, Popov's method and its subsequent development, Eur. J. Control 8 (2002) 200–208.
[3] A. Megretski, A. Rantzer, System analysis via integral quadratic constraints, IEEE Trans. Autom. Control 42 (1997) 819–830.
[4] V. Yakubovich, Necessity in quadratic criterion for absolute stability, Internat. J. Robust Nonlinear Control 10 (2000) 889–907.
[5] D. Altshuller, Frequency Domain Criteria for Absolute Stability, in: Lecture Notes in Control and Information Sciences, Springer London, 2012.
[6] A.I. Lur'e, V.N. Postnikov, On the theory of stability of control systems, Prikl. Mat. Mekh. 8 (1944) 246–248, In Russian.
[7] A. Lur'e, On Some Nonlinear Problems in the Theory of Automatic Control, H.M. Station. Office, London, 1957.
[8] A. Gelig, G. Leonov, V. Yakubovich, Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities, World Scientific Publishing Co., 2004.
[9] G.A. Leonov, I. Burkin, A.I. Shepeljavyi, Frequency Methods in Oscillation Theory, Springer Netherlands, 1996.
[10] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge Univ. Press, New York, 2004.
[11] J. Willems, Dissipative dynamical systems (in 2 parts), Arch. Ration. Mech. Anal. 45 (1972) 321–393.
[12] A. van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, Springer, 2000.
[13] T. Hatanaka, N. Chopra, M. Fujita, M. Spong, Passivity-Based Control and Estimation in Networked Robotics, Springer, 2015.
[14] M. Fazlyab, M. Morari, G.J. Pappas, Safety verification and robustness analysis of neural networks via quadratic constraints and semidefinite programming, IEEE Trans. Autom. Control 67 (2022) 1–15.
[15] A. Davydov, A.V. Proskurnikov, F. Bullo, Non-euclidean contraction analysis of continuous-time neural networks, IEEE Trans. Autom. Control 70 (2025) 235–250.
[16] W. D'Amico, A. La Bella, M. Farina, An incremental input-to-state stability condition for a class of recurrent neural networks, IEEE Trans. Autom. Control 69 (2024) 2221–2236, http://dx.doi.org/10.1109/TAC.2023.3327937.
[17] J.K. Hale, Diffusive coupling, dissipation, and synchronization, J. Dynam. Differential Equations 9 (1997) 1–52.

[18] F. Bullo, Lectures on Network Systems, 2016, published online at http://motion.me.ucsb.edu/book-lns. With contributions by J. Cortes, F. Dorfler, and S. Martinez.
[19] U. Jönsson, C. Kao, H. Fujioka, A Popov criterion for networked systems, Systems Control Lett. 56 (2007) 603–610.
[20] S.A. Plotnikov, A.L. Fradkov, Synchronization of nonlinearly coupled networks based on circle criterion, Chaos 31 (2021) 103110.
[21] S.Z. Khong, E. Lovisari, A. Rantzer, A unifying framework for robust synchronization of heterogeneous networks via integral quadratic constraints, IEEE Trans. Autom. Control 61 (2016) 1297–1309.
[22] Z. Tang, J.H. Park, H. Shen, Finite-time cluster synchronization of Lur'e networks: A nonsmooth approach, IEEE Trans. Syst. Man Cybern.: Syst. 48 (2018) 1213–1224.
[23] B. Wei, F. Xiao, Y. Shi, Fully distributed synchronization of dynamic networked systems with adaptive nonlinear couplings, IEEE Trans. Cybern. 50 (2020) 2926–2934.
[24] F. Zhang, G. Wen, A. Zemouche, W. Yu, J.H. Park, Output-feedback self-synchronization of directed Lur'e networks via global connectivity, IEEE Trans. Cybern. 52 (2022) 6490–6503.
[25] F. Zhang, H. Trentelman, J. Scherpen, Fully distributed robust synchronization of networked Lur'e systems with incremental nonlinearities, Automatica 50 (2014) 2515–2526.
[26] F. Zhang, H.L. Trentelman, J.M.A. Scherpen, Dynamic feedback synchronization of Lur'e networks via incremental sector boundedness, IEEE Trans. Autom. Control 61 (2016) 2579–2584.
[27] F. Zhang, H.L. Trentelman, J.M.A. Scherpen, Robust cooperative output regulation of heterogeneous Lur'e networks, Internat. J. Robust Nonlinear Control 27 (2017) 3061–3078.
[28] F. Zhang, Y. Li, W. Xia, T. Liu, W. Yu, Synchronization of Lur'e networks via heterogeneous unknown interconnections, IEEE Trans. Circuits Syst. II: Express Briefs 71 (2024) 807–811.
[29] N. Chopra, M.W. Spong, On exponential synchronization of Kuramoto oscillators, IEEE Trans. Autom. Control 54 (2009) 353–357.
[30] A. Savkin, T. Cheng, Z. Li, F. Javed, A. Matveev, H. Nguyen, Decentralized Coverage Control Problems For Mobile Robotic Sensor and Actuator Networks, John Wiley & Sons, 2015.
[31] A. Proskurnikov, Nonlinear consensus algorithms with uncertain couplings, Asian J. Control 16 (2014) 1277–1288.
[32] A. Proskurnikov, Consensus in switching networks with sectorial nonlinear couplings: Absolute stability approach, Automatica 49 (2013) 488–495.
[33] A. Proskurnikov, A. Matveev, Popov-type criterion for consensus in nonlinearly coupled networks, IEEE Trans. Cybern. 45 (2015) 1537–1548.
[34] N. Chopra, M. Spong, Passivity-based control of multi-agent systems, in: S. Kawamura, M. Svinin (Eds.), Advances in Robot Control, Springer-Verlag, Berlin, 2006, pp. 107–134.
[35] M. Arcak, Passivity as a design tool for group coordination, IEEE Trans. Autom. Control 52 (2007) 1380–1390.
[36] A. Pogromsky, H. Nijmeijer, Cooperative oscillatory behavior of mutually coupled dynamical systems, IEEE Trans. Circuits Syst. - I 48 (2001) 152–162.
[37] A. Lazri, E. Panteley, A. Loría, Analysis and control of multi-time-scale modular directed heterogeneous networks, IEEE Trans. Control Netw. Syst. 12 (2025) 661–672.
[38] S. Dashkovskiy, H. Ito, F. Wirth, On a small gain theorem for iss networks in dissipative lyapunov form, Eur. J. Control 17 (2011) 357–365.
[39] C. Kawan, A. Mironchenko, M. Zamani, A Lyapunov-based iss small-gain theorem for infinite networks of nonlinear systems, IEEE Trans. Autom. Control 68 (2023) 1447–1462.
[40] L. Scardovi, R. Sepulchre, Synchronization in networks of identical linear systems, Automatica 45 (2009) 2557–2562.
[41] F. Lewis, H. Zhang, K. Hengster-Movris, A. Das, Cooperative Control of Multi-Agent Systems. Optimal and Adaptive Design Approaches, Springer-Verlag, London, 2014.
[42] Z. Li, Z. Duan, G. Chen, L. Huang, Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint, IEEE Trans. Circuits Syst. - I 57 (2010) 213–224.
[43] P. DeLellis, M. di Bernardo, G. Russo, On QUAD, Lipschitz, and contracting vector fields for consensus and synchronization of networks, IEEE Trans. Circuit Syst. - I 58 (2011) 576–583.
[44] A. Proskurnikov, M. Cao, Synchronization of Goodwin's oscillators under boundedness and nonnegativeness constraints for solutions, IEEE Trans. Autom. Control 62 (2017) 372–378.
[45] P. Wieland, R. Sepulchre, F. Allgöwer, An internal model principle is necessary and sufficient for linear output synchronization, Automatica 47 (2011) 1068–1074.
[46] Y. Yan, Z. Chen, R.H. Middleton, Autonomous synchronization of heterogeneous multiagent systems, IEEE Trans. Control Netw. Syst. 8 (2021) 940–950.
[47] G. Fattore, M.E. Valcher, Distributed output synchronization of discrete-time multi-agent systems: A data-driven approach, Automatica 185 (2026) 112789.
[48] Y. Cheng, C. Li, C. Song, S. Xu, Output synchronization of heterogeneous multi-agent systems based on the distributed continuous-discrete state observer, Automatica 158 (2023) 111283.
[49] Z. Liu, D. Nojavanzadeh, A. Saberi, A.A. Stoorvogel, Scale-free collaborative protocol design for output synchronization of heterogeneous multi-agent systems with nonuniform communication delays, IEEE Trans. Netw. Sci. Eng. 9 (2022) 2882–2894.
[50] C. De Persis, B. Jayawardhana, On the internal model principle in the coordination of nonlinear systems, IEEE Trans. Control Netw. Syst. 1 (2014) 272–282.
[51] A. Isidori, L. Marconi, G. Casadei, Robust output synchronization of a network of heterogeneous nonlinear agents via nonlinear regulation theory, IEEE Trans. Autom. Control 59 (2014) 2680–2691.
[52] Y. Tian, Y. Zhang, High-order consensus of heterogeneous multi-agent systems with unknown communication delays, Automatica 48 (2012) 1205–1212.
[53] D. Wang, W. Chen, L. Qiu, Synchronization of diverse agents via phase analysis, Automatica 159 (2024) 111325.
[54] A. Fradkov, Shunt output feedback adaptive controllers for nonlinear plants, IFAC Proc. Vol. 29 (1996) 5304–5309.
[55] L. Torres, J. Hespanha, J. Moehlis, Synchronization of identical oscillators coupled through a symmetric network with dynamics: A constructive approach with applications to parallel operation of inverters, IEEE Trans. Autom. Control 60 (2015) 3226–3241.
[56] Z. Liu, M. Zhang, A. Saberi, A.A. Stoorvogel, State synchronization of multi-agent systems via static or adaptive nonlinear dynamic protocols, Automatica 95 (2018) 316–327.
[57] A.V. Proskurnikov, M. Mazo, Simple synchronization protocols for heterogeneous networks: beyond passivity, IFAC-PapersOnLine 50 (2017) 9426–9431, 20th IFAC World Congress.
[58] H. Khalil, Nonlinear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1996.
[59] A. Fradkov, Passification of non-square linear systems, Eur. J. Control (2003) 573–581.
[60] L. Pecora, T. Carroll, Master stability functions for synchronized coupled systems, Phys. Rev. Lett. 80 (1998) 2109–2112.
[61] W. Ren, R. Beard, Distributed Consensus in Multi-Vehicle Cooperative Control: Theory and Applications, Springer-Verlag, London, 2008.
[62] X. Li, X. Wang, G. Chen, Pinning a complex dynamical network to its equilibrium, IEEE Trans. Circuits Syst. I 51 (2004) 2074–2087.
[63] Y. Zheng, S.E. Li, J. Wang, D. Cao, K. Li, Stability and scalability of homogeneous vehicular platoon: Study on the influence of information flow topologies, IEEE Trans. Intell. Transp. Syst. 17 (2016) 14–26.
[64] S. Öncü, J. Ploeg, N. van de Wouw, H. Nijmeijer, Cooperative adaptive cruise control: Network-aware analysis of string stability, IEEE Trans. Intell. Transp. Syst. 15 (2014) 1527–1537.
[65] S. Santini, A. Salvi, A.S. Valente, A. Pescapé, M. Segata, R. Lo Cigno, A consensus-based approach for platooning with intervehicular communications and its validation in realistic scenarios, IEEE Trans. Veh. Technol. 66 (2017) 1985–1999.
[66] P. Barooah, P. Mehta, J. Hespanha, Mistuning-based control design to improve closed-loop stability margin of vehicular platoons, IEEE Trans. Autom. Control 54 (2009) 2100–2113.
[67] I. Herman, S. Knorn, A. Ahlén, Disturbance scaling in bidirectional vehicle platoons with different asymmetry in position and velocity coupling, Automatica 82 (2017) 13–20.
[68] A. Ghasemi, R. Kazemi, S. Azadi, Stability analysis of bidirectional adaptive cruise control with asymmetric information flow, Proc. Inst. Mech. Eng. C 229 (2015) 216–226.
[69] T. Liu, D. Hill, J. Zhao, Output synchronization of dynamical networks with incrementally-dissipative nodes and switching topology, IEEE Trans. Circuits Syst. I 62 (2015) 2312–2323.
[70] A. Proskurnikov, R. Tempo, M. Cao, N. Friedkin, Opinion evolution in time-varying social influence networks with prejudiced agents, IFAC PapersOnline 50 (2017) 11896–11901.