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Geometric properties of the Bismut and the Obata connections

Beatrice Brienza

Supervisor: Prof. Anna Fino,
Co-Supervisor: Prof. Gueo Grantcharov

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Introduzione

Among the milestones of modern Riemannian geometry, few results have had as profound an impact as Berger's classification of holonomy groups [Ber]. At first glance, the notion of holonomy appears almost elementary: one follows a vector along a loop using parallel transport with respect to the Levi-Civita connection, and records how the vector changes upon returning to the starting point. The collection of all such transformations forms a Lie group, the *Riemannian holonomy group*.

What Berger's theorem reveals is that behind this seemingly simple construction lies a deep structural principle. For simply connected manifolds, once the locally symmetric cases studied by Cartan [Car, Car2] and the product manifolds are set aside, the possible holonomy groups are extraordinarily restricted: they fall into a short, elegant, and now classical list.

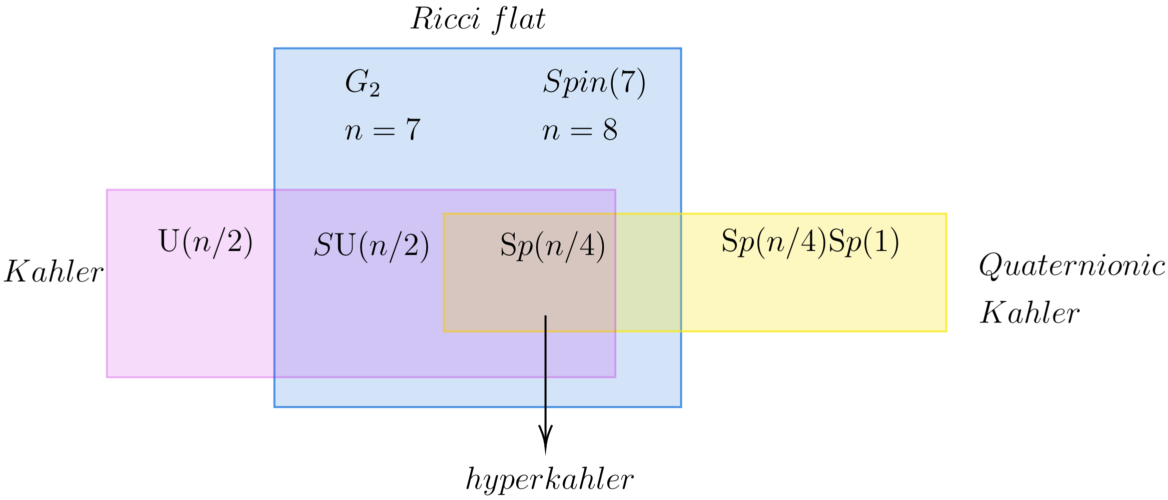
The strength of this classification lies not only in its completeness, but also in the way it links symmetry, geometry, and topology. Holonomy, an object defined purely in terms of connections and loops, governs the existence of special geometric structures and distinguishes metrics of exceptional type. In this sense, Berger's work transformed holonomy from a technical tool into a central organizing principle in Riemannian geometry.

Theorem A ([Ber])**.** *Let M be an oriented, simply connected, n-dimensional Riemannian manifold which is neither locally a product nor locally symmetric. Then its holonomy group must be one of the following:*

$$\mathrm{SO}(n), \quad \mathrm{U}\left(\tfrac{n}{2}\right), \quad \mathrm{SU}\left(\tfrac{n}{2}\right), \quad \mathrm{Sp}\left(\tfrac{n}{4}\right)\mathrm{Sp}(1), \quad \mathrm{G}_2, \quad Spin(7).$$

The original Berger list also included the exceptional group $Spin(9)$, but it was later shown by [Al2] that a manifold with Riemannian holonomy $Spin(9)$ is necessarily a Riemannian symmetric space. When Berger compiled his list, he was unaware of the existence of manifolds with holonomy G_2 and $Spin(7)$; such examples were discovered roughly thirty years later by Bryant and Salamon [BS].

This list is not merely a collection of groups, but also specifies their representations: the action of each group on the tangent space is completely determined. Consequently, each group apart from $\mathrm{SO}(n)$ gives rise to a special geometric structure. These structures can be grouped into three main categories: Kähler manifolds (and hence complex), quaternionic manifolds (necessarily of real dimension a multiple of 4), and manifolds with vanishing Ricci tensor. Strictly speaking, each category is characterized by the inclusion of the holonomy group in one of the indicated groups, and the intersection of these three categories defines the distinguished class of *hyperkähler manifolds*.



Building on Berger's seminal result, two natural and intriguing directions for generalization emerge. Hano and Ozeki [HO] showed that any closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ can occur as the holonomy group of a linear connection, giving rise to a vast array of possibilities. Consequently, since the Levi-Civita connection is both torsion-free and metric, it is more reasonable to focus on two more specific directions.

In the first direction, one may relax the metric requirement, whereas in the second, the torsion-free condition is removed.

Berger's original classification was, in fact, more extensive: in [Ber], he described all irreducible reductive representations that can appear as the holonomy of a torsion-free affine connection, not necessarily arising from a Riemannian metric. He claimed this list to be complete, up to possibly a finite number of missing entries. Subsequent work by several authors [CMS, CMS2, CM] refined these results, and today the most comprehensive classification is due to Merkulov and Schwachhöfer [MS2].

For a hypercomplex manifold, the Obata connection is the unique torsion-free connection preserving the quaternionic structure. By the holonomy principle, its holonomy group is thus contained in $\mathrm{GL}(n, \mathbb{H})$, where n denotes the quaternionic dimension of the manifold.

The Obata connection provides a powerful tool for studying hypercomplex geometry; for instance, a "quaternionic" analogue of the Newlander-Nirenberg theorem generally fails. A celebrated theorem of Obata [Ob] states that a hypercomplex manifold is locally isomorphic to $\mathbb{H}^n$ with its standard hypercomplex structure if and only if the Obata connection is flat, i.e., its restricted holonomy is trivial.

According to Merkulov and Schwachhöfer [MS2], only three subgroups of $\mathrm{GL}(n, \mathbb{H})$

can appear as holonomy groups: $\mathrm{Sp}(n)$, $\mathrm{SL}(n,\mathbb{H})$, and $\mathrm{GL}(n,\mathbb{H})$ itself. The first corresponds to the hyperkähler case, where the Obata connection coincides with the Levi-Civita connection of a hyper-Hermitian metric.

The second case, $\mathrm{SL}(n,\mathbb{H})$, arises for instance in nilmanifolds with invariant hypercomplex structures [BDV]. Nilmanifolds have long been central in differential geometry and topology, as they balance algebraic tractability with rich geometric behavior. Invariant complex structures on nilmanifolds, studied extensively following Salamon [Sal2], exhibit remarkable properties: for example, the existence of a Salamon basis [Sal2, Theorem 1.3], and the fact that the subspace $[\mathfrak{g},\mathfrak{g}] + J[\mathfrak{g},\mathfrak{g}]$ is always a proper ideal of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ [Sal2, Corollary 1.4].

In the hypercomplex setting, many of these properties fail to extend uniformly. Each complex structure J_α ($\alpha = 1,2,3$) may individually satisfy compatibility conditions, but their joint behavior is more rigid. For instance, it remains open whether the subspace

$$\mathfrak{g}_1^{\mathbb{H}} := [\mathfrak{g},\mathfrak{g}] + \sum_{\alpha=1}^{3} J_\alpha[\mathfrak{g},\mathfrak{g}]$$

is a proper ideal of $\mathfrak{g}$, as conjectured in [Gor1, Gor2].

In this thesis we focus on left-invariant hypercomplex structures on 2-step nilpotent Lie groups. Previously known examples in this class are all Obata-flat, naturally raising the question: *Do non-flat 2-step hypercomplex nilpotent structures exist?* We answer this affirmatively, showing that for non-flat examples, each J_α must be 3-step nilpotent [CFGU], thus providing the first known non-flat 2-step hypercomplex nilpotent Lie algebras. We also construct new families of higher-step hypercomplex nilpotent nilmanifolds with non-flat Obata connection.

Moreover, we investigate the Obata holonomy in this setting. For 2-step nilpotent groups, the holonomy algebra is always abelian, and the proof relies on establishing the Gorginyan conjecture in this case. Remarkably, even when the Obata connection is non-flat, the holonomy remains highly constrained, revealing an unexpected rigidity in hypercomplex nilmanifolds and refining the classical result of Barberis–Dotti–Verbitsky [BDV].

Finally, the full-holonomy case $\mathrm{GL}(n,\mathbb{H})$ is particularly intriguing due to the scarcity of examples. The first known example, constructed by Soldatenkov [Sol] on $\mathrm{SU}(3)$, remained unique for nearly a decade. In this thesis, we will present a construction of an hypercomplex structure on $\mathrm{SU}(5)$ with full holonomy.

In both of the aforementioned examples, the hypercomplex structure is left-invariant, and therefore is a Joyce hypercomplex structure [Joy].

Left-invariant hypercomplex structures on Lie groups have appeared in the context of string theory [SSTVP] and, more mathematically, in the seminal work of Joyce [Joy].

Joyce showed that if G is a compact semisimple Lie group of rank r, then the compact Lie group $\mathbb{T}^{2m-r} \times G$ admits, up to an m^2-parameter family, non-equivalent left-invariant hypercomplex structures, where $m \in \mathbb{N}$ is the number of $\mathfrak{su}(2)$-factors in the so-called Joyce decomposition of G.

Subsequently, it was shown in [DT, BGP] that every left-invariant hypercomplex structure on a compact Lie group arises via Joyce's construction. Hence, throughout this thesis, a compact Lie group endowed with such a structure will be referred to as a *Joyce hypercomplex manifold*.

The list of Joyce hypercomplex manifolds $\mathbb{T}^{2m-r} \times G$ with G simple was provided in [SSTVP]:

$$\begin{aligned} &\mathrm{SU}(2k+1), \quad S^1 \times \mathrm{SU}(2k), \quad \mathbb{T}^k \times \mathrm{SO}(2k+1), \\ &\mathbb{T}^k \times \mathrm{Sp}(k), \quad \mathbb{T}^{2k} \times \mathrm{SO}(4k), \quad \mathbb{T}^{2k-1} \times \mathrm{SO}(4k+2), \\ &\mathbb{T}^2 \times \mathrm{E}_6, \quad \mathbb{T}^7 \times \mathrm{E}_7, \quad \mathbb{T}^8 \times \mathrm{E}_8, \quad \mathbb{T}^4 \times \mathrm{F}_4, \quad \mathbb{T}^2 \times \mathrm{G}_2. \end{aligned} \tag{0.0.1}$$

A central question in hypercomplex geometry is determining the holonomy of the associated Obata connection. Beyond low-dimensional cases, very little is known.

In quaternionic dimension one, the only Joyce hypercomplex manifold is the Hopf surface $S^1 \times \mathrm{SU}(2)$, for which the Obata connection is flat, with holonomy group $\mathbb{Z}$ [SV].

In quaternionic dimension two, the situation is richer. Soldatenkov [Sol] showed that the Obata holonomy on $\mathrm{SU}(3)$ is *equal* to $\mathrm{GL}(2, \mathbb{H})$. It was subsequently conjectured (e.g., [SV]) that, except for the Hopf surface, the Obata holonomy of any Joyce hypercomplex manifold in (0.0.1) is always full, i.e., equal to $\mathrm{GL}(n, \mathbb{H})$. Even when M is a product in (0.0.1), the Joyce hypercomplex structure is not a product, so the Obata holonomy need not split.

One of the main results of this thesis addresses this conjecture and provides a more nuanced picture:

Theorem B. *Let M be a compact Lie group from the list (0.0.1), except $\mathrm{SU}(2n+1)$, and let ∇^{Ob} denote the Obata connection of a left-invariant hypercomplex structure on M. Then the Obata holonomy is a proper subgroup of $\mathrm{GL}(n, \mathbb{H})$, i.e., it is strictly contained in $\mathrm{GL}(n, \mathbb{H})$.*

Every left-invariant hypercomplex structure on M arises via Joyce's construction, and Theorem B applies to *all* such structures, even though a complete determination of the holonomy algebra remains challenging. For instance, we explicitly compute it for $\mathbb{T}^2 \times \mathrm{Sp}(2)$.

The case of $\mathrm{SU}(2n+1)$ is more subtle. For $n \geq 2$, we prove that there exist many Joyce hypercomplex structures with Obata holonomy strictly contained in $\mathrm{GL}(n(n+1), \mathbb{H})$, however, there appear to also exist hypercomplex structures for which the holonomy group is full; we provide such an example on $\mathrm{SU}(5)$. We conjecture that numerous additional examples of this kind exist. This dichotomy does not occur for $\mathrm{SU}(3)$, whose one-parameter family of Joyce hypercomplex structures all have full holonomy [Sol].

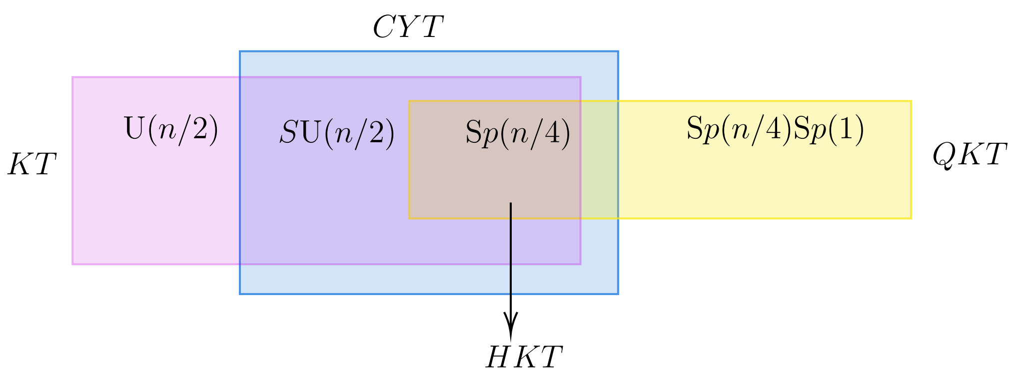
The second natural generalization consists in replacing the Levi-Civita connection with a metric connection that allows for torsion. Any such connection can be written in the form

$$g(\nabla_X Y, Z) = g(\nabla^{\mathrm{LC}}_X Y, Z) + A(X, Y, Z),$$

where A is a $(0,3)$-tensor satisfying $A(X, Y, Z) + A(X, Z, Y) = 0$. In the special case where A is a 3-form, which is the case of our interest, the geodesics of ∇ coincide with those of the Levi-Civita connection.

The holonomy of metric connections with skew torsion exhibits a rich variety of possibilities, including the groups $\mathrm{U}(n/2)$, $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)\mathrm{Sp}(1)$, and $\mathrm{Sp}(n/4)$. Just as in the Riemannian case, each of these groups give rise to a natural class of geometries. These can be organized into three broad families: Kähler with torsion (KT) manifolds, which are Hermitian; quaternionic Kähler with torsion (QKT) manifolds, whose dimension is a multiple of four; and Calabi–Yau with torsion (CYT) manifolds, characterized by a suitable generalization of Ricci-flatness. The intersection of these classes defines hyperkähler with torsion (HKT) manifolds. Observe that when the torsion vanishes, the connection ∇ coincides with the Levi-Civita, and we recover (some of) the groups in the Berger's list.

In this thesis, our attention will focus on KT, CYT, and HKT geometries.



Kähler geometry lies at the intersection of three fundamental geometries: complex, symplectic, and Riemannian.

On compact manifolds, this condition imposes strong topological constraints:

1. odd Betti numbers are even,

2. even Betti numbers are positive,

3. the minimal model is formal in the sense of Sullivan,

4. the Hard Lefschetz condition holds, linking the De Rham cohomology with the symplectic structure.

From the differential-geometric viewpoint, a Kähler structure implies that the Levi-Civita connection preserves the complex structure, $\nabla^{\mathrm{LC}} J = 0$. When $d\omega \neq 0$, this is no longer true, so the Levi-Civita connection is insufficient to encode the Hermitian geometry. This motivates the study of *Hermitian connections*, which preserve both g and J.

On any Hermitian manifold (M, J, g), there exists an affine line of Hermitian connections, known as *Gauduchon* or *canonical connections* [Gau], given by

$$g(\nabla^t_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{t-1}{4}(d^c\omega)(X, Y, Z) + \frac{t+1}{4}(d^c\omega)(X, JY, JZ), \qquad (0.0.2)$$

where $d^c\omega = J d\omega$ and $t \in \mathbb{R}$. When (M, J, g) is Kähler, $d^c\omega = 0$, and the line collapses to the Levi-Civita connection. Otherwise, the connections ∇^t have non-vanishing torsion, leading to the notion of *Hermitian geometry with torsion*.

Two notable instances are the *Chern connection* $\nabla^1 = \nabla^{\mathrm{Ch}}$ and the *Bismut connection* $\nabla^{-1} = \nabla^B$ [Bis]. The Chern connection is closely adapted to the complex geometry, while the Bismut connection preserves the geodesics of the Levi-Civita connection. Its torsion $H = -d^c\omega$ is a 3-form, called the *Bismut torsion*. A Hermitian manifold (M, g, J, ∇^B) is said a Kähler with torsion manifold.

If $dd^c\omega = 0$, the Hermitian structure is called *strong Kähler with torsion* (SKT) or *pluriclosed*.

The Bismut connection is Hermitian, $\nabla^B J = \nabla^B g = 0$, so its holonomy is contained in $\mathrm{U}(n)$. If the (restricted) holonomy reduces to $\mathrm{SU}(n)$, the structure is *Calabi–Yau with torsion* (CYT), and the corresponding Ricci-type form vanishes:

$$\rho^B(X, Y) = \frac{1}{2}\sum_{i=1}^{2n} R^B(X, Y, Je_i, e_i) = 0.$$

CYT structures that are also SKT are sometimes called *Bismut-Hermitian Einstein* (BHE). Unlike the Kähler Ricci-flat case, vanishing first Chern class $c_1(M) = 0$ does not guarantee the existence of a BHE metric [GFJS], highlighting the rigidity of these geometries.

At present, the known examples of compact Bismut–Hermitian Einstein (BHE) manifolds include Bismut-flat manifolds, Kähler Ricci-flat manifolds, and their products. Bismut-flat manifolds have been completely classified [WYZ], and compact BHE manifolds in low dimensions have been partially understood. In particular, in dimension 4, the only compact complex surfaces admitting a BHE metric are either Bismut-flat or Kähler Ricci-flat [GI], while in dimension 6, some rigidity results have been recently established [ABLS].

A natural problem in this context is the construction of compact BHE manifolds that are neither flat nor Kähler and do not split as products. In this thesis, we provide examples in dimension 8, obtained via a suspension construction. The idea is to start with the product of a K3 surface and S^3, and then perform a suspension using the diagonal map (ψ, Id), where ψ is a Kähler isometry of a fixed Kähler Ricci flat structure on the K3 surface. One can then show that the resulting manifold admits a BHE structure.

Although the explicit expression of such metrics on a K3 surface is typically unknown, in this thesis we will show that it is possible to construct such isometries in concrete examples, for instance on the Fermat quartic.

Interestingly, these manifolds have *parallel Bismut torsion*, which motivates the study of compact BHE manifolds with this property. One of the main results of this thesis is the following theorem:

Theorem C. *Let (M, I) be a compact complex manifold admitting an SKT and CYT I-Hermitian metric h with parallel Bismut torsion H, i.e., $\nabla^B H = 0$. Then the Riemannian holomorphic universal cover $(\widetilde{M}, \widetilde{I}, \widetilde{h})$ of (M, I, h) is holomorphically isometric to a product $(M_1, J_1, g_1) \times (M_2, J_2, g_2)$, where (M_1, J_1, g_1) is Kähler Ricci-flat and (M_2, J_2, g_2) is a Samelson space.*

Moreover, using an argument *à la Beauville*, one can show, starting from this result, that any compact BHE manifold with parallel Bismut torsion admits a finite cover that splits as the product of a Kähler Ricci-flat manifold and a Bismut-flat one [BPT].

It remains an open and compelling question whether, in the compact case, the BHE condition itself implies $\nabla^B H = 0$. This conjecture is supported by the absence of known examples without parallel Bismut torsion and by the reduction of the holonomy.

Indeed, since any BHE manifold is Calabi–Yau with torsion (CYT), the holonomy of the Bismut connection is contained in $\mathrm{SU}(n)$. However, examples with parallel torsion cannot have full holonomy by Theorem C. Consequently, if the torsion is always parallel, one expects a general holonomy reduction phenomenon. In fact, we prove that any compact BHE manifold with full holonomy must be Kähler, while in the non-Kähler case the holonomy reduces further to $\mathrm{SU}(n-1)$. This result relies on the existence of a natural Bismut-parallel vector field, defined in terms of the Lee form and the *soliton potential* [GFJS].

Given a hyperhermitian manifold (J_α, g), we say it is HKT if the Bismut torsions H_α coincide. As a result, the Bismut connection is the same for the three Hermitian structures, namely $\nabla^B = \nabla^B_1 = \nabla^B_2 = \nabla^B_3$. By the holonomy principle, the holonomy of ∇^B is inside $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, and therefore each Hermitian structure is CYT.

A strong HKT structure is one for which this common Bismut torsion is closed. It is then immediate that each of the three Hermitian structures in a strong HKT manifold is BHE. This observation has drawn considerable interest, as it allows extending results from the BHE setting to the richer hyperhermitian framework, for instance Theorem C and the holonomy reduction.

In this thesis we mainly focus on dimension 8, where compact simply connected strong HKT manifolds exhibit a particularly rigid geometric structure: they admit an integrable distribution corresponding to a locally free isometric $\mathbb{H}^*$-action. In particular, they have constant positive scalar curvature, and they admit an *Euler vector field*, namely a vector field whose Obata connection is the identity.

These results rely on the interplay among three connections: the Levi-Civita connection, encoding the Riemannian structure; the Obata connection, reflecting the hypercomplex structure; and the Bismut connection, capturing aspects of both. A full understanding of compact strong HKT manifolds requires combining the perspectives provided by all three connections.

SKT geometry, together with its various specializations such as strong HKT and BHE geometries, has become central in recent years, largely due to its role in type II string theory and supersymmetric σ-models. These metrics generalize the Kähler condition within non-Kähler geometry and naturally arise in physics, where they provide solutions compatible with torsionful fluxes [Str].

A fruitful setting for their study is that of locally homogeneous manifolds, where the *symmetrization process* [BE, FG, UG] allows one to reduce the existence problem to invariant structures on the Lie algebra. This reduces analytic difficulties, but the classification problem remains subtle. On nilpotent Lie algebras, existence is severely constrained: as shown in [ArNic, EnFV], a nilpotent Lie algebra carrying an SKT metric must be at most two-step nilpotent. Complete classifications in dimensions 6 and 8 were obtained in [FPS04, EnFV].

In contrast, the solvable non-nilpotent case is richer. While the classification is complete only in dimension 4 [MS], dimension 6 is almost fully understood: the nilradical must have dimension 5 (the *almost nilpotent* case) or 4 (the *codimension 2* case). The first was analyzed in detail in [FP, FP2, FP3, FS, FS2]; in this thesis we address the latter.

Recall that a generalized Kähler structure is given by a bi-Hermitian pair $(g, J_\pm)$ such that (g, J_+) is SKT and the corresponding Bismut torsions satisfy $H_{J_+} = -H_{J_-}$ [Gu]. Since such a structure automatically yields an SKT metric, a broader question concerns the existence of generalized Kähler structures on solvmanifolds. In the nilpotent case, Cavalcanti's rigidity theorem [Cav] shows that only tori admit such structures. For solvmanifolds, the situation is more flexible but still largely unexplored. In dimension 4, the only non-Kähler solvmanifold with a generalized Kähler structure is the Inoue surface [AG]. In dimension 6, examples were constructed in the almost abelian case by Fino and Paradiso [FP, FP2, FP3], extending earlier work [FTo]. All known examples admit an abelian nilradical, leading to the following conjecture:

Conjecture 1. *Any compact solvmanifold admitting an invariant generalized Kähler structure must have an abelian nilradical.*

The compactness assumption is essential: in the non-unimodular case, we will exhibit a counterexample.

The work of Apostolov–Gualtieri [AG] further analyzed the *ambihermitian* case, where the complex structures (J_+, J_-) induce opposite orientations. The only non-Kähler surfaces admitting such structures are the Inoue and Hopf surfaces, but only the Inoue surface admits (left) invariant generalized Kähler structures. Both surfaces satisfy $b_1 = 1$, so by Tischler's theorem [Ti] they can be realized as suspensions. More generally, every solvmanifold admits at least one invariant closed 1-form, hence *every solvmanifold can be topologically realized as a suspension*.

This observation motivates a general suspension construction for generalized Kähler manifolds. Starting from the description of the Inoue surface as the suspension of $\mathbb{T}^3$ via a diffeomorphism ρ, one considers the product $K \times \mathbb{T}^3$, where K is a (hyper)Kähler manifold, and realizes the suspension via the block-diagonal map (ψ, ρ), with ψ a (hyper)Kähler isometry of K. Depending on K, this yields split or non-split generalized Kähler structures.

All known almost abelian generalized Kähler solvmanifolds arise in this way, by taking $K = \mathbb{T}^{2k}$, and non-solvmanifold examples can arise for instance when K is a K3 surface.

An analogous construction can be performed on the Hopf surface by considering the suspension of the product $K \times S^3$, where K is a Kähler manifold. The suspension is realized via the block-diagonal map (ψ, Id), with ψ a Kähler isometry of K. The resulting manifold naturally carries two distinct, non-isomorphic generalized Kähler structures.

This thesis is organized as follows. In Chapter 1, we recall the necessary preliminaries which are essential to understand the constructions and results in the subsequent chapters. In Chapter 2, we complete the classification of 6-dimensional solvmanifolds admitting SKT and generalized Kähler structures, focusing on the codimension-2 case. Chapter 3 is devoted to the construction of new classes of generalized Kähler manifolds via the Inoue and Hopf surfaces, using suspension techniques, providing both split and non-split new examples. In Chapter 4, we apply a similar suspension construction to the Hopf surface to produce the first examples of non-trivial compact Bismut–Hermitian Einstein (BHE) manifolds. We also establish structural results, including a splitting theorem for BHE manifolds with parallel Bismut torsion and analogous results for strong HKT manifolds, particularly in dimension 8, where we describe the Hopf-fibration structure over 4-dimensional orbifolds. In Chapter 5, we investigate the holonomy of the Obata connection on Joyce hypercomplex manifolds, proving that for most compact Lie groups the Obata holonomy is a proper subgroup of $\mathrm{GL}(n, \mathbb{H})$, and providing explicit computations in concrete cases. Finally, in Chapter 6, we study hypercomplex structures on 2-step nilpotent Lie algebras, constructing the first non-flat examples and analyzing their Obata holonomy.

Chapter 1

Preliminaries

1.1 Suspensions and solvmanifolds

1.1.1 Suspensions

Throughout this thesis, we will always assume that all smooth manifolds are connected.

Definition 1.1.1. *Let M be a smooth manifold and $f : M \to M$ a diffeomorphism. The* suspension *(or* mapping torus*) of M via f is the smooth quotient manifold*

$$M_f := M \times \mathbb{R} \,/\, \mathbb{Z},$$

where $\mathbb{Z}$ acts on $M \times \mathbb{R}$ by

$$n \cdot (p, t) = (f^n(p), t + n), \quad n \in \mathbb{Z}.$$

Equivalently, M_f can be described as

$$M_f = \frac{M \times [0, 1]}{(p, 0) \sim (f(p), 1)}.$$

Example 1.1.2. The 2-dimensional torus $\mathbb{T}^2$ and the Klein bottle K can both be realized as mapping tori. Concretely, let $M = S^1$:

- If $f = \mathrm{Id}$, then $M_{\mathrm{Id}} \cong \mathbb{T}^2$.
- If f is the reflection $S^1 \to S^1$, then $M_f \cong K$.

A useful criterion for determining whether a compact smooth manifold is a suspension was given by Tischler [Ti]:

Theorem 1.1.3 ([Ti])**.** *Let M be a compact, connected smooth manifold. Then M admits a non-vanishing closed 1-form if and only if M is a mapping torus.*

The cohomology of a mapping torus M_f can be described in terms of the cohomology of M and the action induced by f on cohomology. More precisely, we have the following result [BFM]:

Theorem 1.1.4 ([BFM]). *Let M be a connected smooth manifold and $f : M \to M$ a diffeomorphism. Then, for each $r \geq 0$,*

$$\mathrm{H}^r(M_f) \cong \ker(f_r^* - \mathrm{Id}) \oplus \operatorname{coker}(f_{r-1}^* - \mathrm{Id}),$$

where $f_r^* : \mathrm{H}^r(M) \to \mathrm{H}^r(M)$ is the map induced by f in cohomology.

1.1.2 Nilmanifolds and solvmanifolds

Let $\mathfrak{g}$ be a Lie algebra. We say that $\mathfrak{g}$ is endowed with a *complex structure* J if $J : \mathfrak{g} \to \mathfrak{g}$ is a linear endomorphism satisfying $J^2 = -\mathrm{Id}$ and such that the *Nijenhuis tensor*

$$N_J(X,Y) := [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

vanishes for all $X, Y \in \mathfrak{g}$.

If G is a Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, then G carries an induced complex structure, also denoted by J, obtained via left translations. In this case, we say that G is endowed with an *invariant complex structure*, meaning that all left multiplications are holomorphic. If G admits a lattice $\Gamma \subset G$, then the quotient $\Gamma\backslash G$ naturally inherits a complex structure, which we will call again invariant, and the projection $G \to \Gamma\backslash G$ is holomorphic.

Assume now that $(\mathfrak{g}, J)$ is as above, and let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$. The pair $(J, \langle\cdot,\cdot\rangle)$ is said to be *Hermitian* if

$$\langle JX, JY\rangle = \langle X, Y\rangle \quad \text{for all } X, Y \in \mathfrak{g}.$$

By left translation, $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ induces an *invariant Hermitian structure* on G, meaning that left multiplications are holomorphic isometries. If G admits a lattice Γ, the quotient $\Gamma\backslash G$ inherits a Hermitian structure, which we will call invariant, such that the natural projection $G \to \Gamma\backslash G$ is a holomorphic isometry.

Lemma 1.1.5 ([Be, FG]). *Let $M = G/\Gamma$ be a compact quotient of a connected and simply connected Lie group G by a lattice Γ. Assume that M is endowed with an invariant complex structure J and a bi-invariant volume form $d\mu$. Due to the symmetrization process, there is a natural one-to-one correspondence*

$$\{\textit{Hermitian metrics on } M\} \longleftrightarrow \{\textit{Hermitian inner products on } \mathfrak{g}\}.$$

Proof. Let g be a Hermitian metric on M, and let $d\mu$ be a bi-invariant volume form on

M. We define a new left-invariant metric $\overline{g}$ on M by

$$\overline{g}(A, B) := \int_M g_m(A_m, B_m)\, d\mu,$$

where A and B are the projections of left-invariant vector fields from G to M.

It is straightforward to check that $\overline{g}$ is Hermitian.

q.e.d.

We recall the following definition.

Definition 1.1.6. *A Lie algebra $\mathfrak{g}$ is said to be* nilpotent *if its lower central series*

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^k \supset \cdots,$$

defined recursively by $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$ for $k \geq 1$, eventually becomes zero; that is, there exists some $k \in \mathbb{N}$ such that

$$\mathfrak{g}^k = 0.$$

The smallest such integer k is called the nilpotency step *of $\mathfrak{g}$.*

Accordingly, a Lie algebra $\mathfrak{g}$ is said to be *k-step nilpotent* if $\mathfrak{g}^k = 0$. A Lie group N is *k-step nilpotent* if its Lie algebra $\mathfrak{g}$ is k-step nilpotent.

Observe that $\mathfrak{g}$ is 2-step nilpotent if its commutator ideal $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ lies in the center $\mathfrak{z}$ of $\mathfrak{g}$.

Definition 1.1.7. *A k-*step nilmanifold *is a compact quotient $\Gamma\backslash N$, where N is a connected, simply connected k-step nilpotent Lie group and $\Gamma \subset N$ is a lattice.*

It is well known that the existence of a lattice for a connected and simply connected nilpotent Lie group N is equivalent to the existence of a rational basis of its Lie algebra $\mathfrak{g}$ [Mal].

Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra equipped with a complex structure J. In this context, it is natural to consider an ascending series adapted to J.

Definition 1.1.8 ([CFGU]). *The ascending series $\{\mathfrak{a}_k\ ;\ k \geq 0\}$ (compatible with J) of $\mathfrak{g}$ is defined inductively by*

$$\mathfrak{a}_0 = 0,\ \mathfrak{a}_k = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{a}_{k-1} \text{ and } [JX, \mathfrak{g}] \subset \mathfrak{a}_{k-1}\},\ k \geq 1. \tag{1.1.1}$$

The complex structure J is said to be *nilpotent* if there exists a $k \in \mathbb{N}$ such that $\mathfrak{a}_k = \mathfrak{g}$. The smallest such k is called the *nilpotency step of J*.

If $\mathfrak{g}$ is 2-step nilpotent, then any complex structure J is nilpotent [Rol, Proposition 3.3], and the nilpotency step of J is either 2 or 3 [GZZ, Theorem 1.3]. We recall from [B3] the following lemma.

Lemma 1.1.9 ([B3, Lemmas 3.3 and 4.1]). *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra endowed with a complex structure J. Then*

(1) *J is 2-step nilpotent if and only if $J\mathfrak{g}^1 \subset \mathfrak{z}$,*

(2) *J is 3-step nilpotent if and only if there exists $X \in \mathfrak{g}^1$ such that $JX \notin \mathfrak{z}$.*

(3) *$J\mathfrak{g}^1$ is an abelian ideal of $\mathfrak{g}$.*

Proof. We prove statements [(1)] and [(2)]. Since any complex structure on $\mathfrak{g}$ is either 2 or 3 step nilpotent [Rol, GZZ], it suffices to prove the first equivalence.

Assume that J is 2-step nilpotent. By definition, $\mathfrak{a}_0 = 0,\ \mathfrak{a}_1 = \mathfrak{z} \cap J\mathfrak{z}$ and $\mathfrak{a}_2 = \mathfrak{g}$. In particular, $\mathfrak{g}^1 \subset \mathfrak{z} \cap J\mathfrak{z}$, which implies that $J\mathfrak{g}^1 \subset \mathfrak{z}$.

Conversely, let us assume that $J\mathfrak{g}^1 \subset \mathfrak{z}$. Since $\mathfrak{g}$ is 2-step nilpotent, $J\mathfrak{g}^1 \subset \mathfrak{z}$ implies that $\mathfrak{g}^1 \subset \mathfrak{z} \cap J\mathfrak{z} = \mathfrak{a}_1$. Let $X, Y \in \mathfrak{g}$. Then $[Y, X] \in \mathfrak{g}^1 \subset \mathfrak{a}_1$ and, analogously, $[Y, JX] \in \mathfrak{g}^1 \subset \mathfrak{a}_1$, which forces $\mathfrak{g} \subset \mathfrak{a}_2$, and so $\mathfrak{a}_2 = \mathfrak{g}$.

The proof of the last statement is an application of the integrability of J. For any $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^1$, $N_J = 0$ implies that

$$[JY, X] = -J[JY, JX]. \tag{1.1.2}$$

In particular, if $X \in J\mathfrak{g}^1$, then $[JY, X] = 0$, as $\mathfrak{g}^1 \subset \mathfrak{z}$.

q.e.d.

Definition 1.1.10. *A Lie algebra $\mathfrak{g}$ is said to be* solvable *if its derived series*

$$\mathfrak{g} = \mathfrak{d}^0 \supset \mathfrak{d}^1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{d}^k \supset \cdots,$$

defined recursively by $\mathfrak{d}^k = [\mathfrak{d}^{k-1}, \mathfrak{d}^{k-1}]$ for $k \geq 1$, eventually becomes zero; that is, there exists some $k \in \mathbb{N}$ such that

$$\mathfrak{d}^k = 0.$$

The smallest such integer k is called the solvable step *of $\mathfrak{g}$.*

By a straightforward inductive argument, any nilpotent Lie algebra is solvable. Consequently, nilpotent Lie groups form a special subclass of solvable Lie groups.

Definition 1.1.11. *A solvable Lie algebra $\mathfrak{g}$ is said to be* unimodular *if*

$$\mathrm{tr}(\mathrm{ad}_X) = 0 \quad \textit{for all } X \in \mathfrak{g}.$$

Every nilpotent Lie algebra is unimodular, but in general a solvable Lie algebra need not be.

Being unimodular is *necessary* for a connected and simply connected solvable Lie group to admit lattices. That is, if such a Lie group G admits a lattice, then its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ must be unimodular [Mil].

Definition 1.1.12. *Given a solvable Lie algebra $\mathfrak{g}$, the maximal nilpotent ideal $\mathfrak{n}$ is called the* nilradical. *When the nilradical has codimension 1 then the corresponding Lie algebra is said* almost nilpotent. *An almost nilpotent Lie algebra is said* almost abelian *if the nilradical is abelian.*

A Lie group G is said to be *almost nilpotent* if its Lie algebra $\mathfrak{g}$ is almost nilpotent. Similarly, G is *almost abelian* if its Lie algebra is almost abelian. The existence of lattices in almost nilpotent and almost abelian Lie groups has been studied in [Bo].

Let $\mathfrak{g}$ be a solvable Lie algebra. Let $\mathfrak{n}$ denote its *nilradical*, i.e., the maximal nilpotent ideal of $\mathfrak{g}$. Then

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}.$$

This is an easy consequence of the fact that the commutator of a solvable Lie algebra is a nilpotent ideal.

We point out that since any solvmanifold admits a non vanishing, closed 1-form, any solvmanifold is a mapping torus by Tischler Theorem.

1.2 The Bismut connection

1.2.1 SKT geometry in a nutshell

Let (M, J, g) be a Hermitian manifold of complex dimension n. We denote by

$$\omega(X, Y) = g(JX, Y) \in \Omega^2(M)$$

the *fundamental form* of (J, g). Since g is a J-compatible Riemannian metric, $\omega \in \Omega^{1,1}(M)$ and $\omega(X, JX) > 0$ for any non zero vector X.

Remark 1.2.1. We point out that if (M, J) is a complex manifold, a standard "partition of unity" argument shows that a Hermitian metric g always exists, and so one can define such a ω. On the other hand, if (M, J) is a complex manifold and ω is a $(1, 1)$ positive real form, one can construct a Hermitian metric $g = -\omega J$.

Definition 1.2.2. *A Hermitian structure (J, g) is said to be Kähler if $d\omega = 0$, i.e., if ω is a symplectic structure.*

From the very definition, we get that Kähler geometry lies at the intersection of three fundamental geometries: complex, symplectic, and Riemannian. In particular, on compact manifolds, the existence of a Kähler structure imposes strong constraints on the topology of the manifold: the odd Betti numbers are even, the even Betti numbers are positive, the minimal model is formal according to Sullivan, and the Hard Lefschetz condition holds.

Given a Hermitian manifold (M, J, g) the Lefschetz operator

$$L : \Omega^\bullet(M) \to \Omega^{\bullet+2}(M), \ \alpha \mapsto \alpha \wedge \omega$$

is such that its $(n-1)$ power

$$L^{n-1} : \Omega^1(M) \to \Omega^{2n-1}(M),$$

is an isomorphism. Therefore, there exists a 1-form θ such that

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

Such 1-form is called *Lee form* and an easy check shows that $\theta = J\delta\omega$. Denoted by $H(X,Y,Z) = -d^c\omega(X,Y,Z) = d\omega(JX,JY,JZ)$ one has that $\theta(X) = \frac{1}{2}\sum_{i=1}^{2n} H(e_i, Je_i, JX)$, where $\{e_1, \dots, e_{2n}\}$ is a orthonormal frame at one point [GFS].

Let us fix (M, J, g) Hermitian manifold.

Definition 1.2.3. *An affine connection ∇ on (M, J, g) is called* Hermitian *if $\nabla g = \nabla J = 0$.*

Proposition 1.2.4 ([Bis])**.** *On a compact Hermitian manifold there exists a unique Hermitian connection ∇^B, called the* Bismut connection*, whose torsion is totally skew symmetric, i.e.,*

$$H(X,Y,Z) = g(T^B(X,Y),Z) \in \Omega^3(M)$$

In general, $\nabla^B \neq \nabla^{LC}$, but we have

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) - \frac{1}{2} d^c\omega(X,Y,Z), \tag{1.2.1}$$

where $-d^c\omega(X,Y,Z) = d\omega(JX,JY,JZ)$.

It follows from (1.2.1) that $\nabla^B = \nabla^{LC}$ if and only if $d\omega = 0$, i.e., if and only if the metric is Kähler, and that $H = -d^c\omega$. In what follows we will call H the *Bismut torsion*.

Remark 1.2.5. Since ∇^B is Hermitian

$$\mathrm{Hol}(\nabla) \subset \mathrm{U}(n),$$

where n is the complex dimension of M.

Remark 1.2.6. It follows from [Gau] that ∇^B belongs to a 1-parameter family of canonical Hermitian connections

$$\nabla^t = \left(1 - \frac{t}{2}\right)\nabla^C + \frac{t}{2}\nabla^B, \quad t \in \mathbb{R},$$

where ∇^C is the Chern connection, i.e., the unique Hermitian connection such that its $(0,1)$-component coincides with the operator $\bar\partial$.

Definition 1.2.7. *A Hermitian structure is called SKT or pluriclosed if $dH = 0$, that is $dd^c\omega = 0$, or, equivalently $\partial\overline{\partial}\omega = 0$.*

Definition 1.2.8. *A Hermitian structure is called Gauduchon if $\partial\overline{\partial}\omega^{n-1} = 0$, or, equivalently $\delta\theta = 0$, where δ is the co-differential.*

Theorem 1.2.9 ([Gau2])**.** *Let (M, J) be a compact complex manifold. Then any conformal class of any given Hermitian metric contains a Gauduchon metric.*

By the Theorem above we immediately obtain that Gauduchon metrics exist on any compact complex manifold. In complex dimension 2, the Gauduchon condition coincides with the SKT one, implying that SKT metrics exist on any compact complex surface. However in higher dimension there are plenty of compact complex manifolds which do not admit any SKT metric; for instance $S^{2p+1} \times S^{2q+1}$ endowed with the Calabi-Eckmann complex structure admits a SKT metric if and only if $(p, q) = (0, 0), (0, 1), (1, 1)$.

The SKT condition is, in a suitable sense, orthogonal to the so called *balanced* condition.

Definition 1.2.10. *A Hermitian metric (J, g) on a smooth manifold of dimension $2n$ is said to be balanced if $d\omega^{n-1} = 0$, or equivalently, if the Lee form vanishes, i.e., $\theta = 0$.*

Theorem 1.2.11 ([AI])**.** *Let (J, g) be a Hermitian metric. Then if (J, g) is both SKT and balanced, then it is Kähler.*

Proof. Following [AI], any Hermitian metric satisfy the following equality

$$\langle dd^c\omega, \omega \wedge \omega \rangle = 2(n-1)^2|\theta|^2 - 2|d\omega|^2 + 2(n-1)\delta\theta.$$

Therefore, since by hypothesis $dd^c\omega = 0$ and $\theta = 0$, the equality above gives $d\omega = 0$. q.e.d.

A natural generalization of the above result leads to what is considered one of the main open conjectures in non-Kähler Hermitian geometry.

Conjecture 1.2.12 ([FV])**.** Every compact complex manifolds admitting a balanced metric ω_1 and a SKT metric ω_2 also admit a Kähler metric ω_3.

As far as the author knows, this conjecture is open.

1.2.2 Generalized Kähler structures

Definition 1.2.13 ([Gu])**.** *A generalized Kähler structure is a bi-Hermitian structure $(J_\pm, g)$ such that*

- *(J_+, g) is SKT;*

- $H_+ + H_- = 0$, *where* $H_\pm$ *are the Bismut torsions of* $(J_\pm, g)$ *respectively.*

Example 1.2.14. If (J, g) is Kähler, then $(\pm J, g)$ is generalized Kähler. Generalized Kähler structures arising from Kähler structures are called *trivial*.

Definition 1.2.15. *A generalized Kähler structure* $(J_\pm, g)$ *is said to be split if* $[J_+, J_-] = 0$.

Generalized complex geometry [Cav2, Gu, Hit] is an extension of classical complex geometry developed to study geometric structures that arise in string theory, especially in the context of mirror symmetry and branes, see, for instance, [CG]. The core idea is to replace geometric structures on the classical tangent bundle TM with structures defined on the *generalized tangent bundle* $E = TM \oplus T^*M$, which carries a natural symmetric bilinear form

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X)),$$

of mixed signature. A *generalized complex structure* is an endomorphism $\mathbb{J} \in End(E)$, orthogonal with respect to $\langle \cdot, \cdot \rangle$, such that $\mathbb{J}^2 = -\mathrm{Id}_E$ and whose $(+i)$-eigenbundle is involutive with respect to the Courant bracket [Cou]. Since complex and symplectic structures naturally induce generalized complex structures on E, generalized complex geometry appears as a bridge between complex and symplectic geometry, and it seems well-motivated to understand its interplay with Kähler geometry.

The definition of generalized Kähler structures can equivalently be stated from the viewpoint of generalized geometry, in fact the generalized complex counterpart of a Kähler structure is a *generalized Kähler structure* [Gu, Gu2], defined as a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that the composition $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric, namely, an endomorphism of the generalized tangent bundle satisfying

1. $\langle \mathcal{G}\cdot, \mathcal{G}\cdot \rangle = \langle \cdot, \cdot \rangle$;
2. $\langle \mathcal{G}\cdot, \cdot \rangle = \langle \cdot, \mathcal{G}\cdot \rangle$;
3. the bilinear pairing $\langle \mathcal{G}\cdot, \cdot \rangle$ is positive definite.

1.2.3 SKT and generalized Kähler structures on solvmanifolds

?? It is worth noting that a large class of examples of SKT metrics occurs on compact quotients of nilpotent and solvable Lie groups, i.e. on nilmanifolds and solvmanifolds, thanks mostly to the symmetrization process.

Theorem 1.2.16 ([UG])**.** *Suppose that* $M = G/\Gamma$ *is a solvmanifold endowed with an invariant complex structure* J *and a SKT metric* g. *Then there exists an invariant SKT inner product* $\overline{g}$ *on* $\mathfrak{g} = Lie(G)$.

Proof. By Milnor [Mil] there exist a bi-invariant volume form on G which induces a volume form $d\mu$ on M. After rescaling, we may assume that $d\mu$ has volume 1. As in the previous section we denote by $\bar{\cdot}$ the corresponding symmetrized tensor.

We define a new left-invariant metric $\overline{g}$ on M by

$$\overline{g}(A, B) := \int_M g_m(A_m, B_m)\, d\mu,$$

where A and B are the projections of left-invariant vector fields from G to M. We have already observed that $\overline{g}$ is Hermitian. Then

$$\overline{\omega}(A, B) := \int_M \omega_m(A_m, B_m)\, d\mu.$$

By [FG], $d\overline{\omega} = \overline{d\omega}$. Moreover, since J is invariant $Jd\overline{\omega} = \overline{Jd\omega}$ and $dJd\overline{\omega} = \overline{dJd\omega} = 0$, as ω is SKT. q.e.d.

Remark 1.2.17. With the same techniques one can easily prove that if $M = G/\Gamma$ is a solvmanifold endowed with invariant complex structures $J_\pm$ and a generalized Kähler metric g, then there exists an invariant generalized Kähler inner product $\overline{g}$ on $\mathfrak{g} = Lie(G)$.

The symmetrization process allows us to reduce the analysis of SKT and generalized Kähler structures to the level of the Lie algebra. We point out, however, that not every Lie algebra $(\mathfrak{g}, J)$ admits a compatible SKT metric. This holds only in real dimension 4, where the SKT condition coincides with the Gauduchon condition. Indeed, using the following lemma:

Lemma 1.2.18. *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension n. Then any $\alpha \in \bigwedge^{n-1}\mathfrak{g}$ satisfies $d\alpha = 0$.*

it follows that $d^c\omega^{n-1}$ is always a closed form, hence $dd^c\omega^{n-1} = 0$. Therefore, any invariant metric is Gauduchon.

The existence of SKT structures on nilpotent Lie algebras imposes severe restrictions on their geometry.

Theorem 1.2.19 ([EnFV, ArNic]). *If $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT with $\mathfrak{g}$ nilpotent, then $\mathfrak{g}$ is at most 2-step nilpotent.*

A special sub-case is given by nilpotent Lie algebras admitting a generalized Kähler structure. It turns out that in this case, the situation is even more restrictive:

Theorem 1.2.20 ([Cav]). *If $(\mathfrak{g}, J_\pm, \langle\cdot,\cdot\rangle)$ is generalized Kähler with $\mathfrak{g}$ nilpotent, then $\mathfrak{g}$ is abelian.*

From this result, we also recover the well-known fact that any Kähler nilpotent Lie algebra is necessarily a torus.

1.2.4 Curvature properties of the Bismut connection

Let (M, J, g) be a Hermitian manifold. We denote by

$$R^B(X,Y)Z = \nabla^B_X \nabla^B_Y Z - \nabla^B_Y \nabla^B_Z X - \nabla^B_{[X,Y]} Z,$$

and

$$R^B(X,Y,Z,U) = g(R^B(X,Y)Z,U)$$

the curvature tensors of type $(1,3)$ and $(0,4)$ respectively associated to the Bismut connection.

Proposition 1.2.21 ([IP2]). *Let (M, g, J, ∇) be a Hermitian manifold. The following identy holds*

$$\begin{aligned} R^g(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)(Y,Z,U) + \frac{1}{2}(\nabla_Y T)(X,Z,U) \\ &- \frac{1}{2} g(T(X,Y),T(Z,U)) \\ &- \frac{1}{4} g(T(Y,Z),T(X,U)) - \frac{1}{4} g(T(Z,X),T(Y,U)) \, . \end{aligned} \tag{1.2.2}$$

Since $\nabla^B J = 0$, it follows that $R^B \in \Omega^2(M) \otimes \Omega^{1,1}(M)$.

In general the Bismut connection does not satisfy the Bianchi identity, however, it holds that [IP2]

$$\sigma_{X,Y,Z} R^B(X,Y,Z,U) = dH(X,Y,Z,U) + \nabla^B_U H(X,Y,Z) - \sigma_{X,Y,Z}\, g(H(X,Y),H(Z,U)).$$

Note that if ∇^B satisfies the first Bianchi, then $R^B \in \Omega^{1,1}(M) \otimes \Omega^{1,1}.(M)$. This is an easy consequence of the fact that

$$\text{First Bianchi identity} \implies R^B(X,Y,Z,U) = R^B(Z,U,X,Y).$$

Definition 1.2.22 ([AOUV]). *The Bismut connection is called Kähler like if it satisfies the first Bianchi identity.*

Theorem 1.2.23. *[ZZ] Let (M, J, g) be a Hermitian manifold. Then*

$$\textit{First Bianchi identity} \iff \nabla^B H = 0, \; dH = 0.$$

Remark 1.2.24. Bismut-Kähler-like manifolds have been completely characterized in [BPT].

Examples of Kähler-like manifolds are given by Kähler manifolds and Bismut flat manifolds. In particular, any Bismut flat Hermitian structure is SKT and has parallel Bismut torsion.

Example 1.2.25. Let $G' = G \times \mathbb{R}^k$ be a simply connected Lie group, where G is a compact semisimple Lie group. Assume that G' is endowed with a left-invariant complex structure J and a compatible bi-invariant metric g, and let ∇ be the connection defined by

$$\nabla X = 0 \quad \text{for any left-invariant vector field } X.$$

It is straightforward to check that ∇ is both metric and complex. Furthermore, on left-invariant vector fields we have

$$T^{\nabla}(X,Y) = -[X,Y],$$

so that

$$T^{\nabla}(X,Y,Z) = -g([X,Y],Z) \in \Omega^3(M),$$

using the bi-invariance of g. By uniqueness, we conclude that $\nabla = \nabla^B$, and it is clear that $R^B = 0$.

These are the only simply connected Bismut-flat manifolds, as the next theorem will show.

Definition 1.2.26 ([WYZ]). *A* Samelson space *is a Hermitian manifold $(G' = G \times \mathbb{R}^n, g' = b + g_E, J_L)$, where G is a compact, connected and simply connected semisimple Lie group, $g' = b + g_E$ is the bi-invariant metric on G' given by the product of the bi-invariant metric b on G and the euclidean metric g_E, and J_L is a left invariant complex structure compatible with g'. A group homomorphism $\rho : \mathbb{Z}^n \to \mathrm{Isom}(G)$ induces a free and properly discontinuous action of $\mathbb{Z}^n$ on G' as isometries via*

$$m \cdot (p,t) \mapsto (\rho(m)p, t+m).$$

If J_L is preserved by this action of $\mathbb{Z}^n$, then the compact quotient $G'/\mathbb{Z}^n$ inherits the structure of a complex manifold by G'. In this case, $G'/\mathbb{Z}^n$ is said to be a local Samelson space.

By [WYZ], if (M,g,J) is a compact Bismut flat Hermitian manifold, then there exists a finite unbranched cover M' of M which is a local Samelson space. Moreover, if (M,g,J) is Bismut-flat complete simply-connected Hermitian manifold, then it is a Samelson space.

Example 1.2.27. Consider $S^1 \times \mathrm{SU}(2)$ with the following invariant Hermitian struture:

$$Je_1 = e_2, Je_3 = e_4, \; g = \sum_{i=1}^{4} (e^i)^2,$$

where e_1 generates $Lie(S^1)$, $\{e_2, e_3, e_4\}$ is a standard basis of $\mathrm{SU}(2)$ and $\{e^i\}$ is a dual basis. Since g is bi-invariant and J is left invarint, the Hermitian structure above is Bismut flat.

The Bismut Ricci tensor is defined as

$$Ric^B(X,Y) = \sum_i R^B(e_i, X, Y, e_i),$$

and by (1.2.2) it is given by

$$Ric^B(X,Y) = Ric^g(X,Y) - \frac{1}{4}H^2(X,Y) - \frac{1}{2}\delta H(X,Y),$$

where $H^2(X,Y) = g(\iota_X H, \iota_Y H)$.

It follows that on a Bismut Ricci flat manifold, both the symmetric and the skew symmetric parts of Ric^B vanish, implying that

$$Ric^g(X,Y) = \frac{1}{4}H^2(X,Y).$$

Therefore, $Ric^g \geq 0$.

In the complex case, many traces of the Bismut curvature tensor are possible. Let

$$\rho^B(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} g(R^B(X,Y)Je_i, e_i) \in \Omega^2(M)$$

be the (first) Bismut Ricci curvature. The tensor ρ^B admits a different interpretation which makes clearer its topological significance. Since the connections ∇^B preserves J, it is clear that it induces a connections on the anticanonical line bundle associated to (M,J). The two form ρ^B is the curvature tensor of this connection, and as such is closed and determines a representative of $c_1(M,J)$.

On a SKT manifold, the Ricci tensor Ric^B is related with ρ^B by

$$\rho^B(X,Y) = -Ric^B(X,JY) - (\nabla^B_X\theta)JY \tag{1.2.3}$$

(see [IP2] or [GFS, Lemma 8.9]).

Example 1.2.28. Let us consider the Bismut flat manifold $(S^1 \times \mathrm{SU}(2), J, g)$ described in Example 1.2.27. Since $\theta = e^1$ and $\nabla^B e^1 = 0$ (recall that ∇^B is such that $\nabla^B X = 0$ for any X left invariant vector field), the Hermitian metric is also Bismut Ricci flat.

Definition 1.2.29. *A Hermitian structure (J,g) is said Calabi Yau with torsion (CYT) if $\rho^B = 0$, or, equivalently, if $\mathrm{Hol}^0(\nabla^B) \subset \mathrm{SU}(n)$, where n is the complex dimension of M and Hol^0 is the restricted holonomy.*

When $\nabla^B\theta = 0$ and the metric is SKT, by (1.2.3), the CYT condition is equivalent to have $Ric^B = 0$.

Obvious examples of CYT manifolds are given by

1. Bismut flat manifolds (which are also SKT and with parallel Bismut torsion)

2. Kähler Ricci flat manifolds (which are also SKT and with parallel Bismut torsion).

We point out that less trivial examples are known. For instance in complex dimension 3, the compact manifolds

$$(k-1)(S^2 \times S^4) \# k(S^3 \times S^3)$$

admit CYT metrics for all $k \geq 1$ [GGP]. It is an open question to determine if the manifolds above admit a CYT structure which is also SKT.

Definition 1.2.30 ([GFS, GFJS]). *A SKT structure (J, g) is said Bismut Hermitian Einstein (BHE) if it is CYT.*

It turns out that Bismut-flat manifolds and Kähler Ricci-flat manifolds (as well as products of these) are the only known examples of BHE manifolds. In Chapter 4, we will present a construction of a class of non-Kähler, non-Bismut-flat manifolds which are not products of the aforementioned types.

Theorem 1.2.31 ([GI]). *A 4-dimensional compact Hermitian manifold (M, J, g) is BHE if it is either the Hopf surface with its flat metric (see Example 1.2.27) or a Kähler Ricci flat manifold.*

In dimension 6, some rigidity results of compact BHE manifolds have been obtained in [ABLS].

1.3 The Obata connection

Let (M^{4n}, I, J, K) be a hypercomplex manifold.

Definition 1.3.1 ([Ob]). *On a hypercomplex manifold (M, I, J, K), there exists a unique torsion-free connection ∇^{Ob} such that*

$$\nabla^{Ob} I = \nabla^{Ob} J = \nabla^{Ob} K = 0.$$

This connection is called the Obata connection.

The Obata connection can be expressed explicitly as [Sol]

$$\nabla^{Ob}_X Y = \frac{1}{2}\big([X,Y] + J_1[J_1X,Y] - J_2[X,J_2Y] + J_3[J_1X,J_2Y]\big). \tag{1.3.1}$$

More generally, if we denote (I, J, K) bt (J_1, J_2, J_3) by applying cyclic permutations (α, β, γ) of $(1, 2, 3)$, the Obata connection can also be written as:

$$\nabla^{Ob}_X Y = \frac{1}{2}\big([X, Y] + J_\alpha[J_\alpha X, Y] - J_\beta[X, J_\beta Y] + J_\gamma[J_\alpha X, J_\beta Y]\big). \tag{1.3.2}$$

By definition, it follows that

$$\mathrm{Hol}(\nabla^{Ob}) \subset \mathrm{GL}(n, \mathbb{H}),$$

so that its holonomy group is typically non-compact. However, if there exists a compatible hyperhermitian metric g, then ∇^{Ob} coincides with the Levi-Civita connection ∇^{LC}, and

$$\mathrm{Hol}(\nabla^{Ob}) \subset \mathrm{Sp}(n).$$

The Obata connection and its holonomy are central objects in hypercomplex geometry, as they depend solely on the fixed hypercomplex structure.

In the hypercomplex setting, a hypercomplex analogue of the Newlander–Nirenberg Theorem [NN] does not hold. However, the fact that a hypercomplex manifold is locally isomorphic to $\mathbb{H}^n$ depends on the (restricted) holonomy of the Obata connection.

Theorem 1.3.2 ([Ob]). *Let (M, I, J, K) be a hypercomplex manifold. The following are equivalent:*

1. The Obata connection ∇^{Ob} is flat

2. M has quaternionic affine transition functions.

Example 1.3.3. Consider $\mathbb{H} \setminus \{0\}$ endowed with its standard hypercomplex structure. Fix $q \in \mathbb{H} \setminus \{0\}$ such that $|q| \neq 1$, and let $\langle q \rangle$ denote the infinite cyclic group generated by right multiplication by q. The quotient manifold

$$(\mathbb{H} \setminus \{0\})/\langle q \rangle \cong S^1 \times \mathrm{SU}(2)$$

inherits the hypercomplex structure from $\mathbb{H} \setminus \{0\}$ and is therefore a hypercomplex manifold endowed with a flat Obata connection.

Remark 1.3.4. For quaternionic dimension $n = 1$, compact manifolds admitting a hypercomplex structure have been classified in [Bo]. Any compact 4-dimensional hypercomplex manifold is either a torus, or a K3 surface, or the Hopf Surface.

A less trivial class of examples, which generalizes the examples above is given by Joyce hypercomplex manifolds.

1.4 Joyce hypercomplex manifolds

1.4.1 Joyce decomposition

The Joyce construction applies to any compact Lie group G. However, since up to a finite cover any compact Lie group splits as a product of a torus and a compact semisimple Lie group, for our purposes here we may assume without loss of generality that G is compact semisimple. We set $r = \operatorname{rank} G$.

Let H be a maximal torus in G, and let $\mathfrak{h}$ and $\mathfrak{g}$ denote their respective Lie algebras*. Choose a system of ordered roots Δ with respect to $\mathfrak{h}_{\mathbb{C}}$ and fix a maximal positive root α_1. Let $\mathfrak{d}_1$ be the $\mathfrak{sp}(1)$-subalgebra of $\mathfrak{g}$ whose complexification is isomorphic to the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$, where $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ are the root spaces for α_1 and $-\alpha_1$, respectively. Let $\mathfrak{b}_1$ denote the centralizer of $\mathfrak{d}_1$. Then there is a real subspace $\mathfrak{f}_1$ of dimension $4d_1$, for some $d_1 \in \mathbb{N}_0$, such that $\mathfrak{g} = \mathfrak{b}_1 \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1$. More precisely,

$$\mathfrak{f}_1 = \mathfrak{g} \cap \bigoplus_{\substack{\alpha_1 \neq \alpha > 0 \\ \langle \alpha, \alpha_1 \rangle \neq 0}} \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}.$$

The subalgebra $\mathfrak{b}_1$ is the direct sum of an abelian Lie algebra and a semisimple Lie algebra $\mathfrak{g}'$. If $\mathfrak{g}'$ is non trivial, then Joyce re-applies the above decomposition to $\mathfrak{g}'$.

By a recursive process, one obtains a decomposition of $\mathfrak{g}$ of the form [Joy, Lemma 4.1]:

$$\mathfrak{g} = \mathfrak{b} \oplus \bigoplus_{i=1}^{m} \mathfrak{d}_i \oplus \bigoplus_{i=1}^{m} \mathfrak{f}_i \,, \tag{1.4.1}$$

where

1. $\mathfrak{b}$ is an abelian subalgebra of dimension $r - m$,

2. $\mathfrak{d}_i \subseteq \mathfrak{g}$ is a subalgebra isomorphic to $\mathfrak{su}(2)$ for each $i = 1, \ldots, m$,

3. $\mathfrak{f}_i \subseteq \mathfrak{g}$ are (possibly trivial) subspaces for each $i = 1, \ldots, m$.

Furthermore, the following commutation relations hold:

(J1) $[\mathfrak{d}_i, \mathfrak{b}] = 0$, for any $i = 1, \ldots, m$;

(J2) $[\mathfrak{d}_i, \mathfrak{d}_j] = 0$, for $i \neq j$;

(J3) $[\mathfrak{d}_i, \mathfrak{f}_j] = 0$, for $i < j$;

(J4) $[\mathfrak{d}_i, \mathfrak{f}_i] \subseteq \mathfrak{f}_i$, for any $i = 1, \ldots, m$; moreover, the action of $\mathfrak{d}_i$ on $\mathfrak{f}_i$ is isomorphic to the direct sum of a finite number of copies of the standard $\mathfrak{su}(2)$-action on $\mathbb{C}^2$ by left matrix multiplication.

*In what follows, we denote the Lie algebra of a Lie group G by the same letter in fraktur font, that is, $\operatorname{Lie}(G) = \mathfrak{g}$.

We will refer to a decomposition as in (1.4.1) as a *Joyce decomposition*. Note that the construction of Joyce involves the choice of a Cartan subalgebra, a system of positive roots, and a maximal root. While different choices lead to different decompositions, they are nonetheless all isomorphic via an inner automorphism of $\mathfrak{g}$. Thus, we may unambiguously talk about *the* Joyce decomposition of a given compact Lie group.

For further purposes, we emphasise the link between the Joyce decomposition and the construction of compact quaternion-Kähler symmetric spaces, so-called Wolf spaces, as given in [Wo]. This relation has also been observed in [OP, SSTVP].

It is shown in [Wo] that for each compact simple Lie group G there is a unique compact quaternion-Kähler symmetric space $M^{4n} = G/H$ obtained as a quotient of G by a subgroup $H = K \cdot \mathrm{Sp}(1)$:

$$M^{4n} := \frac{G}{K \cdot \mathrm{Sp}(1)},$$

where $\mathrm{Sp}(1)$ is the compact real form of the $\mathrm{SL}(2, \mathbb{C})$-triple associated with the maximal root of G, and K is its centralizer in G. This yields the following decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{su}(2) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ corresponds to the isotropy representation. Since M carries a quaternion-Kähler structure, as an $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$-module we have $\mathfrak{m} \cong \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2$ (n copies) cf. [Sal].

The above isotropy decomposition coincides precisely with the Joyce decomposition at the first level: set $\mathfrak{d}_1 = \mathfrak{su}(2)$ and $\mathfrak{m} = \mathfrak{f}_1$. Since K is a compact Lie group (not necessarily simple), one can apply the Joyce decomposition, up to a shift of the index i, to $\mathfrak{k}$. Thus, we can summarize the above observation into:

Lemma 1.4.1. *The decomposition*

$$\mathfrak{g} = \left(\mathfrak{b} \oplus \bigoplus_{i=2}^{m} \mathfrak{d}_i \oplus \bigoplus_{i=2}^{m} \mathfrak{f}_i\right) \oplus \mathfrak{su}(2) \oplus \mathfrak{m} \tag{1.4.2}$$

coincides with the Joyce decomposition (1.4.1) *of* $\mathfrak{g}$.

The latter follows from the construction in [Wo] and is well known to the experts.

1.4.2 Joyce hypercomplex structures

As above, we denote by G a compact semisimple Lie group of rank r, with associated Joyce decomposition (1.4.1). Let $\mathbb{T}^{2m-r} \cong \mathrm{U}(1)^{2m-r}$ be a $(2m-r)$-dimensional torus. To simplify the notation, we set $\ell := 2m - r$. We thus have an isomorphism:

$$\ell\mathfrak{u}(1) \oplus \mathfrak{b} \cong \mathbb{R}^m.$$

Note that the choice of such an isomorphism (that is, the choice of a basis of $\ell\mathfrak{u}(1) \oplus \mathfrak{b}$) depends on m^2 parameters. Let $\mathcal{B} = (e_1^1, e_1^2, \ldots, e_1^m)$ denote a fixed basis of $\ell\mathfrak{u}(1) \oplus \mathfrak{b} \cong \mathbb{R}^m$. The Joyce hypercomplex structure on $\mathbb{T}^\ell \times G$ constructed from $\mathcal{B}$ is defined on each "layer" of the decomposition (1.4.1) as follows:

- For every $i = 1, \ldots, m$, fix a basis (e_2^i, e_3^i, e_4^i) of $\mathfrak{d}_i \cong \mathfrak{su}(2)$ so that e_2^i, e_3^i, e_4^i satisfy

$$[e_2^i, e_3^i] = 2e_4^i\,, \qquad [e_4^i, e_2^i] = 2e_3^i\,, \qquad [e_3^i, e_4^i] = 2e_2^i\,. \tag{1.4.3}$$

  In this way, $(e_1^i, e_2^i, e_3^i, e_4^i)$ is a basis of $\mathfrak{h}_i := \mathbb{R} \oplus \mathfrak{d}_i \cong \mathbb{R} \oplus \mathfrak{su}(2)$, which can be regarded as a copy of the space of quaternions. In particular, each $\mathfrak{h}_i$ has a natural hypercomplex structure

$$Ie_1^i = e_2^i\,,\quad Ie_3^i = e_4^i\,,\quad Je_1^i = e_3^i\,,\quad Je_2^i = -e_4^i\,,\quad Ke_1^i = e_4^i\,,\quad Ke_2^i = e_3^i\,.$$

- The action of I, J, K is extended to each $\mathfrak{f}_i$ by taking advantage of property (J4):

$$If = [e_2^i, f]\,,\quad Jf = [e_3^i, f]\,,\quad Kf = [e_4^i, f]\,,$$

  for each $f \in \mathfrak{f}_i$.

The complex structures $\{I, J, K\}$, defined at the identity, are extended to all of $\mathbb{T}^\ell \times G$ by left translation. By appealing to the construction in [Sam], Joyce shows that these almost complex structures are in fact integrable and define a homogeneous hypercomplex structure on $\mathbb{T}^\ell \times G$ [Joy]. We shall call $(\mathbb{T}^\ell \times G, I, J, K)$ a *Joyce hypercomplex manifold*. We emphasise that the hypercomplex structure depends on the choice of the basis $\mathcal{B}$ and thus, there is a parameter space of dimension m^2 of left-invariant hypercomplex structures on the same underlying manifold $\mathbb{T}^\ell \times G$ and many of them are inequivalent.

For G simple, the list of Joyce hypercomplex manifolds is given in [SSTVP]:

$$\begin{gathered}\mathrm{SU}(2k+1),\ S^1 \times \mathrm{SU}(2k),\ \mathbb{T}^k \times \mathrm{SO}(2k+1),\ \mathbb{T}^k \times \mathrm{Sp}(k),\ \mathbb{T}^{2k} \times \mathrm{SO}(4k),\\ \mathbb{T}^{2k-1} \times \mathrm{SO}(4k+2),\ \mathbb{T}^2 \times \mathrm{E}_6,\ \mathbb{T}^7 \times \mathrm{E}_7,\ \mathbb{T}^8 \times \mathrm{E}_8,\ \mathbb{T}^4 \times \mathrm{F}_4,\ \mathbb{T}^2 \times \mathrm{G}_2.\end{gathered} \tag{1.4.4}$$

Observe that any very left-invariant hypercomplex structure on a compact Lie group arises this way [BGP, DT].

Remark 1.4.2. It is worth pointing out that Joyce also extended the construction of invariant hypercomplex structures to certain homogeneous spaces of the form $\mathbb{T}^s \times (G/H)$ for suitable s and H [Joy]. Although some of our results apply to such homogeneous hypercomplex spaces, we shall restrict to the case of trivial isotropy, i.e. group manifolds.

In quaternionic dimension 1, the only Joyce hypercomplex manifold in the list (1.4.4) is the Hopf surface. It is known that the Obata connection of a Joyce hypercomplex structure (unique up to equivalence) is flat, with holonomy group isomorphic to $\mathbb{Z}$.

In quaternionic dimension 2, the only Joyce hypercomplex manifold in the list (1.4.4) is $\mathrm{SU}(3)$. This case has been studied in [Sol], where the author proved that the holonomy group of the Obata connection (again, unique up to equivalence) satisfies

$$\mathrm{Hol}(\nabla^{Ob}) = \mathrm{GL}(2, \mathbb{H}).$$

The proof proceeds in two steps. First, it is shown that the Obata holonomy group acts irreducibly on the tangent space; this argument relies crucially on properties specific to dimension 8. Second, one appeals to the classification of irreducible holonomy groups of torsion-free affine connections in [MS2]. In this classification, only three candidates appear as possible subgroups of $\mathrm{GL}(n, \mathbb{H})$: $\mathrm{Sp}(n)$, $\mathrm{SL}(n, \mathbb{H})$, and $\mathrm{GL}(n, \mathbb{H})$. The conclusion then follows by eliminating the first two possibilities.

It was conjectured in [SV] that

Conjecture 1.4.3. Given a Joyce hypercomplex manifold in the list (1.4.4), with the exception of $S^1 \times \mathrm{SU}(2)$, the holonomy of the Obata connection is full.

In Chapter 5, we will disprove this conjecture.

1.5 HKT and Strong HKT manifolds

Definition 1.5.1. *A $4n$-dimensional hyperhermitian manifold (M, I, J, K, g) is said to be* hyperkähler with torsion *(HKT, for short) if*

$$\nabla^B_I = \nabla^B_J = \nabla^B_K =: \nabla^B,$$

where ∇^B_I denotes the Bismut connection associated with the Hermitian structure (I, g), and similarly for J and K.

HKT structures were first introduced in the context of string theory, where they arise naturally as target spaces of supersymmetric sigma models. In particular, they appeared in [HP] as target spaces of $(4, 0)$-SUSY sigma models with a Wess–Zumino term.

A first basic property of HKT manifolds is the following.

Remark 1.5.2. By [HP] for an HKT manifold one has

$$\mathrm{Hol}(\nabla^B) \subseteq \mathrm{Sp}(n),$$

since $\nabla^B g = 0$ and

$$\nabla^B I = \nabla^B J = \nabla^B K = 0 \quad .$$

In particular, each of the three Hermitian structures is Calabi–Yau with torsion (CYT) (cfr. Section 1.2.4).

The HKT condition can be expressed in several equivalent ways:

Proposition 1.5.3 ([GP]). *Let (M, I, J, K, g) be a hyperhermitian manifold. The following conditions are equivalent:*

1. $\nabla^B_I = \nabla^B_J = \nabla^B_K$,
2. $I\, d\omega_I = J\, d\omega_J = K\, d\omega_K$,
3. $\partial_I(\omega_J + i\omega_K) = 0$,
4. $\overline{\partial}_I(\omega_J - i\omega_K) = 0$,

where ∂_I and $\overline{\partial}_I$ denote the Dolbeault operators with respect to I.

In quaternionic dimension one, condition 3 shows that every hyperhermitian metric is HKT. In higher dimensions, however, there exist hypercomplex manifolds that admit no compatible HKT metric, as shown in [FG].

A convenient way to encode a hyperhermitian structure (I, J, K, g) is via the fundamental 2-forms

$$\omega_I := g(I\cdot,\cdot), \quad \omega_J := g(J\cdot,\cdot), \quad \omega_K := g(K\cdot,\cdot).$$

The 2-form

$$\Omega := \omega_J + i\omega_K$$

is of type $(2, 0)$ with respect to I, satisfies

$$J\Omega = \overline{\Omega}, \quad \text{and} \quad \Omega(X, JX) > 0 \quad \text{for all non-zero } X,$$

and uniquely determines the hyperhermitian metric by

$$g(X, Y) := \Omega(X, JY).$$

In these terms, Proposition 1.5.3 shows that the HKT condition is equivalent to $\partial_I\Omega = 0$.

We recall the following

Definition 1.5.4. *Let (M^{4n}, I, J, K) be a hypercomplex manifold. A $(2n, 0)$ form α is said to be q-real if $J\overline{\alpha} = \alpha$.*

It follows that Ω is q-real.

Another viewpoint on HKT geometry comes from holonomy considerations. A distinguished class of hypercomplex manifolds arises when the holonomy of the Obata connection reduces. One of the most relevant subgroups of $\mathrm{GL}(n, \mathbb{H})$ is its commutator $\mathrm{SL}(n, \mathbb{H})$, and hypercomplex manifolds for which

$$\mathrm{Hol}(\nabla^{Ob}) \subset \mathrm{SL}(n, \mathbb{H})$$

are called $\mathrm{SL}(n, \mathbb{H})$-manifolds.

On HKT manifolds, the Lee forms of the three Hermitian structures coincide [IP], so we may set $\theta := \theta_I = \theta_J = \theta_K$. This allows one to characterize the Obata holonomy as follows:

Theorem 1.5.5 ([IP]). *Let (M, I, J, K, g) be an HKT manifold of quaternionic dimension n. Then:*

1. $\mathrm{Hol}^0(\nabla^{Ob}) \subset \mathrm{SL}(n, \mathbb{H})$ *if and only if the Lee form θ is closed,*

2. $\mathrm{Hol}(\nabla^{Ob}) \subset \mathrm{SL}(n, \mathbb{H})$ *if and only if the Lee form θ is exact.*

Finally, it is customary to distinguish between *weak* and *strong* HKT structures. Setting

$$H := I\, d\omega_I = J\, d\omega_J = K\, d\omega_K,$$

the structure is called strong if $dH = 0$, and weak otherwise. Equivalently, in the strong case each Hermitian structure (g, I), (g, J), (g, K) is pluriclosed (SKT). Note that when $H \equiv 0$, then $\nabla^B = \nabla^{LC}$ and the metric is hyperkähler. In particular, every strong HKT structure is such that each Hermitian structure (g, L) is Bismut Hermitian–Einstein (BHE), for $L = I, J, K$ (cfr. Section 1.2.4).

The original definition of HKT structures required the strong condition, but this assumption was later relaxed because of the scarcity of examples. To date, the only known compact (non-hyperkähler) strong HKT manifolds arise either from Joyce's construction [Joy] (see next Section) or from the construction of Barberis–Fino [BF], and all of them are homogeneous. In the non-compact setting, a non-homogeneous strong HKT example was obtained in [MV], using moduli spaces of anti-self dual connections.

Remark 1.5.6. The relationship between the Bismut and Obata connections on HKT manifolds can be described explicitly. Let H denote the torsion 3-form of ∇^B and define the tensor

$$2A(X, Y, Z) := -H(X, IY, IZ) - H(IX, IY, Z) - H(X, KY, KZ) - H(IX, KY, JZ). \tag{1.5.1}$$

Then, by [IP, Proposition 3.1],

$$g(\nabla^{Ob}_X Y, Z) = g(\nabla^B_X Y, Z) + A(X, Y, Z). \tag{1.5.2}$$

1.5.1 HKT metrics on Joyce hypercomplex manifolds

Let G be a compact semisimple Lie group. As seen in the previous section, the compact manifold $\mathbb{T}^\ell \times G$ admits a family of (left-invariant) hypercomplex structures, which depends on the choice of a basis $(e_1^1, \dots, e_1^m)$ of the abelian Lie algebra $\ell\mathfrak{u}(1) \oplus \mathfrak{b}$.

Let B denote the (negative of the) Killing–Cartan form of $\mathfrak{g}$. It is shown in [GP] that the Joyce decomposition (1.4.1) is B-orthogonal.

Let (e_2^j, e_3^j, e_4^j) denote a standard basis of $\mathfrak{d}_j$ $\big($i.e., $[e_\alpha^j, e_\beta^j] = 2e_\gamma^j$ for any cyclic permutation (α, β, γ) of $(2,3,4)\big)$ such that

$$B(e_2^j, e_2^j) = B(e_3^j, e_3^j) = B(e_4^j, e_4^j) = \lambda_j^2, \quad j = 1, \dots, m.$$

We consider a basis $(e_1^1, \dots, e_1^\ell, e_1^{\ell+1}, \dots, e_1^m)$ of $\ell\mathfrak{u}(1) \oplus \mathfrak{b} \cong \mathbb{R}^m$, where $(e_1^1, \dots, e_1^\ell)$ is a basis of $\ell\mathfrak{u}(1)$ and $(e_1^{\ell+1}, \dots, e_1^m)$ is a B-orthogonal basis of $\mathfrak{b}$ such that

$$B(e_1^{\ell+j}, e_1^{\ell+j}) = \lambda_{\ell+j}^2, \quad j = 1, \dots, m - \ell. \tag{1.5.3}$$

We then extend B to a positive definite bilinear form g of $\ell\mathfrak{u}(1) \oplus \mathfrak{g}$ by setting:

$$g(e_1^j, e_1^j) = \lambda_j^2, \quad j = 1, \dots, \ell.$$

By construction g is a bi-invariant metric on $\ell\mathfrak{u}(1) \oplus \mathfrak{g}$ which is hyperhermitian with respect to the Joyce hypercomplex structure (I, J, K) obtained from the basis $(e_1^1, \dots, e_1^m)$ of $\ell\mathfrak{u}(1) \oplus \mathfrak{b} \cong \mathbb{R}^m$. More precisely, the metric g satisfies:

1. $g_{|\mathfrak{g}} = B$,

2. the decomposition $\ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{i=1}^m \mathfrak{d}_i \oplus \bigoplus_{i=1}^m \mathfrak{f}_i$ is g-orthogonal,

3. g is bi-invariant,

4. the hyperhermitian structure (g, I, J, K) is strong HKT (see Remark 1.5.7).

This has been observed in [GP, OP].

Remark 1.5.7. Let G be a semisimple Lie group and let $G \times \mathbb{T}^k$ be endowed with a hyperHermitian structure (I, J, K, g), with I, J, K left invariant and g bi-invariant. We have seen in Example 1.2.25 that

$$H_I(X, Y, Z) = -g([X, Y], Z),$$

therefore, the Bismut torsion only depend on the Lie bracket and not on the complex structure chosen, as long as it is left invariant. It follows that

$$H_I = H_J = H_K.$$

Furthermore, since these manifolds are Bismut flat, each structure is SKT. Alternatively, one can easily observe that H_I is bi-invariant, as so is g, and so it is closed.

We emphasize that the requirement (1.5.3) imposes some constraints on the possible choices of $e_1^{\ell+1}, \dots, e_1^m$; indeed, not all Joyce hypercomplex structures are compatible with an extension of the Killing–Cartan form. However, such a choice can always be made, i.e. for each compact simple Lie group there exists at least one Joyce hypercomplex structure compatible with the extension of the Killing-Cartan form as above. We also note that in general the choices of the bases $\{e_1^1, \dots, e_1^\ell\}$ and $\{e_1^{\ell+1}, \dots, e_1^m\}$ are not unique.

When $\mathfrak{b} = 0$ the condition (1.5.3) is vacuous, and thus, any Joyce hypercomplex structure is compatible with an extension of the Killing–Cartan form B, that it is also bi-invariant and strong HKT.

Remark 1.5.8. Observe that when $\mathfrak{b} = 0$, up to equivalence, there is exactly one left-invariant hypercomplex structure on the universal cover group $\mathbb{R}^m \times \widetilde{G}$.

This implies that given two Joyce hypercomplex structures (I, J, K) and $(\tilde{I}, \tilde{J}, \tilde{K})$ with associated Obata connections ∇ and $\tilde{\nabla}$, respectively, one has

$$\mathfrak{hol}(\nabla) \cong \mathfrak{hol}(\tilde{\nabla}).$$

This is a consequence of the well-known fact that the restricted holonomy coincides with the holonomy group of the induced Obata connection on the universal cover $\mathbb{R}^m \times \widetilde{G}$.

1.5.2 Hypercomplex and HKT structures on nilmanifolds

Nilmanifolds have also played a central role in the development of hypercomplex geometry. A milestone in this direction is the theorem of Barberis–Dotti–Verbitsky [BDV], which establishes that any nilmanifold endowed with an invariant hypercomplex structure is an $\mathrm{SL}(n, \mathbb{H})$-manifold.

Beyond holonomy considerations, an equally natural question is the existence of (invariant) HKT structures on nilmanifolds. To address this, one first recalls the notion of abelian complex structure.

Definition 1.5.9. *Let $\mathfrak{g}$ be a Lie algebra. A complex structure J on $\mathfrak{g}$ is called* abelian *if*

$$[JX, JY] = [X, Y], \qquad \forall\, X, Y \in \mathfrak{g}.$$

It follows immediately that any abelian almost complex structure is integrable. Moreover, such structures can occur only on 2-step solvable Lie algebras [Pe]. They enjoy a number of pleasant properties: for instance, the center of $\mathfrak{g}$ is always J-invariant. In the hypercomplex setting, if a solvable Lie algebra carries a triple (I, J, K) with one of the three complex structures, say I, abelian, then automatically the other two are abelian as well [DF1].

This notion is central for the following result.

Theorem 1.5.10 ([BDV]). *Let M be a nilmanifold endowed with an invariant HKT structure. Then the associated hypercomplex structure is necessarily abelian. Moreover, any HKT nilmanifold is balanced.*

A refinement of this picture was obtained in [DF3], where it was proved that if a nilpotent Lie algebra admits a hyperhermitian structure with abelian hypercomplex structure, then it supports an HKT metric, and the Bismut torsion is closed if and only if the Lie algebra itself is abelian. Combining this with the theorem above, one concludes that a nilmanifold admits a strong HKT structure if and only if it is a torus.

Similar rigidity phenomena appear also beyond the nilpotent case. Indeed, an analogue of the above result was established in [AB1] for almost abelian solvmanifolds. As recalled in [BF], it is still an open problem whether more general solvmanifolds can admit strong HKT structures. In Chapter 4 we shall resolve this question in the negative.

Chapter 2

Generalized Kähler structures on solvmanifolds

The study of *strong Kähler with torsion* (SKT) metrics on compact complex manifolds has emerged as an active area of research over the past two decades, primarily due to their connections with type II string theory and two-dimensional supersymmetric σ-models. These metrics are a natural generalization of Kähler condition within non-Kähler geometry and appear naturally in theoretical physics, where they provide geometric solutions compatible with torsionful fluxes.

A particularly fruitful framework for investigating SKT metrics is that of locally homogeneous manifolds, where the *symmetrization process* [BE, FG, UG] plays a key role. More precisely, for a solvmanifold $\Gamma\backslash G$ endowed with an invariant complex structure, the existence of a (not necessarily invariant) SKT metric implies the existence of an invariant one. In this context, *invariant* means that the associated Hermitian structure is defined at the level of the Lie algebra $\mathfrak{g}$ of G. This leads to a major simplification: instead of solving nonlinear partial differential equations on the manifold, one is led to solve algebraic conditions on the Lie algebra itself.

Although this approach simplifies the analytical complexity, the classification problem remains challenging. Even in low dimensions, the solvable case presents greater difficulty than the nilpotent case: the existence of an SKT metric on a nilpotent Lie algebra is heavily constrained. As shown by Arroyo and Nicolini [ArNic], such a Lie algebra must have *nilpotency step at most two*, confirming a conjecture originally proposed in [EnFV]. Moreover, SKT structures have been completely classified on nilpotent Lie algebras in dimensions 6 and 8, in [FPS04] and [EnFV], respectively.

In contrast, SKT structures on non-nilpotent solvable Lie algebras have been intensively studied in literature [FP, FP2, FP3, MS], although a complete classification has been achieved only in dimension 4 [MS]. In dimension 6, the classification is nearly complete. As shown in [Tol], the nilradical must have dimension either 5 (the *almost nilpotent* case) or 4 (the *codimension 2* case). The almost nilpotent case has already been thoroughly investigated in [FP, FP2, FP3, FS, FS2]; in this chapter, we complete the classification by addressing the remaining codimension 2 case.

Since every generalized Kähler structure induces an SKT metric, a broader and deeper question concerns the existence of generalized Kähler structures on solvmanifolds. Unlike the SKT case, however, the generalized Kähler scenario remains largely unexplored. A striking rigidity result by Cavalcanti [Cav] shows that a nilmanifold admits a generalized Kähler structure if and only if it is a torus.

In the *solvable* case, the situation is more flexible, and classification results exist only in low dimensions. In dimension 4, the only non-Kähler solvmanifold admitting a generalized Kähler structure is the Inoue surface [AG]. For dimension 6, a classification of solvmanifolds admitting *invariant* generalized Kähler structures was achieved in the almost nilpotent case by Fino and Paradiso [FP, FP2, FP3], extending an earlier example by Fino and Tomassini [FTo]. A closer analysis of these results reveals that in all such examples, the nilradical is *abelian*, i.e., isomorphic to $\mathbb{R}^5$. Together with Cavalcanti's result and the absence of counterexamples, this observation strongly supports the following natural conjecture:

Conjecture 2. *Any compact solvmanifold admitting an invariant generalized Kähler structure must have an abelian nilradical.*

We emphasize that the compactness assumption is crucial for this conjecture: in the non-unimodular case, counterexamples exist (see Remark 2.2.6 for details).

In this chapter, as previously announced, we complete the classification of six-dimensional solvmanifolds admitting SKT and generalized Kähler structures, by addressing the case where the nilradical has codimension 2. More precisely, we consider solvable Hermitian Lie algebras $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ with nilradical $\mathfrak{h}$ of dimension 2 and examine two distinct cases, depending on the behavior of the complex structure with respect to the nilradical: either $J\mathfrak{h} = \mathfrak{h}$ or $J\mathfrak{h} \neq \mathfrak{h}$.

The results presented in this chapter are part of a published joint work with A. Fino [BrF].

2.1 Case $J\mathfrak{h} = \mathfrak{h}$

Let $\mathfrak{g}$ be a $2n$-dimensional solvable Lie algebra with nilradical $\mathfrak{h}$ of codimension 2. Assume that $\mathfrak{g}$ is equipped with an almost Hermitian structure $(J, \langle\cdot,\cdot\rangle)$ such that $J\mathfrak{h} = \mathfrak{h}$.

Consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Since $J\mathfrak{h} = \mathfrak{h}$, both $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are J-invariant subspaces. Let U be a unit vector in $\mathfrak{h}^\perp$, and denote by JU its image under J. Using the fact that the derived algebra satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^1 \subset \mathfrak{h}$, the Lie bracket on $\mathfrak{g}$ is determined for any $Y, W \in \mathfrak{h}$ by:

$$[U, Y] = A(Y), \quad [JU, Y] = B(Y), \quad [Y, W] = \mu(Y, W), \quad [U, JU] = V,$$

where $A := \mathrm{ad}_U|_{\mathfrak{h}}$ and $B := \mathrm{ad}_{JU}|_{\mathfrak{h}}$ are derivations of the nilpotent Lie algebra $(\mathfrak{h}, \mu)$, and $V \in \mathfrak{h}$. The Jacobi identity implies that $\mathrm{ad}_V|_{\mathfrak{h}} = [A, B]$.

Theorem 2.1.1. *Let $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ be as above. Then the following statements hold:*

(i) The complex structure J is integrable if and only if the restriction $J_{\mathfrak{h}}$ defines a complex structure on $\mathfrak{h}$, and the derivations $A, B \in \mathrm{Der}(\mathfrak{h})$ satisfy:

$$[J_{\mathfrak{h}}, A]J_{\mathfrak{h}} + [J_{\mathfrak{h}}, B] = 0.$$

(ii) If J is integrable, then the associated Hermitian Lie algebra $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is Chern-Ricci flat.

(iii) Suppose J is integrable, $\mathfrak{g}$ is unimodular, and $[U, JU] = 0$. Then the Hermitian structure $(J, \langle\cdot,\cdot\rangle)$ is balanced if and only if its restriction $(J_{\mathfrak{h}}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$ is balanced on $\mathfrak{h}$.

(iv) If J is integrable and $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ admits an SKT structure, then the nilradical $\mathfrak{h}$ must be at most 2-step nilpotent. Moreover, the restrictions of A and B to the center $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ satisfy:

$$A|_{\mathfrak{z}(\mathfrak{h})},\ B|_{\mathfrak{z}(\mathfrak{h})} \in \mathfrak{so}(\mathfrak{z}(\mathfrak{h})), \quad [A, J] = [B, J] = 0 \quad \textit{on} \quad \mathfrak{z}(\mathfrak{h}).$$

Proof.

(i) Since the Nijenhuis tensor N of J satisfies the J-anti-invariance property,

$$N(JX, JY) = -N(X, Y) \quad \forall X, Y \in \mathfrak{g},$$

it suffices to compute $N(U, Y)$ and $N(Y, Z)$ for all $Y, Z \in \mathfrak{h}$.

For $Y, Z \in \mathfrak{h}$, since $J\mathfrak{h} = \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, a direct computation shows that

$$N(Y, Z) = N_{J_{\mathfrak{h}}}(Y, Z),$$

i.e., the Nijenhuis tensor on $\mathfrak{g}$ restricts to the one on $(\mathfrak{h}, J_{\mathfrak{h}})$.

Now consider $N(U, Y)$ for $Y \in \mathfrak{h}$. Using the bracket relations and the fact that J preserves both $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$, we compute:

$$\begin{aligned} N(U, Y) &= [JU, JY] - [U, Y] - J[JU, Y] - J[U, JY] \\ &= [JU, JY] - [U, Y] - J[B(Y)] - J[A(JY)] \\ &= (BJ_{\mathfrak{h}} - A - J_{\mathfrak{h}}B - J_{\mathfrak{h}}AJ_{\mathfrak{h}})Y. \end{aligned}$$

Thus, the vanishing of $N(U, Y)$ for all $Y \in \mathfrak{h}$ is equivalent to:

$$A + J_{\mathfrak{h}}AJ_{\mathfrak{h}} + J_{\mathfrak{h}}B - BJ_{\mathfrak{h}} = 0,$$

which proves point (i).

(ii) Following [VZ] (see also [FP, Formula 5.8]), the Ricci form of the Chern Connection ρ^{Ch} is given by $d\eta^{Ch}$, where

$$\eta^{Ch}(Y) = \frac{1}{2}\big(\mathrm{tr}(ad_Y \circ J) - \mathrm{tr}(ad_{JY})\big),\ \forall Y \in \mathfrak{g}.$$

For any $Y \in \mathfrak{h}$, since $\mathfrak{h}$ is nilpotent, one has $\eta^{Ch}(Y) = \eta^{Ch}_{\mathfrak{h}}(Y) = 0$ by [LRV, Proposition 2.1]. Moreover, for $Z, W \in \mathfrak{g}$, one computes

$$d\eta^{Ch}(Z, W) = -\eta^{Ch}([Z, W]),$$

and since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, where η^{Ch} vanishes, it follows that $\rho^{Ch} = 0$.

(iii) Consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and let $u = U^{\flat}$, $Ju = (JU)^{\flat}$ denote the metric duals. The fundamental 2-form ω of $(J, \langle\cdot,\cdot\rangle)$ can then be written as

$$\omega = \omega_{\mathfrak{h}} + u \wedge Ju.$$

Expanding ω^{n-1} yields:

$$\omega^{n-1} = \omega_{\mathfrak{h}}^{n-1} + (n-1)\omega_{\mathfrak{h}}^{n-2} \wedge u \wedge Ju.$$

To compute $d\omega^{n-1}$, we use the following differential identity for any $\alpha \in \bigwedge^k \mathfrak{h}^*$:

$$d\alpha = u \wedge A^*\alpha + Ju \wedge B^*\alpha - u \wedge Ju \wedge \iota_V \alpha + d_{\mathfrak{h}}\alpha, \tag{2.1.1}$$

where $C^*\gamma = -\sum_{i=1}^k \gamma(\ldots, C\cdot, \ldots)$ and $d_{\mathfrak{h}}$ denotes the Chevalley–Eilenberg differential of the nilpotent Lie algebra $(\mathfrak{h}, \mu)$.

Since $du = dJu = 0$ and $[U, JU] = V = 0$ by assumption, the identity above implies

$$d\omega^{n-1} = d\omega_{\mathfrak{h}}^{n-1} + (n-1)d_{\mathfrak{h}}\omega_{\mathfrak{h}}^{n-2} \wedge u \wedge Ju.$$

We now claim that $d\omega_{\mathfrak{h}}^{n-1} = 0$. Suppose otherwise that

$$d\omega_{\mathfrak{h}}^{n-1} = u \wedge A^*\omega_{\mathfrak{h}}^{n-1} + Ju \wedge B^*\omega_{\mathfrak{h}}^{n-1} \neq 0.$$

If $A^*\omega_{\mathfrak{h}}^{n-1} \neq 0$, consider the $2n-1$ form:

$$d(\omega_{\mathfrak{h}}^{n-1} \wedge Ju) = u \wedge A^*\omega_{\mathfrak{h}}^{n-1} \wedge Ju \neq 0,$$

contradicting the fact that all top-degree forms are closed on a unimodular Lie algebra. A similar contradiction arises from $B^*\omega_{\mathfrak{h}}^{n-1} \neq 0$. Therefore, $d\omega_{\mathfrak{h}}^{n-1} = 0$, and so

$$d\omega^{n-1} = (n-1)d_{\mathfrak{h}}\omega_{\mathfrak{h}}^{n-2} \wedge u \wedge Ju,$$

which vanishes if and only if $d_{\mathfrak{h}}\omega_{\mathfrak{h}}^{n-2} = 0$. This completes the proof of (iii).

(iv) The first part of (iv) follows from [ArNic]. Indeed, if the Hermitian Lie algebra $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT, then the subalgebra $(\mathfrak{h}, J_{\mathfrak{h}}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$ inherits the SKT property (see [ArNic, Proposition 3.1]). In particular, by [ArNic, Theorem 4.8], $\mathfrak{h}$ is at most 2-step nilpotent. Since $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{\mathfrak{h}}, J_{\mathfrak{h}})$ is SKT and at most 2-step nilpotent, we can consider the orthogonal decomposition $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{z}(\mathfrak{h})^{\perp}$, where $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. Each component in this decomposition is $J_{\mathfrak{h}}$-invariant by [EnFV, Proposition 3.5].

Furthermore, since $A, B \in \mathrm{Der}(\mathfrak{h})$, they preserve the center of $\mathfrak{h}$. Indeed, for any $Z \in \mathfrak{z}(\mathfrak{h})$ and all $Y \in \mathfrak{h}$,

$$0 = A\big(\mu(Z,Y)\big) = \mu(AZ,Y) + \mu(Z,AY) = \mu(AZ,Y),$$

from which it follows that $AZ \in \mathfrak{z}(\mathfrak{h})$, and analogously $BZ \in \mathfrak{z}(\mathfrak{h})$.

With respect to the decomposition $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{z}(\mathfrak{h})^{\perp}$, the derivations A and B can be written in block matrix form as:

$$A = \begin{pmatrix} A_{\mathfrak{z}(\mathfrak{h})} & *_A \\ 0 & A_{\mathfrak{z}(\mathfrak{h})^{\perp}} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{\mathfrak{z}(\mathfrak{h})} & *_B \\ 0 & B_{\mathfrak{z}(\mathfrak{h})^{\perp}} \end{pmatrix}.$$

Thus, $\mathfrak{z}(\mathfrak{h})$ is a J-invariant ideal of $\mathfrak{g}$ and the integrability condition involving A, B, and $J_{\mathfrak{h}}$ on $\mathfrak{z}(\mathfrak{h})$ reads:

$$A_{\mathfrak{z}(\mathfrak{h})} + J_{\mathfrak{z}(\mathfrak{h})}A_{\mathfrak{z}(\mathfrak{h})}J_{\mathfrak{z}(\mathfrak{h})} + J_{\mathfrak{z}(\mathfrak{h})}B_{\mathfrak{z}(\mathfrak{h})} - B_{\mathfrak{z}(\mathfrak{h})}J_{\mathfrak{z}(\mathfrak{h})} = 0.$$

Let c denote the Bismut torsion 3-form of $(J, \langle\cdot,\cdot\rangle)$. For any $Z \in \mathfrak{z}(\mathfrak{h})$, using the formula for dc from [DF3] (see also [ArNic, Formula 3]), we get:

$$\begin{aligned} dc(JZ, Z, U, JU) = {} & \|AJZ\|^2 + \|BJZ\|^2 + \|AZ\|^2 + \|BZ\|^2 \\ & - \langle AJAJZ, Z\rangle - \langle JAJAZ, Z\rangle \\ & - \langle BJBJZ, Z\rangle - \langle JBJBZ, Z\rangle. \end{aligned} \tag{2.1.2}$$

Using identity (i), this is equivalent to:

$$\begin{aligned} dc(JZ, Z, X, JX) = {} & \|AJZ\|^2 + \|BJZ\|^2 + \|AZ\|^2 + \|BZ\|^2 \\ & + \langle\big(2(A^2 + B^2 + AJ_{\mathfrak{h}}B - BJ_{\mathfrak{h}}A) \\ & - J_{\mathfrak{h}}[A,B] - [A,B]J_{\mathfrak{h}}\big)Z, Z\rangle. \end{aligned}$$

Since $[A,B] = \mathrm{ad}_V|_{\mathfrak{h}}$ and $Z \in \mathfrak{z}(\mathfrak{h})$, we have $[A,B]Z = 0$. Similarly, since $\mathfrak{z}(\mathfrak{h})$ is $J_{\mathfrak{h}}$-invariant, $[A,B]J_{\mathfrak{h}}Z = 0$. Thus,

$$\begin{aligned} dc(JZ, Z, X, JX) = \|AJZ\|^2 + \|BJZ\|^2 + \|AZ\|^2 + \|BZ\|^2 \\ + \langle 2(A^2 + B^2 + AJ_{\mathfrak{h}}B - BJ_{\mathfrak{h}}A)Z, Z\rangle. \end{aligned}$$

The SKT condition then implies:

$$\|AJZ\|^2 + \|BJZ\|^2 + \|AZ\|^2 + \|BZ\|^2 + \langle 2(A^2 + B^2 + AJ_{\mathfrak{h}}B - BJ_{\mathfrak{h}}A)Z, Z\rangle = 0$$

for all $Z \in \mathfrak{z}(\mathfrak{h})$.
Let $\{e_1, \dots, e_{2r}\}$ be any orthonormal basis of $\mathfrak{z}(\mathfrak{h})$ such that $Je_{2j-1} = e_{2j}$ for all $j = 1, \dots, r$. Using the SKT condition, we compute:

$$\begin{aligned} \sum_{j=1}^{2r} dc(Je_j, e_j, X, JX) &= \sum_{j=1}^{2r} \|AJe_j\|^2 + \|BJe_j\|^2 + \|Ae_j\|^2 + \|Be_j\|^2 \\ &\quad + \langle 2(A^2 + B^2 + AJ_{\mathfrak{h}}B - BJ_{\mathfrak{h}}A)e_j, e_j\rangle \\ &= 2\big(\|A_{\mathfrak{z}(\mathfrak{h})}\|^2 + \|B_{\mathfrak{z}(\mathfrak{h})}\|^2 + \mathrm{tr}(A^2_{\mathfrak{z}(\mathfrak{h})} + B^2_{\mathfrak{z}(\mathfrak{h})}) \\ &\quad + \mathrm{tr}(A_{\mathfrak{z}(\mathfrak{h})}J_{\mathfrak{z}(\mathfrak{h})}B_{\mathfrak{z}(\mathfrak{h})} - B_{\mathfrak{z}(\mathfrak{h})}J_{\mathfrak{z}(\mathfrak{h})}A_{\mathfrak{z}(\mathfrak{h})})\big). \end{aligned}$$

Since $[A,B]_{\mathfrak{z}(\mathfrak{h})} = 0$, the last trace term vanishes, and we obtain:

$$\|A_{\mathfrak{z}(\mathfrak{h})}\|^2 + \|B_{\mathfrak{z}(\mathfrak{h})}\|^2 + \mathrm{tr}(A^2_{\mathfrak{z}(\mathfrak{h})} + B^2_{\mathfrak{z}(\mathfrak{h})}) = 0. \tag{2.1.3}$$

Now we claim that $\|A_{\mathfrak{z}(\mathfrak{h})}\|^2 + \mathrm{tr}(A^2_{\mathfrak{z}(\mathfrak{h})}) \geq 0$ and similarly for $B_{\mathfrak{z}(\mathfrak{h})}$. Indeed, expanding in terms of the components a_{ij}:

$$\|A_{\mathfrak{z}(\mathfrak{h})}\|^2 + \mathrm{tr}(A^2_{\mathfrak{z}(\mathfrak{h})}) = 2\sum_{i=1}^{2r} |a_{ii}|^2 + \sum_{i<j}(|a_{ij}|^2 + |a_{ji}|^2 + 2a_{ij}a_{ji}) = \sum_{i<j}(a_{ij} + a_{ji})^2.$$

Hence, (2.1.3) implies that $A_{\mathfrak{z}(\mathfrak{h})}$ and $B_{\mathfrak{z}(\mathfrak{h})}$ must be skew-symmetric, i.e., lie in $\mathfrak{so}(\mathfrak{z}(\mathfrak{h}))$.

With respect to $\{e_1, \dots, e_{2r}\}$ we may write $A_{\mathfrak{z}(\mathfrak{h})}$ and $B_{\mathfrak{z}(\mathfrak{h})}$ using 2×2 block matrices A_{ij} and B_{ij} as follows

$$A_{\mathfrak{z}(\mathfrak{h})} = \begin{pmatrix} A_{1,1} & \dots & A_{1,r} \\ \vdots & \ddots & \vdots \\ A_{r,1} & \dots & A_{r,r} \end{pmatrix}, \quad B_{\mathfrak{z}(\mathfrak{h})} = \begin{pmatrix} B_{1,1} & \dots & B_{1,r} \\ \vdots & \ddots & \vdots \\ B_{r,1} & \dots & B_{r,r} \end{pmatrix},$$

where, for $i = 1, \dots, r$,

$$A_{i,i} = \begin{pmatrix} 0 & a_{2i-1,2i} \\ -a_{2i-1,2i} & 0 \end{pmatrix}, \quad B_{i,i} = \begin{pmatrix} 0 & b_{2i-1,2i} \\ -b_{2i-1,2i} & 0 \end{pmatrix},$$

and, for $i < j$,

$$A_{i,j} = \begin{pmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{pmatrix}, \; A_{j,i} = -({}^t A_{i,j}),$$
$$B_{i,j} = \begin{pmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ b_{2i,2j-1} & b_{2i,2j} \end{pmatrix}, \; B_{j,i} = -({}^t B_{i,j}).$$

Since we choose $\{e_1, \dots, e_{2r}\}$ satisfying $Je_{2j-1} = e_{2j}$ for each $j = 1, \dots, r$, with respect to such a basis the complex structure $J_{\mathfrak{z}(\mathfrak{h})}$ can be represented by the diagonal block matrix $\mathrm{diag}(\Lambda_1, \dots, \Lambda_r)$ with

$$\Lambda_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If one imposes the integrability condition

$$A_{\mathfrak{z}(\mathfrak{h})} + J_{\mathfrak{z}(\mathfrak{h})} A_{\mathfrak{z}(\mathfrak{h})} J_{\mathfrak{z}(\mathfrak{h})} + J_{\mathfrak{z}(\mathfrak{h})} B_{\mathfrak{z}(\mathfrak{h})} - B_{\mathfrak{z}(\mathfrak{h})} J_{\mathfrak{z}(\mathfrak{h})} = 0$$

then the following linear conditions hold:

$$\begin{cases} a_{2i-1,2j-1} = a_{2i,2j} + b_{2i,2j-1} + b_{2i-1,2j} \\ b_{2i,2j} = a_{2i-1,2j} + a_{2i,2j-1} + b_{2i-1,2j-1}. \end{cases} \tag{2.1.4}$$

We compute the components $dc(e_{2j-1}, Je_{2j-1}, X, JX)$, using (2.1.2). A straightforward computation shows

$$\begin{aligned} dc(e_{2j-1}, Je_{2j-1}, X, JX) = &- 2\sum_{k=1}^{j-1} (b_{2k,2j-1} + b_{2k-1,2j})^2 + (a_{2k-1,2j} + a_{2k,2j-1})^2 \\ &- 2 \sum_{k=j+1}^{r-1} (b_{2j,2k-1} + b_{2j-1,2k})^2 + (a_{2j-1,2k} + a_{2j,2k-1})^2. \end{aligned}$$

Repeating the same argument for each $j = 1, \dots, r$, the vanishing of the expression above leads to the identities $b_{2i,2j-1} = -b_{2i-1,2j}$ and $a_{2i,2j-1} = -a_{2i-1,2j}$ for any $i, j =$

$1, \dots, r$ with $i < j$. Plugging these identities in (2.1.4) we get

$$\begin{cases} a_{2i,2j} = a_{2i-1,2j-1} \\ b_{2i,2j} = b_{2i-1,2j-1}. \end{cases}$$

Hence we have that for any $i < j$, the matrices A_{ij} and B_{ij} are of the kind

$$A_{ij} = \begin{pmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ -a_{2i-1,2j} & a_{2i-1,2j-1} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ -b_{2i-1,2j} & b_{2i-1,2j-1} \end{pmatrix},$$

and it is straightforward to observe that $[A_{\mathfrak{z}(\mathfrak{h})}, J_{\mathfrak{z}(\mathfrak{h})}] = [B_{\mathfrak{z}(\mathfrak{h})}, J_{\mathfrak{z}(\mathfrak{h})}] = 0$.

q.e.d.

Remark 2.1.2. If the Hermitian Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle, J)$ is Kähler, then $\mathfrak{h}$ is abelian. Indeed, since $\omega = \omega_{\mathfrak{h}} + u \wedge Ju$, using the expression of the exterior differential (2.1.1), we get that

$$d\omega = u \wedge A^*\omega_{\mathfrak{h}} + Ju \wedge B^*\omega_{\mathfrak{h}} - u \wedge Ju \wedge \iota_V\omega_{\mathfrak{h}} + d_{\mathfrak{h}}\omega_{\mathfrak{h}}.$$

Therefore, $d\omega = 0$ implies that $d_{\mathfrak{h}}\omega_{\mathfrak{h}} = 0$, and hence the abelianity of $\mathfrak{h}$ as $\mathfrak{h}$ is nilpotent

As a corollary of Theorem 2.1.1, we now consider the special case in which $\mathfrak{h}$ is abelian. In this setting, the complex structure $J_{\mathfrak{h}}$ is trivially integrable, and thus the integrability of J reduces to the algebraic condition

$$A + J_{\mathfrak{h}}AJ_{\mathfrak{h}} + J_{\mathfrak{h}}B - BJ_{\mathfrak{h}} = 0.$$

Hermitian structures on Lie algebras with a J-invariant abelian ideal of codimension two were studied in [GZ]. The following corollary refines those results by showing that, when such an ideal coincides with the nilradical of the Lie algebra, the SKT condition admits a sharper and more explicit characterization.

Corollary 2.1.3. *Let $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ be a Hermitian solvable Lie algebra with a J-invariant, codimension 2 abelian nilradical $\mathfrak{h}$, and let $\{U, JU\}$ be an orthonormal basis of the orthogonal complement $\mathfrak{h}^\perp$ in $\mathfrak{g}$. Then $(J, \langle\cdot,\cdot\rangle)$ is SKT if and only if one of the following condition holds*

(i) $A := ad_U|_{\mathfrak{h}}, B := ad_{JU}|_{\mathfrak{h}} \in \mathfrak{so}(\mathfrak{h})$ *and* $[A, J_{\mathfrak{h}}] = [B, J_{\mathfrak{h}}] = 0$;

(ii) there exists an orthonormal basis $\{e_1, \dots, e_{2n-2}\}$ *of* $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$ *with respect to which*

$$J_{\mathfrak{h}} = \mathrm{diag}(\Lambda_1, \dots, \Lambda_{n-1}), \; A = \mathrm{diag}(A_1, \dots, A_{n-1}), \; B = \mathrm{diag}(B_1, \dots, B_{n-1}),$$

where

$$\Lambda_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \textit{ and } B_i = \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}, \tag{2.1.5}$$

with $a_i, b_i \in \mathbb{R}$.

Moreover, if the Hermitian structure $(J, \langle\cdot,\cdot\rangle)$ *is SKT, then*

(iii) $(J, \langle\cdot,\cdot\rangle)$ *is Kähler if and only if* $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] = 0$*;*

(iv) if $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h} = \{0\}$, *then* $\mathfrak{g}$ *admits also a Kähler metric.*

Proof.

(i) One direction of follows directly from Theorem 2.1.1. Indeed, if $(\langle\cdot,\cdot\rangle, J)$ is SKT and $\mathfrak{h}$ is abelian, then $A, B \in \mathfrak{so}(\mathfrak{h})$ and $[A, J_\mathfrak{h}] = [B, J_\mathfrak{h}] = 0$.

Let us prove the converse. Recall that

$$\omega = \omega_\mathfrak{h} + u \wedge Ju.$$

Then,

$$d\omega = u \wedge A^*\omega_\mathfrak{h} + Ju \wedge B^*\omega_\mathfrak{h} - u \wedge Ju \wedge \iota_V \omega_\mathfrak{h},$$

where $V = [U, JU]$, and for any $Y, W \in \mathfrak{h}$,

$$A^*\omega_\mathfrak{h}(Y, W) = -\langle (J_\mathfrak{h} A + A^t J_\mathfrak{h})Y, W\rangle_\mathfrak{h}, \quad B^*\omega_\mathfrak{h}(Y, W) = -\langle (J_\mathfrak{h} B + B^t J_\mathfrak{h})Y, W\rangle_\mathfrak{h}.$$

Since $[A, J_\mathfrak{h}] = [B, J_\mathfrak{h}] = 0$ and $A, B \in \mathfrak{so}(\mathfrak{h})$, we have

$$J_\mathfrak{h} A + A^t J_\mathfrak{h} = (A + A^t)J_\mathfrak{h} = 0, \quad J_\mathfrak{h} B + B^t J_\mathfrak{h} = (B + B^t)J_\mathfrak{h} = 0.$$

It follows that

$$d\omega = -u \wedge Ju \wedge \iota_V \omega_\mathfrak{h}.$$

The Bismut torsion 3-form c is then given by

$$c = Jd\omega = -u \wedge Ju \wedge J(\iota_V \omega_\mathfrak{h}) = u \wedge Ju \wedge \iota_{JV} \omega_\mathfrak{h},$$

which is clearly closed since $d\mathfrak{h}^* \subset \mathfrak{h}^* \otimes u \oplus \mathfrak{h}^* \otimes Ju$.

(ii) From (i) the SKT condition is equivalent to $[A, J_\mathfrak{h}] = [B, J_\mathfrak{h}] = 0$ and $A, B \in \mathfrak{so}(\mathfrak{h})$. Since $\mathfrak{h}$ is abelian, the three skew-symmetric endomorphisms A, B, and $J_\mathfrak{h}$ commute pairwise. Hence, there exists an orthonormal basis $\{e_1, \ldots, e_{2n-2}\}$ of $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{|\mathfrak{h}})$ such that A, B, and $J_{\mathfrak{h}}$ are simultaneously diagonalizable in their normal forms:

$$J_{\mathfrak{h}} = \mathrm{diag}(\Lambda_1, \dots, \Lambda_{n-1}), \quad A = \mathrm{diag}(A_1, \dots, A_{n-1}), \quad B = \mathrm{diag}(B_1, \dots, B_{n-1}),$$

where Λ_i, A_i, and B_i are as in (2.1.5).

(iii) Part (iii) follows from the fact that if $(J, \langle \cdot, \cdot \rangle)$ is SKT, then

$$d\omega = -u \wedge Ju \wedge \iota_V \omega_{\mathfrak{h}}.$$

(iv) We exploit the condition $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h} = \{0\}$ to prove that $B - AJ_{\mathfrak{h}}$ is invertible. Since $(J, \langle \cdot, \cdot \rangle)$ is SKT, by (ii) there exists an orthonormal basis $\{e_1, \dots, e_{2n-2}\}$ of $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{|\mathfrak{h}})$ such that A, B, and $J_{\mathfrak{h}}$ are in their diagonal normal forms. Then,

$$B - AJ_{\mathfrak{h}} = \mathrm{diag}(C_1, \dots, C_{n-1}), \quad \text{where} \quad C_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}.$$

In particular,

$$\det(B - AJ_{\mathfrak{h}}) = \prod_{i=1}^{n-1} \det(C_i) = \prod_{i=1}^{n-1} (a_i^2 + b_i^2) \neq 0.$$

Indeed, if $\det(B - AJ_{\mathfrak{h}}) = 0$, then there exists an index $\bar{i}$ such that $a_{\bar{i}}^2 + b_{\bar{i}}^2 = 0$, which implies $a_{\bar{i}} = b_{\bar{i}} = 0$. This would mean $e_{2\bar{i}-1}, e_{2\bar{i}} \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h} = \{0\}$, a contradiction.

Since $B - AJ_{\mathfrak{h}}$ is invertible, there exists $Y \in \mathfrak{h}$ such that

$$(B - AJ_{\mathfrak{h}})Y = V.$$

Consider the new J-Hermitian metric

$$\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle_{|\mathfrak{h}} + u'^2 + Ju'^2,$$

where u' and Ju' are dual to $U' = U + Y$ and $JU' = JU + JY$, respectively. Then,

$$A' = \mathrm{ad}_{u'|\mathfrak{h}} = A, \quad B' = \mathrm{ad}_{Ju'|\mathfrak{h}} = B,$$

implying that $(J, \langle \cdot, \cdot \rangle')$ is again SKT (noting that $\langle \cdot, \cdot \rangle'_{|\mathfrak{h}} = \langle \cdot, \cdot \rangle_{|\mathfrak{h}}$). Moreover, since $[U', JU'] = 0$, the Hermitian structure $(\langle \cdot, \cdot \rangle', J)$ is Kähler by (iii).

q.e.d.

Regarding the existence of generalized Kähler structures we can prove the following

Theorem 2.1.4. *Let $\mathfrak{g}$ be a solvable Lie algebra with nilradical $\mathfrak{h}$ of codimension 2. The following are equivalent*

(i) $\mathfrak{g}$ *admits a generalized Kähler structure* $(J_\pm, \langle\cdot,\cdot\rangle)$ *such that* $J_\pm\mathfrak{h} = \mathfrak{h}$*;*

(ii) $\mathfrak{g}$ *admits a Kähler structure* $(J_+, \langle\cdot,\cdot\rangle)$ *such that* $J_+\mathfrak{h} = \mathfrak{h}$*.*

Proof.
(ii)$\Rightarrow$ (i). It suffices to take $J_- = -J_+$.
(i)$\Rightarrow$ (ii). Let us fix an orthonormal basis $\{U, U'\}$ of $\mathfrak{h}^\perp$ with dual basis $\{u, u'\}$. Without loss of generality, we may assume that $U' = J_+U$ and $J_-U = \varepsilon U'$, for $\varepsilon \in \{-1, +1\}$. Let $A = ad_U|_\mathfrak{h}$, $B = ad_{U'}|_\mathfrak{h}$ and $V = [U, U']$.
Since the Bismut torsions 3-forms $c_\pm$ of the Hermitian structures $(J_\pm, \langle\cdot,\cdot\rangle)$ satisfy $c_+ = -c_-$, then there exist $\alpha, \beta \in \bigwedge^2\mathfrak{h}^*$ and $\gamma \in \mathfrak{h}^*$ such that

$$c_\pm = \pm\alpha \wedge u \pm \beta \wedge u' \pm \gamma \wedge u \wedge u' + c_{\mathfrak{h}_\pm},$$

where $c_{\mathfrak{h}_\pm}$ are the torsion 3-forms of the Hermitian structures $(J_{\pm\mathfrak{h}}, \langle\cdot,\cdot\rangle_\mathfrak{h})$ on $\mathfrak{h}$, respectively. Since $c_+ = -c_-$, we clearly have $c_{\mathfrak{h}+} = -c_{\mathfrak{h}-}$.
Using (2.1.1) and the SKT condition $dc_+ = 0$, we get

$$\begin{aligned} dc_+ &= (d_\mathfrak{h}\, \alpha - A^* c_{\mathfrak{h}+}) \wedge u + (d_\mathfrak{h}\, \beta - B^* c_{\mathfrak{h}+})u' \\ &\quad + (-B^*\alpha + A^*\beta + d_\mathfrak{h}\gamma - \iota_V c_{\mathfrak{h}+}) \wedge u \wedge u' + d_\mathfrak{h} c_{\mathfrak{h}+} = 0, \end{aligned}$$

which forces $d_\mathfrak{h} c_{\mathfrak{h}+} = 0$, i.e., $(\mathfrak{h}, J_{\pm\mathfrak{h}}, \langle\cdot,\cdot\rangle_\mathfrak{h})$ is a generalized Kähler Lie algebra. Since $\mathfrak{h}$ is nilpotent, $\mathfrak{h}$ has to be abelian by [Cav].
For any $W \in \mathfrak{h}$:

$$\begin{aligned} c_+(W, U, U') &= -\langle [J_+W, J_+U], U'\rangle - \langle [J_+U, J_+U'], W\rangle - \langle [J_+U', J_+W], U\rangle \\ &= -\langle V, W\rangle \end{aligned}$$

and, analogously, $c_-(W, U, U') = -\varepsilon^2\langle V, W\rangle = -\langle V, W\rangle$. Since $c_+ = -c_-$, we must have $V = 0$, i.e., $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] = 0$. Moreover, this implies that the SKT structure $(J_+, \langle\cdot,\cdot\rangle)$ is Kähler by Corollary 2.1.3, statement (iv). q.e.d.

Remark 2.1.5. One can show that if a solvable Lie algebra with nilradical $\mathfrak{h}$ of even-codimension admits a generalized Kähler structure $(\langle\cdot,\cdot\rangle, J_\pm)$ such that $J_\pm\mathfrak{h} = \mathfrak{h}$, then $\mathfrak{h}$ is abelian. The proof proceeds in the same way as before, using a generalization of (2.1.1).

2.2 Case $J\mathfrak{h} \neq \mathfrak{h}$

Let $\mathfrak{g}$ be a $2n$-dimensional solvable Lie algebra with nilradical $\mathfrak{h}$ of codimension 2. Suppose that $\mathfrak{g}$ is endowed with an almost Hermitian structure $(J, \langle\cdot,\cdot\rangle)$ such that $J\mathfrak{h} \neq \mathfrak{h}$. As a consequence, we have

$$\mathfrak{g} = \mathfrak{h} + J\mathfrak{h}.$$

We set

$$\mathfrak{h}_J := \mathfrak{h} \cap J\mathfrak{h},$$

and we consider the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h}_J \oplus (\mathfrak{h}_J)^{\perp},$$

where both summands are J-invariant. Since $\mathfrak{h}$ has codimension 2, it follows that $\dim(\mathfrak{h}_J) = 2n - 4$.*

We now focus on the 4-dimensional J-invariant space

$$(\mathfrak{h}_J)^{\perp} = \mathfrak{k} \oplus \mathfrak{h}^{\perp},$$

where $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{h}_J$ in $\mathfrak{h}$. Let $\{e_{2n-1}, e_{2n}\}$ be any orthonormal basis of $\mathfrak{h}^{\perp}$. Then,

$$Je_{2n-1} = J_{34}e_{2n} + h_{2n-2}, \qquad Je_{2n} = -J_{34}e_{2n-1} + h_{2n-3},$$

where $J_{34} \in \mathbb{R}$ and h_{2n-2}, h_{2n-3} are non-zero, orthogonal vectors in $\mathfrak{k}$ such that

$$J^2e_{2n-1} = -e_{2n-1}, \qquad J^2e_{2n} = -e_{2n}.$$

Define

$$e_{2n-3} := \frac{h_{2n-3}}{\|h_{2n-3}\|}, \qquad e_{2n-2} := \frac{h_{2n-2}}{\|h_{2n-2}\|},$$

so that the set $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ is an orthonormal basis of the J-invariant subspace $(\mathfrak{h}_J)^{\perp}$.

With respect to the decomposition $\mathfrak{g} = \mathfrak{h}_J \oplus (\mathfrak{h}_J)^{\perp}$, the almost complex structure J splits as

$$J = \begin{pmatrix} J_{\mathfrak{h}_J} & 0 \\ 0 & J_{(\mathfrak{h}_J)^{\perp}} \end{pmatrix}.$$

With respect to the orthonormal basis $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ of $(\mathfrak{h}_J)^{\perp}$, the restriction

*Observe that when $2n = 4$, the decomposition above is trivial. Moreover, since SKT structures on 4-dimensional solvable Lie algebras have been fully described in [MS], we may restrict to the case $2n > 4$.

$J_{(\mathfrak{h}_J)^\perp}$ is represented by the skew-symmetric matrix

$$J_{(\mathfrak{h}_J)^\perp} = \begin{pmatrix} 0 & J_{12} & 0 & J_{14} \\ -J_{12} & 0 & J_{23} & 0 \\ 0 & -J_{23} & 0 & J_{34} \\ -J_{14} & 0 & -J_{34} & 0 \end{pmatrix},$$

where the entries J_{ij} satisfy the relations

$$\begin{cases} J_{12}^2 + J_{14}^2 = 1, \\ J_{12}^2 + J_{23}^2 = 1, \\ J_{23}^2 + J_{34}^2 = 1, \\ J_{14}^2 + J_{34}^2 = 1, \\ -J_{14}J_{34} + J_{12}J_{23} = 0, \\ -J_{14}J_{12} + J_{23}J_{34} = 0. \end{cases} \tag{2.2.1}$$

Note that $J_{14} \neq 0$ and $J_{23} \neq 0$, since $J\mathfrak{h} \neq \mathfrak{h}$. By solving the system (2.2.1), we obtain two distinct but equivalent almost complex structures on $(\mathfrak{h}_J)^\perp$, corresponding to either

$$J_{12} = -J_{34}, \quad J_{14} = -J_{23} \quad \text{or} \quad J_{12} = J_{34}, \quad J_{14} = J_{23}.$$

We will consider only the first case, as the two are equivalent up to a change of basis in $(\mathfrak{h}_J)^\perp$.

Remark 2.2.1. To sum it up, we may always endow $(\mathfrak{h}_J)^\perp$ with an orthonormal basis $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ such that $\{e_{2n-3}, e_{2n-2}\}$ and $\{e_{2n-1}, e_{2n}\}$ are unitary basis of $\mathfrak{k}$ and $\mathfrak{h}^\perp$ respectively, and the restricted almost complex structure $J_{(\mathfrak{h}_J)^\perp}$ can be written with respect to such a basis as

$$J_{(\mathfrak{h}_J)^\perp} = \begin{pmatrix} 0 & J_{12} & 0 & J_{14} \\ -J_{12} & 0 & -J_{14} & 0 \\ 0 & J_{14} & 0 & -J_{12} \\ -J_{14} & 0 & J_{12} & 0 \end{pmatrix}, \tag{2.2.2}$$

with $J_{12}^2 + J_{14}^2 = 1$ and $J_{14} \neq 0$.

For further purposes, we restrict to the case where $\mathfrak{h}$ is abelian, as this allows a more tractable investigation of the existence of generalized Kähler structures. We begin with a preliminary result:

Lemma 2.2.2. *Let $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ be a Hermitian solvable Lie algebra with codimension-2 abelian nilradical $\mathfrak{h}$ such that $J\mathfrak{h} \neq \mathfrak{h}$. Then $\mathfrak{h}_J := \mathfrak{h} \cap J\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Moreover, if $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ *is an orthonormal basis of* $(\mathfrak{h}_J)^\perp$ *as in Remark 2.2.1, then*

$$[A_{\mathfrak{h}_J}, J_{\mathfrak{h}_J}] = [B_{\mathfrak{h}_J}, J_{\mathfrak{h}_J}] = 0,$$

where $A_{\mathfrak{h}_J}, B_{\mathfrak{h}_J}, J_{\mathfrak{h}_J}$ *denote the restrictions of* $A := \mathrm{ad}_{e_{2n-1}}|_{\mathfrak{h}}, B := \mathrm{ad}_{e_{2n}}|_{\mathfrak{h}}$*, and* J *to* $\mathfrak{h}_J$*, respectively.*

Proof. Let $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ be an orthonormal basis of $(\mathfrak{h}_J)^\perp$ as in Remark 2.2.1. Since $\mathfrak{h}$ is abelian and $\mathfrak{g}^1 \subset \mathfrak{h}$, to prove that $\mathfrak{h}_J$ is an ideal it suffices to show that for any $X \in \mathfrak{h}_J$, we have $[X, e_{2n-1}], [X, e_{2n}] \in J\mathfrak{h}$.

By the integrability of J,

$$\begin{aligned} N_J(X, e_{2n-2}) &= J[X, Je_{2n-2}] - [JX, Je_{2n-2}] \\ &= J[X, J_{12}e_{2n-3} + J_{14}e_{2n-1}] - [JX, J_{12}e_{2n-3} + J_{14}e_{2n-1}] \\ &= J_{14}(J[X, e_{2n-1}] - [JX, e_{2n-1}]) = -J_{14}(JAX - AJX) = 0. \end{aligned}$$

Thus, $[X, e_{2n-1}] = J[-JX, e_{2n-1}] \in J\mathfrak{h}$. Analogously, $[X, e_{2n}] = J[-JX, e_{2n}] \in J\mathfrak{h}$. The final commutator identities follow directly from this and the integrability condition. q.e.d.

Proposition 2.2.3. *Let* $\mathfrak{g}$ *be a solvable Lie algebra with abelian nilradical* $\mathfrak{h}$ *of codimension* 2*. Suppose* $\mathfrak{g}$ *is endowed with an SKT structure* $(J, \langle \cdot, \cdot \rangle)$ *such that* $J\mathfrak{h} = \mathfrak{h}$*. If there exists another complex structure* I *compatible with* $\langle \cdot, \cdot \rangle$ *and such that* $I\mathfrak{h} \neq \mathfrak{h}$*, then* $(J, \langle \cdot, \cdot \rangle)$ *is Kähler.*

In particular, $\mathfrak{g}$ *does not admit any non-Kähler generalized Kähler structure* $(I, J, \langle \cdot, \cdot \rangle)$ *with* $I\mathfrak{h} \neq \mathfrak{h}$ *and* $J\mathfrak{h} = \mathfrak{h}$*.*

Proof. Let $\{e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ be an orthonormal basis of $(\mathfrak{h}_I)^\perp := (\mathfrak{h} \cap I\mathfrak{h})^\perp$ as in Remark 2.2.1. Let $\{e_1, \ldots, e_{2n-4}\}$ be any orthonormal basis of $\mathfrak{h}_I$, so that

$$\mathcal{B} = \{e_1, \ldots, e_{2n-4}, e_{2n-3}, e_{2n-2}\}$$

is an orthonormal basis of $\mathfrak{h}$. By Lemma 2.2.2, $\mathfrak{h}_I$ is an ideal of $\mathfrak{g}$.

Since $(\langle \cdot, \cdot \rangle, J)$ is SKT and $\mathfrak{h}$ is abelian, we have that $A := \mathrm{ad}_{e_{2n-1}}|_{\mathfrak{h}}, B := \mathrm{ad}_{e_{2n}}|_{\mathfrak{h}}$ lie in $\mathfrak{so}(\mathfrak{h})$ by Corollary 2.1.3, statement (i). With respect to the basis $\mathcal{B}$,

$$A = \left(\begin{array}{c|cc} A_{\mathfrak{h}_I} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & c_{12} \\ & -c_{12} & 0 \end{array}\right), \quad B = \left(\begin{array}{c|cc} B_{\mathfrak{h}_I} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & d_{12} \\ & -d_{12} & 0 \end{array}\right)$$

where $A_{\mathfrak{h}_I}, B_{\mathfrak{h}_I} \in \mathfrak{so}(\mathfrak{h}_I)$.

We now evaluate the Nijenhuis tensor N_I as a $(0,3)$-tensor using the inner product:

$$N_I(X,Y,Z) := \langle N_I(X,Y), Z\rangle.$$

Then:

$$N_I(e_{2n-2}, e_{2n-3}, e_{2n-1}) = -I_{14}c_{12}, \qquad N_I(e_{2n-2}, e_{2n-3}, e_{2n}) = -I_{14}d_{12}.$$

By integrability of I and the fact that $I_{14} \neq 0$ (cf. Remark 2.2.1), we conclude that $c_{12} = d_{12} = 0$.

Therefore,

$$[e_{2n-1}, e_{2n}] = -\big(I[Ie_{2n-1}, e_{2n}] + I[e_{2n-1}, Ie_{2n}]\big) + [Ie_{2n-1}, Ie_{2n}] = I_{12}^2[e_{2n-1}, e_{2n}],$$

which implies that

$$(1 - I_{12}^2)[e_{2n-1}, e_{2n}] = 0.$$

Using the identity $I_{12}^2 + I_{14}^2 = 1$, and using that $I_{14} \neq 0$, it follows that $[e_{2n-1}, e_{2n}] = 0$, i.e., $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] = 0$.

The conclusion then follows from Corollary 2.1.3, statement (iv). q.e.d.

Using the Symmetrization process ([BE, FG, UG]), one can show that if $M = \Gamma\backslash G$ is a compact solvmanifold endowed with a pair of invariant complex structures $J_\pm$, then M admits a generalized Kähler structure $(J_\pm, g)$ if and only if the Lie algebra $\mathfrak{g}$ of G admits a generalized Kähler inner product $(J_\pm, \langle\cdot,\cdot\rangle)$. Hence, we can prove the following result:

Theorem 2.2.4. *Let $M = \Gamma\backslash G$ be a $2n$-dimensional solvmanifold, and let $J_\pm$ be invariant complex structures on M. Assume that the nilradical $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G has codimension 2. Then:*

(i) If $J_\pm\mathfrak{h} = \mathfrak{h}$, then $(M, J_\pm)$ admits a generalized Kähler metric if and only if (M, J_+) admits a Kähler metric.

(ii) Assume that $\mathfrak{h}$ is abelian and that $J_+\mathfrak{h} \neq \mathfrak{h}$ while $J_-\mathfrak{h} = \mathfrak{h}$. If $(M, J_\pm)$ admits a generalized Kähler metric, then (M, J_-) also admits a Kähler metric.

Proof. To prove (i), observe that the implication from right to left is immediate. For the converse, by the Symmetrization process, $M = \Gamma\backslash G$ admits a generalized Kähler structure $(J_\pm, g)$ if and only if $\mathfrak{g}$ admits a generalized Kähler structure $(J_\pm, \langle\cdot,\cdot\rangle)$. The conclusion follows by applying Theorem 2.1.4.

To prove (ii), again by the Symmetrization process, there must exist a left-invariant generalized Kähler structure $(J_\pm, \langle\cdot,\cdot\rangle)$ on the Lie algebra $\mathfrak{g}$, with $J_+\mathfrak{h} \neq \mathfrak{h}$ and $J_-\mathfrak{h} = \mathfrak{h}$. Then the result follows by applying Proposition 2.2.3. q.e.d.

Corollary 2.2.5. *Let $M = \Gamma\backslash G$ be a $2n$-dimensional solvmanifold and let $I_\pm$ be left-invariant complex structures on M. Assume that the nilradical $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G is abelian and of codimension 2. If $(M, J_\pm)$ admits a (non Kähler) generalized Kähler metric, then $J_\pm\mathfrak{h} \neq \mathfrak{h}$.*

Remark 2.2.6. As seen in the previous section, if $\mathfrak{g}$ is a solvable Lie algebra with codimension 2 nilradical $\mathfrak{h}$ that admits a generalized Kähler structure $(J_\pm, \langle\cdot,\cdot\rangle)$ such that $J_\pm\mathfrak{h} = \mathfrak{h}$, then $\mathfrak{h}$ must be abelian. However, this is not necessarily true when $J_\pm\mathfrak{h} \neq \mathfrak{h}$.

Indeed, consider the Lie algebra

$$\mathfrak{s} \cong (\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^2 = \left(e^{23} + e^{17}, \tfrac{1}{2}e^{27}, \tfrac{1}{2}e^{37}, -e^{48}, \tfrac{1}{2}e^{58} - e^{67}, \tfrac{1}{2}e^{68} + e^{57}, 0, 0\right),$$

where $\{e_1, \dots, e_6\}$ is a basis of $\mathfrak{h}_3 \oplus \mathbb{R}^3$, $\{e_7, e_8\}$ is a basis of $\mathbb{R}^2$, and e^{ij} denotes $e^i \wedge e^j$.

From the structure equations, we see that the nilradical $\mathfrak{h}$ is spanned by $\{e_1, \dots, e_6\}$, and clearly $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}^3$.

Define the bi-Hermitian structure $(\langle\cdot,\cdot\rangle, J_\pm)$ by:

$$J_\pm e_1 = e_7, \quad J_\pm e_2 = e_3, \quad J_\pm e_5 = \pm e_6, \quad J_\pm e_4 = e_8, \qquad \langle\cdot,\cdot\rangle = \sum_{i=1}^{8} (e^i)^2.$$

The corresponding fundamental forms are given by

$$\omega_\pm = e^1 \wedge e^7 + e^2 \wedge e^3 \pm e^5 \wedge e^6 + e^4 \wedge e^8,$$

and satisfy:

$$d^c_\pm \omega_\pm = \pm e^4 \wedge e^5 \wedge e^6.$$

Since $e^4 \wedge e^5 \wedge e^6$ is closed, it follows that $(J_\pm, \langle\cdot,\cdot\rangle)$ defines a generalized Kähler structure.

However, we note that since the Lie algebra $\mathfrak{s}$ is not unimodular, the corresponding simply connected Lie group does not admit lattices, i.e., there exists no compact quotient.

2.3 Classification in dimension six

In this section we provide a classification of six-dimensional unimodular solvable Lie algebras with nilradical of codimension 2 that admit a SKT structure.

Theorem 2.3.1. *A unimodular six-dimensional solvable Lie algebra $\mathfrak{g}$ with nilradical $\mathfrak{h}$ of codimension 2 admits a SKT structure $(J, \langle\cdot,\cdot\rangle)$ if and only if $\mathfrak{g}$ is isomorphic to one of the* *following Lie algebras:*

$$\tau_{3,0}\times\tau_{3,0}=(-f^{25},f^{15},-f^{46},f^{36},0,0),$$
$$\mathfrak{g}_{5.35}^{-2,0}\oplus\mathbb{R}=(2f^{15},-f^{25}-f^{36},-f^{35}+f^{26},0,0,0).$$

Hence, in particular, $\mathfrak{h}$ *must be abelian. An explicit example of SKT structure is given respectively by*

$$Jf_1=f_2,\ Jf_3=f_4,\ Jf_5=f_6,\ \langle\cdot,\cdot\rangle=\sum_{i=1}^{6}(f^i)^2$$
$$Jf_1=f_5,\ Jf_2=f_3,\ Jf_4=f_6,\ \langle\cdot,\cdot\rangle=\sum_{i=1}^{6}(f^i)^2.$$

Proof. We discuss separately the cases $J\mathfrak{h}=\mathfrak{h}$ and $J\mathfrak{h}\neq\mathfrak{h}$.

1. $J\mathfrak{h}=\mathfrak{h}$**.** Assume that $\mathfrak{g}$ is endowed with a SKT structure $(J,\langle\cdot,\cdot\rangle)$ such that $J\mathfrak{h}=\mathfrak{h}$. By Theorem 2.1.1, $\mathfrak{h}$ is a 4-dimensional nilpotent SKT Lie algebra, so we have that either $\mathfrak{h}=\mathbb{R}^4$ or $\mathfrak{h}=\mathfrak{h}_3\oplus\mathbb{R}$.

(a) *Case* $\mathfrak{h}=\mathbb{R}^4$. In this case, one can easily prove that $\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h}=0$ (otherwise we get a contradiction with the fact that $\mathfrak{h}$ has codimension 2), and so by the proof of statement (iv) in Corollary 2.1.3, we get that $\mathfrak{h}=[\mathfrak{g},\mathfrak{g}]$. The classification follows by [FS, FS2] and the Lie algebra $\mathfrak{g}$ must be isomorphic to $\tau_{3,0}\times\tau_{3,0}$.

(b) *Case* $\mathfrak{h}=\mathfrak{h}_3\oplus\mathbb{R}$. We prove that this case cannot occur. Assume by contradiction that $\mathfrak{g}$ admits a SKT structure $(\langle\cdot,\cdot\rangle,J)$ such that $J\mathfrak{h}=\mathfrak{h}$. Then

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^\perp=\mathfrak{z}(\mathfrak{h})\oplus\mathfrak{z}(\mathfrak{h})^\perp\oplus\mathfrak{h}^\perp.$$

Let e_1 be a generator of $\mathfrak{h}^1=[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{z}(\mathfrak{h})$ which, up to rescaling, we may assume to be unitary. Then an orthonormal basis of $\mathfrak{g}$ is provided by $\mathcal{B}=\{e_1,e_2=Je_1,e_3,e_4=Je_3,e_5,e_6=Je_5\}$, where:

- $\{e_1,e_2\}$ is a basis of $\mathfrak{z}(\mathfrak{h})$,
- $\{e_3,e_4\}$ is a basis of $\mathfrak{z}(\mathfrak{h})^\perp$,
- $\{e_5,e_6\}$ is a basis of $\mathfrak{h}^\perp$.

The Lie algebra $\mathfrak{g}$ is completely determined by the data:

$$[e_5,X]=AX,\quad [e_6,X]=BX,\quad \forall X\in\mathfrak{h},\quad [e_3,e_4]=\eta e_1,\quad [e_5,e_6]=V,$$

where $\eta\in\mathbb{R}\setminus\{0\}$, and $A=\mathrm{ad}_{e_5}|_{\mathfrak{h}}, B=\mathrm{ad}_{e_6}|_{\mathfrak{h}}$ are derivations of $\mathfrak{h}$ satisfying $[A,B]=\mathrm{ad}_V|_{\mathfrak{h}}$.

From earlier results, we may decompose A and B as:

$$A = \begin{pmatrix} A_{\mathfrak{z}(\mathfrak{h})} & *_A \\ 0 & A_{\mathfrak{z}(\mathfrak{h})^\perp} \end{pmatrix}, \quad B = \begin{pmatrix} B_{\mathfrak{z}(\mathfrak{h})} & *_B \\ 0 & B_{\mathfrak{z}(\mathfrak{h})^\perp} \end{pmatrix},$$

with $A_{\mathfrak{z}(\mathfrak{h})}, B_{\mathfrak{z}(\mathfrak{h})} \in \mathfrak{so}(\mathfrak{z}(\mathfrak{h}))$ such that $[A_{\mathfrak{z}(\mathfrak{h})}, J_{\mathfrak{z}(\mathfrak{h})}] = [B_{\mathfrak{z}(\mathfrak{h})}, J_{\mathfrak{z}(\mathfrak{h})}] = 0$ by Theorem 2.1.1.

Moreover, since A and B are derivations of $\mathfrak{h}$, we have $A\mathfrak{h}^1 \subset \mathfrak{h}^1$ and $B\mathfrak{h}^1 \subset \mathfrak{h}^1$, hence $A_{\mathfrak{z}(\mathfrak{h})} = B_{\mathfrak{z}(\mathfrak{h})} = 0$. In particular, this implies $\operatorname{tr}(A_{\mathfrak{z}(\mathfrak{h})^\perp}) = \operatorname{tr}(B_{\mathfrak{z}(\mathfrak{h})^\perp}) = 0$.

Let us write:

$$A_{\mathfrak{z}(\mathfrak{h})^\perp} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & -a_{33} \end{pmatrix}, \quad B_{\mathfrak{z}(\mathfrak{h})^\perp} = \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & -b_{33} \end{pmatrix}.$$

Using that $A_{\mathfrak{z}(\mathfrak{h})} = B_{\mathfrak{z}(\mathfrak{h})} = 0$, the integrability condition

$$[J_\mathfrak{h}, A]J_\mathfrak{h} + [J_\mathfrak{h}, B] = 0$$

together with $[A, B] = \operatorname{ad}_V|_\mathfrak{h}$ yield:

$$[J_\mathfrak{h}, A_{\mathfrak{z}(\mathfrak{h})^\perp}]J_\mathfrak{h} + [J_\mathfrak{h}, B_{\mathfrak{z}(\mathfrak{h})^\perp}] = 0 \iff 2a_{33} = b_{34} + b_{43},\ 2b_{33} = -(a_{34} + a_{43}),$$
$$[A_{\mathfrak{z}(\mathfrak{h})^\perp}, B_{\mathfrak{z}(\mathfrak{h})^\perp}] = 0 \iff 2a_{33}(b_{34} + b_{43}) - 2b_{33}(a_{34} + a_{43}) = 0.$$

These conditions together imply that $A_{\mathfrak{z}(\mathfrak{h})^\perp}$ and $B_{\mathfrak{z}(\mathfrak{h})^\perp}$ are skew-symmetric.

If any of a_{34} or b_{34} were zero, the corresponding matrix would be nilpotent, which contradicts the assumption that $\mathfrak{h}$ has codimension 2. Hence, we must have $a_{34}, b_{34} \neq 0$.

Finally, consider the subspace generated by

$$\left\{ e_1, \ldots, e_4,\ e_5 - \frac{a_{34}}{b_{34}} e_6 \right\}.$$

This subspace is a nilpotent ideal strictly containing $\mathfrak{h}$, contradicting the fact that $\mathfrak{h}$ is the nilradical. Thus, this case cannot occur.

2. Case $J\mathfrak{h} \neq \mathfrak{h}$ In this case, the four-dimensional nilradical is isomorphic to one of $\mathbb{R}^4, \mathfrak{h}_3 \oplus \mathbb{R}$ and $\mathfrak{h}_4 = (-24, -34, 0, 0)$. Moreover, the only six-dimensional solvable Lie algebra with nilradical $\mathfrak{h}_4$ is $\mathfrak{g}_{6.28} = (0, 0, 46 - 13 - 2.25, 56 - 24, 15, -16 + 26)$ (see [RT]), which is not unimodular.

Hence, as in the previous case, we may restrict to consider either $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}$ or $\mathfrak{h} = \mathbb{R}^4$. We start by considering the first case, which is more involved.

(a) *Case* $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}$. Firstly, we prove that if $\mathfrak{g}$ admits a complex structure J such that $J\mathfrak{h} \neq \mathfrak{h}$, then $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 1$. Since the center $\mathfrak{z}(\mathfrak{h})$ has dimension 2, it suffices to check that if $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 0, 2$ we get a contradiction.

i. $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 0$.
Let Z_1 be a generator of $\mathfrak{h}^1 = [\mathfrak{h}, \mathfrak{h}]$, and let Z_2 be such that

$$\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{R}}\langle Z_1, Z_2\rangle.$$

Since $\mathfrak{g} = \mathfrak{h} + J\mathfrak{h}$, we may fix a basis

$$\{Z_1, Z_2, X, JX, JZ_1, JZ_2\}$$

for $\mathfrak{g}$, where $X \in \mathfrak{h}_J := \mathfrak{h} \cap J\mathfrak{h}$.
Observe that $\mathrm{ad}_{JZ_1}|_{\mathfrak{h}}$ and $\mathrm{ad}_{JZ_2}|_{\mathfrak{h}}$ are derivations of $\mathfrak{h}$, and hence preserve both $\mathfrak{h}^1$ and $\mathfrak{z}(\mathfrak{h})$. Exploiting this, together with the integrability of J and the inclusion $\mathfrak{g}^1 \subset \mathfrak{h}$, we obtain the following structure equations:

$$\begin{cases} [X, JX] &= \eta Z_1, \\ [Z_1, JZ_1] &= a_{11} Z_1, \\ [Z_1, JZ_2] &= a_{12} Z_1, \\ [Z_2, JZ_1] &= a_{12} Z_1, \\ [Z_2, JZ_2] &= b_{12} Z_1 + b_{22} Z_2, \\ [X, JZ_1] &= a_{33} X + a_{43} JX, \\ [X, JZ_2] &= b_{33} X + b_{43} JX, \\ [JX, JZ_1] &= -a_{43} X + a_{33} JX, \\ [JX, JZ_2] &= -b_{43} X + b_{33} JX, \\ [JZ_1, JZ_2] &= 0. \end{cases}$$

Since

$$\mathrm{ad}_{JZ_1}[X, JX] = [\mathrm{ad}_{JZ_1}(X), JX] + [X, \mathrm{ad}_{JZ_1}(JX)]$$

and similarly for $\mathrm{ad}_{JZ_2}[X, JX]$, we deduce the identities

$$a_{11} = 2a_{33}, \qquad a_{12} = 2b_{33}.$$

In particular, unimodularity implies

$$a_{11} = a_{33} = 0,$$

and consequently
$$b_{22} = -4b_{33}.$$
Moreover, the condition
$$[\mathrm{ad}_{JZ_1}, \mathrm{ad}_{JZ_2}] = \mathrm{ad}_{[JZ_1, JZ_2]} = 0$$
holds if and only if
$$b_{33} = 0.$$

In summary, the Lie algebra $\mathfrak{g}$ is determined by the data
$$\mathrm{ad}_{JZ_1}|_{\mathfrak{h}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{43} \\ 0 & 0 & -a_{43} & 0 \end{pmatrix}, \quad \mathrm{ad}_{JZ_2}|_{\mathfrak{h}} = \begin{pmatrix} 0 & -b_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{43} \\ 0 & 0 & -b_{43} & 0 \end{pmatrix},$$
and
$$[X, JX] = \eta Z_1 \neq 0.$$

Note that $a_{43} \neq 0$, since otherwise the Lie bracket structure would contradict previous assumptions. In particular, consider the subspace generated by
$$\{Z_1, Z_2, X, JX, JZ_2 - \tfrac{b_{43}}{a_{43}} JZ_1\}.$$
This subspace defines a nilpotent ideal strictly containing $\mathfrak{h}$, which contradicts the assumption on the codimension. The conclusion follows.

ii. $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 2$. In this case, $\mathfrak{h}_J = \mathfrak{z}(\mathfrak{h})$, so we may fix a basis $\{Z, JZ\}$ of $\mathfrak{z}(\mathfrak{h})$, where Z is a generator of $\mathfrak{h}^1$. Since $\mathfrak{g} = \mathfrak{h} + J\mathfrak{h}$, we complete Z, JZ to a basis
$$\{Z, JZ, X, Y, JX, JY\}$$
of $\mathfrak{g}$, where $\{Z, JZ, X, Y\}$ is a basis of $\mathfrak{h}$. Analogously to the previous case,

the structure equations are

$$\begin{cases} [X,Y] = \eta Z, \\ [Z,JX] = a_{11}Z, \\ [Z,JY] = b_{11}Z, \\ [JZ,JX] = a_{11}JZ, \\ [JZ,JY] = b_{11}JZ, \\ [X,JX] = a_{13}Z + a_{23}JZ + a_{33}X + a_{43}Y, \\ [X,JY] = b_{13}Z + b_{23}JZ + a_{34}X + a_{44}Y, \\ [Y,JX] = a_{14}Z + a_{24}JZ + a_{34}X + a_{44}Y, \\ [Y,JY] = b_{14}Z + b_{24}JZ + b_{34}X + b_{44}Y, \\ [JX,JY] = (\eta + (a_{24} - b_{23}))Z + (b_{13} - a_{14})JZ. \end{cases}$$

Exploiting $\mathrm{ad}_{JX}[X,Y] = [\mathrm{ad}_{JX}X,Y]+[X,\mathrm{ad}_{JX}Y]$, and $\mathrm{ad}_{JY}[X,Y] = [\mathrm{ad}_{JY}X,Y]+[X,\mathrm{ad}_{JY}Y]$ we further obtain the relations

$$a_{11} = a_{33} + a_{44}, \quad b_{11} = a_{34} + b_{44}.$$

Combining these with the unimodularity condition, we get

$$a_{11} = 0, \quad a_{33} = -a_{44}, \quad b_{11} = 0, \quad a_{34} = -b_{44}.$$

Hence,

$$\mathrm{ad}_{JX}\big|_{\mathfrak{h}} = \begin{pmatrix} 0 & 0 & -a_{13} & -a_{14} \\ 0 & 0 & -a_{23} & -a_{24} \\ 0 & 0 & -a_{33} & -a_{34} \\ 0 & 0 & -a_{43} & a_{33} \end{pmatrix}, \quad \mathrm{ad}_{JY}\big|_{\mathfrak{h}} = \begin{pmatrix} 0 & 0 & -b_{13} & -b_{14} \\ 0 & 0 & -b_{23} & -b_{24} \\ 0 & 0 & -a_{34} & -b_{34} \\ 0 & 0 & a_{33} & a_{34} \end{pmatrix},$$

with $[X,Y] = \eta Z$, and $[JX,JY] = (\eta+(a_{24}-b_{23}))Z+(b_{13}-a_{14})JZ \in \mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{g})$. Since $[JX,JY] \in \mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{g})$, as before $[\mathrm{ad}_{JX}, \mathrm{ad}_{JY}] = \mathrm{ad}_{[JX,JY]} = 0$. Using this, if one computes the spectrum of ad_{JX}, one obtains that ad_{JX} has only the eigenvalue 0, so it is nilpotent. Hence, the subspace generated by $\{Z, JZ, X, Y, JX\}$ is a nilpotent ideal which strictly contains $\mathfrak{h}$, giving a contradiction.

Hence, we may restrict to consider the case of $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 1$, and we prove that if $\dim(J\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}) = 1$, then $\mathfrak{g}$ cannot admit any SKT structure $(\langle\cdot,\cdot\rangle, J)$ such that $J\mathfrak{h} \neq \mathfrak{h}$. Let Z_1, Z_2 be a basis of $\mathfrak{z}(\mathfrak{h})$ such that $JZ_1 \in \mathfrak{h}$ and $JZ_2 \notin \mathfrak{h}$. Since

$\mathfrak{g} = \mathfrak{h} + J\mathfrak{h}$, there always exists a vector $X \in \mathfrak{h}$ such that

$$\{Z_1, JZ_1, Z_2, X, JZ_2, JX\}$$

is a basis of $\mathfrak{g}$. We proceed as before. Indeed, using that ad_{JZ_2} and ad_{JX} preserve $\mathfrak{z}(\mathfrak{h})$, the integrability of J, and the inclusion $\mathfrak{g}^1 \subset \mathfrak{h}$, we get

$$\begin{cases} [JZ_1, X] = \eta_1 Z_1 + \eta_2 Z_2, \\ [Z_1, JZ_2] = a_{11} Z_1, \\ [Z_1, JX] = b_{11} Z_1 - \eta_2 Z_2, \\ [JZ_1, JZ_2] = a_{11} JZ_1, \\ [JZ_1, JX] = (\eta_1 + b_{11}) JZ_1, \\ [Z_2, JZ_2] = a_{13} Z_1 + a_{33} Z_2, \\ [Z_2, JX] = b_{13} Z_1 + a_{34} Z_2, \\ [X, JZ_2] = a_{14} Z_1 + a_{24} JZ_1 + a_{34} Z_2, \\ [X, JX] = b_{14} Z_1 + b_{24} JZ_1 + b_{34} Z_2 + b_{44} X, \\ [JZ_2, JX] = a_{24} Z_1 + (b_{13} - a_{14}) JZ_1. \end{cases}$$

By the unimodularity of $\mathfrak{g}$,

$$a_{33} = -2a_{11} \quad \text{and} \quad 2b_{11} + \eta_1 + a_{34} + b_{44} = 0.$$

Observe that if $a_{11} = 0$, then ad_{JZ_1} is strictly upper triangular, hence nilpotent. Thus $a_{11} \neq 0$, and, exploiting

$$\mathrm{ad}_{JZ_2}[JZ_1, X] = [\mathrm{ad}_{JZ_2}(JZ_1), X] + [JZ_1, \mathrm{ad}_{JZ_2}(X)],$$

we get $\eta_2 = 0$.

Analogously, computing

$$\mathrm{ad}_{JX}[JZ_1, X] = [\mathrm{ad}_{JX}(JZ_1), X] + [JZ_1, \mathrm{ad}_{JX}(X)],$$

one obtains

$$b_{44} = -\eta_1.$$

Plugging this into the unimodularity condition $2b_{11} + \eta_1 + a_{34} + b_{44} = 0$, we also get

$$2b_{11} + a_{34} = 0.$$

To sum it up,

$$\mathrm{ad}_{JZ_2}|_{\mathfrak{h}} = \begin{pmatrix} -a_{11} & 0 & -a_{13} & -a_{14} \\ 0 & -a_{11} & 0 & -a_{24} \\ 0 & 0 & 2a_{11} & 2b_{11} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{JX}|_{\mathfrak{h}} = \begin{pmatrix} -b_{11} & 0 & -b_{13} & -b_{14} \\ 0 & -(\eta_1 + b_{11}) & 0 & -b_{24} \\ 0 & 0 & 2b_{11} & -b_{34} \\ 0 & 0 & 0 & \eta_1 \end{pmatrix},$$

$$[JZ_2, JX] = a_{24}Z_1 + (b_{13} - a_{14})JZ_1.$$

Consider any J-Hermitian inner product $\langle \cdot, \cdot \rangle$ and denote by c the Bismut torsion 3-form of $(J, \langle \cdot, \cdot \rangle, J)$. Then,

$$dc(Z_1, JZ_1, X, JZ_2) = -2a_{11}\eta_1 \|Z_1\|^2 \neq 0.$$

Indeed, this can be zero only if $\eta_1 = 0$. Moreover, in this case, $\mathfrak{h} = \mathbb{R}^4$, giving a contradiction.

Hence, we have proved that if $\mathfrak{g}$ is endowed with an SKT structure $(J, \langle \cdot, \cdot \rangle)$ such that $J\mathfrak{h} \neq \mathfrak{h}$, then $\mathfrak{h}$ must be abelian.

(b) *Case* $\mathfrak{h} = \mathbb{R}^4$ Let us assume that $\mathfrak{g}$ admits a SKT structure $(J, \langle \cdot, \cdot \rangle)$ such that $J\mathfrak{h} \neq \mathfrak{h}$, with $\mathfrak{h} = \mathbb{R}^4$. We fix an orthonormal basis $\mathcal{B} = \{e_1, \dots, e_6\}$ of $\mathfrak{g}$ such that $\{e_1, e_2\}$ is an orthonormal basis of $\mathfrak{h}_J$ such that $Je_1 = e_2$ and $\{e_3, \dots, e_6\}$ is an orthonormal basis of $(\mathfrak{h}_J)^{\perp}$ as in Remark 2.2.1, namely, with respect to $\mathcal{B}$ the complex structure J can be written as

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{12} & 0 & J_{14} \\ 0 & 0 & -J_{12} & 0 & -J_{14} & 0 \\ 0 & 0 & 0 & J_{14} & 0 & -J_{12} \\ 0 & 0 & -J_{14} & 0 & J_{12} & 0 \end{pmatrix}.$$

By Proposition 2.2.2, $\mathfrak{h}_J$ is an ideal of $\mathfrak{g}$ and $[A_{\mathfrak{h}_J}, J] = [B_{\mathfrak{h}_J}, J] = 0$, where $A_{\mathfrak{h}_J}, B_{\mathfrak{h}_J}$ denote the restrictions of $A := ad_{e_5}|_{\mathfrak{h}}, B := ad_{e_6}|_{\mathfrak{h}}$ to $\mathfrak{h}_J$, respectively. By the integrability of J, we have that

$$A = \left(\begin{array}{cc|cc} a_{11} & a_{12} & \multicolumn{2}{c}{*} \\ -a_{12} & a_{11} & & \\ \hline \multicolumn{2}{c|}{0} & c_{11} & c_{12} \\ & & c_{21} & c_{22} \end{array}\right), \quad B = \left(\begin{array}{cc|cc} b_{11} & b_{12} & \multicolumn{2}{c}{*} \\ -b_{12} & b_{11} & & \\ \hline \multicolumn{2}{c|}{0} & d_{11} & -c_{11} \\ & & d_{21} & -c_{21} \end{array}\right),$$

$$[e_5, e_6] = V_{\mathfrak{h}_J} - \frac{J_{12}}{J_{14}}(c_{12} - d_{11})e_3 - \frac{J_{12}}{J_{14}}(c_{22} - d_{21})e_4,$$

with $2a_{11} + c_{11} + c_{22} = 0$ and $2b_{11} + d_{11} - c_{21} = 0$ by the unimodularity of $\mathfrak{g}$. In the following we will denote by

$$C := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \text{ and } D := \begin{pmatrix} d_{11} & -c_{11} \\ d_{21} & -c_{21} \end{pmatrix}.$$

Since by the Jacobi identity $[A, B] = 0$, then also $[C, D] = 0$.
By the SKT condition, we get

$$\begin{aligned} dc(e_1, e_2, e_3, e_6) &= 2J_{14}(-b_{11}c_{21} + a_{11}d_{21}) = 0, \\ dc(e_1, e_2, e_3, e_5) &= 2J_{14}(b_{11}c_{22} + a_{11}c_{21}) = 0, \\ dc(e_1, e_2, e_4, e_5) &= -2J_{14}(b_{11}c_{12} + a_{11}c_{11}) = 0 \\ dc(e_1, e_2, e_4, e_6) &= 2J_{14}(b_{11}c_{11} - a_{11}d_{11}) = 0 \end{aligned}$$

Since $J_{14} \neq 0$, the coefficients of A and B must obey to

$$\begin{aligned} b_{11}c_{21} - a_{11}d_{21} &= 0, \\ b_{11}c_{22} + a_{11}c_{21} &= 0, \\ b_{11}c_{12} + a_{11}c_{11} &= 0, \\ b_{11}c_{11} - a_{11}d_{11} &= 0. \end{aligned} \tag{2.3.1}$$

We distinguish two cases depending on whether a_{11} is zero or not. We will do in details the first case, as the second one is analogous.
Let $a_{11} = 0$. The conditions (2.3.1) becomes

$$b_{11}c_{21} = 0, \quad b_{11}c_{22} = 0, \quad b_{11}c_{12} = 0, \quad b_{11}c_{11} = 0,$$

and furthermore $c_{11} = -c_{22}$ in order to have $\mathrm{tr}(A) = 0$.
We claim that $b_{11} \neq 0$. Assume by contradiction that $b_{11} = 0$. Then (2.3.1) are satisfied and $d_{11} = c_{21}$ by the unimodularity condition. Using $[C, D] = 0$, one can show that C and D are nilpotent matrices and so we must have $a_{12}, b_{12} \neq 0$. Indeed, if for instance $a_{12} = 0$, then the subspace generated by $\{e_1, e_2, e_3, e_4, e_5\}$

would be a nilpotent ideal which strictly contains $\mathfrak{h}$, which is a contradiction with the maximality of the nilradical.
If $a_{12}, b_{12} \neq 0$, then we may consider $e_5' = e_5 - \frac{a_{12}}{b_{12}} e_6$ with

$$ad_{e_5'}|_{\mathfrak{h}} = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & E \end{array}\right),$$

where $E = C - \frac{a_{12}}{b_{12}} D$. Moreover, since C, D are nilpotent and $[C, D] = 0$, so is E and, hence, also $\mathrm{ad}_{e_5'}|_{\mathfrak{h}}$. Therefore, if we consider the ideal $\{e_1, e_2, e_3, e_4, e_5'\}$, then again this is a nilpotent ideal which strictly contains the nilradical, proving the claim.
Since $b_{11} \neq 0$, by (2.3.1) we must have that

$$A = \left(\begin{array}{cc|cc} 0 & a_{12} & \multicolumn{2}{c}{\multirow{2}{*}{$*$}} \\ -a_{12} & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 0 \end{array}\right), \quad B = \left(\begin{array}{cc|cc} b_{11} & b_{12} & \multicolumn{2}{c}{\multirow{2}{*}{$*$}} \\ -b_{12} & b_{11} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -2b_{11} & 0 \\ & & d_{21} & 0 \end{array}\right),$$

$$[e_5, e_6] = V_{\mathfrak{h}_J} - 2\frac{J_{12}}{J_{14}} b_{11} e_3 + \frac{J_{12}}{J_{14}} d_{21} e_4,$$

with $a_{12} \neq 0$ (otherwise A would be nilpotent). In order to kill the components of V along e_3 and e_4, we take $e_5' := e_5 + \frac{J_{12}}{J_{14}} e_3$. In such a way $[e_5', e_6] \in \mathfrak{h}_J$ and $\{e_5', e_6\}$ define a new complement of $\mathfrak{h}$ inside $\mathfrak{g}$. Observe that since $\mathfrak{h}$ is abelian, $A' := \mathrm{ad}_{e_5'}|_{\mathfrak{h}} = A$.
Let e_3' and e_4' be eigenvectors of B associated to the eigenvalues $-2b_{11}$ and 0, respectively. Since the eigenspaces $V_{-2b_{11}}$ and V_0 are 1-dimensional and $[A, B] = 0$, we must have that e_3' and e_4' are also eigenvectors of A with eigenvalue 0. Hence, with respect the new basis $\{e_1, e_2, e_3', e_4', e_5', e_6\}$ the Lie algebra $\mathfrak{g}$ is determined by the data

$$A' = \left(\begin{array}{cc|cc} 0 & a_{12} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -a_{12} & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 0 \end{array}\right), B = \left(\begin{array}{cc|cc} b_{11} & b_{12} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -b_{12} & b_{11} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -2b_{11} & 0 \\ & & 0 & 0 \end{array}\right),$$

$$[e_5', e_6] \in \mathfrak{h}_J.$$

Finally, let $X \in \mathfrak{h}_J$ be such that $A'X = -[e_5', e_6]$ (observe that it is possible since $\mathrm{Im} A' = \mathfrak{h}_J$). The basis $\{e_1', \dots, e_5', e_6' = e_6 + X\}$ is such that $B := \mathrm{ad}_{e_6'}|_{\mathfrak{h}} = B'$ and $[e_5', e_6'] = 0$. The isomorphism between $\mathfrak{g}$ and $\mathfrak{g}_{5.35}^{-2,0} \oplus \mathbb{R}$ is immediate.

q.e.d.

Remark 2.3.2. The Lie algebra $\mathfrak{g}_{5.35}^{-2,0}$ firstly appeared in [Bo].

Remark 2.3.3. The connected and simply connected solvable Lie groups corresponding to $\tau_{3,0} \times \tau_{3,0}$ and $\mathfrak{g}_{5.35}^{-2,0} \oplus \mathbb{R}$ admit lattices.

Corollary 2.3.4. *A unimodular six-dimensional solvable Lie algebra with nilradical $\mathfrak{h}$ of codimension 2 admits a generalized Kähler structure $(J_\pm, \langle \cdot, \cdot \rangle)$ if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

$$\begin{aligned}
&\tau_{3,0} \times \tau_{3,0} = (-f^{25}, f^{15}, -f^{46}, f^{36}, 0, 0),\\
&\mathfrak{g}_{5.35}^{-2,0} \oplus \mathbb{R} = (2f^{15}, -f^{25} - f^{36}, -f^{35} + f^{26}, 0, 0, 0).
\end{aligned}$$

An explicit example of generalized Kähler structure is given respectively by

$$J_\pm f_1 = \pm f_2,\ J_\pm f_3 = \pm f_4,\ J_\pm f_5 = \pm f_6,\ \langle \cdot, \cdot \rangle = \sum_{i=1}^{6} (f^i)^2$$

$$J_\pm f_1 = f_5,\ J_\pm f_2 = \pm f_3,\ J_\pm f_4 = f_6,\ \langle \cdot, \cdot \rangle = \sum_{i=1}^{6} (f^i)^2.$$

Chapter 3

Generalized Kähler manifolds via suspensions

In [AG], the authors study the existence of *ambihermitian* generalized Kähler structures on compact complex surfaces. Recall that a generalized Kähler structure $(J_{\pm}, g)$ is said to be *ambihermitian* if the complex structures J_+ and J_- induce opposite orientations. The only non-Kähler surfaces admitting such structures are the Inoue surface and the Hopf surface (cf. [AG, Theorem 3]). Among these, only the Inoue surface admits *invariant* generalized Kähler structures.

It is well known that both the Inoue and Hopf surfaces satisfy $b_1 = 1$, and that the corresponding non-trivial cohomology class is represented by a nowhere-vanishing closed 1-form. By Tischler's theorem [Ti], this implies that both surfaces admit a structure of *suspensions* or *mapping tori*.

The case of $\mathrm{SU}(2) \times S^1$ is straightforward, as it is the suspension of $\mathrm{SU}(2)$ via the identity map. The Inoue surface, on the other hand, admits a description as the suspension of the 3-torus $\mathbb{T}^3$ via a suitable diffeomorphism ρ, and is also known to be diffeomorphic to a compact 4-dimensional solvmanifold (cf. Section 3.1). More generally, any solvmanifold admits at least one invariant closed 1-form; hence, *every solvmanifold can be topologically realized as a suspension*.

As recalled in the Preliminaries, the problem of constructing generalized Kähler structures on solvmanifolds has received considerable attention over the past two decades. Explicit examples have been constructed in [FTo, FP, FP2, FP3, BrF]; all of them admit an abelian nilradical and can be realized as torus bundles over the Inoue surface.

Among these, the *almost abelian* case plays a particularly prominent role. In fact, with the exception of the examples presented in [BrF], all known constructions fall within this class. In the almost abelian setting, both the suspension structure of the solvmanifolds and their interpretation as torus bundles over the Inoue surface are especially transparent. This observation motivates the central question addressed in this chapter:

Can all known examples of almost abelian generalized Kähler solvmanifolds be recovered from a general construction via suspensions?

The construction we propose is as follows. As mentioned, the Inoue surface arises as the suspension of $\mathbb{T}^3$ via a diffeomorphism ρ, which will be described explicitly in Section 3.1. We then consider a (hyper)Kähler manifold K, and define the suspension of $K \times \mathbb{T}^3$ via the block-diagonal diffeomorphism (ψ, ρ), where ψ is a (hyper)Kähler isometry of K. Depending on the nature of K, the resulting structure will be split or non-split generalized Kähler.

We shall focus on explicit examples. Although all known almost-abelian generalized Kähler solvmanifolds arise from this construction by taking $K = \mathbb{T}^{2k}$ and a suitable ψ, we will analyze two illustrative cases in detail.

More importantly, this construction also produces examples that are not diffeomorphic to solvmanifolds; for instance, when K is a K3 surface. However, identifying explicit hyperkähler isometries of a K3 surface is challenging, primarily because the hyperkähler metric itself is not explicitly known. Nonetheless, we will show that for the Fermat quartic, it is possible to find an explicit expression for such an isometry, even without knowing the exact form of the metric (cf. Example 3.1.14).

An analogous construction applies to the Hopf surface, which admits two different (and non-isomorphic) generalized Kähler structures. We will show that, depending on the choice of structure, one obtains different generalized Kähler structures on suspensions of $K \times \mathrm{SU}(2)$. This construction can be further generalized by replacing the Hopf surface with arbitrary compact Lie groups (of even dimension). As we shall see in the next chapter, these examples based on the Hopf surface also yield non-trivial examples of Bismut Hermitian Einstein manifolds.

The results presented in this chapter are contained in the published papers [BrF2, BFG].

3.1 Construction via the Inoue surface

3.1.1 Split examples

We begin this section by describing the suspension structure of the Inoue surface, as well as its generalized Kähler structure.

Example 3.1.1. Let p and t_0 be non-zero real numbers such that the matrix

$$\rho(t_0) = \begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & e^{-\frac{t_0}{2}}\cos(t_0 p) & e^{-\frac{t_0}{2}}\sin(t_0 p) \\ 0 & -e^{-\frac{t_0}{2}}\sin(t_0 p) & e^{-\frac{t_0}{2}}\cos(t_0 p) \end{pmatrix} \tag{3.1.1}$$

is similar to an integer matrix A. The existence of such a pair (p, t_0) is ensured by [AO, Section 3.2.2]. Thus, there exists an invertible matrix P such that

$$PA = \rho(t_0)P.$$

We define a lattice $\Gamma_0 \subset \mathbb{R}^3$ by

$$\Gamma_0 := P\mathbb{Z}^3.$$

Since $\rho(t_0)P = PA$ and A is integer, it follows that $\rho(t_0)$ preserves the lattice Γ_0, and hence descends to a diffeomorphism of the torus $\mathbb{T}^3 := \Gamma_0 \backslash \mathbb{R}^3$. From now on, we denote $\rho(t_0)$ simply by ρ.

For each $t \in [0, t_0]$, consider the following time-dependent frame on $\mathbb{T}^3$:

$$\begin{aligned}
e_1(t) &= e^t \frac{\partial}{\partial x^1}, \\
e_2(t) &= e^{-\frac{t}{2}} \cos(pt) \frac{\partial}{\partial x^2} - e^{-\frac{t}{2}} \sin(pt) \frac{\partial}{\partial x^3}, \\
e_3(t) &= e^{-\frac{t}{2}} \sin(pt) \frac{\partial}{\partial x^2} + e^{-\frac{t}{2}} \cos(pt) \frac{\partial}{\partial x^3},
\end{aligned}$$

with dual coframe:

$$\begin{aligned}
e^1(t) &= e^{-t} dx^1, \\
e^2(t) &= e^{\frac{t}{2}} \cos(pt) dx^2 - e^{\frac{t}{2}} \sin(pt) dx^3, \\
e^3(t) &= e^{\frac{t}{2}} \sin(pt) dx^2 + e^{\frac{t}{2}} \cos(pt) dx^3,
\end{aligned}$$

where $\{\frac{\partial}{\partial x^i}\}$ are the standard coordinate vector fields on $\mathbb{T}^3$.

It is straightforward to verify that

$$\rho_* \big(e_i(0)_{\underline{x}}\big) = e_i(t_0)_{\rho(\underline{x})},$$

and therefore also

$$\rho^* \big(e^i(t_0)_{\rho(\underline{x})}\big) = e^i(0)_{\underline{x}}.$$

The frame $\{e^i(t)\}$ induces a set of 1-forms $\{e^i\}$ on the product $\mathbb{T}^3 \times [0, t_0]$, and the above computation shows that this set is preserved by the smooth map

$$\rho : \mathbb{T}^3 \times \{0\} \to \mathbb{T}^3 \times \{t_0\}, \quad (\underline{x}, 0) \mapsto (\rho(\underline{x}), t_0).$$

Hence, $\{e^i\}$ descends to a well-defined set of 1-forms on the suspension

$$\mathbb{T}^3_\rho := \frac{\mathbb{T}^3 \times [0, t_0]}{(\underline{x}, 0) \sim (\rho(\underline{x}), t_0)}.$$

Analogously, the vector fields $\{e_i\}$ descend to global vector fields on $\mathbb{T}^3_\rho$.

Let

$$\pi : \mathbb{T}^3_\rho \longrightarrow S^1, \quad [(\underline{x}, t)] \longmapsto e^{2\pi i \frac{t}{t_0}},$$

and denote by θ the pullback via π of the standard volume form on S^1. Observe that, up to a constant rescaling, θ is locally given by dt.

As discussed above, the forms $\{e^1, e^2, e^3, \theta\}$ define a global coframe on $\mathbb{T}^3_\rho$. Dually, the corresponding global frame is given by $\{e_1, e_2, e_3, \frac{\partial}{\partial\theta}\}$, where $\frac{\partial}{\partial\theta}$ denotes the vector field on $\mathbb{T}^3_\rho$ induced by the circle action on the base, whose local expression is $\frac{\partial}{\partial t}$.

The triple $(J_\pm, g)$ on $\mathbb{T}^3_\rho$ given by

$$\begin{aligned} J_\pm(e_1) &= \frac{\partial}{\partial\theta}, \quad J_\pm e_2 := \pm e_3, \quad J_\pm e_3 := \mp e_2, \\ g &= \sum_{i=1}^{3} (e^i)^2 + \theta^2, \end{aligned}$$

with associated fundamental forms $\omega_\pm = e^1 \wedge \theta + \pm e^2 \wedge e^3$, defines a generalized Kähler structure of split type.

Indeed, $N_{J_\pm}(e_2, e_3) = 0$,

$$\begin{aligned} N_{J_\pm}(e_1, e_2) &= \left[\tfrac{\partial}{\partial t}, J_\pm e_2\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_2\right]\right) \\ &= \pm\left[\tfrac{\partial}{\partial t}, e_3\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_2\right]\right) \\ &= \pm(pe_2 - \tfrac{1}{2}e_3) - J_\pm(-\tfrac{1}{2}e_2 - pe_3) = 0 \end{aligned}$$

and similarly

$$\begin{aligned} N_{J_\pm}(e_1, e_3) &= = \left[\tfrac{\partial}{\partial t}, J_\pm e_3\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_3\right]\right) \\ &= \mp\left[\tfrac{\partial}{\partial t}, e_2\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_3\right]\right) \\ &= \mp\left[\tfrac{\partial}{\partial t}, e_2\right] \pm \left[\tfrac{\partial}{\partial t}, e_2\right] = 0. \end{aligned}$$

Moreover, in local coordinates,

$$\begin{aligned} d^c_\pm \omega_\pm &= J_\pm^{-1} d\omega_\pm \\ &= \pm J_\pm^{-1} d(e^2 \wedge e^3) \\ &= \pm J_\pm^{-1} dt \wedge e^2 \wedge e^3 \\ &= \mp e^1 \wedge e^2 \wedge e^3 \\ &= \mp dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

and, clearly, $dd^c\omega_\pm = 0$.

We further observe that $J_\pm$ induce opposite orientations on $\mathbb{T}^3_\rho$, so that the generalized Kähler structure $(g, J_\pm)$ is of *ambihermitian* type. In order to prove that $\mathbb{T}^3_\rho$ is the Inoue surface, we appeal to the Apostolov-Gualtieri classification [AG]. Firstly we determine that $\mathbb{T}^3_\rho$ is non Kähler. This suffices to conclude the claim, since the only non

Kähler surfaces in the classification are the Inoue surface and the Hopf one, but only the Inoue surface has the structure of a $\mathbb{T}^3$ bundle over S^1.

The first Betti number b_1 of $\mathbb{T}^3_\rho$ is given by

$$b_1(\mathbb{T}^3_\rho) = \dim(\ker(\rho_1^* - Id)) + \dim(\operatorname{coker}(\rho_0^* - Id)) = \dim(\ker(\rho_1^* - Id)) + 1,$$

where ρ_1^* is the isomorphism induced by ρ on $H^1(\mathbb{T}^3)$. With respect to the standard basis $\{[dx^1], [dx^2], [dx^3]\}$ of $H^1(\mathbb{T}^3)$ we have

$$\rho_1^* - Id = \begin{pmatrix} e^{t_0} - 1 & 0 & 0 \\ 0 & e^{\frac{-t_0}{2}}\cos(t_0 p) - 1 & e^{\frac{-t_0}{2}}\sin(t_0 p) \\ 0 & -e^{\frac{-t_0}{2}}\sin(t_0 p) & e^{\frac{-t_0}{2}}\cos(t_0 p) - 1 \end{pmatrix}$$

and so for any values of p in $\mathbb{R} \setminus \{0\}$

$$\det(\rho_1^* - Id) = (e^{t_0} - 1)(e^{-t_0} - 2e^{\frac{-t_0}{2}}\cos(t_0 p) + 1) = 0 \iff t_0 = 0,$$

which is an excluded value by hypothesis. It follows that $b_1(\mathbb{T}^3_\rho) = 1$ and as a consequence $\mathbb{T}^3_\rho$ is an Inoue surface in the family S_M.

The suspension $\mathbb{T}^3_\rho$ can be also described as the compact almost abelian solvmanifold $\Gamma\backslash(\mathbb{R}^3 \rtimes_\varphi \mathbb{R})$, where $\varphi(t)$ is given by

$$\varphi(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-\frac{t}{2}}\cos(tp) & e^{-\frac{t}{2}}\sin(tp) \\ 0 & -e^{-\frac{t}{2}}\sin(tp) & e^{-\frac{t}{2}}\cos(tp) \end{pmatrix},$$

and $\Gamma = \Gamma_0 \rtimes t_0\mathbb{Z}$. Note that the almost abelian Lie group $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ has structure equations

$$\begin{aligned} &df^1 = f^1 \wedge f^4\ ,\ df^2 = -\frac{1}{2} f^2 \wedge f^4 + p f^3 \wedge f^{4n}, \\ &df^3 = -p f^2 \wedge f^4 - \frac{1}{2} f^3 \wedge f^4,\ df^4 = 0. \end{aligned} \tag{3.1.2}$$

and $e^1 = f^1$, $e^2 = f^2$, $e^3 = f^3$, $\theta = f^4$.

We now show that the previous example arises as a special case of a more general construction.

Let $a_1 = a_1(t),\ b_2 = b_2(t),\ b_3 = b_3(t)$ be smooth real-valued functions depending on a time parameter $t \in [0, 1]$. For each fixed t, consider the following family of vector fields on the 3-torus $\mathbb{T}^3$, viewed as the quotient $\mathbb{R}^3/\mathbb{Z}^3$, with local coordinates (x^1, x^2, x^3):

$$
\begin{aligned}
e_1(t) &= a_1(t)\frac{\partial}{\partial x^1}, \\
e_2(t) &= b_2(t)\frac{\partial}{\partial x^2} + b_3(t)\frac{\partial}{\partial x^3}, \\
e_3(t) &= -b_3(t)\frac{\partial}{\partial x^2} + b_2(t)\frac{\partial}{\partial x^3}.
\end{aligned} \tag{3.1.3}
$$

These vector fields define a global frame for each fixed t. The corresponding dual coframe $\{e^i(t)\}$ is given by:

$$
\begin{aligned}
e^1(t) &= \frac{1}{a_1(t)}dx^1, \\
e^2(t) &= \frac{b_2(t)}{l(t)}dx^2 + \frac{b_3(t)}{l(t)}dx^3, \\
e^3(t) &= -\frac{b_3(t)}{l(t)}dx^2 + \frac{b_2(t)}{l(t)}dx^3,
\end{aligned} \tag{3.1.4}
$$

where $l(t) := b_2^2(t) + b_3^2(t) > 0$.

Define the auxiliary functions

$$
v(t) := b_2(t)b_2'(t) + b_3(t)b_3'(t) = \frac{1}{2}l'(t), \tag{3.1.5}
$$

and

$$
w(t) := b_2(t)b_3'(t) - b_3(t)b_2'(t).
$$

We further require the periodicity conditions:

$$
\frac{v(1)}{l(1)} = \frac{v(0)}{l(0)}, \tag{3.1.6}
$$

$$
\frac{w(1)}{l(1)} = \frac{w(0)}{l(0)}. \tag{3.1.7}
$$

We construct on $\mathbb{T}^3$ the following pair of 1-parameter family of normal almost contact metric structures $(\xi(t), \eta(t), \varphi_\pm(t), h(t))$ given by

$$
\xi(t) := e_1(t), \quad \eta(t) := e^1(t), \quad h(t) := \sum_{i=1}^{3}(e^i(t))^2,
$$

$$
\varphi_\pm(t)(e_1(t)) := 0, \quad \varphi_\pm(t)(e_2(t)) := \pm e_3(t), \quad \varphi_\pm(t)(e_2(t)) := \mp e_3(t),
$$

with fundamental forms

$$
F_\pm(t) = \pm e^2(t) \wedge e^3(t) = \pm\frac{1}{l(t)}\, dx^2 \wedge dx^3.
$$

Starting from the frame $\{e_i(t)\}$ defined in (3.1.3), with functions a_1, b_2, b_3 satisfying the compatibility conditions (3.1.6) and (3.1.7), we can construct a generalized Kähler suspension of $\mathbb{T}^3$. The resulting 4-manifold is either a Kähler surface or an Inoue surface of type S_M, depending on whether the function $v(t)$ vanishes identically. In this paper, we focus on the latter case.

Lemma 3.1.2. *Let $\{e^i(t)\}$ be the basis of 1-forms on $\mathbb{T}^3$ as in (3.1.4) with coefficients a_1, b_2, b_3 satisfying (3.1.6) and (3.1.7). Suppose that there exist a diffeomorphism ρ of $\mathbb{T}^3$ such that*

$$\begin{aligned} &\rho^*\left(e^i(1)_{\rho(\underline{x})}\right) = e^i(0)_{\underline{x}}, \\ &\left(\tfrac{1}{l(t)}\right)' \cdot \tfrac{1}{a_1} = const \neq 0. \end{aligned} \tag{3.1.8}$$

Then the suspension $\mathbb{T}^3_\rho$ admits a split generalized Kähler structure and it is biholomorphic to an Inoue surface in the family S_M.

Proof. A geometric structure on $\mathbb{T}^3 \times [0,1]$ descends to the suspension $\mathbb{T}^3_\rho$ if it is preserved under the identification $(\underline{x}, 0) \sim (\rho(\underline{x}), 1)$. Therefore, by the first condition of (3.1.8), the forms $\{e^i(t)\}$ define a globally well-defined set of 1-forms on $\mathbb{T}^3_\rho$. Moreover, both $F_\pm(t)$ and the frame $\{e_i(t)\}$ descend to $\mathbb{T}^3_\rho$, since the pullback ρ^* distributes over the wedge product, and one can verify that $\rho_*\left(e_i(0)_{\underline{x}}\right) = e_i(1)_{\rho(\underline{x})}$. From now on, we denote by e^i and e_i the global 1-forms and vector fields on $\mathbb{T}^3_\rho$, respectively.

Let θ denote the pullback of the standard volume form on S^1 via the fibration

$$\pi : \mathbb{T}^3_\rho \to S^1, \quad [(\underline{x}, t)] \mapsto e^{2\pi i t},$$

which, up to rescaling, locally corresponds to dt. It follows that $d(e^j \wedge \theta) = 0$ for every $j = 1, 2, 3$, since the coefficients of dx^i in the expressions for e^j depend only on the t-coordinate.

To prove the lemma, we construct a pair of complex structures $J_\pm$ on $\mathbb{T}^3_\rho$ and a Riemannian metric g compatible with both $J_\pm$ such that $[J_+, J_-] = 0$, $d^c_- \omega_- = -d^c_+ \omega_+$, and $dd^c_+ \omega_+ = 0$, where $\omega_\pm$ denote the fundamental forms associated with the pairs $(g, J_\pm)$.

Consider the globally defined non-degenerate 2-forms

$$\omega_+ = e^1 \wedge \theta + F_+, \quad \omega_- = e^1 \wedge \theta + F_-.$$

We compute

$$d\omega_\pm = d(e^1 \wedge \theta) + dF_\pm = dF_\pm,$$

since $d(e^1 \wedge \theta) = 0$. The 3-forms $dF_\pm$ are given by

$$dF_\pm = \pm d\left(\frac{1}{l(t)} dx^2 \wedge dx^3\right) = \mp \frac{l'(t)}{l(t)}\, dt \wedge e^2 \wedge e^3 = \mp \frac{2v(t)}{l(t)}\, dt \wedge e^2 \wedge e^3,$$

where $2\frac{v(t)}{l(t)}$ is the smooth function on $\mathbb{T}^3_\rho$ assigning to $[(\underline{x},t)]$ the value $2\frac{v(t)}{l(t)}$. By condition (3.1.6), this expression is well defined on $\mathbb{T}^3_\rho$, and in terms of global 1-forms,

$$dF_\pm = \mp\frac{2v}{l}\,\theta\wedge e^2\wedge e^3.$$

The 2-forms $\omega_\pm$ are of type $(1,1)$ with respect to the almost complex structures $J_\pm$ defined by:

$$J_\pm(e_1) = \frac{\partial}{\partial\theta},$$
$$J_\pm(e_2) = \varphi_\pm(e_2) = \pm e_3, \qquad J_\pm(e_3) = \varphi_\pm(e_3) = \mp e_2,$$

where $\frac{\partial}{\partial\theta}$ denotes the S^1-vector field, whose local expression is $\frac{\partial}{\partial t}$. It is a trivial check that the structures $(\omega_\pm, J_\pm)$ induce the same Hermitian metric g defined as:

$$g = \sum_{i=1}^{3}(e^i)^2 + \theta^2.$$

We now verify that $J_\pm$ are integrable by checking that their Nijenhuis tensors vanish, i.e., $N_{J_\pm}(X,Y) = 0$ for all vector fields X, Y. It suffices to verify this locally in a trivialization for the nontrivial cases:

$$X = e_1, \quad Y\in\{e_2,e_3\},$$
$$X = \frac{\partial}{\partial\theta}, \quad Y\in\{e_2,e_3\}.$$

Using the local expression $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial t}$, we compute:

$$\begin{aligned} N_{J_\pm}(e_1,e_2) &= [J_\pm e_1, J_\pm e_2] - J_\pm([J_\pm e_1, e_2] + [e_1, J_\pm e_2]) - [e_1,\ e_2] \\ &= \left[\tfrac{\partial}{\partial t}, J_\pm e_2\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_2\right]\right) \\ &= \pm\left[\tfrac{\partial}{\partial t}, e_3\right] - J_\pm\left(\left[\tfrac{\partial}{\partial t}, e_2\right]\right). \end{aligned}$$

Since

$$\left[\frac{\partial}{\partial t}, e_2\right] = b_2'(t)\frac{\partial}{\partial x^2} + b_3'(t)\frac{\partial}{\partial x^3} = \frac{v(t)}{l(t)}e_2 + \frac{w(t)}{l(t)}e_3$$

and

$$\left[\frac{\partial}{\partial t}, e_3\right] = -b_3'(t)\frac{\partial}{\partial x^2} + b_2'(t)\frac{\partial}{\partial x^3} = \frac{-w(t)}{l(t)}e_2 + \frac{v(t)}{l(t)}e_3,$$

we obtain

$$N_{J_\pm}(e_1,e_2) = \pm\left(\frac{-w(t)}{l(t)}e_2 + \frac{v(t)}{l(t)}e_3\right) - J_\pm\left(\frac{v(t)}{l(t)}e_2 + \frac{w(t)}{l(t)}e_3\right) = 0.$$

A similar computation shows $N_{J_\pm}(e_1, e_3) = 0$, and for $X = \dfrac{\partial}{\partial t}$, $Y = e_i$, $i = 2, 3$, one finds:

$$N_{J_\pm}\left(\tfrac{\partial}{\partial t}, e_i\right) = \left[\tfrac{\partial}{\partial t}, e_i\right] - \left[\tfrac{\partial}{\partial t}, e_i\right] = 0.$$

Lastly, $X = \frac{\partial}{\partial \theta}$ and $Y = e_i$, with $i = 2, 3$, we have

$$\begin{aligned}
N_{J_\pm}\left(\frac{\partial}{\partial t}, e_i\right) &= \left[J_\pm \frac{\partial}{\partial t}, J_\pm e_i\right] - J_\pm\left(\left[J_\pm \frac{\partial}{\partial t}, e_i\right] + \left[\frac{\partial}{\partial t}, J_\pm e_i\right]\right) - \left[\frac{\partial}{\partial t}, e_i\right] \\
&= -[e_1, J_\pm e_i] - J_\pm\left(-[e_1, e_i] + \left[\frac{\partial}{\partial t}, J_\pm e_i\right]\right) - \left[\frac{\partial}{\partial t}, e_i\right] \\
&= -J_\pm\left(\left[\frac{\partial}{\partial t}, J_\pm e_i\right]\right) - \left[\frac{\partial}{\partial t}, e_i\right] \\
&= \left[\frac{\partial}{\partial t}, e_i\right] - \left[\frac{\partial}{\partial t}, e_i\right] = 0.
\end{aligned}$$

We now compute the real Dolbeault differentials of $\omega_\pm$. From $d\omega_\pm = dF_\pm = \mp\dfrac{2v}{l}\,\theta \wedge e^2 \wedge e^3$, we obtain:

$$\begin{aligned}
d^c_+\omega_+ &= J_+^{-1}(dF_+) \\
&= \frac{2v}{l}\, J_+\theta \wedge J_+e^2 \wedge J_+e^3 \\
&= \frac{2v}{l}\, e^1 \wedge e^2 \wedge e^3 \\
&= -\left(\frac{1}{l}\right)' \cdot \frac{1}{a_1}\, dx^1 \wedge dx^2 \wedge dx^3,
\end{aligned}$$

and similarly,

$$\begin{aligned}
d^c_-\omega_- &= -\frac{2v}{l}\, J_-\theta \wedge J_-e^2 \wedge J_-e^3 \\
&= -\frac{2v}{l}\, e^1 \wedge e^2 \wedge e^3 \\
&= \left(\frac{1}{l}\right)' \cdot \frac{1}{a_1}\, dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Thus,

$$d^c_+\omega_+ = -d^c_-\omega_- = \frac{2v}{l}\, e^1 \wedge e^2 \wedge e^3. \tag{3.1.9}$$

By assumption, $\left(\frac{1}{l}\right)' \cdot \frac{1}{a_1}$ is constant, implying $dd^c_\pm\omega_\pm = 0$.

Since J_+ and J_- induce opposite orientations, the resulting generalized Kähler structure is of ambihermitian type. To conclude, it remains to verify that $b_1(\mathbb{T}^3_\rho)$ is odd. As discussed in Example 3.1.1, it suffices to show that the de Rham cohomology class $[H]$ is non-trivial.

Assume by contradiction that H is exact. Since v is not identically zero, there exists $\bar{t} \in [0, 1]$ with $v(\bar{t}) \neq 0$. Consider the inclusion:

$$\iota_{\bar{t}} : \mathbb{T}^3 \to \mathbb{T}^3_\rho, \quad \underline{x} \mapsto [(\underline{x}, \bar{t})],$$

and set $\alpha := \iota_{\bar t}^* H \in \Omega^3(\mathbb{T}^3)$. Since H is exact, so is α. However,

$$\alpha = \frac{2v(\bar t)}{l(\bar t)}\, e^1(\bar t) \wedge e^2(\bar t) \wedge e^3(\bar t) = \frac{2v(\bar t)}{l(\bar t)}\, \mathrm{vol}_{\mathbb{T}^3}.$$

By Stokes' Theorem:

$$0 = \int_{\mathbb{T}^3} \alpha = \frac{2v(\bar t)}{l(\bar t)} \int_{\mathbb{T}^3} \mathrm{vol}_{\mathbb{T}^3} \neq 0,$$

a contradiction. This completes the proof.

q.e.d.

We can extend the previous construction considering suspensions of $\mathbb{T}^3 \times K$, where $\mathbb{T}^3$ is endowed with the same geometric structures as before and K is any compact Kähler manifold.

Theorem 3.1.3. *Let (N, J, k, ω) be a compact Kähler manifold and let $\{e^i(t)\}$ be the 1-forms given in (3.1.4) satisfying (3.1.6) and (3.1.7). Suppose that there exists a diffeomorphism ρ of $\mathbb{T}^3$ such that,*

$$\begin{aligned} &\rho^*\left(e^i(1)_{\rho(\underline{x})}\right) = e^i(0)_{\underline{x}}, \\ &\left(\tfrac{1}{l(t)}\right)' \cdot \tfrac{1}{a_1} = const \neq 0, \end{aligned} \tag{3.1.10}$$

and a diffeomorphism ψ of N preserving the Kähler structure, i.e. holomorphic with respect to the complex structure J, and such that $\psi^*(\omega) = \omega$. Then the suspension M_f of $M = \mathbb{T}^3 \times N$ via the diffeomorphism

$$f : (\underline{x}, p) \to (\rho(\underline{x}), \psi(p)),$$

admits a split generalized Kähler structure $(I_\pm, g, \omega_\pm)$.

Proof. We have already observed that a geometric structure on $M \times [0,1]$ descends to the suspension if it glues up correctly in the quotient, i.e., if it is preserved by the map sending $(\underline{x}, p, 0)$ to $(\rho(\underline{x}), \psi(p), 1)$.

Moreover, $\{e^i(t)\}$ and ω are well defined on M_f as f has a block structure. Indeed, by hypothesis, conditions (3.1.10) are satisfied and ψ preserves the Kähler structure. As in the proof of Lemma 3.1.3, $F_\pm(t)$ and $\{e_i(t)\}$ descend to M_f too.

From now on, we denote by e^i and e_i the corresponding global 1-forms and vector fields on M_f, respectively.

Again, we denote by θ the pull-back of the standard volume form of S^1. With the same argument as before, $d(e^j \wedge \theta) = 0$.

To prove the theorem, we have to construct a pair of complex structures $I_\pm$ on M_f and a Riemannian metric g compatible with both $I_\pm$ such that $d^c_- \omega_- = -d^c_+ \omega_+$ and $dd^c_+ \omega_+ = 0$, where we denote by $\omega_\pm$ the fundamental forms of the pairs $(g, I_\pm)$.

Note that, by previous remarks, the following non-degenerate 2-forms

$$\omega_+ = e^1 \wedge \theta + F_+ + \omega, \quad \omega_- = e^1 \wedge \theta + F_- + \omega$$

are well defined on M_f.

Observe that the exterior derivative of both $\omega_\pm$ is a 3-form globally defined on the suspension. Indeed,

$$d\omega_\pm = d(e^1 \wedge \theta) + dF_\pm + d\omega = dF_\pm,$$

where the last equality holds since $d(e^1 \wedge \theta) = 0$ and ω is closed by the Kähler hypothesis on N. If we compute the local expression of $dF_\pm$, we get

$$dF_\pm = \mp \frac{2v(t)}{l(t)} dt \wedge e^2 \wedge e^3.$$

Since $\frac{2v}{l}$ is a smooth globally well-defined function on M_f by condition (3.1.6), the global expression of $dF_\pm$ is actually $\mp \frac{2v}{l} \theta \wedge e^2 \wedge e^3$.

Let us denote by D the smooth involutive distribution $\ker(e^1) \cap \ker(e^2) \cap \ker(e^3) \cap \ker(\theta) \subset \mathfrak{X}(M_f)$. If a vector field X is in D, then

$$e^i(X) = 0 \quad \text{and} \quad \theta(X) = 0.$$

Recall that a trivialization U of M_f is of the form $U_1 \times W_1 \times I_1$, if $\{0,1\} \not\subset I_1$, and of the form

$$\pi\left(U_1 \times W_1 \times [0, \tfrac{1}{2}) \sqcup \rho(U_1) \times \psi(W_1) \times (\tfrac{1}{2}, 1]\right)$$

otherwise, with U_1, W_1, and I_1 being open sets of $\mathbb{T}^3$, N, and $(0,1)$ respectively, and $\pi : \mathbb{T}^3 \times N \times [0,1] \to M$ the quotient map.

On such a trivialization, we may write

$$X_{|U} = \sum_{i=1}^{3} X^i(x_i, y_j, t)\, e_i + \sum_{j=1}^{2k} Y^j(x_i, y_j, t)\, \frac{\partial}{\partial y^j} + Z(x_i, y_j, t)\, \frac{\partial}{\partial t},$$

where (x_i, y_j, t) are local coordinates on U and X^i, Y^j, Z are smooth functions on U. Since $X \in D$, we have

$$X_{|U} = \sum_{j=1}^{2k} Y^j \frac{\partial}{\partial y^j}.$$

By hypothesis, the complex structure J satisfies

$$\psi_* \circ J = J \circ \psi_*,$$

and thus induces a well-defined complex structure on D by $C^\infty(M_f)$-linearity:

$$J\left(Y^j \frac{\partial}{\partial y^j}\right) := Y^j J\left(\frac{\partial}{\partial y^j}\right).$$

Define the almost complex structures $I_\pm$ on M_f as follows:

$$\begin{aligned} I_\pm(e_1) &= \frac{\partial}{\partial\theta}, \\ I_\pm(e_2) &= \varphi_\pm e_2 = \pm e_3, \\ I_\pm(e_3) &= \varphi_\pm e_3 = \mp e_2, \\ I_+ &= J, \quad I_- = J \quad \text{on } D, \end{aligned}$$

where $\frac{\partial}{\partial\theta}$ is the S^1-vector field, whose local expression is $\frac{\partial}{\partial t}$.

The 2-forms $\omega_\pm$ are of type $(1,1)$ with respect to $I_\pm$ and induce the same Hermitian metric g, where

$$g = \sum_{i=1}^{3}(e^i)^2 + k + \theta^2.$$

We verify integrability of $I_\pm$ by checking that the Nijenhuis tensor

$$N_{I_\pm} = [I_\pm\cdot, I_\pm\cdot] - I_\pm([I_\pm\cdot,\cdot] + [\cdot, I_\pm\cdot]) - [\cdot,\cdot]$$

vanishes.

Since D is involutive and $I_\pm$-invariant, if $X, Y \in D$, then $N_{I_\pm}(X,Y) = N_J(X,Y) = 0$. Also, since $D^\perp = \langle e_1, e_2, e_3, \frac{\partial}{\partial\theta}\rangle$ is involutive and $I_\pm$-invariant, $N_{I_\pm}$ coincides with the Nijenhuis tensor $N_{J_\pm}$ from Lemma 3.1.3.

It remains to check that $N_{I_\pm}(X,Y) = 0$ for $X \in \{e_1, e_2, e_3, \frac{\partial}{\partial\theta}\}$ and $Y \in D$. Since $N_{I_\pm}$ is a tensor, we compute locally:

$$N_{I_\pm}(X,Y) = N_{I_\pm}\left(X, Y^j\frac{\partial}{\partial y^j}\right) = Y^j N_{I_\pm}\left(X, \frac{\partial}{\partial y^j}\right) = 0.$$

We compute $d^c_\pm = I_\pm^{-1} d I_\pm$. From earlier,

$$d\omega_\pm = \mp\frac{2v}{l}\theta\wedge e^2\wedge e^3,$$

and then

$$\begin{aligned} d^c_+\omega_+ &= I_+^{-1} d\omega_+ = -\frac{2v}{l} I_+^{-1}(\theta\wedge e^2\wedge e^3) \\ &= \frac{2v}{l}\, e^1\wedge e^2\wedge e^3 = -\left(\frac{1}{l}\right)' \cdot \frac{1}{a_1} dx^1\wedge dx^2\wedge dx^3, \end{aligned}$$

while

$$d^c_-\omega_- = I_-^{-1}d\omega_- = +\frac{2v}{l}I_-^{-1}(\theta\wedge e^2\wedge e^3)$$
$$= -\frac{2v}{l}\,e^1\wedge e^2\wedge e^3 = \left(\frac{1}{l}\right)'\cdot\frac{1}{a_1}dx^1\wedge dx^2\wedge dx^3.$$

Hence,

$$d^c_+\omega_+ = -d^c_-\omega_-,$$

and since $\left(\frac{1}{l}\right)'\cdot\frac{1}{a_1}$ is constant by hypothesis, we have $dd^c_\pm\omega_\pm = 0$.

Clearly, the generalized Kähler structure $(I_\pm, g, \omega_\pm)$ constructed above is split, concluding the proof. q.e.d.

3.1.2 Non-split examples

In this section, we further generalize the previously discussed construction to describe examples of non-split generalized Kähler suspensions. The key idea is to replace the Kähler condition with the *hyperkähler* one, leveraging the fact that the three complex structures in a hypercomplex structure satisfy the relation $J_1J_2 = -J_2J_1$. Accordingly, we define two distinct complex structures on the integrable distribution D, which will lead to the non-split property.

Theorem 3.1.4. *Let $(N, J_1, J_2, J_3, \omega_1, \omega_2, \omega_3, k)$ be a compact hyperkähler manifold and let $\{e^i(t)\}$ be the 1-forms given in (3.1.4) satisfying (3.1.6) and (3.1.7). Suppose that there exists a diffeomorphism ρ of $\mathbb{T}^3$ such that*

$$\begin{aligned}&\rho^*\left(e^i(1)_{\rho(\underline{x})}\right) = e^i(0)_{\underline{x}},\\ &\left(\tfrac{1}{l(t)}\right)'\cdot\tfrac{1}{a_1} = const \neq 0,\end{aligned} \tag{3.1.11}$$

and a diffeomorphism ψ of N preserving the hyperkähler structure, i.e. holomorphic with respect to every complex structure J_i, and such that

$$\psi^*(\omega_i) = \omega_i, \quad i = 1, 2, 3. \tag{3.1.12}$$

Then the suspension M_f of $M = \mathbb{T}^3\times N$ via the diffeomorphism

$$f : (\underline{x}, p) \to (\rho(\underline{x}), \psi(p)),$$

admits a non-split generalized Kähler structure $(I_\pm, g, \omega_\pm)$.

Proof. The proof is similar to the previous one, so we highlight the differences and omit details already seen.

To define a non-split generalized Kähler structure, we consider the following non-degenerate 2-forms:

$$\omega_+ = e^1 \wedge \theta + F_+ + \omega_1, \quad \omega_- = e^1 \wedge \theta + F_- + \omega_2,$$

which are well defined on M_f.

The exterior derivative of both $\omega_\pm$ is a globally defined 3-form on the suspension. Indeed,

$$d\omega_\pm = d(e^1 \wedge \theta) + dF_\pm + d\omega_i = dF_\pm, \quad \text{with } i = 1, 2,$$

where the last equality holds since $d(e^1 \wedge \theta) = 0$ and the ω_i are closed by the hyperkähler hypothesis on N. More precisely, we have:

$$dF_\pm = \mp \frac{2v}{l}\, \theta \wedge e^2 \wedge e^3.$$

Let us denote by D the smooth involutive distribution

$$D := \ker(e^1) \cap \ker(e^2) \cap \ker(e^3) \cap \ker(\theta) \subset \mathfrak{X}(M_f).$$

If a vector field X is in D, then

$$e^i(X) = 0 \quad \text{and} \quad \theta(X) = 0,$$

and hence, on a trivialization U of M_f, we may write:

$$X|_U = \sum_{j=1}^{4k} Y^j(x_i, y_j, t) \frac{\partial}{\partial y^j},$$

where (x_i, y_j, t) are local coordinates on U and Y^j are smooth functions on U.

By hypothesis, the complex structures J_i satisfy

$$\psi_* \circ J_i = J_i \circ \psi_*,$$

and thus induce well-defined complex structures on D by $C^\infty(M_f)$-linearity:

$$J_i \left(\sum_{j=1}^{4k} Y^j \frac{\partial}{\partial y^j} \right) := \sum_{j=1}^{4k} Y^j J_i \left(\frac{\partial}{\partial y^j} \right).$$

Let the two almost complex structures $I_\pm$ on M_f be defined as follows:

$$
\begin{aligned}
&I_\pm(e_1) = \frac{\partial}{\partial\theta},\\
&I_\pm(e_2) = \varphi_\pm e_2 = \pm e_3,\\
&I_\pm(e_3) = \varphi_\pm e_3 = \mp e_2,\\
&I_+ = J_1, \quad I_- = J_2 \qquad \text{on } D.
\end{aligned}
$$

The 2-forms $\omega_\pm$ are of type $(1,1)$ with respect to $I_\pm$, and satisfy:

$$-\omega_+ I_+ = -\omega_- I_- = g,$$

where

$$g = \sum_{i=1}^{3}(e^i)^2 + k + \theta^2.$$

We claim that the almost complex structures $I_\pm$ are integrable by verifying that their Nijenhuis tensors vanish. This proof is the same as in Theorem 3.1.3, so we omit it to avoid repetition.

It remains to verify the condition on the real Dolbeault operators of $\omega_\pm$. By the previous remarks,

$$d\omega_\pm = dF_\pm = \mp\frac{2v}{l}\,\theta\wedge e^2\wedge e^3,$$

and we have already computed that

$$d^c_+\omega_+ = -d^c_-\omega_-.$$

As before, since by hypothesis $\left(\frac{1}{l}\right)'\cdot\frac{1}{a_1}$ is constant, we obtain

$$dd^c_\pm\omega_\pm = 0.$$

The generalized Kähler structure $(I_\pm, g, \omega_\pm)$ constructed above is *non-split*. Indeed, since $I_\pm$ both have block structure, it suffices to check that $I_+I_- \neq I_-I_+$ on a vector field $X \in D$. Let $X \in D$. Then:

$$
\begin{aligned}
I_+I_-X &= J_1J_2X = J_3X,\\
I_-I_+X &= J_2J_1X = -J_3X.
\end{aligned}
$$

Alternatively, one may compute the Poisson tensor:

$$\sigma = \frac{1}{2}[I_+, I_-]g^{-1} = -\omega_3^{-1} \neq 0.$$

q.e.d.

Remark 3.1.5. The previous proof also adapts to the case where N is a generalized Kähler manifold $(N, J_\pm, \sigma_\pm, k)$, and ψ is a diffeomorphism preserving the generalized Kähler structure, i.e., holomorphic with respect to $J_\pm$ and such that $\psi^*\sigma_\pm = \sigma_\pm$.

In this case, we define:

$$\omega_\pm = e^1 \wedge \theta \,\pm\, dF_\pm + \sigma_\pm, \quad \text{and} \quad I_\pm = J_\pm \text{ on } D.$$

Since $[I_+, I_-] = [J_+, J_-]$ on D, the pair $(I_\pm, g)$ defines a *split* generalized Kähler structure on M_f if $(J_\pm, k)$ is split on N, and a *non-split* one otherwise.

3.1.3 Formality

In this section, we prove some results concerning the formality of the generalized Kähler manifolds constructed in this chapter.

We begin by establishing the following proposition.

Proposition 3.1.6. *Let M_f be a generalized Kähler suspension constructed as in Theorems 3.1.3 and 3.1.4. Then the first Betti number of M_f is odd. As a consequence, M_f does not admit any Kähler metric and does not satisfy the dd^c-lemma.*

Proof. Recall that

$$b_1(M_f) = 1 + \dim(\ker(\rho_1^* - \mathrm{Id})) + \dim(\ker(\psi_1^* - \mathrm{Id})).$$

Since $\mathbb{T}^3_\rho$ is biholomorphic to an Inoue surface in the family S_M, it follows that $\dim(\ker(\rho_1^* - \mathrm{Id})) = 0$. Hence, it remains to show that the vector subspace

$$\ker(\psi_1^* - \mathrm{Id}) = \{[\alpha] \in H^1(N) \mid \psi_1^*([\alpha]) = [\alpha]\} \subset H^1(N)$$

has even dimension.

Let $[\alpha] \in \ker(\psi_1^* - \mathrm{Id})$. Without loss of generality, assume α is the harmonic representative of its cohomology class, as N is compact. Since $[\alpha]$ is fixed under ψ_1^*, we have $\psi^*\alpha = \alpha + d\eta$ for some smooth function η on N. Denote by $(\cdot,\cdot)$ the L^2 inner product on (N, k), where k is the Kähler (respectively, hyperkähler) metric on N.

We claim that $d\eta = 0$. Indeed,

$$\begin{aligned} 0 &= (\psi^*(\Delta\alpha), d\eta) = (\Delta(\psi^*\alpha), d\eta) = (\Delta(\alpha + d\eta), d\eta) \\ &= (\Delta(d\eta), d\eta) = (d\delta d\eta, d\eta) = (\delta d\eta, \delta d\eta) = \|\delta d\eta\|^2, \end{aligned}$$

where the second equality holds because ψ is an isometry and pullback commutes with the Laplacian.

Hence $\delta d\eta = 0$, and thus $\Delta(d\eta) = 0$. So α and $\alpha + d\eta$ are both harmonic representatives of the same class, and by uniqueness, $d\eta = 0$.

Therefore, if $[\alpha] \in \ker(\psi_1^* - \mathrm{Id})$ and α is harmonic, then $\psi_1^*\alpha = \alpha$.

On a Kähler manifold, the complex structure J induces an endomorphism of the space of harmonic 1-forms:

$$J : \mathcal{H}^1(N) \to \mathcal{H}^1(N), \quad \beta \mapsto J\beta, \quad \text{with } (J\beta)(X) = \beta(JX).$$

This is well defined because J commutes with the Laplacian. In the hyperkähler case, we may take $J = J_1$.

If α is harmonic and $[\alpha] \in \ker(\psi_1^* - \mathrm{Id})$, then so is $[J\alpha]$. Indeed,

$$\psi_1^*(J\alpha) = J(\psi_1^*\alpha) = J\alpha.$$

This proves the first statement, as the space $\ker(\psi_1^* - \mathrm{Id})$ is preserved under J, and hence has even dimension.

The second statement follows immediately: by [Gau], a compact complex manifold satisfies the dd^c-lemma only if its first Betti number is even. Since $b_1(M_f)$ is odd, the dd^c-lemma cannot hold, and M_f cannot admit a Kähler metric. q.e.d.

In order to study the formality of M_f, we first prove the following lemma.

Lemma 3.1.7. *The 4-dimensional generalized Kähler suspensions $\mathbb{T}^3_\rho$ constructed in Lemma 3.1.2 are formal.*

Proof. This follows from the fact that $(\mathbb{T}^3_\rho, \omega_\pm, J_\pm)$ is biholomorphic to an Inoue surface in the family S_M. The de Rham cohomology of $\mathbb{T}^3_\rho$ is computed, for example, in [AS] and is given by:

$$H^\bullet(\mathbb{T}^3_\rho) = \mathbb{R}\langle 1\rangle \oplus \mathbb{R}\langle[\theta]\rangle \oplus \mathbb{R}\langle[e^{123}]\rangle \oplus \mathbb{R}\langle[\theta \wedge e^{123}]\rangle. \tag{3.1.13}$$

Therefore, a minimal model is

$$\left(\bigwedge V = \bigwedge(a) \otimes \bigwedge(b), d\right), \quad \text{with } |a| = 1,\ |b| = 3,\ da = db = 0,$$

and

$$\varphi(a) = \theta, \quad \varphi(b) = e^1 \wedge e^2 \wedge e^3.$$

Define a CDGA quasi-isomorphism $\nu : (\bigwedge V, d) \to (H^\bullet(\bigwedge V, d), 0)$ by $\nu(a) = [a]$, $\nu(b) = [b]$. Then ν induces the identity map in cohomology. Thus, $\mathbb{T}^3_\rho$ is formal. q.e.d.

Using the previous lemma, we can now prove the following:

Corollary 3.1.8. *If the diffeomorphism ψ in Theorem 3.1.3 is the identity map, then $(M_f, I_\pm)$ is biholomorphic to $(N \times \mathbb{T}^3_\rho, J \oplus J_\pm)$, where J is the complex structure on the Kähler manifold N, and $J_\pm$ are the complex structures on $\mathbb{T}^3_\rho$ described in Lemma 3.1.2. Therefore, M_f is formal, and*

$$H^k(M_f) = \bigoplus_{i+j=k} H^i(N) \otimes H^j(\mathbb{T}^3_\rho), \quad H^{p,q}_{\bar\partial_\pm}(M_f) = \bigoplus_{\substack{a+c=p \\ b+d=q}} H^{a,b}_{\bar\partial_i}(N) \otimes H^{c,d}_{\bar\partial_\pm}(\mathbb{T}^3_\rho).$$

Proof. Clear. q.e.d.

Remark 3.1.9. The corollary above also holds in the hyperkähler case. Indeed, if in Theorem 3.1.4 we take $\psi = \mathrm{Id}$, then $(M_f, I_\pm)$ is biholomorphic to $(N \times \mathbb{T}^3_\rho, J_i \oplus J_\pm)$ with $i = 1, 2$.

3.1.4 Dolbeault cohomology

In this section, we show that the generalized Kähler suspension M_f, constructed as in Theorems 3.1.3 and 3.1.4, is the total space of a holomorphic fibre bundle

$$p : M_f \to \mathbb{T}^3_\rho$$

with fibre N. Such a fibration gives rise to the Borel spectral sequence, which relates the Dolbeault cohomology of the total space M_f to that of the base $\mathbb{T}^3_\rho$ and the fibre N.

This generalizes the trivial case where the diffeomorphism ψ is the identity. We begin by recalling the following theorem of A. Borel, as stated in [HBS, Appendix II].

Theorem 3.1.10 (Borel [HBS]). *Let $p : T \to B$ be a holomorphic fibre bundle with compact, connected fibre F, and suppose that T and B are connected complex manifolds. Assume further that F is Kähler. Then there exists a spectral sequence (E_r, d_r), where d_r is the restriction of the Dolbeault differential $\bar\partial$ on T to E_r, with the following properties:*

- *E_r is 4-graded by fibre degree, base degree, and Dolbeault type. Let ${}^{p,q}E_r^{u,v}$ denote the component of E_r with type (p, q), fibre degree u, and base degree v. Then ${}^{p,q}E_r^{u,v} = 0$ if $p + q \neq u + v$, or if any of p, q, u, v is negative. Moreover,*

$$d_r : {}^{p,q}E_r^{u,v} \longrightarrow {}^{p,q+1}E_r^{u+r,v-r+1}.$$

- *If $p + q = u + v$, then the E_2-page is given by*

$${}^{p,q}E_2^{u,v} = \bigoplus_k H^{k,u-k}_{\bar\partial}(B) \otimes H^{p-k,q-u+k}_{\bar\partial}(F).$$

- *The spectral sequence converges to the Dolbeault cohomology $H^{p,q}_{\bar\partial}(T)$.*

Let us now define $p : (M_f, I_\pm) \to (\mathbb{T}^3_\rho, J_\pm)$ to be the holomorphic projection

$$[(\underline{x}, q, t)] \mapsto [(\underline{x}, t)].$$

We claim that p is a holomorphic fibre bundle map.

To show this, we construct local holomorphic trivializations. Specifically, for each point $[(\underline{x}, t)]$ in $\mathbb{T}^3_\rho$, we must find a neighbourhood U and a biholomorphism

$$\varphi_U : (p^{-1}(U), I_\pm) \to (U, J_\pm) \times (N, J_i)$$

such that p coincides with the projection onto the first factor. We consider two cases, depending if $t = 0$ or not.

Case 1: $t \neq 0, 1$**.** A neighbourhood of $[(\underline{x}, t)]$ is given by $U = V \times I$, where $V \subset \mathbb{T}^3$ is a small open set around $\underline{x}$ and $I \subset (0, 1)$ is an open interval. Then,

$$p^{-1}(U) = \{[(\underline{x}, q, t)] \mid q \in N,\ [(\underline{x}, t)] \in U\},$$

and the trivialization is defined by

$$\varphi_U([(\underline{x}, q, t)]) = ([(\underline{x}, t)], q).$$

This is clearly a biholomorphism.

Case 2: $t = 0$**.** A neighbourhood of $[(\underline{x}, 0)]$ is given by

$$U = \pi_{\mathbb{T}^3_\rho}(V \times [0, \varepsilon) \cup \rho(V) \times (1 - \varepsilon, 1]),$$

where $\pi_{\mathbb{T}^3_\rho}$ denotes the quotient map, and $V \subset \mathbb{T}^3$ is an open set around $\underline{x}$. Then,

$$p^{-1}(U) = \pi\left(V \times N \times [0, \varepsilon) \cup \rho(V) \times N \times (1 - \varepsilon, 1]\right).$$

We define the trivialization φ_U on representatives by:

$$\varphi_U([(\underline{x}, q, t)]) = \begin{cases} ([(\underline{x}, t)], q) & \text{if } t \in [0, \varepsilon), \\ ([(\underline{x}, t)], \psi^{-1}(q)) & \text{if } t \in (1 - \varepsilon, 1]. \end{cases}$$

This is well-defined, because

$$\varphi_U([(\rho(\underline{x}), \psi(q), 1)]) = ([(\underline{x}, 0)], q) = \varphi_U([(\underline{x}, q, 0)]).$$

Its inverse is given by:

$$\varphi_U^{-1}([(\underline{x}, t)], q) = \begin{cases} [(\underline{x}, q, t)] & \text{if } t \in [0, \varepsilon), \\ [(\underline{x}, \psi(q), t)] & \text{if } t \in (1-\varepsilon, 1]. \end{cases}$$

Again, this map is well-defined:

$$\varphi_U^{-1}([(\rho(\underline{x}), 1)], q) = [(\rho(\underline{x}), \psi(q), 1)] = [(\underline{x}, q, 0)].$$

The holomorphicity of φ_U follows from the holomorphic structures on N, $\mathbb{T}^3_\rho$, and the fact that ψ is holomorphic.

We conclude the following:

Theorem 3.1.11. *Let M_f be the generalized Kähler suspension constructed as in Theorems 3.1.3 and 3.1.4. Then M_f is the total space of a holomorphic fibre bundle*

$$N \longrightarrow M_f \longrightarrow \mathbb{T}^3_\rho.$$

In particular, there exist two Borel spectral sequences $(E_r^{\pm}, d_r^{\pm})$ such that:

- *If $p+q = u+v$, then*

$${}^{p,q}E_2^{u,v\,\pm} = \bigoplus_k H^{k,u-k}_{\bar{\partial}_\pm}(\mathbb{T}^3_\rho) \otimes H^{p-k,q-u+k}_{\bar{\partial}_i}(N).$$

- *These spectral sequences converge to the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}_\pm}(M_f)$.*

When $\psi = \mathrm{Id}$, we have already seen in Corollary 3.1.8 (and in the remark that follows) that $(M_f, I_\pm)$ is biholomorphic to the product

$$(N \times \mathbb{T}^3_\rho, J_i \oplus J_\pm),$$

so the fibration is trivial. In this case, the Borel spectral sequence degenerates at the E_2 page, and we recover the Dolbeault cohomology via the Künneth formula, as in Corollary 3.1.8.

As a consequence of the holomorphic fibration structure, we also observe that $(M_f, I_\pm)$ does not admit any balanced metric, since the Inoue surface $\mathbb{T}^3_\rho$ is non-Kähler [Mic].

3.1.5 Explicit constructions

We now present some explicit examples obtained by applying the construction described in the previous sections.

Example 3.1.12. Let ρ be the diffeomorphism of $\mathbb{T}^3$ defined by (3.1.1), and let $\{e_i\}$ and $\{e^i\}$ be the bases introduced in Example 3.1.1. As already noted in that example, ρ preserves the basis $e^i(t)$, and one can easily verify that

$$\left(\frac{1}{l(t)}\right)' \cdot \frac{1}{a_1} = (e^t)' \cdot e^{-t} = 1 \neq 0.$$

We apply Theorem 3.1.4 in the case where $N = \mathbb{T}^4$ is the 4-torus endowed with the standard flat hyperkähler structure $(J_1, J_2, J_3, \omega_1, \omega_2, \omega_3)$, defined by

$$\begin{aligned} &J_1\left(\frac{\partial}{\partial x^4}\right) = \frac{\partial}{\partial x^5}, \quad J_1\left(\frac{\partial}{\partial x^6}\right) = -\frac{\partial}{\partial x^7}, \\ &J_2\left(\frac{\partial}{\partial x^4}\right) = -\frac{\partial}{\partial x^7}, \quad J_2\left(\frac{\partial}{\partial x^5}\right) = \frac{\partial}{\partial x^6}, \\ &J_3 = J_1 J_2, \quad \text{and} \quad k = \sum_{i=4}^{7} (dx^i)^2, \end{aligned}$$

where (x^4, x^5, x^6, x^7) are standard coordinates on $\mathbb{T}^4 \cong \mathbb{Z}^4\backslash\mathbb{R}^4$, and ψ is the identity map.

Note that

$$M_f = \frac{\mathbb{T}^3 \times \mathbb{T}^4 \times [0, t_0]}{(q, 0) \sim (f(q), t_0)},$$

where $f(\underline{x}, \underline{y}) = (\rho(\underline{x}), \underline{y})$. The only difference from the general case is that the Cartesian product is now taken with the interval $[0, t_0]$.

By Theorem 3.1.4, M_f can be endowed with a non-split generalized Kähler structure $(g, I_\pm)$. Moreover, Corollary 3.1.8 implies that $(M_f, I_\pm)$ is a formal manifold biholomorphic to $(\mathbb{T}^4 \times \mathbb{T}^3_\rho, J_i \oplus J_\pm)$. Furthermore, M_f does not admit any Kähler metric and does not satisfy the dd^c-lemma.

We can also describe M_f as the compact generalized Kähler solvmanifold introduced in [FP]. Consider the 8-dimensional unimodular almost abelian Lie group $G_8^{p,q}$, for $q \in \mathbb{R} \setminus \{0\}$, with structure equations:

$$\begin{aligned} &df^1 = f^1 \wedge f^8, \quad df^2 = -\tfrac{1}{2} f^2 \wedge f^8 + p f^3 \wedge f^8, \quad df^3 = -p f^2 \wedge f^8 - \tfrac{1}{2} f^3 \wedge f^8, \\ &df^4 = q f^5 \wedge f^8, \quad df^5 = -q f^4 \wedge f^8, \\ &df^6 = q f^7 \wedge f^8, \quad df^7 = -q f^6 \wedge f^8, \\ &df^8 = 0. \end{aligned} \tag{3.1.14}$$

The group $G_8^{p,q}$ is the semidirect product $\mathbb{R}^7 \rtimes_{\widetilde{\varphi}} \mathbb{R}$, where

$$\widetilde{\varphi}(t) = \begin{pmatrix} \varphi(t) & 0 & 0 \\ 0 & R_q(t) & 0 \\ 0 & 0 & R_q(t) \end{pmatrix},$$

with $\varphi(t)$ defined as in Example 3.1.1, and $R_q(t)$ the rotation matrix

$$R_q(t) = \begin{pmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{pmatrix}.$$

As previously observed, for $t = t_0$, $\varphi = \rho(t_0)$ is similar to an integer matrix A. Thus, by choosing $q = \frac{2\pi}{t_0}$, $\widetilde{\varphi}(t_0)$ is similar to the integer matrix $\operatorname{diag}(A, I_2, I_2)$ via an invertible matrix $\widetilde{P}$. The suspension described above is biholomorphic to the quotient $\Gamma\backslash G_8^{p,q}$, where $\Gamma = \widetilde{P}\mathbb{Z}^7 \rtimes t_0\mathbb{Z}$ [Br].

We now present an example which is not biholomorphic to the product $N \times \mathbb{T}^3_\rho$.

Example 3.1.13. Let ρ, $\{e^i\}$, and $(\mathbb{T}^4, k, J_i)$ be defined as in the previous example, and let ψ be the $\mathbb{R}^4$-rotation

$$(x^4, x^5, x^6, x^7) \mapsto (x^5, -x^4, x^7, -x^6).$$

Since ψ is represented by an integer matrix, it descends to a diffeomorphism of the flat torus $\mathbb{T}^4$.

We now prove that ψ is holomorphic with respect to each J_i, and preserves the hyperkähler structure $(\mathbb{T}^4, k, J_i)$. Indeed,

$$[\psi_*, J_1] = [\psi_*, J_2] = 0,$$

and

$$\begin{aligned} \psi^* k &= \psi^*\left(\sum_{i=4}^{7} (dx^i)^2\right) = (dx^5)^2 + (dx^4)^2 + (dx^7)^2 + (dx^6)^2 = k, \\ \psi^*\omega_1 &= \psi^*(dx^4 \wedge dx^5 - dx^6 \wedge dx^7) = \omega_1, \\ \psi^*\omega_2 &= \psi^*(-dx^4 \wedge dx^7 + dx^5 \wedge dx^6) = \omega_2, \\ \psi^*\omega_3 &= \psi^*(-dx^4 \wedge dx^6 - dx^5 \wedge dx^7) = \omega_3. \end{aligned}$$

Consider now the suspension

$$M_f = \frac{\mathbb{T}^3 \times \mathbb{T}^4 \times [0, t_0]}{(q, 0) \sim (f(q), t_0)},$$

where $f(\underline{x}, \underline{y}) = (\rho(\underline{x}), \psi(\underline{y}))$. By Theorem 3.1.4, M_f is a non-split generalized Kähler manifold which admits no Kähler metric and does not satisfy the dd^c-lemma.

Observe that M_f can also be realized as a compact quotient of $G^8_{p,q}$ for $q = \frac{\pi}{2t_0}$, via a lattice Γ, i.e.,

$$M_f \cong \Gamma\backslash G^8_{p,q},$$

where $\Gamma = \widetilde{P}\mathbb{Z}^7 \rtimes t_0\mathbb{Z}$, and $\widetilde{P}$ is the change-of-basis matrix from $\widetilde{\varphi}_{\frac{\pi}{2t_0}}(t_0)$ to $\operatorname{diag}(A, \Lambda, \Lambda)$, with

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We now compute the de Rham cohomology of M_f, using the fact that it is the suspension of $\mathbb{T}^3 \times \mathbb{T}^4 \cong \mathbb{T}^7$ by the diffeomorphism $f = (\rho, \psi)$.

The cohomology in degree r is given by

$$H^r(M_f) = K^r \oplus C^{r-1},$$

where $K^r = \ker(f_r^* - \mathrm{Id})$ and $C^r = \operatorname{coker}(f_r^* - \mathrm{Id})$.

We fix a basis for each $H^r(\mathbb{T}^3 \times \mathbb{T}^4) \cong H^r(\mathbb{T}^7)$ given by forms $[dx^I \wedge dx^J]$, where I and J are multi-indices of lengths $|I|$ and $|J|$, with indices in $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$ respectively, such that $|I| + |J| = r$.

Note that on such forms,

$$f^*(dx^I \wedge dx^J) = \rho^*(dx^I) \wedge \psi^*(dx^J).$$

We define

$$K^r_{\mathbb{T}^3} = \ker(\rho_r^* - \mathrm{Id}), \qquad C^r_{\mathbb{T}^3} = \operatorname{coker}(\rho_r^* - \mathrm{Id}),$$
$$K^r_{\mathbb{T}^4} = \ker(\psi_r^* - \mathrm{Id}), \qquad C^r_{\mathbb{T}^4} = \operatorname{coker}(\psi_r^* - \mathrm{Id}).$$

A degree-by-degree analysis shows

$$K^r = \bigoplus_{i+j=r} K^i_{\mathbb{T}^3} \wedge K^j_{\mathbb{T}^4}.$$

Since $\mathbb{T}^3_\rho$ is an Inoue surface in the family S_M, we have:

$$K^0_{\mathbb{T}^3} = C^0_{\mathbb{T}^3} = \langle 1 \rangle, \qquad K^1_{\mathbb{T}^3} = C^1_{\mathbb{T}^3} = 0,$$
$$K^2_{\mathbb{T}^3} = C^2_{\mathbb{T}^3} = 0, \qquad K^3_{\mathbb{T}^3} = C^3_{\mathbb{T}^3} = \langle [e^{123}] \rangle.$$

Similarly, for the $\mathbb{T}^4$ component, we obtain:

$$K^0_{\mathbb{T}^4} = C^0_{\mathbb{T}^4} = \langle 1 \rangle, \quad K^1_{\mathbb{T}^4} = C^1_{\mathbb{T}^4} = 0,$$
$$K^2_{\mathbb{T}^4} = C^2_{\mathbb{T}^4} = \langle [dx^{45}], [dx^{46} + dx^{57}], [dx^{47} - dx^{56}], [dx^{67}] \rangle,$$
$$K^3_{\mathbb{T}^4} = C^3_{\mathbb{T}^4} = 0, \quad K^4_{\mathbb{T}^4} = C^4_{\mathbb{T}^4} = \langle [dx^{4567}] \rangle.$$

Combining these, the full cohomology of M_f is:

Cohomology group	**Generators**
$H^1(M_f)$	$\langle[\theta]\rangle$
$H^2(M_f)$	$\langle[dx^{45}],\ [dx^{46}+dx^{57}],\ [dx^{47}-dx^{56}],\ [dx^{67}]\rangle$
$H^3(M_f)$	$\langle[e^{123}],\ [\theta\wedge dx^{45}],\ [\theta\wedge(dx^{46}+dx^{57})],$ $[\theta\wedge(dx^{47}-dx^{56})],\ [\theta\wedge dx^{67}]\rangle$
$H^4(M_f)$	$\langle[dx^{4567}],\ [\theta\wedge e^{123}]\rangle$
$H^5(M_f)$	$\langle[e^{123}\wedge dx^{45}],\ [e^{123}\wedge(dx^{46}+dx^{57})],$ $[e^{123}\wedge(dx^{47}-dx^{56})],\ [e^{123}\wedge dx^{67}],\ [\theta\wedge dx^{4567}]\rangle$
$H^6(M_f)$	$\langle[\theta\wedge e^{123}\wedge dx^{45}],\ [\theta\wedge e^{123}\wedge(dx^{46}+dx^{57})],$ $[\theta\wedge e^{123}\wedge(dx^{47}-dx^{56})],\ [\theta\wedge e^{123}\wedge dx^{67}]\rangle$
$H^7(M_f)$	$\langle[e^{123}\wedge dx^{4567}]\rangle$
$H^8(M_f)$	$\langle[\theta\wedge e^{123}\wedge dx^{4567}]\rangle$

Table 3.1: Generators of the cohomology groups $H^k(M_f)$.

Let $I=\{1,2,3,4\}$. The minimal model of M_f is given by the differential graded algebra $(\bigwedge V,d,\varphi)$, where

$$\bigwedge V=\bigwedge(a)\otimes\bigwedge_{i\in I}(b_i)\otimes\bigwedge_{(i,j)\in I\times I\setminus\{(1,4)\}}(c,\lambda_{ij}),$$

with degrees

$$|a|=1,\quad |b_i|=2,\quad |c|=|\lambda_{ij}|=3,$$

and differential

$$da=db_i=dc=0,\quad d\lambda_{ij}=b_i\wedge b_j.$$

Note that all wedge products $b_i\wedge b_j$ are exact, except for $b_1\wedge b_4$, which represents a non-trivial cohomology class.

The quasi-isomorphism $\varphi:(\bigwedge V,d)\to(\Omega^*(M_f),d)$ is defined on generators by:

$$\begin{aligned}
\varphi(a)&=\theta, & \varphi(b_1)&=dx^{45},\\
\varphi(b_2)&=dx^{46}+dx^{57}, & \varphi(b_3)&=dx^{47}-dx^{56},\\
\varphi(b_4)&=dx^{67}, & \varphi(c)&=e^{123},\\
\varphi(\lambda_{ij})&=0.
\end{aligned}$$

Under the identification given by φ, the cohomology of $(\bigwedge V,d)$ matches that of M_f.

We claim that M_f is formal. Define a quasi-isomorphism $\nu:(\bigwedge V,d)\to(H^*(\bigwedge V),0)$ by setting:

$$\nu(a)=[a],\quad \nu(b_i)=[b_i],\quad \nu(c)=[c],\quad \nu(\lambda_{ij})=0.$$

Then the induced map on cohomology is the identity, and formality follows.

Since $\mathbb{T}^3_\rho$ is an Inoue surface in the family S_M, it admits the following basis of $(1,0)$-forms:

$$\varphi^1_\pm = e^2 + ie^3, \qquad \varphi^2_\pm = e^1 + i\theta,$$

with differentials:

$$d\varphi^1_\pm = \frac{\alpha - i\beta_\pm}{2i}\,\varphi^{12} - \frac{\alpha - i\beta_\pm}{2i}\,\varphi^{1\bar{2}},$$
$$d\varphi^2_\pm = -i\alpha\,\varphi^{2\bar{2}},$$

where $\alpha = -\frac{1}{2}$ and $\beta_\pm = \pm p$.

Then:

$$H^{\bullet,\bullet}_{\bar\partial_\pm}(\mathbb{T}^3_\rho) = \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle[\varphi^{\bar{2}}_\pm]\rangle \oplus \mathbb{C}\langle[\varphi^{12\bar{1}}_\pm]\rangle \oplus \mathbb{C}\langle[\varphi^{12\bar{1}\bar{2}}_\pm]\rangle.$$

Let $\{\eta^1_i, \eta^2_i\}$ be invariant coframes for $T^{1,0}_i\mathbb{T}^4$, given by:

$$\eta^1_1 = dx^4 + idx^5, \qquad \eta^2_1 = dx^7 + idx^6,$$
$$\eta^1_2 = dx^5 + idx^6, \qquad \eta^2_2 = dx^7 + idx^4.$$

Since $d\eta^j_i = 0$, the Dolbeault cohomology of $\mathbb{T}^4$ is computed directly:

$$H^{\bullet,\bullet}_i(\mathbb{T}^4) = \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle[\eta^k_i]\rangle \oplus \mathbb{C}\langle[\eta^{\bar{k}}_i]\rangle \oplus \mathbb{C}\langle[\eta^{12}_i]\rangle \oplus \mathbb{C}\langle[\eta^{\bar{1}\bar{2}}_i]\rangle$$
$$\oplus\, \mathbb{C}\langle[\eta^{k\bar{h}}_i]\rangle \oplus \mathbb{C}\langle[\eta^{12\bar{k}}_i]\rangle \oplus \mathbb{C}\langle[\eta^{k\bar{1}\bar{2}}_i]\rangle \oplus \mathbb{C}\langle[\eta^{12\bar{1}\bar{2}}_i]\rangle,$$

for $k, h \in \{1,2\}$.

By Theorem 4.1.9, for $p+q = u+v$, the second page of the Borel spectral sequence is:

$${}^{p,q}E^{u,v,\pm}_2 = \sum_k H^{k,u-k}_{\bar\partial_\pm}(\mathbb{T}^3_\rho) \otimes H^{p-k,q-u+k}_{\bar\partial_i}(\mathbb{T}^4),$$

with differential

$$d_2 : {}^{p,q}E^{u,v}_2 \to {}^{p,q+1}E^{u+2,v-1}_2.$$

We claim that the Borel spectral sequences degenerate at $r = 2$. The only nontrivial cases to check are:

$$(p,q,u,v) = (2,1,1,2) \quad \text{and} \quad (2,2,1,3).$$

Case $(2,1,1,2)$**:** We have:

$${}^{2,1}E^{1,2}_2 = H^{0,1}_\pm(\mathbb{T}^3_\rho) \otimes H^{2,0}_i(\mathbb{T}^4) = \langle[\varphi^{\bar{2}}_\pm] \otimes [\eta^{12}_i]\rangle.$$

The target is:

$${}^{2,2}E^{3,1}_2 = H^{2,1}_\pm(\mathbb{T}^3_\rho) \otimes H^{0,1}_i(\mathbb{T}^4) = \langle[\varphi^{12\bar{1}}_\pm] \otimes [\eta^{\bar{k}}_i]\rangle.$$

Since the fibration $p : M_f \to \mathbb{T}^3_\rho$ is holomorphic, d_2 acts as $\mathrm{id} \otimes d_2$ (we used the Liebinitz rule for d_2).

$$d_2({}^{2,1}E_2^{1,2}) = H_\pm^{0,1}(\mathbb{T}^3_\rho) \otimes d_2(H_i^{2,0}(\mathbb{T}^4)) = H_\pm^{0,1}(\mathbb{T}^3_\rho) \otimes d_2({}^{2,0}E_2^{0,2\pm}) = 0.$$

Case $(2,2,1,3)$**:** The argument is analogous and also yields $d_2 = 0$.

Hence, the Borel spectral sequences degenerate at the second page.

As already observed, the previous examples are both diffeomorphic to solvmanifolds. A class of examples not diffeomorphic to solvmanifolds can be constructed by taking a $K3$ surface as the (hyper)Kähler manifold N. Indeed, in this latter case, by Theorem 4.1.9 we have that the holomorphic fibration

$$N \to M_f \to \mathbb{T}^3_\rho$$

induces the long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(\mathbb{T}^3_\rho) \to \pi_k(N) \to \pi_k(M_f) \to \pi_k(\mathbb{T}^3_\rho) \to \cdots .$$

Focusing on $k = 1$, we get

$$\cdots \to \pi_2(\mathbb{T}^3_\rho) \to \pi_1(N) \to \pi_1(M_f) \to \pi_1(\mathbb{T}^3_\rho) \to \pi_0(N) \to \cdots . \tag{3.1.15}$$

Observe that (3.1.15) reduces to the short exact sequence

$$0 \to \pi_1(N) \to \pi_1(M_f) \to \pi_1(\mathbb{T}^3_\rho) \to 0,$$

since the Inoue surfaces in the family S_M are solvmanifolds (and hence in particular they are aspherical) and N is path-connected.

Since by assumption N is a $K3$ surface, $\pi_1(N)$ is trivial and thus

$$\pi_1(M_f) \cong \pi_1(\mathbb{T}^3_\rho).$$

By contradiction, assume that M_f is a solvmanifold. Since $\mathbb{T}^3_\rho$ is another solvmanifold with isomorphic fundamental group, M_f is diffeomorphic to $\mathbb{T}^3_\rho$. However, this is absurd because

$$\dim(\mathbb{T}^3_\rho) = 4 < \dim(M_f).$$

Here, we give an example of a hyperkähler isometry of a K3 surface.

Example 3.1.14. The *Fermat quartic* is the complex surface in $\mathbb{CP}^3$ defined by the equation

$$\mathcal{F} = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\},$$

where z_0, z_1, z_2, z_3 are the standard homogeneous coordinates on $\mathbb{CP}^3$. It is well known that $\mathcal{F}$ is a K3 surface. We will always assume that $\mathcal{F}$ is endowed with the standard complex structure J_1, i.e., J_1 is such that the natural embedding $\iota : \mathcal{F} \hookrightarrow \mathbb{CP}^3$ is holomorphic.

Let $\iota^*\omega_{FS}$ be the Kähler form on $\mathcal{F}$ induced by the Fubini–Study metric on $\mathbb{CP}^3$, and define g to be the unique Kähler Ricci-flat metric on $\mathcal{F}$ whose Kähler form is cohomologous to $\iota^*\omega_{FS}$. Such a metric g is hyperkähler. We denote the hyperkähler structure by (J_1, J_2, J_3, g).

Although the explicit form of g is not known, the group of (not necessarily holomorphic) isometries of g is characterized in [AlGr]. In particular, $\mathrm{Isom}(g)$ is identified with the group of all holomorphic and anti-holomorphic isometries of $\mathbb{CP}^3$ which preserve $\mathcal{F}$.

Define $\sigma : \mathbb{CP}^3 \to \mathbb{CP}^3$ by

$$[z_0, z_1, z_2, z_3] \mapsto [-z_0, -z_1, z_2, z_3].$$

Then, by the identification above, the restriction $\psi := \sigma|_{\mathcal{F}}$ is clearly an isometry of $(\mathcal{F}, g)$. It is also straightforward to verify that ψ is holomorphic with respect to I, and hence $\psi \in \mathrm{Isom}_{\mathrm{hol}}(g)$. It remains to prove that ψ is a hyperkähler isometry, i.e., that ψ preserves the fundamental forms $\omega_2 = gJ_2$ and $\omega_3 = gJ_3$.

Since $\mathcal{F}$ is a K3 surface, there exists (up to complex scalar multiple) a unique holomorphic $(2,0)$-form Ω on $\mathcal{F}$. The fixed locus of σ in $\mathbb{CP}^3$ consists of the lines $\{z_0 = z_1 = 0\}$ and $\{z_2 = z_3 = 0\}$, each of which intersects the Fermat quartic in four isolated fixed points. Therefore, ψ has exactly eight isolated fixed points.

According to [Nik], since the fixed locus consists of isolated points, ψ is symplectic. In fact, since ψ is a holomorphic involution, it must send Ω either to Ω or to $-\Omega$.

Let p be one of the eight fixed points of ψ. Then the differential $d\psi$ acts on $T_p\mathcal{F}$ with eigenvalues ± 1 (since ψ is an involution). If 1 is an eigenvalue, then the corresponding geodesics would be pointwise fixed by ψ, contradicting the assumption that all fixed points are isolated. Therefore, $d\psi = -\,\mathrm{Id}$ on $T_p\mathcal{F}$ and hence $\psi^*\Omega = \Omega$.

Let $k \in \mathbb{C}^*$ be such that $|k|^2\,\Omega \wedge \overline{\Omega} = 2\omega_1^2$. Then,

$$\omega_2 = \mathrm{Re}(k\Omega), \quad \omega_3 = \mathrm{Im}(k\Omega).$$

Since $\psi^*\Omega = \Omega$, we have $\psi^*(k\Omega) = k\Omega$. Moreover, since ψ is holomorphic, we compute:

$$\begin{aligned}\psi^*(k\Omega) &= \psi^*(\mathrm{Re}(k\Omega) + i\,\mathrm{Im}(k\Omega)) \\ &= \psi^*\,\mathrm{Re}(k\Omega) + i\psi^*\,\mathrm{Im}(k\Omega),\end{aligned}$$

which implies that

$$\psi^*\omega_2 = \omega_2, \quad \psi^*\omega_3 = \omega_3.$$

Hence, ψ is indeed a hyperkähler isometry.

Example 3.1.15. Let ρ be the diffeomorphism of $\mathbb{T}^3$ defined by (3.1.1), and let $\{e_i\}$ and $\{e^i\}$ be the bases introduced in Example 3.1.1. As already noted there, ρ preserves the basis $e^i(t)$, and one can easily verify that

$$\left(\frac{1}{l(t)}\right)' \cdot \frac{1}{a_1} = (e^t)' \cdot e^{-t} = 1 \neq 0.$$

We now apply Theorem 3.1.4 in the case where $N = \mathcal{F}$ is the Fermat quartic endowed with the hyperkähler structure (J_1, J_2, J_3, g), where g is the unique Ricci-flat Kähler metric cohomologous to $\iota^*\omega_{FS}$, as constructed in Example 3.1.14. Let ψ be the hyperkähler isometry described there, and define the suspension

$$M_f = \frac{\mathbb{T}^3 \times \mathbb{T}^4 \times [0, t_0]}{(q, 0) \sim (f(q), t_0)}, \quad \text{where} \quad f(\underline{x}, q) = (\rho(\underline{x}), \psi(q)).$$

By Theorem 3.1.4, the manifold M_f admits a split generalized Kähler structure.

3.2 Construction via the Hopf surface

In this section we describe a construction of generalized Kähler suspensions, analogous to the previous one, obtained by replacing the role of the Inoue surface with that of the Hopf surface. Since this will be useful in the sequel, we first describe the construction of an SKT metric, and then show that this SKT metric arises as the SKT metric of a generalized Kähler structure.

Lemma 3.2.1. *Let (K, J, g) be a compact Kähler manifold of complex dimension k and let ψ be a Kähler isometry, i.e., ψ is an holomorphic diffeomorphism of K satisfying $\psi^*(g) = g$. Then, the suspension*

$$M_f = (K \times \mathbb{S}^3)_f,$$

with $f = (\psi, Id_{S^3})$, admits a SKT structure (I, h).

Proof. Let us fix on $K \times \mathbb{C}^2 \setminus \{(0,0)\}$ the product complex structure $J \times J_-$, where J_- is the standard complex structure on $\mathbb{C}^2 \setminus \{(0,0)\}$.

Consider the following free and proper discontinuous $\mathbb{Z}$-action on $K \times \mathbb{C}^2 \setminus \{(0,0)\}$

$$n \cdot (p, \underline{x}) \mapsto (\psi^n(p), 2^n \underline{x}). \tag{3.2.1}$$

The associated automorphisms φ_n are manifestly holomorphic with respect to $J \times J_-$, and so, $J \times J_-$ descends to a complex structure I on the quotient $K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}}$.

We claim that $K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}}$ is diffeomorphic to $M_f = (K \times \mathbb{S}^3)_f$. Let

$$\alpha : K \times S^3 \times \mathbb{R} \to K \times \mathbb{C}^2 \setminus \{(0,0)\}, \quad (p,q,t) \mapsto (p, 2^t \cdot q)$$

with inverse

$$\alpha^{-1} : K \times \mathbb{C}^2 \setminus \{(0,0)\} \to K \times S^3 \times \mathbb{R}, \quad (p, \underline{x}) \mapsto (p, \frac{\underline{x}}{\|\underline{x}\|}, \log_2 \|\underline{x}\|).$$

The induced maps on the quotient

$$\alpha : M_f \to K \times \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}, \quad \alpha^{-1} : K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}} \to M_f,$$

are well defined and give the claimed identification $K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}} \cong M_f$. Therefore, M_f inherits from $K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}}$ the complex structure I.

Consider on $K \times \mathbb{C}^2 \setminus \{(0,0)\}$ the following 2-form $\Omega = \omega + \omega_-$, where in the coordinates (z_1, z_2) on $\mathbb{C}^2 \setminus \{(0,0)\}$

$$\omega_- = \frac{i}{R^2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \tag{3.2.2}$$

with $R^2 = z_1\overline{z}_1 + z_2\overline{z}_2$.
Observe that Ω induces a well defined non-degenerate global 2-form Ω * on

$$K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}} \cong M_f,$$

as it is preserved by φ_n^*, for each n in $\mathbb{Z}$. Furthermore, Ω is of type $(1,1)$ with respect to I.

The Hermitian metric $h = \Omega I$ is hence given by

$$h = g + \frac{1}{R^2}(dz_1 d\overline{z}_1 + dz_2 d\overline{z}_2). \tag{3.2.3}$$

We compute $d^c\Omega$, where $d^c = -IdI$, and $dd^c\Omega$.

$$d^c\Omega = -\frac{1}{R^4}\left[(\overline{z}_2 dz_2 - z_2 d\overline{z}_2) \wedge dz_1 \wedge d\overline{z}_1 + (\overline{z}_1 dz_1 - z_1 d\overline{z}_1) \wedge dz_2 \wedge d\overline{z}_2\right], \tag{3.2.4}$$

$$d(d^c\Omega) = d\left(\frac{1}{R^4}\left[(\overline{z}_2 dz_2 - z_2 d\overline{z}_2) \wedge dz_1 \wedge d\overline{z}_1 + (\overline{z}^1 dz_1 - z_1 d\overline{z}_1) \wedge dz_2 \wedge d\overline{z}_2\right]\right) = 0.$$

Therefore, (I, h) is a SKT structure on M_f. q.e.d.

In the next proposition we investigate some cohomological properties of the manifold M_f.

*We denote the induced form by Ω, with a little abuse of notation

Proposition 3.2.2. *Let $M_f = (K \times \mathbb{S}^3)_f$ be the suspension constructed as in Lemma 3.2.1. Then*

1. *M_f is diffeomorphic to the product of $K_\psi \times S^3$, where K_ψ is the suspension of the Kähler manifold K with respect to the Kähler isometry ψ;*
2. *M_f is formal;*
3. *M_f is non-Kähler.*

Proof. Consider the map

$$\beta : M_f \to K_\psi \times S^3, \quad [(p, q, t)] \mapsto ([(p, t)], q)$$

with inverse

$$\beta^{-1} : K_\psi \times S^3 \to M_f, \quad ([(p, t)], q) \mapsto [(p, q, t)].$$

It is immediate to observe that β and β^{-1} are well defined diffeomorphisms. Indeed,

$$\beta[(p, q, 0)] = ([(p, 0)], q) = ([(\psi(p), 1], q) = \beta[(\psi(p), q, 1)],$$
$$\beta^{-1}([(p, 0)], q) = [(p, q, 0)] = [(\psi(p), q, 1)] = \beta^{-1}([(\psi(p), 1)], q).$$

The second statement follows by the identification $M_f \cong K_\psi \times S^3$ proved above. In fact, by [Li], the suspensions of compact Kähler manifolds are compact co-Kähler manifolds and so they are formal in the sense of Sullivan (see for instance [CDLM]). Since S^3 is formal, M_f is the product of formal manifolds and, hence, formal.

To prove that M_f is non-Kähler is suffices to show that the first Betti number $b_1(M_f) = b_1(K_\psi)$ is odd.

Using the identification proved above, $b_1(M_f) = b_1(K_\psi \times S^3) = b_1(K_\psi)$, where the last equality follows by Künneth formula. Moreover, since K_ψ is co-Käher, its first Betti number is odd [CDLM]. q.e.d.

In the next theorem we show that, on the SKT suspension $(M_f, I = I_-, h)$, there exists another complex structure I_+ such that the pair $(I_\pm, h)$ defines a generalized Kähler structure. In particular, this result, together with Proposition 3.2.2, provides the existence of a non-trivial generalized Kähler structure, extending the SKT structure constructed in Lemma 3.2.1.

Before proceeding with the statement of the theorem, we recall the following definition.

Definition 3.2.3. *A generalized Kähler structure $(I_\pm, g)$ is said to be twisted if $[d^c_+\omega_+] \neq 0 \in H^3(M)$ and untwisted otherwise.*

Theorem 3.2.4. *Let (K, J, g) be a compact Kähler manifold of complex dimension k and let ψ be a Kähler isometry, i.e., $\psi : K \to K$ is an holomorphic diffeomorphism satisfying*

$\psi^*(g) = g$.
Then, the suspension $M_f = (K \times S^3)_f$, with $f = (\psi, Id_{S^3})$, admits a split twisted generalized Kähler structure $(I_\pm, h)$.
Moreover, I_+ and I_- induce opposite orientations on M_f.

Proof. Let $(I_- := I, h)$ the SKT structure constructed in Lemma 3.2.1.

We consider on $K \times \mathbb{C}^2 \setminus \{(0,0)\} \cong K \times S^3 \times \mathbb{R}$ another product complex structure $J \times J_+$, where J_+ is the complex structure on $\mathbb{C}^2 \setminus \{(0,0)\}$ obtained by changing the orientation of the z_2 plane. More precisely, if (z_1, z_2) and (ζ^1, ζ^2) are the holomorphic coordinates associated to $J_\pm$ respectively, $\zeta_1 = z_1$ and $\zeta_2 = \overline{z_2}$.

The automorphisms φ_n associated to the $\mathbb{Z}$-action described in Lemma 3.2.1 are holomorphic also with respect to $J \times J_+$, implying that $J \times J_+$ descends to a complex structure I_+ on the quotient $M_f \cong K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}}$, which satisfy $[I_+, I_-] = 0$.
We denote by $\Omega_+ = \omega + \omega_+$ the fundamental form of the Hermitian structure (I_+, h), where h is the Riemannian metric explicitly given in (3.2.3) and ω_+ can be written in the coordinates (ζ_1, ζ_2) and (z_1, z_2), respectively, as

$$\omega_+ = \frac{i}{R^2}(d\zeta_1 \wedge d\overline{\zeta}_1 + d\zeta_2 \wedge d\overline{\zeta}_2) = \frac{i}{R^2}(dz_1 \wedge d\overline{z}_1 - dz_2 \wedge d\overline{z}_2).$$

We compute $d^c_+ \Omega_+$.

$$\begin{aligned} d^c_+\Omega_+ &= -\frac{1}{R^4}\left[(\overline{\zeta}_2 d\zeta_2 - \zeta_2 d\overline{\zeta}_2) \wedge d\zeta_1 \wedge d\overline{\zeta}_1 + (\overline{\zeta}_1 d\zeta_1 - \zeta_1 d\overline{\zeta}_1) \wedge d\zeta_2 \wedge d\overline{\zeta}_2\right] \\ &= \frac{1}{R^4}\left[(\overline{z}_2 dz_2 - z_2 d\overline{z}_2) \wedge dz_1 \wedge d\overline{z}_1 + (\overline{z}_1 dz_1 - z_1 d\overline{z}_1) \wedge dz_2 \wedge d\overline{z}_2\right] \\ &= -d^c_-\Omega_-. \end{aligned}$$

It follows that $(h, I_\pm)$ is a split generalized Kähler structure.

We claim that $(h, I_\pm)$ is twisted. Let $H = d^c_+\Omega_+$. If one considers the radial projection

$$\pi : K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}} \to S^3, \quad [(p, \underline{x})] \mapsto \frac{\underline{x}}{\|\underline{x}\|},$$

then it is straightforward to observe that $H = 2\pi^* vol_{S^3}$.
By contradiction, let us assume that H is exact. Fixed any $p \in K$ and $t \in (0,1)$, we define $\iota_{p,t} : S^3 \to K \times \mathbb{C}^2 \setminus \{(0,0)\} /_{\mathbb{Z}}, \;\; q \mapsto [(p, 2^t \cdot q)]$. Then, by Stokes Theorem,

$$0 = \int_{S^3} \iota^*_{p,t} H = 2\int_{S^3} \iota^*_{p,t}(\pi^* vol_{S^3}) = 2\int_{S^3} (\pi \circ \iota_{p,t})^* vol_{S^3} = 2\int_{S^3} vol_{S^3} \neq 0.$$

Clearly, this leads to a contradiction.

We conclude the proof by proving that $I_\pm$ induces opposite orientations on M_f. Consider the volume forms associated to the pairs $(h, I_\pm)$, which are respectively

$$\frac{1}{k+2!}\Omega_-^{k+2} = -\frac{1}{k!R^4}\omega^k \wedge dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$
$$\frac{1}{k+2!}\Omega_+^{k+2} = -\frac{1}{k!R^4}\omega^k \wedge d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2.$$

Since $d\zeta_1 = dz_1$ and $d\zeta_2 = d\bar{z}_2$, $\frac{1}{k+2!}\Omega_-^{k+2} = -\frac{1}{k+2!}\Omega_+^{k+2}$. The last statement follows.
q.e.d.

We exhibit an explicit Example fitting in the hypothesis of Theorem 3.2.4.

3.2.1 Generalization to compact Lie groups

In this section, we present a natural generalization of the construction discussed in the previous section. Previously, the construction was based on the Hopf surface, which is the lowest-dimensional example of a compact complex manifold endowed with a non-Kähler Hermitian structure.

Here, we generalize this idea by replacing the Hopf surface (or, more precisely, its universal cover) with an arbitrary even-dimensional compact Lie group. As in the previous case, our approach proceeds in two steps: first, we construct an SKT metric on the total space, and then we show that this SKT metric arises as part of a generalized Kähler structure.

Consider a $2r$-dimensional Lie group

$$G' = G \times \mathbb{R}^n,$$

where G is a compact, connected, and simply connected semisimple Lie group. Throughout this Section, we will always assume that G is non-trivial. Assume that G' is endowed with a bi-invariant metric

$$g' = b + g_E,$$

where b is a bi-invariant metric on G, and g_E is the Euclidean metric on $\mathbb{R}^n$. We further observe that the Lie algebra of left-invariant vector fields of G' coincides with that of

$$G'' = G \times \mathbb{T}^n,$$

which inherits the bi-invariant metric g' from G'. Since G'' is compact, there exists a left-invariant complex structure J_L compatible with g'.

As J_L is left-invariant, it is naturally defined on G' as well, so that (J_L, g') defines a Hermitian structure on G'.

Let ρ be a group homomorphism

$$\rho : \mathbb{Z}^n \longrightarrow \mathrm{Isom}(G).$$

Then, for each $m \in \mathbb{Z}^n$, the diffeomorphism

$$m \cdot (q,t) = \big(\rho(m)q,\ t+m\big)$$

is an isometry of g'.

Furthermore, consider a compact Kähler manifold (K, g, J) and a group homomorphism

$$\psi : \mathbb{Z}^n \longrightarrow \mathrm{Isom}_{hol}(K),$$

where $\mathrm{Isom}_{hol}(K)$ denotes the group of holomorphic Kähler isometries of (K, g, J).

For clarity, we give an example of such a homomorphism. Let φ be a Kähler isometry of K. Define

$$\psi : \mathbb{Z}^n \to \mathrm{Isom}_{hol}(K), \qquad m = (z_1, \dots, z_n) \mapsto \varphi^{|m|} := \varphi^{z_1+\cdots+z_n}.$$

Then $\psi(0) = \varphi^0 = \mathrm{Id}$, and

$$\psi(m+m') = \varphi^{|m+m'|} = \varphi^{|m|} \circ \varphi^{|m'|},$$

so that ψ is indeed a group homomorphism.

We now define the following $\mathbb{Z}^n$-action on $K \times G \times \mathbb{R}^n$:

$$m \cdot (p,q,t) = \big(\psi(m)p,\ \rho(m)q,\ t+m\big).$$

This action is trivial if $n = 0$ and non-trivial otherwise. We focus on the case $n \neq 0$.

In this case, the action is clearly free, since

$$(\psi(m)p, \rho(m)q, t+m) = (p,q,t) \quad \Longrightarrow \quad t+m = t \quad \Longrightarrow \quad m = 0.$$

We claim that the action is also properly discontinuous. First, note that the action is manifestly smooth, as $\mathbb{Z}^n$ acts by isometries. To check properness, let $(p,q,t), (p',q',t') \in K \times G'$. Since the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translations is proper, there exist open neighborhoods I of t and I' of t' such that the set

$$\Gamma := \big\{ m \in \mathbb{Z}^n \bigm| (m+I) \cap I' \neq \emptyset \big\}$$

is finite.

Let U, U' be open neighborhoods of p, p', and V, V' open neighborhoods of q, q', respectively. For $m \notin \Gamma$, we compute

$$\begin{aligned}\big(m \cdot (U \times V \times I)\big) \cap (U' \times V' \times I') &= \big(\psi(m)U \times \rho(m)V \times (m+I)\big) \cap (U' \times V' \times I') \\ &= \big(\psi(m)U \cap U'\big) \times \big(\rho(m)V \cap V'\big) \times \big((m+I) \cap I'\big) \\ &= \emptyset.\end{aligned}$$

Thus

$$\Lambda := \big\{m \in \mathbb{Z}^n \;\big|\; \big(m \cdot (U \times V \times I)\big) \cap (U' \times V' \times I') \neq \emptyset\big\} \subset \Gamma,$$

and therefore, since Γ is finite, so is Λ. This shows that the action is proper.

Lemma 3.2.5. *Let $(G' = G \times \mathbb{R}^n, g' = b + g_E, J_L)$ be as above and let ρ be a group homomorphism $\rho : \mathbb{Z}^n \to Isom(G)$ such that for each $m \in \mathbb{Z}^n$ the diffeomorphism*

$$m : G' \to G', \; (q,t) \mapsto (\rho(m)q, t+m)$$

is an holomorphic isometry of (g', J_L). Let (K, g, J) be a compact Kähler manifold and let ψ be a group homomorphism $\psi : \mathbb{Z}^n \to Isom_{hol}(K)$, where $Isom_{hol}(K)$ is the group of holomorphic Kähler isometries of (K, g, J).
Then the quotient

$$M_{\psi,\rho} = (K \times G \times \mathbb{R}^n)_{\psi,\rho} = {}^{K \times G \times \mathbb{R}^n}/_{\mathbb{Z}^n},$$

where $\mathbb{Z}^n$ acts freely and properly discontinuously on $K \times G \times \mathbb{R}^n$ as

$$m \cdot (p,q,t) = (\psi(m)p, \rho(m)q, t+m),$$

admits a SKT structure (I, h).

Proof. We prove the result by constructing a SKT structure $(\widetilde{g}, \widetilde{J})$ on $K \times G \times \mathbb{R}^n$ preserved by the action of $\mathbb{Z}^n$.

Consider the Hermitian structure

$$\widetilde{J} = J \times J_L, \qquad \widetilde{g} = g + b + g_E$$

on $K \times G \times \mathbb{R}^n$, with corresponding fundamental form $\widetilde{\omega} = \omega + \omega_L$. Since $d\omega = 0$, we have

$$d^c_{\widetilde{J}}\widetilde{\omega} = d^c_{J_L}\omega_L.$$

We now compute $d^c_{J_L}\omega_L$. Let X, Y, Z be left-invariant vector fields on G'. Using the integrability of J_L and the bi-invariance of the metric g', we obtain

$$d^c_{J_L}\omega_L(X,Y,Z) = g'([X,Y],Z),$$

(see for a more detailed computation [Gu, Example 2.25]).

Since $d^c_{J_L}\omega_L$ is $Ad(G')$-invariant, it is a bi-invariant form on G', and hence closed. This shows that $(\widetilde{g}, \widetilde{J})$ is SKT on $K \times G \times \mathbb{R}^n$.

To conclude the proof, we show that the Hermitian structure $(\widetilde{g}, \widetilde{J})$ is preserved by the action of $\mathbb{Z}^n$. Consider the diffeomorphism induced by $m \in \mathbb{Z}^n$:

$$m \cdot (p, q, t) = (\psi(m)p, \rho(m)q, t + m).$$

We need to check that this action is holomorphic with respect to $\widetilde{J}$. By hypothesis, $\psi(m)$ is holomorphic with respect to J and $\rho(m)$ is such that $(\rho(m)q, t + m)$ is holomorphic with respect to J_L. Hence $\widetilde{J}$ descends to a complex structure I on the quotient

$$M_{\psi,\rho} := (K \times G \times \mathbb{R}^n)/\mathbb{Z}^n.$$

Moreover, since both group homomorphisms ψ and ρ have image in the isometry groups of K and G, respectively, and $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translations, the product metric $\widetilde{g} = g + b + g_E$ is preserved by $\mathbb{Z}^n$. Hence it descends to a metric h on $M_{\psi,\rho}$, which by construction is compatible with I. Therefore, (I, h) defines a SKT structure on $M_{\psi,\rho}$.
q.e.d.

Denote by $\mathfrak{g}^L(G')$ and $\mathfrak{g}^R(G')$ the Lie algebras of left- and right-invariant vector fields on G', respectively. Then

$$\mathfrak{g}^L(G') = \mathfrak{g}^L \oplus \mathbb{R}^n = \mathfrak{g}^L(G''), \qquad \mathfrak{g}^R(G') = \mathfrak{g}^R \oplus \mathbb{R}^n = \mathfrak{g}^R(G''),$$

where $G'' = G \times \mathbb{T}^n$.

Therefore, G'' inherits from G' the bi-invariant metric g'. Since G'' is compact, it admits both a left- and a right-invariant complex structure, denoted by J_L and J_R, which are compatible with the bi-invariant metric g'. As G' and G'' share the same Lie algebras, it follows that (g', J_L, J_R) is also a bi-Hermitian structure on G', where J_L and J_R are left- and right-invariant complex structures, respectively.

Theorem 3.2.6. *Consider $(G' = G \times \mathbb{R}^n, g', J_L, J_R)$ be as above and let ρ be a group homomorphism $\rho : \mathbb{Z}^n \to \mathrm{Isom}(G)$ such that for each $m \in \mathbb{Z}^n$ the diffeomorphism*

$$m \cdot (q, t) = (\rho(m)q, t + m)$$

is a holomorphic isometry of g' with respect to J_L and J_R. Let (K, g, J) be a compact Kähler manifold and let ψ be a group homomorphism $\psi : \mathbb{Z}^n \to \mathrm{Isom}_{hol}(K)$, where $\mathrm{Isom}_{hol}(K)$ is the group of holomorphic Kähler isometries of (K, g, J). Then the quotient

$$M_{\psi,\rho} = (K \times G \times \mathbb{R}^n)_{\psi,\rho} = {K \times G \times \mathbb{R}^n}/{\mathbb{Z}^n},$$

where $\mathbb{Z}^n$ acts freely and properly discontinuously on $K \times G \times \mathbb{R}^n$ as

$$m \cdot (p, q, t) = (\psi(m)p, \rho(m)q, t + m),$$

admits a generalized Kähler structure.

Proof. As already observed in the proof of Lemma 3.2.5,

$$(\widetilde{g} = g + g', \ \widetilde{J}_- = \widetilde{J} = J \times J_L)$$

defines an SKT structure on $K \times G \times \mathbb{R}^n$ that is preserved by the $\mathbb{Z}^n$-action

$$m \cdot (p, q, t) = (\psi(m)p, \rho(m)q, t + m),$$

and hence induces an SKT structure $(h, I_- = I)$ on the quotient $M_{\psi,\rho}$ (see Lemma 3.2.5 for notation).

Consider the different product complex structure on $K \times G \times \mathbb{R}^n$

$$\widetilde{J}_+ = J \times J_R,$$

which is compatible with $\widetilde{g}$ by construction. Its fundamental form is

$$\widetilde{\omega}_+ = \omega + \omega_R.$$

Since $d\omega = 0$, we have

$$d^c_{\widetilde{J}_+} \widetilde{\omega}_+ = d^c_{J_R} \omega_R.$$

Recalling that

$$d^c_{J_L} \omega_L(X, Y, Z) = g'([X, Y], Z)$$

on left-invariant vector fields, it follows that

$$d^c_{\widetilde{J}_-} \widetilde{\omega}_- = d^c_{J_L} \omega_L = -d^c_{J_R} \omega_R,$$

since the right Lie algebra is anti-isomorphic to the left one.

To conclude the proof, it remains to show that $\widetilde{J}_+$ is preserved by the $\mathbb{Z}^n$-action. By hypothesis, $\psi(m)$ is holomorphic with respect to J, and $\rho(m)$ is such that $(\rho(m)q, t+m)$ is holomorphic with respect to both J_L and J_R. Hence $\widetilde{J}_+$ descends to a complex structure I_+ on the quotient $M_{\psi,\rho}$. Therefore, $M_{\psi,\rho}$ inherits from $K \times G \times \mathbb{R}^n$ a generalized Kähler structure (I_-, I_+, h).

q.e.d.

Remark 3.2.7. If ρ is the trivial homomorphism, i.e. $\rho(m) = Id$, for each $m \in \mathbb{Z}^n$, then the induced diffeomorphism $(q, t) \mapsto (q, t + m)$ is necessarily holomorphic with respect to J_L and J_R.

Remark 3.2.8. The complex structures of the generalized Kähler metric $(h, I_\pm)$ induce the same orientation. Indeed, $I_\pm$ are induced by $\widetilde{J}_- = J \times J_L$ and $\widetilde{J}_+ = J \times J_R$, and the structures $\widetilde{J}_\pm$ define the same orientation since J_L and J_R are isomorphic as complex manifolds via the group inversion.

In the special case $G = SU(2)$ and $n = 1$, Theorem 3.2.6 produces a generalized Kähler structure on the mapping torus M_f that is distinct from the one obtained in Theorem 3.2.4.

Corollary 3.2.9. *Let $(M_{\psi,\rho}, h, I_\pm)$ be the generalized Kähler manifold constructed as in Theorem 3.2.6. Then the generalized Kähler structure $(h, I_\pm)$ is twisted, I_+ and I_- both fail to satisfy the $dd^c_\pm$-Lemma and, in particular, they do not admit any compatible Kähler metric.*

Proof. We prove the result by contradiction. Assume that the torsion 3-form H of the generalized Kähler structure $(h, I_\pm)$ on $M_{\psi,\rho}$ is exact. Let $\pi : K \times G \times \mathbb{R}^n \to M_{\psi,\rho} = (K \times G \times \mathbb{R}^n)_{\psi,\rho}$ be the covering map. Then $\pi^* H$ is also exact.

By construction, we have

$$\pi^* H = \widetilde{H}, \qquad \widetilde{H} = d^c_{\widetilde{J}_+} \widetilde{\omega}_+ = d^c_{J_R} \omega_R = -g'([\cdot,\cdot],\cdot) = -b([\cdot,\cdot]_G,\cdot),$$

where $[\cdot,\cdot]_G$ denotes the Lie bracket on G. Hence, $\widetilde{H}$ can be identified with a 3-form on G.

Since H is exact, we have $[\widetilde{H}] = 0 \in H^3(G)$. Moreover, as G is compact, the third de Rham cohomology satisfies

$$H^3(G) \cong \Omega^3_I(G),$$

where $\Omega^3_I(G)$ denotes the space of bi-invariant 3-forms. Because $\widetilde{H}$ is bi-invariant, it follows that $\widetilde{H} = 0$.

In particular, this implies that $[X, Y] = 0$ for any pair of left-invariant vector fields X, Y on G. Thus the Lie algebra of G would be abelian, which contradicts the fact that G is semisimple.

The claim then follows by applying [Gu, Corollary 2.19].

q.e.d.

Chapter 4

Bismut Hermitian Einstein and Strong HKT manifolds with parallel Bismut torsion

The main goal of this chapter is to construct compact BHE manifolds in dimension 8 which are neither Bismut–flat nor Kähler, and which are not products of such manifolds (see Theorem 4.1.1). The idea is to show that, for certain Kähler manifolds (K, J, g), the pluriclosed metric constructed in Lemma 3.2.1 on the manifold M_f is BHE and non-flat. The most interesting case occurs when (K, J, g) is a K3 surface with its Ricci–flat Kähler metric and $\psi \neq \mathrm{Id}$. In this situation, M_f does not split topologically as a product of $S^3 \times S^1$ (which is Bismut–flat) and the K3 surface, because of topological obstructions. Hence M_f is not a product of a Bismut–flat manifold and a Ricci–flat Kähler one (see Proposition 4.1.6). A further generalization is obtained by replacing the Hopf surface with a compact Lie group, in the spirit of Lemma 3.2.5 (see Proposition 4.1.2).

All the examples constructed in this way have *parallel Bismut torsion*, namely $\nabla^B H = 0$. This motivates the study of compact BHE manifolds with parallel Bismut torsion. We characterize their universal covers in Theorem 4.2.5. The same result was later obtained by different methods in [BPT], where it was shown that a compact BHE manifold has parallel Bismut torsion if and only if it admits a finite cover splitting as such a product.

The absence of examples with $\nabla^B H \neq 0$ suggests that, in the compact case, the BHE condition may force the torsion H to be parallel.

Since any BHE manifold is CYT, the restricted holonomy of the Bismut connection is contained in $\mathrm{SU}(n)$, where n is the complex dimension. However, by the splitting result above, BHE manifolds with parallel Bismut torsion cannot have full holonomy. Therefore, if compact BHE manifolds always have parallel Bismut torsion, it follows that their holonomy can never be full.

The lack of compact, non-Kähler examples with full holonomy (that is, restricted holonomy equal to $\mathrm{SU}(n)$) led to the question of whether such examples exist. We show that they do not: a compact BHE manifold with full holonomy must be Kähler. The proof relies on the fact that in the non-Kähler case, the restricted holonomy reduces further to $\mathrm{SU}(n-1)$ (see Theorem 6.4.1; see also [Pap]). More precisely, every compact BHE manifold admits a unique smooth function f, defined up to a constant, called the *potential function*, such that

$$V = \theta^\sharp - \nabla f$$

is Bismut–parallel and nowhere vanishing [GFJS].

As noted in the preliminaries, BHE metrics also appear in the context of strong HKT geometry. This observation has recently attracted considerable interest, since it suggests extending results from the BHE setting to the richer hyperhermitian framework.

The existence of strong HKT structures has been studied in the locally homogeneous case. Combining the results of [BDV, DF3], one finds that a nilmanifold admits a strong HKT structure if and only if it is hyperkähler, and an analogous result for almost abelian solvmanifolds was proved in [AB1]. The existence of strong HKT structures on solvmanifolds remained open for ten years. In Corollary 4.2.19 we give a negative answer, proving a more general non-existence result for non-Kähler BHE structures on solvmanifolds. The key step is showing that the potential function f is constant, after which the result follows from curvature properties of left-invariant metrics on solvable Lie algebras.

Given a compact strong HKT structure, each Hermitian structure is BHE and the Lee forms θ_L coincide. This allows one to define Bismut–parallel vector fields $V_L = \theta^\sharp - \nabla f_L$. Moreover, the potential functions f_L differ only by constants, so that the vector fields coincide:

$$V := V_I = V_J = V_K$$

(see Lemma 4.2.17). As in the BHE case, this leads to a further reduction of the holonomy of the Bismut connection from $Sp(n)$ to $Sp(n-1)$. More precisely, a compact strong HKT manifold has holonomy $Sp(n)$ if and only if it is hyperkähler (see Theorem 6.4.1).

The existence of compact strong HKT manifolds with $\nabla^B H \neq 0$ remains an open problem. In analogy with the splitting result above, we prove in Theorem 4.2.11 that if a compact strong HKT manifold has parallel Bismut torsion, then up to finite cover it splits as the product of a compact hyperkähler manifold and a compact Bismut–flat one.

We conclude this chapter with the study of 8-dimensional compact simply connected strong HKT manifolds. In this case these manifolds admit an integrable distribution isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, corresponding to an isometric $\mathbb{H}^*$-action. In particular, we show that the potential function f must be constant, leading to a very rigid geometric structure. Recently, such HKT geometries with $\mathbb{H}^*$-action have been considered in [PS, Pap, PW, Wit]. Finally, we characterize the conditions under which the Bismut torsion is parallel. By the splitting result of mentioned above, this forces the manifold to be the Lie group $\mathrm{SU}(3)$.

The results in this chapter rely crucially on the interplay among the Levi-Civita, Obata, and Bismut connections. Each captures a different geometric aspect: the Levi-Civita connection encodes the Riemannian structure, the Obata connection the hypercomplex structure, and the Bismut connection sits between them, reflecting features of both. A full understanding of these manifolds requires bringing together the perspectives provided by all three.

The results presented in this chapter are based on two papers. The first, written jointly with A. Fino and G. Grantcharov, has already been published [BFG]. The second, authored together with A. Fino, G. Grantcharov, and M. Verbitsky, has been submitted for publication [BFGV].

4.1 Construction of the Examples

We begin this chapter by presenting a general construction of BHE metrics.

Theorem 4.1.1. *Let M_f be the mapping torus $M_f = (K \times \mathbb{S}^3)_f$ constructed as in Lemma 3.2.1. If the Kähler metric g on (K, J) is (non-flat) Ricci flat, then the Bismut connection associated to the SKT metric h on (M_f, I), constructed in Lemma 3.2.1 is CYT with non-flat Bismut connection.*

Proof. Consider the product metric

$$\widetilde{g} := g + \frac{g_E}{R^2}$$

on $K \times (\mathbb{C}^2 \setminus \{0\})$, where g is the Kähler metric on K, g_E is the Euclidean metric on $\mathbb{C}^2$, and $R^2 = |z_1|^2 + |z_2|^2$ in terms of the standard coordinates (z_1, z_2) on $\mathbb{C}^2$. As observed in the proof of Lemma 3.2.1, the complex structure $\widetilde{J} = J \times J_-$, with J_- the standard complex structure on $\mathbb{C}^2 \setminus \{0\}$, makes $(\widetilde{g}, \widetilde{J})$ into a Hermitian structure.

Since $(\widetilde{g}, \widetilde{J})$ is a product Hermitian structure, the corresponding Bismut connection $\widetilde{\nabla}^B$ splits on decomposable vector fields $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ as

$$\widetilde{\nabla}^B_X Y = (\nabla^B_1)_{X_1} Y_1 + (\nabla^B_2)_{X_2} Y_2 = (\nabla^{LC}_1)_{X_1} Y_1 + (\nabla^B_2)_{X_2} Y_2,$$

where ∇^B_1 is the Bismut connection of (J, g) and ∇^B_2 is the Bismut connection of $(J_-, g_E/R^2)$. The first equality holds since (J, g) is Kähler. Since $\mathbb{C}^2 \setminus \{0\}$ is well known to be Bismut flat, we have $R^B_2 = 0$, so that the Bismut curvature of the product reduces to

$$\widetilde{R}^B = R^B_1 + R^B_2 = R^{LC}_1.$$

Consequently, the Bismut Ricci form satisfies

$$\widetilde{\rho}^B = \rho_1^{LC} = 0,$$

since g is Ricci-flat.

As seen in the proof of Lemma 3.2.1, for each $n \in \mathbb{Z}$ the diffeomorphism φ_n is a holomorphic isometry of $(\widetilde{g}, \widetilde{J})$. Therefore, the structure $(\widetilde{g}, \widetilde{J})$ descends to the mapping torus M_f as (h, I), where (h, I) is the SKT structure constructed in Lemma 3.2.1. It follows that (M_f, h, I) is a SKT and CYT manifold with non-flat Bismut connection.

q.e.d.

With an analogues argument, the following generalization holds

Proposition 4.1.2. *Let $M_{\psi,\rho} = (K \times G \times \mathbb{R}^n)_{\psi,\rho}$ be constructed as in Lemma 3.2.5 and we assume that (K, g) is a (non-flat) Ricci flat Kähler manifold. Then the SKT structure (I, h) is CYT and the Bismut connection is non-flat.*

Remark 4.1.3. Observe that the Bismut torsion is parallel with respect to the Bismut connection. Indeed, the Bismut curvature tensor satisfies the first Bianchi identity (which holds at the level of the universal cover), and the Hermitian structure is SKT. It then follows from Theorem [FT] that the torsion is ∇^B-parallel.

We are mainly interested in the case of K being a $K3$ surface. In particular, we want to show that when K is a $K3$ surface, the BHE mapping tori M_f are not trivial, i.e., they do not split to a product of a Ricci flat Kähler manifold with a Bismut flat one. To do so, we briefly recall the following renowned result in the Theory of $K3$ surfaces.

Theorem 4.1.4 (Torelli Theorem [BR, PS])**.** *Let K, K' be $K3$ surfaces and let $\Omega_K, \Omega_{K'}$ be nowhere vanishing holomorphic 2-forms on K and K', respectively. Assume that there exists an isometry of lattices*

$$\alpha : H^2(K, \mathbb{Z}) \to H^2(K', \mathbb{Z})$$

satisfying

1. $\alpha([\Omega_k]) = c \cdot [\Omega_{K'}]$*, for some* $c \in \mathbb{C}^*$*,*

2. α *sends a Kähler class of K to a Kähler class of K'.*

Then there exists a unique isomorphism of $K3$ surfaces $f : K' \to K$ such that $f^* = \alpha$ on $H^2(K, \mathbb{Z})$.

We use the previous Theorem to show the following Proposition.

Proposition 4.1.5. *Let K be a $K3$ surface admitting an automorphism $\psi \neq Id_K$. Then* $\dim(N_\psi^2) < 22$*, where* $N_\psi^2 = \ker(\psi_2^* - Id)$*.*

Proof. By contradiction, assume that $\dim(N^2_\psi) = 22$, i.e., $\psi^* = Id$ on $H^2(K, \mathbb{R})$. Then, the restriction $\psi^*_{|H^2(K,\mathbb{Z})} = Id_{H^2(K,\mathbb{Z})}$ satisfies the hypothesis of Theorem 4.1.4. Hence, there exists a unique automorphism f of K such that $f^*_{|H^2(K,\mathbb{Z})} = Id_{H^2(K,\mathbb{Z})} = \psi^*_{|H^2(K,\mathbb{Z})}$. By uniqueness, $f = Id_K = \psi$. The contradiction follows. q.e.d.

Corollary 4.1.6. *Assume that K is a $K3$ surface admitting a non-trivial Kähler isometry ψ, i.e., $\psi \neq Id_K$. Then the mapping torus $M_f = (K \times S^3)_{(\psi, Id_{S^3})}$ constructed in Lemma 3.2.1 is never trivial.*

Proof. By the Künneth formula, $H^2(K \times S^3 \times S^1, \mathbb{R}) \cong H^2(K, \mathbb{Z}) \cong \mathbb{R}^{22}$.
We claim that $\dim(H^2(M_f, \mathbb{R})) < 22$. Using Proposition 3.2.2, $M_f \cong K_\psi \times S^3$ and so, again by Künneth formula, $H^2(M_f, \mathbb{R}) \cong H^2(K_\psi, \mathbb{Z}) \cong N^2_\psi \oplus C^1_\psi$, where $N^2_\psi = \ker(\psi^*_2 - Id)$ and $C^1_\psi = \operatorname{coker}(\psi^*_1 - Id)$. Since ψ is an automorphism of K different from the identity, $\dim(N^2_\psi) < 22$ by Proposition 4.1.5, and $\dim(C^1_\psi) = 0$ as $H^1(K, \mathbb{R}) = 0$. It then follows that $\dim(H^2(M_f, \mathbb{R})) < 22$, concluding the proof. q.e.d.

Example 4.1.7. Let $\mathcal{F}$ be the Fermat quartic and let ψ be the hypeerkähler isometry constructed as in Example 3.1.14. Then the product manifold $\mathcal{F}_\psi \times S^3$ admits a BHE metric.

Remark 4.1.8. The fundamental group of the mapping torus M_f is given by $\pi_1(K \times S^3) \rtimes_{f_*} \mathbb{Z}$. Therefore, when K is simply connected $\pi_1(M_f) = \mathbb{Z}$. Let K be a K3 surface endowed with its Kähler Ricci flat metric and let ψ be a Kähler isometry. Since any normal subgroup of $\pi_1(M_f) \cong \mathbb{Z}$ of finite index is of the kind $k\mathbb{Z}$ for some integer $k > 1$, it follows that any finite cover of M_f corresponds to a quotient of $K \times \mathbb{C}^2 \setminus (0,0)$ by the action of $k\mathbb{Z}$ given by

$$kn \cdot (p, \underline{x}) = (\psi^{kn} p, 2^{kn} \underline{x}).$$

Moreover, since ψ has finite order, there exists a $\bar{k} > 1$ such that $\psi^{\bar{k}} = Id$, i.e., there exists a finite cover of M_f which splits as a product of K and the Hopf surface. In Section 4.2, we will see that this is a general behaviour.

We now describe M_f as the total space of a holomorphic fibre bundle

$$p : M_f \longrightarrow S^3 \times S^1$$

with fibre K. To such a fibre bundle one associates the Borel spectral sequence, which relates the Dolbeault cohomology of the total space M_f to that of the base $S^3 \times S^1$ and the fibre K. We proceed as in Section 3.1.4.

Let π be the natural projection

$$\pi : M_f \to S^3 \times S^1, \qquad [(p, q, t)] \mapsto [(q, t)].$$

We always assume that M_f carries the standard complex structure I, induced by $\widetilde{J} = J \times J_-$, and that $S^3 \times S^1$ is endowed with the complex structure induced by J_-, i.e. the standard complex structure of its universal cover. With respect to these complex structures, the map π is holomorphic.

We now construct local trivializations around points of $S^3 \times S^1$.

Case 1: Points $[(q,t)]$ with $t \neq 0, 1$. Let $U = V \times (t-\varepsilon, t+\varepsilon)$, where V is a neighborhood of q in S^3 and ε is chosen so that the interval does not meet $\{0,1\}$. Then

$$\pi^{-1}(U) = K \times V \times (t-\varepsilon, t+\varepsilon).$$

Define

$$\varphi_U : \pi^{-1}(U) \to U \times K, \qquad [(p,q,t)] \mapsto ([(q,t)], p),$$

which is clearly a biholomorphism.

Case 2: Points $[(q,0)]$. A neighborhood of $[(q,0)]$ is

$$U = \pi'(V \times [0,\varepsilon) \sqcup V \times (1-\varepsilon, 1]),$$

where V is a neighborhood of q in S^3, $\varepsilon < \frac{1}{2}$, and $\pi' : S^3 \times \mathbb{R} \to S^3 \times S^1$ is the standard quotient map. Then

$$\pi^{-1}(U) = \pi''(K \times V \times [0,\varepsilon) \sqcup K \times V \times (1-\varepsilon, 1]),$$

where $\pi'' : K \times S^3 \times S^1 \to M_f$ is the mapping torus map. We define

$$\varphi_U : \pi^{-1}(U) \to U \times K, \qquad [(p,q,t)] \mapsto \begin{cases} ([(q,t)], p), & t \in [0,\varepsilon), \\ ([(q,t)], \psi^{-1}(p)), & t \in (1-\varepsilon, 1]. \end{cases}$$

This is well defined: for instance,

$$\varphi_U[(p,q,0)] = ([(q,0)], p) = ([(q,1)], p) = \varphi_U[(\psi(p), q, 1)].$$

Since ψ is holomorphic with respect to J, φ_U is holomorphic. Its inverse is given by

$$\varphi_U^{-1} : U \times K \to \pi^{-1}(U), \qquad ([(q,t)], p) \mapsto \begin{cases} [(p,q,t)], & t \in [0,\varepsilon), \\ [(\psi(p), q, t)], & t \in (1-\varepsilon, 1]. \end{cases}$$

This map is again well defined and holomorphic, and one checks directly that φ_U^{-1} is the inverse of φ_U. Although it is a straightforward check, we report it here for the sake of completeness.

$$[(p,q,t)] \xrightarrow{\varphi_U} \begin{cases} ([(q,t)],p) \text{ if } t \in [0,\varepsilon) \\ ([(q,t)],\psi^{-1}(p)) \text{ if } t \in (1-\varepsilon,1]. \end{cases} \xrightarrow{\varphi_U^{-1}} [(p,q,t)]$$

and

$$([(q,t)],p) \xrightarrow{\varphi_U^{-1}} \begin{cases} [(p,q,t)] \text{ if } t \in [0,\varepsilon) \\ [(\psi(p),q,t)] \text{ if } t \in (1-\varepsilon,1] \end{cases} \xrightarrow{\varphi_U} ([(q,t)],p).$$

We may now conclude:

Theorem 4.1.9. *If M_f is the mapping torus constructed as in Theorem 3.2.4, then (M_f, I) is the total space of the holomorphic fiber bundle*

$$(K,J) \longrightarrow (M_f, I) \longrightarrow (S^3 \times S^1, J_-).$$

The associated Borel spectral sequence (E_r, d_r) satisfies:

1. *For $p+q = u+v$,*

$$^{p,q}E_2^{u,v} = \sum_k H_{\bar\partial}^{k,u-k}(S^3 \times S^1) \otimes H_{\bar\partial}^{p-k,q-u+k}(K).$$

2. *The Borel spectral sequence converges to $H_{\bar\partial}^{\bullet,\bullet}(M_f)$.*

4.2 BHE and strong HKT manifolds with parallel Bismut torsion

In this section, we completely characterize compact BHE and strong HKT manifolds with parallel Bismut torsion.

One of the main tools in our proof is the following theorem of Zhao and Zheng.

Theorem 4.2.1 ([ZZ]). *Let (M, g, J) be a compact BKL Hermitian manifold without any Kähler de Rham factor. If the (first) Bismut Ricci curvature vanishes, then (g, J) is Bismut flat.*

Remark 4.2.2. The above theorem holds with the same proof in the case where the metric g is complete and M is not necessarily compact.

Lemma 4.2.3. *Let (M, I) be a compact complex manifold with parallel Bismut torsion H, i.e. $\nabla^B H = 0$. Define*

$$\mathcal{K} := \ker H = \{X \in \mathfrak{X}(M) \mid \iota_X H = 0\}.$$

Then $\mathcal{K}$ is an integrable distribution on M, preserved by both the Levi-Civita connection ∇^{LC} and the Bismut connection ∇^B.

Proof. Let $p, q \in M$ and let $\gamma : [0,1] \to M$ be a smooth regular curve with $\gamma(0) = p$ and $\gamma(1) = q$. Consider $X_0 \in \mathcal{K}_p$, $Y_0 \in T_pM$, and let $X(t), Y(t)$ be their ∇^B–parallel extensions along γ. Define

$$Z(t) := H_{\gamma(t)}(X(t), Y(t)), \qquad t \in [0,1].$$

Since H is ∇^B–parallel, $Z(t)$ is ∇^B–parallel along γ. Thus, if $P_\gamma : T_pM \to T_qM$ denotes ∇^B–parallel transport along γ, one has

$$P_\gamma(Z(0)) = Z(1).$$

But $Z(0) = H_p(X_0, Y_0) = 0$, hence $Z(1) = 0$. Therefore

$$H_q(P_\gamma X_0,\ P_\gamma Y_0) = 0$$

for all $Y_0 \in T_pM$. Since P_γ is an isomorphism, this implies $P_\gamma X_0 \in \mathcal{K}_q$. Hence the map

$$P_\gamma|_{\mathcal{K}_p} : \mathcal{K}_p \to \mathcal{K}_q$$

is injective. By symmetry (reversing the curve γ), it is also surjective, so

$$\dim \mathcal{K}_p = \dim \mathcal{K}_q.$$

Thus the rank of $\mathcal{K}$ is constant on connected components, and $\mathcal{K}$ defines a smooth distribution.

Next, we prove preservation by ∇^B. For $X \in \mathcal{K}$ and arbitrary $W, Y, Z \in \mathfrak{X}(M)$, the covariant derivative of H yields

$$(\nabla^B_W H)(X, Y, Z) = -H(\nabla^B_W X, Y, Z),$$

since $H(X, \cdot, \cdot) = 0$. As $\nabla^B H = 0$, it follows that $H(\nabla^B_W X, Y, Z) = 0$ for all Y, Z, i.e. $\nabla^B_W X \in \mathcal{K}$. Hence $\mathcal{K}$ is preserved by ∇^B.

Now observe that the difference between ∇^B and ∇^{LC} is expressed in terms of H. If $X, Y \in \mathcal{K}$, then $H(X, Y, \cdot) = 0$, and consequently

$$\nabla^B_X Y = \nabla^{LC}_X Y.$$

Thus ∇^{LC} also preserves $\mathcal{K}$.

Finally, integrability follows because ∇^{LC} is torsion-free: for $X, Y \in \mathcal{K}$,

$$[X,Y] \;=\; \nabla^{LC}_X Y - \nabla^{LC}_Y X \;\in\; \mathcal{K}.$$

Therefore $\mathcal{K}$ is an involutive distribution, and hence integrable. q.e.d.

Using the previous lemma, in the case where (M, I, h) is a compact non-Kähler complex manifold with Bismut-parallel torsion H, we may orthogonally decompose

$$TM = \mathcal{K} \oplus \mathcal{K}^{\perp},$$

where $\mathcal{K}^{\perp}$ is the orthogonal complement of $\mathcal{K}$. Since both the Levi-Civita and Bismut connections are metric, it follows that $\mathcal{K}^{\perp}$ is also preserved by both ∇^B and ∇^{LC}.

Since (M, I, h) is non-Kähler, the orthogonal complement $\mathcal{K}^{\perp}$ is non-trivial.

Let $\mathcal{F}_1$ be the maximal sub-distribution of $\mathcal{K}$ preserved by both $\nabla^{LC} = \nabla^B$ and by the complex structure I, i.e., $I(\mathcal{F}_1) = \mathcal{F}_1$. Then we may further decompose

$$\mathcal{K} = \mathcal{F}_1 \oplus \mathcal{W},$$

where $\mathcal{W}$ is the orthogonal complement of $\mathcal{F}_1$ in $\mathcal{K}$ with respect to the restricted metric $h_{|\mathcal{K}}$.

Hence, the tangent bundle splits isometrically as the direct sum of three mutually orthogonal subbundles:

$$TM = \mathcal{K} \oplus \mathcal{K}^{\perp} = \mathcal{F}_1 \oplus \mathcal{W} \oplus \mathcal{K}^{\perp} = \mathcal{F}_1 \oplus \mathcal{F}_2,$$

where $\mathcal{F}_2 = \mathcal{W} \oplus \mathcal{K}^{\perp}$. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are preserved by ∇^B and ∇^{LC}, and are also I-invariant.

Lemma 4.2.4. *Let (M, h, I) be a compact non-Kähler complex manifold with parallel Bismut torsion H, and consider the orthogonal splitting $TM = \mathcal{F}_1 \oplus \mathcal{F}_2$ described above. Then $\mathcal{F}_2$ does not contain any non-trivial sub-distribution $\mathcal{F}_3 \subset \mathcal{F}_2$ such that $\mathcal{F}_3 \subset \mathcal{K}$, $\nabla^{LC}\mathcal{F}_3 \subset \mathcal{F}_3$, and $I(\mathcal{F}_3) = \mathcal{F}_3$.*

Proof. Suppose, for contradiction, that such a sub-distribution $\mathcal{F}_3$ exists. Then $\mathcal{F}_1 \oplus \mathcal{F}_3$ would be a sub-distribution of $\mathcal{K}$ preserved by both ∇^{LC} and I, contradicting the maximality of $\mathcal{F}_1$. q.e.d.

Theorem 4.2.5. *Let (M, I) be a compact complex manifold admitting a SKT and CYT I-Hermitian metric h with parallel Bismut torsion H, i.e., $\nabla^B H = 0$. Then the Riemannian holomorphic universal cover $(\widetilde{M}, \widetilde{I}, \widetilde{h})$ of (M, I, h) is holomorphically isometric to a prod-*

uct $(M_1, J_1, g_1) \times (M_2, J_2, g_2)$*, where* (M_1, J_1, g_1) *is Kähler Ricci-flat and* (M_2, J_2, g_2) *is a Samelson space.*

Proof. The theorem is trivial in the Kähler case, so we assume the metric is non-Kähler.

Let $\widetilde{M}$ be the holomorphic Riemannian universal cover of M. By a standard argument, $\widetilde{M}$ is complete. The Bismut Hermitian-Einstein structure (h, I) with parallel torsion H lifts to $(\widetilde{h}, \widetilde{I}, \widetilde{H})$ on $\widetilde{M}$.

By the de Rham splitting theorem, $(\widetilde{M}, \widetilde{h}, \widetilde{I})$ is holomorphically isometric to

$$(M_1, J_1, g_1) \times (M_2, J_2, g_2),$$

and the decomposition $TM = \mathcal{F}_1 \oplus \mathcal{F}_2$ pulls back to $T\widetilde{M} = TM_1 \oplus TM_2$. This follows from the I-invariance of $\mathcal{F}_1$ and $\mathcal{F}_2$, which ensures that the complex structures on the distributions pull back to J_1 and J_2 on TM_1 and TM_2, respectively.

The splitting of the metric and complex structure induces a splitting of the torsion $\widetilde{H} = H_1 + H_2$. Since $\mathcal{F}_1$ lies in the kernel of H, we have $H_1 \equiv 0$, so (M_1, g_1, J_1) is Kähler Ricci-flat.

Furthermore, (M_2, h_2, J_2) is a Bismut Hermitian-Einstein structure with parallel torsion, and by Lemma 4.2.4, (M_2, h_2, J_2) contains no Kähler de Rham factor. Hence, (M_2, h_2, J_2) is Bismut flat by Theorem 4.2.1 in the complete case. The last statement follows from the classification of simply connected Bismut flat manifolds [WYZ, Theorem 5] and by the fact that the metric is complete. q.e.d.

Remark 4.2.6. Observe that we may always assume the Kähler factor to be compact. In fact, the universal cover of a compact Kähler Ricci flat manifold splits uniquely as a product of the kind $\mathbb{C}^k$ and N, where N is the product of manifolds with holonomy in $\mathrm{Sp}(m)$ and $\mathrm{SU}(r)$ [Be]. Now, since by the assumption of the Theorem 4.2.5 (M, J, g) is compact, BHE and with parallel Bismut torsion, it has non-negative Ricci curvature, and so its universal cover splits as a product of a compact manifold and a vectorial factor [ChGr]. Hence, N is compact and the Kähler vectorial factor can be absorbed by the Bismut flat part.

Remark 4.2.7. If $\dim_{\mathbb{C}}(\mathcal{F}_1) \leq 1$, then M is Bismut flat. Indeed, in this case, M has no Kähler de Rham factor of dimension greater than 1, so Bismut flatness follows from [ZZ, Theorem 3].

Corollary 4.2.8. *Any compact BHE manifold with parallel Bismut torsion is formal according to Sullivan.*

Proof. Let (M, g) be a compact manifold with non-negative Ricci and let $\widetilde{M}$ be its universal cover. By [Mi], if $\widetilde{M}$ is formal, then M is formal. Since any BHE metric g with parallel Bismut torsion has non-negative Ricci, it suffices to look at the formality of the

universal cover.
By Theorem 4.2.5, the universal cover of a compact BHE manifold with parallel Bismut torsion is diffeomorphic to a product of a compact Kähler manifold and a Samelson space which are both formal. Then the result follows by the fact that a product manifold is formal if and only if each factor is formal. q.e.d.

As a Corollary of the result above, Barbaro, Pediconi and Tardini obtained the following

Theorem 4.2.9 ([BPT]). *Let (M, I) be a compact complex manifold admitting a SKT and CYT I-Hermitian metric h with parallel Bismut torsion H, i.e., $\nabla^B H = 0$. Then, up to a finite cover, (M, I, h) splits as a product of a compact Kähler Ricci flat manifold and a Bismut flat one.*

Theorem 4.2.5 admits a natural generalization to strong HKT manifolds. Recall that for a strong HKT manifold (M, I, J, K, g), each Hermitian structure (L, g) is BHE. We exploit this observation to prove the next theorem.

Definition 4.2.10. *A hyperhermitian manifold $(G' = G \times \mathbb{R}^k, I, J, K, b')$ is a* hyperhermitian Samelson space *if G is a 1-connected compact semisimple Lie group, b' is a bi-invariant metric on G' and I, J, K are left invariant complex structures compatible with b'.*

Just for the proof of the following two Theorems we will denote the hyper-complex structure by J_1, J_2, J_3 instead of I, J, K since the notation with the subscript suits better for the proofs.

Theorem 4.2.11. *Let (M, J_1, J_2, J_3, g) be a compact strong HKT manifold satisfying $\nabla^B H = 0$ and let $(\widetilde{M}, \widetilde{J}_a, \widetilde{g})$ its universal cover. Then, $(\widetilde{M}, \widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3, \widetilde{g})$ splits as a product of a compact hyperkähler manifold and a hyperhermitian Samelson space.*

Proof. If we apply Theorem 4.2.5 to each Hermitian structure (J_a, g), we get that for any $a = 1, 2, 3$,

$$(\widetilde{M}, \widetilde{J}_a, \widetilde{g}) = (K_a, I_a, k_a) \times (G'_a, I_{L_a}, b'_a),$$

where (K_a, I_a, k_a) is the compact Kähler Ricci flat factor and (G'_a, I_{L_a}, b'_a) is a Samelson space. We recall that $G'_a = G_a \times \mathbb{R}^{s_a}$, where G_a is a 1-connected compact semisimple Lie group and that the bi-invariant metric b'_a is the product of a bi-invariant metric b_a on G_a and a flat metric h_a on $\mathbb{R}^{s_a}$ (see [Mil]).

We first observe that $s := s_1 = s_2 = s_3$ and $h := h_1 = h_2 = h_3$. This is a straightforward consequence of the uniqueness of de De Rham splitting and the Cheeger-Gromoll Theorem [ChGr]. Analogously, $(K_1, k_1) = (K_2, k_2) = (K_3, k_3)$ and $(G_1, b_1) = (G_2, b_2) = (G_3, b_3)$. We explain the first in detail, the second is analogous.

We prove that $(K_1, k_1) = (K_2, k_2)$, the proof proceeds in the same way for $(K_2, k_2) = (K_3, k_3)$. Since any irreducible de Rham factor of (K_1, k_1) is still Kähler, by the uniqueness of de De Rham splitting, any irreducible de Rham factor of (K_1, k_1) is identified with a unique irreducible de Rham factor of (K_2, k_2). This forces $(K, k) := (K_1, k_1) = (K_2, k_2) = (K_3, k_3)$, and so (K, k) is hyperkähler as it is preserved by I_1, I_2, I_3, which satisfy $I_1 I_2 = -I_2 I_1 = I_3$.

With the same argument, one can prove that $(G, b) := (G_1, b_1) = (G_2, b_2) = (G_3, b_3)$, and so $(G' = G \times \mathbb{R}^s, b' = b + h, I_{L_a})$ is a hyperhermitian Samelson space. q.e.d.

As an application of this result we may characterize a finite cover of a strong HKT manifold with parallel Bismut torsion.

Theorem 4.2.12. *Any compact strong HKT manifold with parallel Bismut torsion* (M, J_1, J_2, J_3, g) *is, up to a finite cover, the product of a compact hyperkähler manifold and a compact Bismut flat one.*

Proof. Let $(\widetilde{M}, \widetilde{J}_a, \widetilde{g}) = (K, I_a, k) \times (G', I_{L_a}, b')$ be the universal cover of the strong HKT manifold with parallel Bismut torsion. By the previous Theorem, $\widetilde{M} \cong K \times G \times \mathbb{R}^s$, where K is a compact hyperkähler manifold, G is a compact semisimple Lie group and $\mathbb{R}^s$ is endowed with its flat metric. Since K is hyperkähler, any de Rham factor of K is hyperkähler, implying that no irreducible de Rham factor of K can be isometric to an irreducible de Rham factor of G. Furthermore, since K is compact, no irreducible de Rham factor of K can be isometric to some $\mathbb{R}^{4r}$. From this we get that

$$\mathrm{Iso}(\widetilde{M}, \widetilde{g}) \cong \mathrm{Iso}(K) \times \mathrm{Iso}(G) \times \mathrm{Iso}(\mathbb{R}^s).$$

The fundamental group $\pi_1(M)$ acts on $(\widetilde{M}, \widetilde{J}_a, \widetilde{g})$ via hyper-holomorphic isometries. Let σ be such an automorphism; by the discussion above, there exist three isometries σ_1, σ_2 and σ_3 such that

$$\sigma(p, g, t) = (\sigma_1(p), \sigma_2(g), \sigma_3(t)), \text{ for } p \in K,\ g \in G \text{ and } t \in \mathbb{R}^s,$$

and we point out that σ_1 is a hyperkähler isometry of (K, I_a, k).
We consider the group homomorphism $\tau : \sigma \to \sigma_1$ from $\pi_1(M)$ to the group of hyper-holomorphic isometries of (K, I_a, k). Let Γ to be its kernel. Then Γ is normal in $\pi_1(M)$, as it is the kernel of a group homomorphism and, furthermore, it is finite. In fact $\frac{\pi_1(M)}{\Gamma} \cong \mathrm{Im}(\tau)$ which is finite, as so is the group of hyper-holomorphic isometries of (K, I_a, k).
Observe that if Γ is trivial, then $\pi_1(M)$ is itself finite, and, hence, the covering $\widetilde{M} \to M$.

Since Γ is a finite normal subgroup of $\pi_1(M)$, we may consider the finite (and hence compact) cover $M' = \widetilde{M}/\Gamma \to M$, which splits as a product of (K, I_a, k) and a Bismut flat space. q.e.d.

As a Corollary, we immediately get that

Corollary 4.2.13. *Any compact strong HKT manifold with parallel Bismut torsion is formal according to Sullivan.*

Remark 4.2.14. We point out that an analogous result does not hold for compact balanced HKT manifolds. Indeed, it has been shown by Dotti and Fino [DF3] that there exist a (non hyperkähler) 8-dimensional balanced HKT nilmanifold with parallel Bismut torsion.

As remarked in the introduction of this thesis, it remains an open question whether there exist compact BHE or, more generally, strong HKT manifolds whose Bismut torsion is not parallel. Evidence suggesting that the torsion must be parallel arises from the holonomy reduction, which we now discuss. In fact, by Theorems 4.2.5 and 4.2.11, any BHE (or sHKT) manifold with parallel Bismut torsion cannot have holonomy exactly $\mathrm{SU}(n)$ $(\mathrm{Sp}(n))$ unless it is (hyper)Kähler. This naturally raises the broader question of whether compact non-(hyper)Kähler manifolds with holonomy precisely $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$ can exist. In this section, we prove that no such examples occur.

Firstly, we introduce some notations: given a Hermitian structure (J, g) and let θ be the associated Lee form. We will denote by $\theta^\sharp$ the vector field dual to θ via g, i.e., $\theta(Y) = g(\theta^\sharp, Y)$, for any $Y \in \Gamma(\mathrm{TM})$.

Definition 4.2.15. *A SKT manifold (M, J, g) is called*

1. steady pluriclosed soliton *if there exists a smooth function f such that*

$$Ric^{LC} - \frac{1}{4}H^2 + \nabla^2 f = 0, \quad \delta H + \iota_{\mathrm{grad}\, f} H = 0,$$

where H^2 is defined as $H^2(X,Y) = g(\iota_X H, \iota_Y H)$ and $\delta = \delta$ is the codifferential of g.

2. generalized Einstein *if $Ric^B = 0$, namely, $Ric^{LC} = \frac{1}{4}H^2$, $\delta H = 0$.*

In what follows, we will need the following Proposition

Proposition 4.2.16 ([ABLS, GFJS, Le, Le2, SU])**.** *Any compact BHE manifold (M, J, g) is a steady pluriclosed soliton with a unique normalized potential f. The vector fields $V := \frac{1}{2}(\theta^\sharp - \mathrm{grad}\, f)$ and JV are holomorphic Killing and Bismut parallel and they vanish if and only if (J, g) is Kähler. Moreover, $dV^\flat = \frac{1}{2}d\theta$ and $dJV^\flat = \frac{1}{2}(dJ\theta - dJdf)$ are $(1,1)$, and, furthermore, they satisfy $\iota_V dV^\flat = \iota_{JV} dV^\flat = 0$ and $\iota_V dJV^\flat = \iota_{JV} dJV^\flat = 0$. Finally, $\iota_V df = \iota_{JV} df = 0$.*

Given a hyperhermitian manifold (M, I, J, K, g), the Lee forms $\theta_I, \theta_J, \theta_K$ coincide [FG2], and so, also the functions $\delta\theta_I, \delta\theta_J, \delta\theta_K$. Also in this case, we will set $\theta := \theta_I = \theta_J = \theta_K$.

In what follows, we will need the following Proposition

Proposition 4.2.17. *Let (M, I, J, K, g) be a compact strong HKT manifold, and let $V_I = \frac{1}{2}(\theta^\sharp - \operatorname{grad} f_I), V_J = \frac{1}{2}(\theta^\sharp - \operatorname{grad} f_J), V_K = \frac{1}{2}(\theta^\sharp - \operatorname{grad} f_K)$ be the vector fields respectively associated to $(I, g), (J, g), (K, g)$ defined in Proposition 4.2.16. Then $V_I = V_J = V_K := V$.*

Proof. Since $V_L = \frac{1}{2}(\theta^\sharp - \operatorname{grad} f_L)$ is Killing (see Proposition 4.2.16), $\mathcal{L}_{V_L} g = 0$, for any $L = I, J, K$. Fixed any local orthonormal basis $\{e_1, \dots, e_{4n}\}$, if we take the trace $\sum_i \mathcal{L}_{V_L} g(e_i, e_i) = 0$ we get

$$\delta\theta + \Delta f_L = 0, \tag{4.2.1}$$

for each $L = I, J, K$. Hence, $\Delta f_I = \Delta f_J = \Delta f_K$, and so using the maximum principle the functions f_L must differ each other by a constant. q.e.d.

Without loss of generality, we may assume that $f_I = f_J = f_K := f$.

Theorem 4.2.18. ***1.*** *Let (M, J, g) be a compact non-Kähler BHE manifold of complex dimension n. Then $\mathrm{Hol}^0(\nabla^B) \subseteq \mathrm{SU}(n-1)$. In particular, if a compact BHE manifold has restricted holonomy exactly $\mathrm{SU}(n)$, then it is Kähler.*

2. *Let (M, I, J, K, g) be a compact strong HKT manifold which is not hyperkähler of quaternionic dimension n. Then $\mathrm{Hol}(\nabla^B) \subseteq \mathrm{Sp}(n-1)$. In particular, if a strong HKT manifold has holonomy exactly $\mathrm{Sp}(n)$, then it is hyperkähler.*

Proof. We prove the first assert in details. Given a compact non-Kähler BHE manifold (M, J, g), the vectors V and JV satisfy $\nabla^B V = 0$ and $\nabla^B JV = 0$ and, furthermore, they have constant and non-zero norm (see Proposition 4.2.16).

Let us fix any point $p \in M$ and any null-homotopic loop γ centered at p. The vectors $Z := V_p$ and $JZ = (JV)_p$ are always non-zero and, exploiting that $\nabla^B V = 0$ and $\nabla^B JV = 0$, they satisfy $P_\gamma Z = Z$ and $P_\gamma(JZ) = JZ$. Therefore, if (J, g) is non Kähler, at any point p there exists a complex vector W which is non zero and such that $P_\gamma W = W$, for any null-homotopic loop γ centered at p. This implies that the holonomy of the Bismut connection of any non-Kähler BHE manifold reduces from $\mathrm{SU}(n)$ to $\mathrm{SU}(n-1)$, where, to the sake of clarity, we specify that we consider $\mathrm{SU}(n-1)$ as a subgroup of $\mathrm{SU}(n)$ in the following way:

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n), \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

In particular, if there exists a compact BHE manifold with $\mathrm{Hol}^0(\nabla^B) = SU(n)$, it has to be necessarily Kähler. The proof of the second statement is analogous. In fact, exploiting that $\nabla^B V = 0$ and $\nabla^B IV = \nabla^B JV = \nabla^B KV = 0$, we get that $\|V\|^2, \|IV\|^2, \|JV\|^2, \|KV\|^2$ are constant and non zero, as the manifold is not hyperkähler. We fix any point $p \in M$ and we set $Z := V_p,\ IZ = (IV)_p\ JZ = (JV)_p,\ KZ = (KV)_p \in \mathrm{T_pM} \setminus \{0\}$. Then, for any loop γ based at p, $P_\gamma Z = Z$, $P_\gamma IZ = IZ, P_\gamma JZ = JZ$, $P_\gamma KZ = KZ$. From here the proof is straightforward. q.e.d.

As an application of the previous results, we establish the non-existence of BHE metrics, and therefore of strong HKT metrics, on solvmanifolds. The existence of strong HKT metrics on solvmanifolds has remained an open problem for the past decade. It was previously known [BDV, DF3] that a nilmanifold admits a strong HKT metric if and only if it is a torus, and a similar statement was proved in the almost abelian case [AB1]. Here, we prove the result in fully generality.

Corollary 4.2.19. *Let $\Gamma\backslash G$ be a solvmanifold endowed with an invariant Hermitian structure (J, g). Then the following are equivalent:*

1. *(J, g) is BHE,*
2. *(J, g) is Kähler.*

In particular, given a solvmanifold $\Gamma\backslash G$ endowed with an invariant hyperhermitian structure (I, J, K, g), then (I, J, K, g) is strong HKT if and only if it is hyperkähler.

Proof. the implication from right to left follows by the characterization of Kähler solvmanifolds given in [Ha], which are flat. For the other implication, assume that (J, g) is BHE. Then, taking the trace of $\mathcal{L}_V g = 0$ as before, we get that $\Delta f + \delta\theta = 0$. Since (g, J) is left-invariant, then it is Gauduchon, and so f must be constant by the maximum principle, implying that

$$\nabla^B \theta^\sharp = 2\nabla^B V + \nabla^B \operatorname{grad} f = 2\nabla^B V = 0.$$

By (1.2.3), on a compact Hermitian SKT manifold

$$\rho^B(X, Y) = -Ric^B(X, JY) - (\nabla^B_X \theta)JY.$$

Exploiting that $\rho^B = 0$ and $\nabla^B\theta = 0$, we get that $Ric^B = 0$.
Moreover, if $Ric^B = 0$ then the symmetric part of Ric^B vanishes, i.e., $Ric^{LC} = \frac{1}{4}H^2$. Therefore

$$scal^{LC} = \frac{1}{4}\|H\|^2 \geq 0.$$

Since for any left-invariant metric on a solvable Lie group $scal^{LC} \leq 0$ [Je, Mil], this forces $scal^{LC} = 0$. Hence $H = 0$, concluding the proof. q.e.d.

Remark 4.2.20. We observe that with the same proof one can prove that any compact BHE manifold (M, J, g) such that (J, g) is Gauduchon and g has non positive scalar curvature must be Kähler.

4.3 Ricci foliation of a Hypercomplex manifold

Let (M^{4n}, I, J, K) be a compact hypercomplex manifold and let ∇^{Ob} be the Obata connection. Let $F(\mathrm{TM})$ be the frame bundle of M and consider the character

$$det : \mathrm{GL}(n, \mathbb{H}) \to \mathbb{R}_{>0}$$

induced by the Dieudonnè determinant [Di]. The associated bundle $K_{\mathbb{R}}$ is the real line bundle of $(2n, 0)$ q-real forms on M. More precisely, fixed any hyperhermitian metric g on M, $K_{\mathbb{R}}$ is the real line bundle generated by the $(2n, 0)$ form Ω^n, where $\Omega = \omega_J + i\omega_K$, where $\omega_J = g(J\cdot, \cdot)$ and $\omega_K = g(K\cdot, \cdot)$. Clearly, the canonical bundle $K_{(M,I)}$ is obtained as a complexification of $K_{\mathbb{R}}$.

Definition 4.3.1. *The* Obata Ricci curvature Θ *is the curvature of the Obata connection on* $K_{\mathbb{R}}$.

Proposition 4.3.2. *The Obata Ricci curvature Θ of a compact hypercomplex manifold* (M, I, J, K) *is* $\mathrm{SU}(2)$*-invariant and exact.*

Proof. Since $K_{(M,I)} = K_{\mathbb{R}} \otimes \mathbb{C}$, Θ is the curvature of the Obata connection on the canonical bundle of (M, I). Fixed any hyperhermitian metric g, $K_{(M,I)}$ admits a non-vanishing section Ω^n as above, and $\nabla^{Ob}_{K(M,I)}\Omega^n = \alpha\, \Omega^n$, where $\alpha = \gamma + \overline{\gamma}$, $\gamma \in \Omega^{1,0}M$ and $K_{(M,I)}$ is the canonical bundle obtained as a complexification of $K_{\mathbb{R}}$. Then,

$$\Theta = d\alpha + \alpha \wedge \alpha = d\alpha \tag{4.3.1}$$

coincides with the Ricci of the Obata connection on TM, and, therefore, it is $(1, 1)$ with respect to any complex structure $L = I, J, K$, i.e., it is $\mathrm{SU}(2)$-invariant. A clear explanation of the $\mathrm{SU}(2)$-invariance of the Obata Ricci tensor can be found in [Sol, Proposition 2.2]. Since Ω is a non-degenerate $(2, 0)$ form, $c_1(M, I) = 0$ and Θ is exact. Alternatively, Θ is exact by equation (4.3.1). q.e.d.

We recall the following Lemma

Lemma 4.3.3. *Let* (V, I, J, K) *be a real vector space equipped with a quaternionic action and let* $\eta \in \bigwedge_I^{1,1} V$ *be a real* $\mathrm{SU}(2)$*-invariant* $(1, 1)$ *form. Then the eigenvalues of* η *occur in pairs* $\sigma_i,\ -\sigma_i$.

Proof. Let us take the unitary basis $\{\psi_1 = \varphi_1 + iI\varphi_1, \ldots, \psi_{2n} = \varphi_{2n} + iI\varphi_{2n}\}$ of $V^{1,0^*}$ which diagonalizes η, i.e., such that

$$\eta = i\sum_{j=1}^{2n} \lambda_j \psi^j \wedge \psi^{\overline{j}} = 2\sum_{j=1}^{2n} \lambda_j \varphi^j \wedge I\varphi^j = J\eta = -2\sum_{j=1}^{2n} \lambda_j J\varphi^j \wedge I(J\varphi^j).$$

Since J takes the positive form $\varphi^j \wedge I\varphi^j$ in the negative form $-J\varphi^j \wedge I(J\varphi^j)$ and since η and $J\eta = \eta$ have the same eigenvalues, they must occur with opposite signs. q.e.d.

Let r be the rank of Θ at a general point of M. Since by previous Lemma the eigenvalues of Θ goes in pairs, then $r < 2n$. In fact, if $r = 2n$, then Θ^{2n} would be a positive $(2n, 2n)$ form if n is even and a negative $(2n, 2n)$ form otherwise, implying that

$$\int_M \Theta^{2n} > 0 \text{ in the first case, and } \int_M \Theta^{2n} < 0 \text{ in the second one,}$$

which is impossible, as Θ is exact.

Definition 4.3.4. *The sub-sheaf* $\ker(\Theta) \subset \mathrm{TM}$ *is involutive, as* $d\Theta = 0$. *We call* $\ker(\Theta)$ *the* Ricci Foliation *of a hypercomplex manifold* (M, I, J, K).

Remark 4.3.5. The leaves of the Ricci foliation are always hypercomplex manifolds with restricted Obata holonomy contained in $\mathrm{SL}(n, \mathbb{H})$.

4.4 Ricci foliation of strong HKT manifolds

Let us consider an HKT manifold (I, J, K, g). It has been shown in [IP] that the Ricci tensor of the Obata connection coincides with the differential of the Lee form, and hence, the Obata Ricci curvature of $K_{\mathbb{R}}$ satisfies $\Theta = d\theta$. In this case the Ricci foliation is hence given by the kernel of $d\theta$.

Proposition 4.4.1. *Let* (M, I, J, K, g) *be a compact simply connected strong HKT manifold. If there exists a point* p *and an open neighborhood* $\mathcal{U}$ *of* p *on which* $d\theta = 0$*, then the manifold is hyperkähler.*

Proof. Locally, any strong HKT metric Ω has a potential φ which satisfies $dd^I d^J d^K \varphi = 0$. In particular, φ is real analytic and so Ω is real analytic (see [MV2, Lemma 9.4]), where Ω is the $(2, 0)$ form defining the strong HKT structure. Therefore, θ is real analytic and so $d\theta$ must vanish everywhere. Since M is simply connected, the result follows by [IP, Corollary 5.3]. q.e.d.

We recall the following well known

Lemma 4.4.2. *Let* (M, J, g) *be a Hermitian manifold. Then, for any* $X, Y, Z \in \Gamma(TM)$,

$$H(X, Y, Z) = H(JX, JY, Z) + H(JX, Y, JZ) + H(X, JY, JZ).$$

The equality immediately follows by $H^{3,0} = 0$.

Remark 4.4.3. If (M, I, J, K, g) is a compact (non-hyperkähler) strong HKT we have already noticed that (I, g), (J, g), (K, g) are BHE structures with the same soliton potential f, namely, the vector fields V_I, V_J, V_K defined in Proposition 4.2.16 coincide (see Proposition 4.2.17). In particular, we get for free the following useful properties:

1. V, IV, JV, KV are non zero, Killing and Bismut parallel,
2. $\mathcal{L}_V I = \mathcal{L}_V J = \mathcal{L}_V K = 0$,
3. $\mathcal{L}_{IV} I = \mathcal{L}_{JV} J = \mathcal{L}_{KV} K = 0$,
4. $\iota_V dV^\flat = \iota_{IV} dV^\flat = \iota_{JV} dV^\flat = \iota_{KV} dV^\flat = 0$. Furthermore, $dV^\flat$ is $\mathrm{SU}(2)$-invariant
5. $\iota_V dIV^\flat = \iota_{IV} dIV^\flat$ and $dIV^\flat$ is $(1,1)$ with respect to I,
6. $\iota_V dJV^\flat = \iota_{IV} dJV^\flat$ and $dJV^\flat$ is $(1,1)$ with respect to J,
7. $\iota_V dKV^\flat = \iota_{IV} dKV^\flat$ and $dKV^\flat$ is $(1,1)$ with respect to K,
8. $\iota_V df = \iota_{IV} df = \iota_{JV} df = \iota_{KV} df = 0$.

Theorem 4.4.4. *Let (M, I, J, K, g) be a compact simply connected 8-dimensional strong HKT manifold which is not hyperkähler. Then:*

1. *the distribution spanned by $\{V, IV, JV, KV\}$ is closed under the Lie bracket and its Lie algebra is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$,*
2. *The vector fields IV, JV, KV satisfy $\mathcal{L}_{IV}\omega_J = -\omega_K$, $\mathcal{L}_{JV}\omega_K = -\omega_I$ and $\mathcal{L}_{KV}\omega_I = -\omega_J$.*
3. *The soliton potential f is constant.*

Proof. Since by statement (4) in Remark 4.4.3, $d\theta = 2dV^\flat = \iota_V d\theta = \iota_{IV} d\theta = \iota_{JV} d\theta = \iota_{KV} d\theta = 0$, the vector fields $\{V, IV, JV, KV\}$ span a sub-distribution $\mathcal{F}$ of $\ker(d\theta)$. We decompose $\mathrm{TM} = \mathcal{F} \oplus \mathcal{F}^\perp$, where $\mathcal{F}$ is spanned by V, IV, JV, KV and $\mathcal{F}^\perp$ is the orthogonal complement of $\mathcal{F}$ with respect to the metric g. *.

The fact that $\mathcal{F}$ is closed under the Lie bracket is a straightforward consequence of Proposition 4.4.1.

Since V, IV, JV, KV are Bismut parallel, they have constant (non-zero) norm we may assume to be 1, up to rescaling. Hence, we may decompose $\mathrm{TM} = \mathcal{F} \oplus^\perp \mathcal{F}^\perp$, where $\mathcal{F}^\perp$ is the horizontal distribution.

By statement (3) of Remark 4.4.3, $\mathcal{L}_V I = \mathcal{L}_V J = \mathcal{L}_V K = 0$, forcing $[V, IV] = [V, JV] = [V, KV] = 0$. This, together with $\nabla^B V = \nabla^B IV = \nabla^B JV = \nabla^B KV = 0$,

*In what follows we will say that a tensor field T is *horizontal* if $\iota_X T = 0$ for any $X \in \mathcal{F}$

implies that

$$\begin{aligned} g([IV,JV],V) &= g(\nabla^B_{IV}JV - \nabla^B_{JV}IV - T^B(IV,JV)) \\ &= -H(IV,JV,V) = 0, \\ g([IV,JV],IV) &= -H(IV,JV,IV) = 0, \\ g([IV,JV],JV) &= -H(IV,JV,JV) = 0, \\ g([IV,JV],KV) &= -H(IV,JV,KV), \end{aligned} \tag{4.4.1}$$

and therefore $[IV, JV] = aKV$, with $a = -H(IV, JV, KV)$. Analogously, $[JV, KV] = aIV$ and $[KV, IV] = aJV$.

We claim that

$$dV^\flat = \iota_V H,\ d(IV^\flat) = \iota_{IV} H,\ d(JV^\flat) = \iota_{JV} H \text{ and } d(KV^\flat) = \iota_{KV} H. \tag{4.4.2}$$

We prove in details one of the assertions, since the proof of the remaining ones is similar.

$$\begin{aligned} d(IV^\flat)(X,Y) &= \mathcal{L}_X(IV^\flat(Y)) - \mathcal{L}_Y(IV^\flat(X)) - IV^\flat([X,Y]) \\ &= \mathcal{L}_X(g(IV,Y)) - \mathcal{L}_Y(g(IV,X)) - g(IV,[X,Y]) \\ &= g(\nabla^B_X IV,Y) + g(IV,\nabla^B_X Y) - g(\nabla^B_Y IV,X) - g(IV,\nabla^B_Y X) - g(IV,[X,Y]) \\ &= g(IV,\nabla^B_X Y) - g(IV,\nabla^B_Y X) - g(IV,[X,Y]) \\ &= g(IV,T^B(X,Y)) \\ &= H(IV,X,Y), \end{aligned}$$

where the fourth line follows by the fact that $\nabla^B(IV) = 0$.

Using that $d(JV^\flat) = \iota_{JV} H$, we are going to show that

$$d(JV^\flat) = -aKV^\flat \wedge IV^\flat + \beta_J, \tag{4.4.3}$$

where β_J is a horizontal 2-form, namely $\beta_I \in \bigwedge \mathcal{F}^\perp$, and $a \in \mathcal{C}^\infty(M)$ a smooth function. Indeed, since $d(JV^\flat) = \iota_{JV^\flat} H$ we get

$$\begin{aligned} &d(JV^\flat)(V,X) = H(JV,V,X) = -g([JV,V],X) = 0,\ \forall X \in \Gamma(\mathrm{TM}), \\ &d(JV^\flat)(IV,X) = H(JV,IV,X) = -g([JV,IV],X) = ag(KV,X),\ \forall X \in \Gamma(\mathrm{TM}), \\ &d(JV^\flat)(JV,X) = H(JV,JV,X) = 0,\ \forall X \in \Gamma(\mathrm{TM}), \\ &d(JV^\flat)(KV,X) = H(JV,KV,X) = -g([JV,KV],X) = -ag(IV,X),\ \forall X \in \Gamma(\mathrm{TM}), \end{aligned}$$

where we used the same trick as in (4.4.1). This shows (4.4.3).

Applying the exterior differential to both sides of (4.4.3), we get

$$0 = d^2(JV^\flat) = -da \wedge KV^\flat \wedge IV^\flat - ad(KV^\flat) \wedge IV^\flat + aKV^\flat \wedge d(IV^\flat) + d\beta_J$$

and hence

$$da \wedge KV^\flat \wedge IV^\flat = -ad(KV^\flat) \wedge IV^\flat + aKV^\flat \wedge d(IV^\flat) + d\beta_J. \tag{4.4.4}$$

By Remark 4.4.3, statements 5 and 7, $\iota_{IV} dIV^\flat = 0$ and $\iota_{KV} dKV^\flat = 0$. Therefore, if we evaluate equation (4.4.4) on KV, IV and $Z \in \mathcal{F}^\perp$ we get

$$da(Z) = d\beta_J(KV, IV, Z) = 0$$

since

$$\begin{aligned} d\beta_J(KV, I, V, Z) = \; & \mathcal{L}_{KV}(\beta_J(IV, Z)) - \mathcal{L}_{IV}(\beta_J(KV, Z)) - \mathcal{L}_Z(\beta_J(KV, IV)) \\ & - \beta_J([KV, IV], Z) + \beta_J([KV, Z], IV) + \beta_J([IV, Z], KV) = 0, \end{aligned}$$

as β_J is horizontal and $[KV, IV] = aJV$.
It remains to prove that $da(V) = da(IV) = da(JV) = da(KV) = 0$. Applying the Jacobi identity to V, IV, KV and using statement (2) of Remark 4.4.3, we get

$$0 = [[V, IV], KV] + [[IV, KV], V] + [[KV, V], IV] = [aJV, V] = V(a)JV,$$

which forces $da(V) = 0$. Analogously, the Jacobi identity applied to IV, JV, KV implies that

$$da(IV)IV + da(JV)JV + da(KV)KV = 0,$$

forcing a to be constant. So the sub-algebra of $\Gamma(\mathrm{TM})$ spanned by $\{V, IV, JV, KV\}$ is therefore isomorphic to $\mathbb{R}^4$ if $a = 0$ and to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, otherwise. To finish the proof it remains to show that $a \neq 0$.

Let us consider a as a constant, without assigning a specific value. We will obtain $a = -1$ when we will be proven that the vector fields IV, JV and KV satisfy the identities

$$\mathcal{L}_{IV}\omega_J = -\omega_K, \quad \mathcal{L}_{JV}\omega_K = -\omega_I, \quad \mathcal{L}_{KV}\omega_I = -\omega_J.$$

We will focus to prove that $\mathcal{L}_{IV}\omega_J = -\omega_K$, but once this will be proven, also the other identities will immediately follow.

We fix the following notation: with the same technique we adopted to compute $d(JV^\flat) = -aKV^\flat \wedge IV^\flat + \beta_J$, one can prove that $d(KV^\flat) = -aIV^\flat \wedge JV^\flat + \beta_K$ and $d(IV^\flat) = -aJV^\flat \wedge KV^\flat + \beta_I$, where β_I, β_J and β_K are horizontal 2-forms. Furthermore since $d(IV^\flat), d(JV^\flat), d(KV^\flat)$ are of type $(1,1)$ with respect to I, J, K respectively (see statements (5)-(7) of Remark 4.4.3), we also have that $\beta_I, \beta_J, \beta_K$ are of type $(1,1)$ with respect to I, J, K respectively. Indeed, for instance

$$\beta_I = dIV + aJV^\flat \wedge KV^\flat,$$

and both dIV and $JV^\flat \wedge KV^\flat$ are of type $(1,1)$ with respect to I.
Also, exploiting that V, IV, JV, KV are a basis of orthonormal vectors in $\mathcal{F}$ we will write

$$\begin{aligned}
\omega_I &= V^\flat \wedge IV^\flat + JV^\flat \wedge KV^\flat + \omega_I^T, \\
\omega_J &= V^\flat \wedge JV^\flat + KV^\flat \wedge IV^\flat + \omega_J^T, \\
\omega_K &= V^\flat \wedge KV^\flat + IV^\flat \wedge JV^\flat + \omega_K^T.
\end{aligned}$$

By Cartan's magic formula $\mathcal{L}_{IV}\omega_J = d(\iota_{IV}\omega_J) + \iota_{IV}(d\omega_J)$. For any vector field X

$$\iota_{IV}\omega_J(X) = \omega_J(IV, X) = g(JIV, X) = -KV^\flat(X)$$

and

$$\begin{aligned}
\iota_{IV} d\omega_J(X,Y) &= (\iota_{IV} JH)(X,Y) \\
&= JH(IV, X, Y) \\
&= H(-JIV, JX, JY) \\
&= H(KV, JX, JY) \\
&= H(KV, IX, IY) + H(V, IX, JY) + H(V, JX, IY),
\end{aligned}$$

where the last line follows by 4.4.2. Exploiting the equalities written in (4.4.2), we get

$$\begin{aligned}
\iota_{IV} d\omega_J(X,Y) &= d(KV^\flat)(IX, IY) + dV^\flat(IX, JY) + dV^\flat(JX, IY) \\
&= d(KV^\flat)(IX, IY),
\end{aligned}$$

where the last equality follows by the $\mathrm{SU}(2)$-invariance of $dV^\flat = \frac{1}{2}d\theta$.

Therefore,

$$\mathcal{L}_{IV}\omega_J = IdKV^\flat - dKV^\flat = a(V^\flat \wedge KV^\flat + IV^\flat \wedge JV^\flat) + I\beta_K - \beta_K. \tag{4.4.5}$$

On the other hand, using that $\omega_J = V^\flat \wedge JV^\flat + KV^\flat \wedge IV^\flat + \omega_J^T$,

$$\mathcal{L}_{IV}\omega_J = \mathcal{L}_{IV}(V^\flat \wedge JV^\flat + KV^\flat \wedge IV^\flat + \omega_J^T) = a(V^\flat \wedge KV^\flat + IV^\flat \wedge JV^\flat) + \mathcal{L}_{IV}\omega_J^T. \tag{4.4.6}$$

If we compare equations (4.4.5) and (4.4.6) we get that

$$\mathcal{L}_{IV}\omega_J^T = \iota_{IV} d\omega_J^T = I\beta_K - \beta_K.$$

Therefore, to prove the second statement of the Theorem it suffices to show that $\mathcal{L}_{IV}\omega_J^T = \iota_{IV}d\omega_J^T = I\beta_K - \beta_K = a\omega_K^T$, with $a = -1$. Since β_K, and hence $I\beta_K$, are $(1,1)$-forms with respect to K, it follows that also $\mathcal{L}_{IV}\omega_J^T$ is of type $(1,1)$ with respect to K. Now we argue as follows: the horizontal distribution $\mathcal{F}^\perp$ has dimension 4, and so the bundle $\bigwedge^2 \mathcal{F}^\perp$ splits as $\bigwedge^+ \mathcal{F}^\perp \oplus \bigwedge^- \mathcal{F}^\perp$ (with respect to g^T) as the direct sum of self dual and anti-self dual horizontal two-forms. It is straightforward to see that $\bigwedge^+ \mathcal{F}^\perp$ is generated by $\omega_I^T, \omega_J^T, \omega_K^T$, while the bundle $\bigwedge^- \mathcal{F}^\perp$ of anti self dual 2-forms coincides with the space of horizontal forms which are $\mathrm{SU}(2)$-invariant (see Remark 4.4.3, statement 4). We also point out that the 2-form $dV^\flat$ lies in $\bigwedge^- \mathcal{F}^\perp$, as it is horizontal and $\mathrm{SU}(2)$-invariant (see Remark 4.4.3). Using that V, IV, JV, KV are Killing vector fields (see Remark 4.4.3, statement 1), they generate isometries which preserve the Riemannian volume. Since the Lie derivatives along V, IV, JV, KV preserve the Hodge star of the horizontal metric g^T, we get that $\mathcal{L}_{IV}\omega_J^T$ is self dual, and so in the span of $\omega_I^T, \omega_J^T, \omega_K^T$. Since $\mathcal{L}_{IV}\omega_J^T$ is of type $(1,1)$ with respect to K, there exists a $b_{IJ} \in C^\infty(M)$ such that $\mathcal{L}_{IV}\omega_J^T = b_{IJ}\omega_K^T$.

With analogous computations we then get the following equalities

$$\begin{aligned}
\mathcal{L}_{IV}\omega_J^T &= \iota_{IV}d\omega_J^T = I\beta_K - \beta_K = b_{IJ}\omega_K^T,\\
\mathcal{L}_{IV}\omega_K^T &= \iota_{IV}d\omega_K^T = -I\beta_J + \beta_J = b_{IK}\omega_J^T,\\
\mathcal{L}_{JV}\omega_I^T &= \iota_{JV}d\omega_I^T = -J\beta_K + \beta_K = b_{JI}\omega_K^T,\\
\mathcal{L}_{JV}\omega_K^T &= \iota_{JV}d\omega_K^T = J\beta_I - \beta_I = b_{JK}\omega_I^T,\\
\mathcal{L}_{KV}\omega_I^T &= \iota_{KV}d\omega_I^T = K\beta_J - \beta_J = b_{KI}\omega_J^T,\\
\mathcal{L}_{KV}\omega_J^T &= \iota_{KV}d\omega_J^T = -K\beta_I + \beta_I = b_{KJ}\omega_I^T.
\end{aligned} \tag{4.4.7}$$

Using that β_I, β_J and β_K are of type $(1,1)$ with respect to I, J, K respectively, we may write them as $\beta_I = \lambda_I\omega_I^T + \eta_I$, $\beta_J = \lambda_J\omega_J^T + \eta_J$ and $\beta_K = \lambda_K\omega_K^T + \eta_K$, where η_I, η_J, η_K are anti self dual horizontal forms and $\lambda_I, \lambda_K, \lambda_K$ are smooth functions.
We are going to show that $b_{IJ} = b_{JK} = b_{KI} = b_{JI} = b_{KJ} = b_{IK}$. Since η_L are $\mathrm{SU}(2)$-invariant

$$I\beta_K - \beta_K = -\lambda_K\omega_K^T + \eta_K - \lambda_K\omega_K^T - \eta_K = -2\lambda_K\omega_K^T = b_{IJ}\omega_K^T \tag{4.4.8}$$

and

$$-J\beta_K + \beta_K = 2\lambda_K\omega_K^T = b_{JI}\omega_K^T,$$

from which follows that $b_{IJ} = -b_{JI}$. Analogously we get $b_{JK} = -b_{KJ}$ and $b_{KI} = -b_{IK}$. On the other hand for any horizontal vector fields X, Y

$$H(IV, X, Y) = H(IV, JX, JY) + H(JIV, JX, Y) + H(JIV, X, JY),$$

from which, applying (4.4.2), we get that

$$\begin{aligned}(\beta_I - J\beta_I)(X,Y) &= H(IV,X,Y) - H(IV,JX,JY)\\ &= H(JIV,JX,Y) + H(JIV,X,JY) = (J\beta_K - \beta_K)(X,JY).\end{aligned}$$

Contracting both side of (4.4.7) with X, Y, we obtain

$$-b_{JK}g(IX,Y) = -b_{JK}\omega_I^T(X,Y) = b_{IJ}\omega_K^T(X,JY) = b_{IJ}g(KX,JY) = -b_{IJ}g(IX,Y),$$

from which we get $b_{IJ} = b_{JK}$. The last equality $b_{JK} = b_{KI}$ follows in a similar way. We also point out that from equation (4.4.8) we also get that $\lambda_I = \lambda_J = \lambda_K = -\frac{b}{2}$, where $b := b_{IJ} = b_{JK} = b_{KI} = b_{JI} = b_{KJ} = b_{IK}$.
To summarize

$$\begin{aligned}
\mathcal{L}_{IV}\omega_J^T &= \iota_{IV}d\omega_J^T = I\beta_K - \beta_K = b\omega_K^T,\\
\mathcal{L}_{IV}\omega_K^T &= \iota_{IV}d\omega_K^T = -I\beta_J + \beta_J = -b\omega_J^T,\\
\mathcal{L}_{JV}\omega_I^T &= \iota_{JV}d\omega_I^T = -J\beta_K + \beta_K = -b\omega_K^T,\\
\mathcal{L}_{JV}\omega_K^T &= \iota_{JV}d\omega_K^T = J\beta_I - \beta_I = b\omega_I^T,\\
\mathcal{L}_{KV}\omega_I^T &= \iota_{KV}d\omega_I^T = K\beta_J - \beta_J = b\omega_J^T,\\
\mathcal{L}_{KV}\omega_J^T &= \iota_{KV}d\omega_J^T = -K\beta_I + \beta_I = -b\omega_I^T.
\end{aligned} \tag{4.4.9}$$

By Remark 4.4.3 statements (1),(2) and (3), $\mathcal{L}_V\omega_I = \mathcal{L}_V\omega_J = \mathcal{L}_V\omega_K = \mathcal{L}_{IV}\omega_I = \mathcal{L}_{JV}\omega_J = \mathcal{L}_{KV}\omega_K = 0$. In particular, this proves that $\iota_V(d\omega_I^T) = \iota_{IV}(d\omega_I^T) = 0$, and an analogous formula holds for ω_J^T and ω_K^T. Indeed, using that V, IV, JV, KV are Killing,

$$\begin{aligned}
0 &= \mathcal{L}_V\omega_I = \mathcal{L}_V(V^\flat \wedge IV^\flat + JV^\flat \wedge KV^\flat + \omega_I^T) = \mathcal{L}_V(\omega_I^T) = \iota_V(d\omega_I^T),\\
0 &= \mathcal{L}_{IV}\omega_I = \mathcal{L}_{IV}(V^\flat \wedge IV^\flat + JV^\flat \wedge KV^\flat + \omega_I^T) = \mathcal{L}_{IV}(\omega_I^T) = \iota_{IV}(d\omega_I^T).
\end{aligned}$$

Combining all the above properties with equations (4.4.9) we get that

$$\begin{aligned}
d\omega_I^T &= bKV^\flat \wedge \omega_J^T - bJV^\flat \wedge \omega_K^T + \gamma_1,\\
d\omega_J^T &= bIV^\flat \wedge \omega_K^T - bKV^\flat \wedge \omega_I^T + \gamma_2,\\
d\omega_K^T &= bJV^\flat \wedge \omega_I^T - bIV^\flat \wedge \omega_J^T + \gamma_3,
\end{aligned}$$

where $\gamma_1, \gamma_2, \gamma_3$ are the horizontal components of $d\omega_I^T, d\omega_J^T$ and $d\omega_K^T$ respectively. Since the horizontal distribution is 4-dimensional, $\gamma_I = \theta_I^T \wedge \omega_I^T$, where θ_I^T is the transverse Lee form, and analogously for γ_J and γ_K. Let us compute $\theta_I^T, \theta_J^T, \theta_K^T$. Recall that the Lee form

$$\theta_I(X) = \theta(X) = \frac{1}{2}\sum_{i=1}^{8} H(e_i, Ie_i, IX) \tag{4.4.10}$$

(see for instance [GFS, Lemma 8.6]), where $\{e_1, \dots, e_8\}$ is any local orthonormal frame. Without loss of generality we may chose as a local orthonormal frame the frame $\{V, \dots, KV, \xi_1, \dots, \xi_4\}$ where $\{\xi_1, \dots, \xi_4\}$ is a local horizontal orthonormal frame adapted to $\{I, J, K\}$, namely $\xi_2 = I\xi_1,\ \xi_3 = J\xi_1$ and $\xi_4 = K\xi_1$. Let us deduce

$$\theta_I = -(a+b)V^\flat + \theta_I^T. \tag{4.4.11}$$

We have

$$\begin{aligned}\theta_I(V) =& H(V, IV, IV) + H(JV, KV, IV) + H(\xi_1, \xi_2, IV) + H(\xi_3, \xi_4, IV)\\ =& -a + dIV^\flat(\xi_1, \xi_2) + dIV^\flat(\xi_3, \xi_4)\\ =& -a + dIV^\flat(\xi_1, I\xi_1) + dIV^\flat(J\xi_1, K\xi_1)\\ =& -a + (dIV^\flat - JdIV^\flat)(\xi_1, I\xi_1)\\ =& -a + (\beta_I - J\beta_I)(\xi_1, I\xi_1)\\ =& -a - b\omega_I^T(\xi_1, I\xi_1)\\ =& -(a+b),\end{aligned}$$

where we used equation (4.4.2), and equations (4.4.9). Similarly

$$\begin{aligned}\theta_I(IV) =& -H(V, IV, V) - H(JV, KV, V) - H(\xi_1, \xi_2, V) - H(\xi_3, \xi_4, V)\\ =& -(dV^\flat - JdV^\flat)(\xi_1, I\xi_1)\\ =& 0,\end{aligned}$$

$$\begin{aligned}\theta_I(JV) =& H(V, IV, KV) + H(JV, KV, KV) + H(\xi_1, \xi_2, KV) + H(\xi_3, \xi_4, KV)\\ =& (dKV^\flat - JdKV^\flat)(\xi_1, I\xi_1)\\ =& (\beta_K - J\beta_K)(\xi_1, I\xi_1)\\ =& -b\omega_K^T(\xi_1, I\xi_1)\\ =& 0,\end{aligned}$$

and with analogous computations also $\theta_I(KV) = 0$ follows. Lastly, for any $X \in \mathcal{F}^\perp$,

$$\begin{aligned}\theta_I(X) =& H(V, IV, IX) + H(JV, KV, IX) + H(\xi_1, \xi_2, IX) + H(\xi_3, \xi_4, IX)\\ =& H^T(\xi_1, \xi_2, IX) + H^T(\xi_3, \xi_4, IX)\\ =& \theta_I^T,\end{aligned}$$

where H^T denotes the Bismut torsion of the transverse hyperhermitian geometry.

By definition of V and using (4.4.11), $V^\flat = \frac{1}{2}(\theta - df) = -\frac{1}{2}(a+b)V^\flat + \frac{1}{2}(\theta_I^T - df)$, forcing $a + b = -2$ and $\theta_I^T = df$, where here we use the fact that df is horizontal (see Remark 4.4.3, statement 8). In particular, b must be constant. Analogously one gets also $\theta_J^T = df$ and $\theta_K^T = df$. To sum it up:

$$
\begin{aligned}
d\omega_I^T &= bKV^\flat \wedge \omega_J^T - bJV^\flat \wedge \omega_K^T + df \wedge \omega_I^T, \\
d\omega_J^T &= bIV^\flat \wedge \omega_K^T - bKV^\flat \wedge \omega_I^T + df \wedge \omega_J^T, \\
d\omega_K^T &= bJV^\flat \wedge \omega_I^T - bIV^\flat \wedge \omega_J^T + df \wedge \omega_K^T.
\end{aligned} \tag{4.4.12}
$$

Taking the exterior differential in both side of the first line of (4.4.12), we get $b(b-a) = 0$. If $a = 0$, then $b = 0$ and so the relation $a+b = -2$ cannot hold. So a must be non zero, concluding the proof of the first statement of the theorem. We are left with two cases, either $a = b = -1$ or $b = 0, a = -2$. If $a = b = -1$ then by equations (4.4.9) the proof of the second statement follows. Therefore, we have to exclude the case $b = 0, a = -2$.

We prove that $b = 0$ leads to a contradiction, as a consequence of the fact that the potential f is constant, whose proof will occupy the remaining part of the theorem. Using that $\omega_I = V^\flat \wedge IV^\flat + JV^\flat \wedge KV^\flat + \omega_I^T$, we compute

$$
H = V^\flat \wedge dV^\flat - aIV^\flat \wedge JV^\flat \wedge KV^\flat + IV^\flat \wedge \beta_I + JV^\flat \wedge \beta_J + KV^\flat \wedge \beta_K - Idf \wedge \omega_I^T \tag{4.4.13}
$$

and

$$
\begin{aligned}
0 = dH = \; & dV^\flat \wedge dV^\flat + \beta_I \wedge \beta_I + \beta_J \wedge \beta_J + \beta_K \wedge \beta_K \\
& - dIdf \wedge \omega_I^T + Idf \wedge df \wedge \omega_I^T + bIdf \wedge KV^\flat \wedge \omega_J^T - bIdf \wedge JV^\flat \wedge \omega_K^T.
\end{aligned}
$$

We further observe that $H^T = -Idf \wedge \omega_I^T$.
Exploiting that $\beta_I = -\frac{b}{2}\omega_I^T + \eta_I$, $\beta_J = -\frac{b}{2}\omega_J^T + \eta_J$, $\beta_K = -\frac{b}{2}\omega_K^T + \eta_K$, where η_I, η_J and η_K are anti self dual, we get

$$
\begin{aligned}
dH = & \left[-\frac{1}{2} \left(\left\| dV^\flat \right\|^2 + \|\eta_I\|^2 + \|\eta_J\|^2 + \|\eta_K\|^2 \right) + \frac{3}{2} b^2 \right] vol^T \\
& - dIdf \wedge \omega_I^T + Idf \wedge df \wedge \omega_I^T + bIdf \wedge KV^\flat \wedge \omega_J^T - bIdf \wedge JV^\flat \wedge \omega_K^T,
\end{aligned} \tag{4.4.14}
$$

where vol^T is the volume form of the transverse hyperhermitian structure.

To lighten the notation we set $l := \left\| dV^\flat \right\|^2 + \|\eta_I\|^2 + \|\eta_J\|^2 + \|\eta_K\|^2$. Using [GFS, Proposition 4.33], we have that

$$
\frac{1}{6} \|H\|^2 + \Delta f - \|\mathrm{grad}\, f\|^2 = const. \tag{4.4.15}
$$

Using equation (4.4.13) and the fact that for any $L = I, J, K$ $\|\beta_L\|^2 = \frac{1}{4} \left\| \omega_L^T \right\|^2 + \|\eta_L\|^2 = 1 + \|\eta_L\|^2$, we immediately get that $\|H\|^2 = \left\| H^T \right\|^2 + 3m + 6a^2$, where $m = l + const$ (for further details see the proof of [ABLS, Proposition 2.14]). Furthermore [†]

$$\left\|H^T\right\|^2 = 6\left\|\mathrm{grad}\, f\right\|^2.$$

Pluggin this in (4.4.15) we get $\frac{1}{2}l + \Delta f = \mathrm{const}$.

Let $\{e_1, e_2, e_3, e_4\}$ be a local unitary orthonormal horizontal frame with respect to I. We compute $dH(e_1, e_2, e_3, e_4)$.

$$dH(e_1, e_2, e_3, e_4) = \left(-\frac{1}{2}l + \frac{3}{2}b^2\right) - dIdf \wedge \omega_I^T(e_1, e_2, e_3, e_4) + Idf \wedge df \wedge \omega_I(e_1, e_2, e_3, e_4).$$

Now,

$$\begin{aligned} dIdf \wedge \omega_I^T(e_1, e_2, e_3, e_4) =& d(Idf)(e_1, e_2) + d(Idf)(e_3, e_4) \\ =& \sum_{j=1}^4 \mathcal{L}_{e_j}(df(e_j)) - \sum_{i=1}^2 Idf([e_{2i-1}, e_{2i}]) \\ =& \sum_{j=1}^4 \mathcal{L}_{e_j}(df(e_j)) - \sum_{i=1}^2 Idf(\nabla^{Ob}_{e_{2i-1}} e_{2i} - \nabla^{Ob}_{e_{2i}} e_{2i-1}) \\ =& \sum_{j=1}^4 \mathcal{L}_{e_j}(df(e_j)) - df(\nabla^{Ob}_{e_j} e_j) \\ =& \sum_{j=1}^4 \mathcal{L}_{e_j}(df(e_j)) - df(\nabla^{B}_{e_j} e_j) - A(e_j, e_j, \mathrm{grad}\, f), \end{aligned}$$

where the last equation follows by (5.2.12). We observe that since e_j and $\mathrm{grad}\, f$ are horizontal vector fields, by (1.5.1),

$$\begin{aligned} 2A(e_j, e_j, \mathrm{grad}\, f) =& - H(e_j, Ie_j, I\,\mathrm{grad}\, f) - H(Ie_j, Ie_j, \mathrm{grad}\, f) \\ & - H(e_j, Ke_j, K\,\mathrm{grad}\, f) - H(Ie_j, Ke_j, J\,\mathrm{grad}\, f) \\ =& - H^T(e_j, Ie_j, I\,\mathrm{grad}\, f) - H^T(Ie_j, Ie_j, \mathrm{grad}\, f) \\ & - H^T(e_j, Ke_j, K\,\mathrm{grad}\, f) - H^T(Ie_j, Ke_j, J\,\mathrm{grad}\, f). \end{aligned}$$

[†] the next computation is also included in an early version of [ABLS].

Furthermore, a straightforward computation shows

$$
\begin{aligned}
\sum_{j=1}^{4} A(e_j, e_j, \operatorname{grad} f) &= \sum_{j=1}^{4} -\frac{1}{2}H^T(e_j, Ie_j, I \operatorname{grad} f) - \frac{1}{2}H^T(e_j, Ke_j, K \operatorname{grad} f) \\
&\quad - \frac{1}{2}H^T(Ie_j, Ke_j, J \operatorname{grad} f) \\
&= \sum_{j=1}^{4} -\frac{1}{2}H^T(e_j, Ie_j, I \operatorname{grad} f) - \frac{1}{2}H^T(e_j, Ke_j, K \operatorname{grad} f) + \\
&\quad \sum_{j=1}^{4} \frac{1}{2}H^T(e_j, Je_j, J \operatorname{grad} f) \\
&= -\,\theta^T(\operatorname{grad} f) \\
&= -\left\|\operatorname{grad} f\right\|^2,
\end{aligned}
$$

where we used the definition of the transverse Lee form θ^T and the fact that $\theta^T = df$. Therefore,

$$
\begin{aligned}
(dIdf \wedge \omega_I^T)(e_1, e_2, e_3, e_4) &= \sum_{j=1}^{4} \mathcal{L}_{e_j}(df(e_j)) - df(\nabla^{LC}_{e_j} e_j) + \left\|\operatorname{grad} f\right\|^2 \\
&= \Delta f + \left\|\operatorname{grad} f\right\|^2.
\end{aligned}
$$

Straightforward computations also show that $(Idf \wedge df \wedge \omega_I)(e_1, e_2, e_3, e_4) = -\left\|\operatorname{grad} f\right\|^2$.

To sum it up,

$$
dH(e_1, e_2, e_3, e_4) = \left(-\frac{1}{2}l + \frac{3}{2}b^2\right) - \Delta f - 2\left\|\operatorname{grad} f\right\|^2 = 0.
$$

Using that $-\frac{1}{2}l - \Delta f$ is constant, this leads to $2\left\|\operatorname{grad} f\right\|^2 = \text{const}$, which forces $f = \text{const}$ applying the maximum principle.

To conclude the proof we observe that the value $b = 0$ leads to a contradiction. In fact, for $b = 0$, exploiting that $dH = 0$, then by equation (4.4.14) one gets $\left\|dV^\flat\right\|^2 + \left\|\eta_I\right\|^2 + \left\|\eta_J\right\|^2 + \left\|\eta_K\right\|^2 = 0$, which implies that $d\theta = 0$, as $2dV^\flat = d\theta$. The contradiction follows by Proposition 4.4.1. q.e.d.

By statements (1) and (2) of the previous Theorem, we may immediately prove the following

Corollary 4.4.5. *Let (M, I, J, K, g) be a strong HKT compact and simply connected 8-dimensional manifold which is not hyperkähler. Then the following statements holds true*

1) $\mathbb{H}^*$ acts on M isometrically.

2) V, IV, JV, KV are infinitesimal automorphisms preserving $\mathcal{F}^\perp \in \mathrm{TM}$, namely $[W, \mathcal{F}^\perp] \subset$

$\mathcal{F}^\perp$ for any $W \in \{V, IV, JV, KV\}$, where $\mathcal{F}$ is the distribution generated by V, IV, JV, KV and $\mathcal{F}^\perp$ is its orthogonal complement with respect to g.

3) $\nabla^{Ob}V = \frac{1}{2}Id$, $\nabla^{Ob}IV = \frac{1}{2}I$, $\nabla^{Ob}JV = \frac{1}{2}J$, $\nabla^{Ob}KV = \frac{1}{2}K$.

4) *All the components of the Obata curvature tensor involving the quaternionic span of V vanishes.*

Proof. By the previous Theorem the Lie algebra of the distribution spanned by $\{V, IV, JV, KV\}$ is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. Furthermore, since V, IV, JV, KV are Killing (see Remark 4.4.3), they generate an isometric action of $\mathbb{H}^*$ on M. This concludes the proof of the first claim.
For the second claim we only have to check that $[W, \mathcal{F}^\perp] \subset \mathcal{F}^\perp$ for any $W \in \{V, IV, JV, KV\}$. Let us fix the following notation, let $V = I_0V$, $IV = I_1V$, $JV = I_2V$ and $KV = I_3V$. Using that I_aV is Killing for $a = 0, \dots, 3$ we get that for any $a, b = 0, \dots, 3$ and $X \in \mathcal{F}^\perp$

$$0 = \mathcal{L}_{I_aV}g(X, I_bV) = g([I_aV, X], I_bV),$$

forcing $[I_aV, X] \in \mathcal{F}^\perp$.
The third and fourth claim are consequences of [PPS, Lemma 2.2 and Proposition 2.3], where the authors proved that if we have a hyper-complex manifold endowed with a vector field W such that W, IW, JW, KW span a copy of $U(2)$ where W is in the center $U(1)$, then if W is hyper-holomorphic and $\mathcal{L}_{IW}J = K$, $\mathcal{L}_{JW}K = I$, $\mathcal{L}_{KW}I = J$, this leads to $\nabla^{Ob}W = -\frac{1}{2}Id$. Applying this to $-V$ we immediately get the statements $3)$ and $4)$. q.e.d.

The vector field V is commonly referred to as the *Euler* vector field, and the corresponding hyper-complex structure is described as *conical* [CH2].

Remark 4.4.6. Under the hypothesis of Theorem 4.4.4, as a consequence of the previous Proposition we get that the orbits of the foliation $\mathcal{F}$ are totally geodesic and Obata flat. Furthermore, as a consequence of the fact that $\nabla^{Ob}V = \frac{1}{2}Id$, we observe that the Obata connection can only preserve tensors of the kind (k, k), as pointed out in [Sol].

4.5 Characterization of $\nabla^B H = 0$

Since it will be useful in the following, we summarize some facts from the proof of Theorem 4.4.4: the expression of H is given by

$$H = V^\flat \wedge dV^\flat + IV^\flat \wedge JV^\flat \wedge KV^\flat + IV^\flat \wedge \beta_I + JV^\flat \wedge \beta_J + KV^\flat \wedge \beta_K, \qquad (4.5.1)$$

whereas the differential of $IV^\flat, JV^\flat, KV^\flat$ can be written as

$$dIV^\flat = JV^\flat \wedge KV^\flat + \beta_I,\ \ dJV^\flat = KV^\flat \wedge IV^\flat + \beta_J,\ \ dKV^\flat = IV^\flat \wedge JV^\flat + \beta_K \quad (4.5.2)$$

where

$$\beta_I = \frac{1}{2}\omega_I^T + \eta_I,\ \beta_J = \frac{1}{2}\omega_J^T + \eta_J,\ \beta_K = \frac{1}{2}\omega_K^T + \eta_K, \quad (4.5.3)$$

and η_I, η_J, η_K are anti self dual horizontal 2-forms. Using equations (4.4.12) and $d^2IV^\flat = d^2JV^\flat = d^2KV^\flat = 0$, we recover that

$$d\eta_I = -\eta_J \wedge KV^\flat + \eta_K \wedge JV^\flat,\ d\eta_J = -\eta_K \wedge IV^\flat + \eta_I \wedge KV^\flat,\ \ d\eta_K = -\eta_I \wedge JV^\flat + \eta_J \wedge IV^\flat. \quad (4.5.4)$$

Furthermore, the strong HKT condition implies that

$$0 = dH = \left[-\frac{1}{2}\left(\left\| dV^\flat \right\|^2 + \|\eta_I\|^2 + \|\eta_J\|^2 + \|\eta_K\|^2 \right) + \frac{3}{2} \right] vol^T. \quad (4.5.5)$$

We recall that a HKT structure (I, J, K, g) is said to be Einstein HKT if there exists a smooth function λ such that

$$\frac{\rho_I^C - J\rho_I^C}{2} = \lambda\,\omega_I,$$

where ρ_I^C is the (first) Chern Ricci curvature of (I, g). Examples of strong HKT Einstein metrics have been constructed in [FG2] on Joyce's examples [Joy].

Proposition 4.5.1. *Any* 8*-dimensional compact simply connected strong HKT manifold* (M, I, J, K, g) *is HKT Einstein with constant* λ*.*

Proof. If (M, I, J, K, g) is hyperkähler, then it is HKT Einstein with $\lambda = 0$. Let us assume then that (M, I, J, K, g) is not hyperkähler. Since $\rho^B = 0$, then $\rho_I^C = dI\theta = 2dIV^\flat$. Therefore

$$\begin{aligned}
\frac{\rho_I^C - J\rho_I^C}{2} =& dIV^\flat - J(dIV^\flat) \\
=& JV^\flat \wedge KV^\flat + \frac{1}{2}\omega_I^T + \eta_I - J(JV^\flat \wedge KV^\flat + \frac{1}{2}\omega_I^T + \eta_I) \\
=& JV^\flat \wedge KV^\flat + \frac{1}{2}\omega_I^T + \eta_I + V^\flat \wedge IV^\flat + \frac{1}{2}\omega_I^T - \eta_I \\
=& V^\flat \wedge IV^\flat + JV^\flat \wedge KV^\flat + \omega_I^T \\
=& \omega_I,
\end{aligned}$$

concluding the proof. q.e.d.

We determine a sufficient condition to have parallel Bismut torsion in terms of η_I, η_J, η_K.

Theorem 4.5.2. *Let (M, I, J, K, g) be a strong HKT compact and simply connected 8-dimensional manifold which is not hyperkähler. If any of η_I, η_J, η_K is zero, then all of them are zero and $\nabla^B H = 0$. In particular, (M, I, J, K, g) is the hyperhermitian Samelson space $SU(3)$.*

Proof. By (4.5.4), we have that if any of η_I, η_J, η_K is zero, then all of them must vanish on M. Indeed, if for instance $\eta_I = 0$ then by (4.5.4)

$$0 = d\eta_I = -\eta_J \wedge KV^\flat + \eta_K \wedge JV^\flat,$$

which implies that $\eta_J = \eta_K = 0$.

If η_L are zero, then $\beta_L = \frac{1}{2}\omega_L^T$ by (4.5.3) and, furthermore, $\nabla^B \beta_L = 0$.

Using the expression of the Bismut torsion in (4.5.1), we get that for any vector field X

$$\nabla^B_X H = (\nabla^B_X dV^\flat) \wedge V^\flat. \tag{4.5.6}$$

In order to prove that $\nabla^B_X H$ is zero, we only need to prove that $\nabla^B_X dV^\flat(Y, Z)$ vanishes for any $X \in \Gamma(\mathrm{TM})$ and Y, Z vector fields different from (a multiple of) V.

We observe that if we take $Y, Z \in \{IV, JV, KV\}$ and X any vector field, then the statement is trivially true, since $dV^\flat$ is horizontal and IV, JV, KV are Bismut parallel. The same holds for $Y \in \{IV, JV, KV\}$ and Z horizontal. It remains to check the condition for Y, Z both horizontal. If $X \in \{V, IV, JV, KV\}$ is vertical, then

$$\begin{aligned}
\nabla^B_X dV^\flat(Y, Z) =& X(dV^\flat(Y, Z) - dV^\flat(\nabla^B_X Y, Z) - dV^\flat(Y, \nabla^B_X Z) \\
=& - Xg([Y, Z], V) - dV^\flat([X, Y] + T^B(X, Y), Z) \\
& - dV^\flat(Y, [XZ] + T^B(X, Z)) \\
=& - g([X, [Y, Z]] + [[X, Y], Z] + [Y, [X, Y]], V) \\
& - dV^\flat(T^B(X, Y), Z) - dV^\flat(Y, T^B(X, Z)) \\
=& - dV^\flat((\iota_Y dX^\flat)^\sharp, Z) - dV^\flat(Y, (\iota_Z dX^\flat)^\sharp).
\end{aligned} \tag{4.5.7}$$

We are going to show that this is zero in any point p of M. Let us fix a orthonormal basis $\{e_1, \ldots, e_4\}$ of horizontal vectors in $\mathrm{T_pM}$ adapted to I, J, K. Then, denoted by $\{e^1, \ldots, e^4\}$ the dual basis

$$\omega_I^T = e^{12} + e^{34},\ \omega_J^T = e^{13} + e^{42},\ \omega_K^T = e^{14} + e^{23}.$$

Then we may write Y and Z at p as $Y = Y^i e_i$ and $Z = Z^j e_j$. The equality (4.5.7) at p yields

$$\nabla^B_X dV^\flat(Y, Z)(p) = Y^i Z^j \left[(dX^\flat)_{ik}(dV^\flat)_{jk} - (dX^\flat)_{jk}(dV^\flat)_{ik}\right]. \tag{4.5.8}$$

If $X = V$ then the above expression is clearly zero. For $X = IV$ we use the following fact: since $dV^\flat$ is horizontal and anti self dual we may write $dV^\flat$ at p as $\lambda_1(e^{12}-e^{34})+\lambda_2(e^{13}-e^{42})+\lambda_3(e^{14}-e^{23})$, with $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ (see the proof of Theorem 14.4.4). The expression (4.5.8) is hence given by

$$\nabla^B_{IV}dV^\flat(Y,Z)(p)=\frac{1}{2}Y^iZ^j[(\omega^T_I)_{ik}(dV^\flat)_{jk}-(\omega^T_I)_{jk}(dV^\flat)_{ik}],$$

as $dIV^\flat(Y,Z)=\beta_I(Y,Z)=\frac{1}{2}\omega^T_I(Y,Z)$. It is an easy computation that for any (i,j), $(\omega^T_I)_{ik}(dV^\flat)_{jk}-(\omega^T_I)_{jk}(dV^\flat)_{ik}=0$. The other cases follow in analogous way.

It remains to check that $\nabla^B_X dV^\flat(Y,Z)=0$ for any X,Y,Z horizontal vector fields. In this case

$$\nabla^B_X dV^\flat(Y,Z)=\nabla^T_X dV^\flat(Y,Z),$$

where ∇^T is the transverse Levi-Civita and the equality follows since for any X,Y,Z horizontal vector fields $g(\nabla^B_XY,Z)=g(\nabla^{LC}_XY,Z)+\frac{1}{2}H(X,Y,Z)=g(\nabla^T_XY,Z)$.

We observe that by the condition $dH=0$ (4.5.5), we get that $dV^\flat$ has constant non zero norm.

We are going to define a transverse almost complex structure $\mathbb{J}^T$ on the bundle $\mathcal{F}^\perp$ using the metric g^T and the basic 2-form $dV^\flat$[‡]. In particular, since $\mathcal{L}_Vg^T=\mathcal{L}_{IV}g^T=\mathcal{L}_{JV}g^T=\mathcal{L}_{KV}g^T=0$, the pair $(\mathbb{J}^T,g^T)$ will induce an almost Hermitian structure on any local quotient of M by the flows of V,IV,JV,KV.

Let $\mathbb{J}^T$ be the endomorphism defined on horizontal vector fields by the relation

$$g^T(\mathbb{J}^TX,Y)=\frac{1}{\|dV^\flat\|}dV^\flat(X,Y).$$

Since $dV^\flat$ is skew-symmetric we immediately get that $g^T(\mathbb{J}^TX,Y)=-g^T(X,\mathbb{J}^TY)$. At any point p in M we may fix an orthonormal horizontal basis of $\mathcal{F}^\perp_p$ adapted to I,J,K so that $dV^\flat(p)=\lambda_1(e^{12}-e^{34})+\lambda_2(e^{13}-e^{42})+\lambda_3(e^{14}-e^{23})$. By definition of $\mathbb{J}^T$ we get

$$\mathbb{J}^T_p=\frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{pmatrix}0&-\lambda_1&-\lambda_2&-\lambda_3\\ \lambda_1&0&\lambda_3&-\lambda_2\\ \lambda_2&-\lambda_3&0&\lambda_1\\ \lambda_3&\lambda_2&-\lambda_1&0\end{pmatrix}$$

which immediately implies that $(\mathbb{J}^T)^2=-\mathrm{Id}$ in any p in M. We point out that the basic Hermitian structure $(\mathbb{J}^T,g^T)$ is only almost Kähler. Moreover we adapt the proof of [ABLS] in our case, which relay on some results in [ADM].

We first observe that the transverse metric is Einstein. By O'Neill formulas [ON] (see

[‡]By Remark 4.4.3 it is not difficult to check that $\mathcal{L}_VdV^\flat=\mathcal{L}_{IV}dV^\flat=\mathcal{L}_{JV}dV^\flat=\mathcal{L}_{KV}dV^\flat=0$.

also [BG, Section 2.5.1]),

$$Ric^T(X,Y) = Ric^{LC}(X,Y) + 2\big[g(A_X V, A_Y V) + g(A_X IV, A_Y IV) + \\ g(A_X JV, A_Y JV) + g(A_X KV, A_Y KV)\big],$$

where $A_X V = \mathrm{pr}(\nabla^{LC}_X V)$ and $\mathrm{pr} : \mathrm{TM} \to \mathcal{F}^\perp$ is the standard projection.

Since by Theorem 4.4.4 $\nabla^B V = 2\nabla^B\theta = 0$, by formula (1.2.3) $Ric^B = 0$. In particular, its symmetric part has to vanish, implying that $Ric^{LC} = \frac{1}{4}H^2$, where $H^2(X,Y) = g(\iota_X H, \iota_Y H)$. On the other hand, for any X, Z horizontal vector fields

$$g(A_X V, Z) = g(\nabla^{LC}_X V, Z) = g(\nabla^B_X V, Z) - \frac{1}{2}H(X,V,Z) = \frac{1}{2}dV^\flat(X,Z),$$

implying that $A_X V = \frac{1}{2}(\iota_X dV^\flat)^\sharp$. Similarly, $A_X IV = \frac{1}{4}(\iota_X \omega_I^T)^\sharp$, $A_X JV = \frac{1}{4}(\iota_X \omega_J^T)^\sharp$ and $A_X KV = \frac{1}{4}(\iota_X \omega_K^T)^\sharp$. In particular,

$$Ric^T(X,Y) = \frac{3}{4}\big[(dV^\flat)^2(X,Y) + \frac{1}{4}(\omega_I^T)^2(X,Y) + \frac{1}{4}(\omega_J^T)^2(X,Y) + \frac{1}{4}(\omega_K^T)^2(X,Y)\big],$$

where again $\alpha^2(X,Y) = g(\iota_X\alpha, \iota_Y\alpha)$.

We fix a local orthonormal horizontal basis $\{e_1, \dots, e_4\}$ of $\mathcal{F}^\perp$ adapted to I, J, K in a open set $\mathcal{U}$. With respect to the dual basis $\{e^1, \dots, e^4\}$ we may write $\omega_I = e^{12} + e^{34}, \omega_J = e^{13} + e^{42}, \omega_K = e^{14} + e^{23}$ and $dV^\flat = \lambda_1(e^{12} - e^{34}) + \lambda_2(e^{13} - e^{42}) + \lambda_3(e^{14} - e^{23})$, with λ_i smooth function defined in a neighborhood of p. It is a straightforward computation that for any $q \in \mathcal{U}$

$$Ric^T_q(e_i, e_j) = \frac{3}{4}\left(\frac{3}{4} + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(q)\right)\delta^i_j.$$

Using that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is a constant multiple of $\|dV\|^2$ we get that the transverse metric g_T is Einstein with strictly positive Einstein constant.

Now we argue as in [ABLS]. In fact, as pointed out in [ABLS], the integral identity in [ADM, Corollary 2.2] also extends in our situation, in fact, also in our case it holds that the L^2 product on (M,g) satisfies $< \delta_g \pi^*\alpha, \pi^*\beta > = < \pi^*\delta_{g^T}\alpha, \pi^*\beta >$ for any α, β transversal forms. In particular, using that the transverse metric is Einstein with positive Einstein constant, we get that $\nabla^T dV^\flat = 0$ (see [ADM, Remark 2.3] for further details).

Therefore, by (4.5.6), $\nabla^B H = 0$ and the result follows by Theorem 4.2.11. q.e.d.

We are able now to fully characterize the condition $\nabla^B H$ for compact and simply connected 8-dimensional strong HKT manifolds.

Theorem 4.5.3. *Let (M^8, g, I, J, K) be a compact simply connected strong HKT manifold which is not hyperkähler. The following are equivalent:*

1. $\nabla^B H = 0$, i.e., (M^8, g, I, J, K) is the hyperhermitian Samelson space $SU(3)$;

2. *for any* $L \in \{I, J, K\}$, $R^{LC}(V, LV, X, Y) = 0$ *for any* X, Y *horizontal vector fields;*

3. *there exists* $L \in \{I, J, K\}$ *such that* $R^{LC}(V, LV, X, Y) = 0$ *for any* X, Y *horizontal vector fields;*

4. *for any* $L \in \{I, J, K\}$, $\eta_L = 0$*;*

5. *there exists* $L \in \{I, J, K\}$ *such that* $\eta_L = 0$.

Proof.

$4 \Rightarrow 5$ Obvious.

$5 \Rightarrow 4$ It is a straightforward consequence of equation (4.5.4).

$5 \Rightarrow 1$ It is Theorem 4.5.2.

$1 \Rightarrow 2$ If $\nabla^B H = 0$, and so (M^8, g, I, J, K) is the hyperhermitian Samelson space $SU(3)$, then (M^8, g, I, J, K) is Bismut flat. In particular, fixed a local horizontal orthonormal frame $\{e_1, \dots, e_4\}$, we have that for any $L \in \{I, J, K\}$, $R^B(V, LV, e_i, e_j) = 0$. By Lemma 4.A.1,

$$R^{LC}(V, LV, e_i, e_j) = \frac{1}{4} R^B(V, LV, e_i, e_j) = 0.$$

$2 \Rightarrow 3$ Obvious.

$3 \Rightarrow 4$ To lighten the notation we may assume that $L = I$, the other proofs are analogues. [§]. Firstly we have to show that fixed any local horizontal orthonormal frame $\{e_1, \dots, e_4\}$ adapted to I, J, K we have that for any $k = 1, \dots, 4$,

$$\sum_{1 \le i < j \le 4} dV^\flat(e_i, e_j)(\nabla^{LC}_{e_k} \eta_I)(e_i, e_j) - \eta_I(e_i, e_j)(\nabla^{LC}_{e_k} dV^\flat)(e_i, e_j) = 0. \tag{4.5.9}$$

We prove this equality as a consequence of the second Bianchi identity

$$\nabla^{LC}_{e_i} R^{LC}(V, IV, e_l, e_j) + \nabla^{LC}_{e_l} R^{LC}(V, IV, e_j, e_i) + \nabla^{LC}_{e_j} R^{LC}(V, IV, e_i, e_l) = 0. \tag{4.5.10}$$

We study in details the first term $\nabla^{LC}_{e_i} R^{LC}(V, IV, e_l, e_j)$, as the other ones behave similarly.

We observe that since $R^{LC}(V, IV, e_l, e_j) = 0$ by hypothesis,

$$\begin{aligned} \nabla^{LC}_{e_i} R^{LC}(V, IV, e_l, e_j) = & - R^{LC}(\nabla^{LC}_{e_i} V, IV, e_l, e_j) - R^{LC}(V, \nabla^{LC}_{e_i} IV, e_l, e_j) \\ & - R^{LC}(V, IV, \nabla^{LC}_{e_i} e_l, e_j) - R^{LC}(V, IV, e_l, \nabla^{LC}_{e_i} e_j). \end{aligned}$$

We examine each term of the above expression.

[§]The strategy of the proof is inspired by an argument appearing in a preliminary version of the work [ABLS]

Since e_i is horizontal, for any vertical vector field U,

$$g(\nabla^{LC}_{e_i}V, U) = g(\nabla^{B}_{e_i}V, U) - \frac{1}{2}H(e_i, V, U) = 0,$$

as V is Bismut parallel and by the expression of the torsion H given in (4.5.1). On the other hand, if U is instead horizontal,

$$g(\nabla^{LC}_{e_i}V, U) = g(\nabla^{B}_{e_i}V, U) - \frac{1}{2}H(e_i, V, U) = \frac{1}{2}dV^\flat(e_i, U),$$

implying that $\nabla^{LC}_{e_i}V = \frac{1}{2}(\iota_{e_i}dV^\flat)^\sharp$. Hence,

$$\begin{aligned} R^{LC}(\nabla^{LC}_{e_i}V, IV, e_l, e_j) &= \frac{1}{2}R^{LC}((\iota_{e_i}dV^\flat)^\sharp, IV, e_l, e_j) \\ &= \frac{1}{2}\sum_{k=1}^{4} dV^\flat(e_i, e_k)R^{LC}(e_k, IV, e_l, e_j). \end{aligned}$$

The same argument applies for $R^{LC}(V, \nabla^{LC}_{e_i}IV, e_l, e_j)$.
For the term $R^{LC}(V, IV, \nabla^{LC}_{e_i}e_l, e_j)$ we may argue in this way. By hypothesis,

$$R^{LC}(V, IV, \nabla^{LC}_{e_i}e_l, e_j) = R^{LC}(V, IV, \pi_V(\nabla^{LC}_{e_i}e_l), e_j),$$

where π_V is the vertical projection. Moreover, for any vertical vector U,

$$R^{LC}(V, IV, U, Y) = 0,$$

again by using Lemma 4.A.1. Therefore

$$R^{LC}(V, IV, \nabla^{LC}_{e_i}e_l, e_j) = 0.$$

The last term can be done similarly.
To summarize, using Lemma 4.A.1 we have that

$$\begin{aligned} \nabla^{LC}_{e_i}R^{LC}(V, IV, e_l, e_j) = \frac{1}{2}\sum_{k=1}^{4} & dV^\flat(e_i, e_k)R^{LC}(IV, e_k, e_l, e_j) \\ & - (\beta_I)(e_i, e_k)R^{LC}(V, e_k, e_l, e_j) \end{aligned}$$

$$\begin{aligned} \nabla^{LC}_{e_i}R^{LC}(V, IV, e_l, e_j) = \frac{1}{4}\sum_{k=1}^{4} & - dV^\flat(e_i, e_k)\nabla^{LC}_{e_k}\eta_I(e_l, e_j) \\ & + (\beta_I)(e_i, e_k)\nabla^{LC}_{e_k}dV^\flat(e_l, e_j). \end{aligned}$$

Dealing with the other terms in a similar fashion, we have that the second

Bianchi identity (4.5.10) reads as

$$\begin{aligned}0=\sum_{k=1}^{4}&-(dV^\flat)_{ik}(\nabla^{LC}_{e_k}\eta_I)_{lj}-(dV^\flat)_{lk}(\nabla^{LC}_{e_k}\eta_I)_{ji}-(dV^\flat)_{jk}(\nabla^{LC}_{e_k}\eta_I)_{il}\\&+(\beta_I)_{ik}(\nabla^{LC}_{e_k}dV^\flat)_{lj}+(\beta_I)_{lk}(\nabla^{LC}_{e_k}dV^\flat)_{ji}+(\beta_I)_{jk}(\nabla^{LC}_{e_k}dV^\flat)_{il},\end{aligned} \tag{4.5.11}$$

where we omitted the constant factor $\frac{1}{4}$. We also point out that since $R^B(V,X,Y,Z)=R^B(V,X,IY,IZ)=R^B(V,X,JY,JZ)=R^B(V,X,KY,KZ)$ (this holds in general for any HKT manifold since R^B has values in $(1,1)$ forms), we have the following identities

$$\begin{aligned}\nabla^{LC}_X dV^\flat(e_1,e_2)&=-\nabla^{LC}_X dV^\flat(e_3,e_4),\ \nabla^{LC}_X\eta_I(e_1,e_2)=-\nabla^{LC}_X\eta_I(e_3,e_4)\\\nabla^{LC}_X dV^\flat(e_1,e_3)&=\ \ \nabla^{LC}_X dV^\flat(e_2,e_4),\ \nabla^{LC}_X\eta_I(e_1,e_3)=\ \ \nabla^{LC}_X\eta_I(e_2,e_4)\\\nabla^{LC}_X dV^\flat(e_1,e_4)&=-\nabla^{LC}_X dV^\flat(e_2,e_3),\ \nabla^{LC}_X\eta_I(e_1,e_4)=-\nabla^{LC}_X\eta_I(e_2,e_3).\end{aligned}$$

We start by taking $i=1,l=2,j=3$ in (4.5.11). The first line of (4.5.11) can be written as

$$\begin{aligned}&-dV^\flat_{12}[(\nabla^{LC}_2\eta_I)_{23}+(\nabla^{LC}_1\eta_I)_{13}-(\nabla^{LC}_4\eta_I)_{12}]-dV^\flat_{13}[(-\nabla^{LC}_3\eta_I)_{32}-(\nabla^{LC}_4\eta_I)_{13}\\&-(\nabla^{LC}_1\eta_I)_{12}]-dV^\flat_{14}[(-\nabla^{LC}_4\eta_I)_{14}-(\nabla^{LC}_3\eta_I)_{31}-(\nabla^{LC}_2\eta_I)_{21}].\end{aligned} \tag{4.5.12}$$

Using that for any $j=1,\dots,4$, the codifferential $\delta\eta_I(e_j)=0$ and that for any $U\in\{V,IV,JV,KV\}$ we have $\nabla^{LC}_U\eta_I(U,e_j)=-\eta_I(\nabla^{LC}_U U,e_j)=0$, we get that

$$\delta\eta_I(e_j)=-\sum_{i=1}^{4}\nabla^{LC}_{e_i}\eta_I(e_i,e_j)=0.$$

In particular, (4.5.12) can be written also as

$$\begin{aligned}&-dV^\flat_{12}[(-\nabla^{LC}_4\eta_I)_{43}-(\nabla^{LC}_4\eta_I)_{12}]-dV^\flat_{13}[(\nabla^{LC}_4\eta_I)_{42}-(\nabla^{LC}_4\eta_I)_{13}]\\&-dV^\flat_{14}[(-\nabla^{LC}_4\eta_I)_{14}+(\nabla^{LC}_4\eta_I)_{41}]\\&=-dV^\flat_{12}[-2(\nabla^{LC}_4\eta_I)_{12}]-dV^\flat_{13}[-2(\nabla^{LC}_4\eta_I)_{13}]-dV^\flat_{14}[-2(\nabla^{LC}_4\eta_I)_{14}]=\\&=2[dV^\flat_{12}(\nabla^{LC}_4\eta_I)_{12}+dV^\flat_{13}(\nabla^{LC}_4\eta_I)_{13}+dV^\flat_{14}(\nabla^{LC}_4\eta_I)_{14}]=\\&=2\sum_{1\le i<j\le 4}dV^\flat_{ij}(\nabla^{LC}_4\eta_I)_{ij}.\end{aligned}$$

Analogously, the second line of (4.5.11) is

$$\begin{aligned}
&\eta_{I_{12}}[(\nabla^{LC}_2 dV^\flat)_{23} + (\nabla^{LC}_1 dV^\flat)_{13} - (\nabla^{LC}_4 dV^\flat)_{12}] + \eta_{I_{13}}[(-\nabla^{LC}_3 dV^\flat)_{32} - (\nabla^{LC}_4 dV^\flat)_{13} \\
&- (\nabla^{LC}_1 dV^\flat)_{12}] + \eta_{I_{14}}[(-\nabla^{LC}_4 dV^\flat)_{14} - (\nabla^{LC}_3 dV^\flat)_{31} - (\nabla^{LC}_2 dV^\flat)_{21}] \\
&+ \frac{1}{2}[(\nabla^{LC}_2 dV^\flat)_{34} + (\nabla^{LC}_4 dV^\flat_I)_{41} - (\nabla^{LC}_3 dV^\flat)_{13}] \\
&= \eta_{I_{12}}[(\nabla^{LC}_2 dV^\flat)_{23} + (\nabla^{LC}_1 dV^\flat)_{13} - (\nabla^{LC}_4 dV^\flat)_{12}] + \eta_{I_{13}}[(-\nabla^{LC}_3 dV^\flat)_{32} - (\nabla^{LC}_4 dV^\flat)_{13} \\
&- (\nabla^{LC}_1 dV^\flat)_{12}] + \eta_{I_{14}}[(-\nabla^{LC}_4 dV^\flat)_{14} - (\nabla^{LC}_3 dV^\flat)_{31} - (\nabla^{LC}_2 dV^\flat)_{21}] \\
&+ \frac{1}{2}[(\nabla^{LC}_2 dV^\flat)_{34} + (\nabla^{LC}_2 dV^\flat)_{12}] \\
&= -2 \sum_{1\le i<j\le 4} \eta_{I_{ij}} (\nabla^{LC}_4 dV^\flat)_{ij},
\end{aligned}$$

where we used again that $\delta dV^\flat(e_j) = 0$, and so $\delta dV^\flat_I(e_j) = -\sum_{i=1}^4 \nabla^{LC}_{e_i} dV^\flat_I(e_i, e_j) = 0$.
Hence, for $i = 1, j = 2, l = 3$ in (4.5.11) we get that

$$\frac{1}{2}\sum_{1\le i<j\le 4} dV^\flat_{ij}(\nabla^{LC}_4 \eta_I)_{ij} - \frac{1}{2}\sum_{1\le i<j\le 4} \eta_{I_{ij}}(\nabla^{LC}_4 dV^\flat)_{ij} = 0.$$

The same result for the other values of k follows by analogues computations. Having showed equation (4.5.9), we may argue as follows. With respect to the dual frame $\{e^1, \dots, e^4\}$ we may write

$$dV^\flat = a_1(e^{12} - e^{34}) + a_2(e^{13} - e^{42}) + a_3(e^{14} - e^{23}),$$

and

$$\eta_I = b_1(e^{12} - e^{34}) + b_2(e^{13} - e^{42}) + b_3(e^{14} - e^{23}),$$

where a_i and b_i are smooth functions. Applying Lemma 4.A.1,

$$\begin{aligned}
0 =& R^{LC}(V, LV, e_1, e_2) = 2(b_2a_3 - b_3a_2), \\
0 =& R^{LC}(V, LV, e_1, e_3) = 2(b_3a_1 - b_1a_3), \\
0 =& R^{LC}(V, LV, e_1, e_4) = 2(b_1a_2 - b_2a_1).
\end{aligned} \tag{4.5.13}$$

By Proposition 4.4.1, we have that $dV^\flat$ has only isolated zeros. Let $Z := \{p \in M \mid dV^\flat(p) = 0\}$ and let p be any point of $M \setminus Z$. Then, either $\eta_I(p) = 0$ or $\eta_I(p) \neq 0$. If $\eta_I(p) \neq 0$ there exists an open neighborhood $\mathcal{U}$ of p such that η_I is never vanishing on $\mathcal{U}$, and if we take $\mathcal{V} := \mathcal{U} \cap M \setminus Z$, then $\mathcal{V}$ is an open neighborhood of p on which both η_I and $dV^\flat$ are never vanishing.

By looking more carefully at (4.5.13), we get that on $\mathcal{V}$ there exists a smooth function λ such that $dV^\flat = \lambda\eta_I$, with λ being never vanishing, otherwise $dV^\flat$

would have a zero on $\mathcal{V}$.

On $\mathcal{V}$ we apply (4.5.9) and we get that for any $k = 1, \ldots, 4$,

$$\begin{aligned} 0 =& \sum_{1\le i<j\le 4} \lambda(\eta_I)_{ij}(\nabla^{LC}_{e_k}\eta_I)_{ij} - (\eta_I)_{ij}(\nabla^{LC}_{e_k}\lambda\eta_I)_{ij} \\ =& \sum_{1\le i<j\le 4} \lambda(\eta_I)_{ij}(\nabla^{LC}_{e_k}\eta_I)_{ij} - (\eta_I)_{ij}e_k(\lambda)(\eta_I)_{ij} - \lambda(\eta_I)_{ij}(\nabla^{LC}_{e_k}\eta_I)_{ij} \\ =& - \sum_{1\le i<j\le 4} ((\eta_I)_{ij})^2 e_k(\lambda) \\ =& - \left\|\eta_I\right\|^2 e_k(\lambda), \end{aligned}$$

which forces λ to be constant on $\mathcal{V}$ in the horizontal directions. In particular

$$d\lambda = \lambda_1 V^\flat + \lambda_2 IV^\flat + \lambda_3 JV^\flat + \lambda_4 KV^\flat,$$

where λ_i are smooth functions. Differentiating again we get

$$\begin{aligned} 0 =& d\lambda_1 \wedge V^\flat + \lambda_1 dV^\flat + d\lambda_2 \wedge IV^\flat + \lambda_2 JV^\flat \wedge KV^\flat + \lambda_2\beta_I + d\lambda_3 \wedge JV^\flat \\ &+ \lambda_3 KV^\flat \wedge IV^\flat + \lambda_3\beta_J + d\lambda_4 \wedge KV^\flat + \lambda_4 IV^\flat \wedge JV^\flat + \lambda_4\beta_K, \end{aligned}$$

which forces

$$\lambda_1 dV^\flat + \lambda_2\beta_I + \lambda_3\beta_J + \lambda_4\beta_K = 0.$$

In particular, the self dual part and the anti self dual parts of the above expression have to be zero, which means that

$$\lambda_2\omega_I^T + \lambda_3\omega_J^T + \lambda_4\omega_K^T = 0 \text{ and } \lambda_1 dV^\flat + \lambda_2\eta_I + \lambda_3\eta_J + \lambda_4\eta_K = 0,$$

i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, since $dV^\flat$ is never vanishing on $\mathcal{V}$. In particular, λ is constant and non zero. By differentiating $dV^\flat = \lambda\eta_I$, we get

$$0 = \lambda d\eta_I = \lambda(-\eta_J \wedge KV^\flat + \eta_K \wedge JV^\flat),$$

which forces $\lambda = 0$, a contradiction. In fact, since η_I is non zero on $\mathcal{V}$, so are η_J and η_K.

Therefore, η_I must vanish on $M \setminus Z$ and so it must vanish on M, as Z is a set of isolated points (see Proposition 4.4.1) and η_I is smooth. This concludes the proof. q.e.d.

4.A Appendix: Curvature components

In this Appendix, we compute explicitly some components of the Bismut and the Levi-Civita curvatures.

Lemma 4.A.1. *Let (M^8, g, I, J, K) be a compact simply connected strong HKT manifold which is not hyperkähler. Let $\mathcal{F}$ be the distribution generated by V, IV, JV, KV and let $\mathcal{F}^\perp$ the orthogonal complement of $\mathcal{F}$ with respect to g. Then fixed any local horizontal orthonormal frame $\{e_1, \dots, e_4\}$ adapted to I, J, K and any $L \in \{I, J, K\}$ we get*

1. $R^B(V, LV, e_i, e_j) = dV^\flat((\iota_{e_i}\eta_L)^\sharp, e_j) + dV^\flat(e_i, (\iota_{e_j}\eta_L)^\sharp)$
2. $R^B(V, e_i, e_j, e_k) = -(\nabla^{LC}_{e_i} dV^\flat)(e_j, e_k)$
3. $R^B(LV, e_i, e_j, e_k) = -(\nabla^{LC}_{e_i} \eta_L)(e_j, e_k)$
4. $R^{LC}(V, LV, e_i, e_j) = \frac{1}{4} R^B(V, LV, e_i, e_j)$
5. $R^{LC}(V, e_i, e_j, e_k) = \frac{1}{2} R^B(V, e_i, e_j, e_k)$
6. $R^{LC}(LV, e_i, e_j, e_k) = \frac{1}{2} R^B(LV, e_i, e_j, e_k)$
7. $R^{LC}(V, LV, U, e_i) = 0$*, for any U vertical vector field.*

Proof. Since for any $L = I, J, K$ the Hermitian structure (g, L) is BHE, by [ABLS, Lemma 2.13] we have that for any $X, Y, Z \in \ker(V^\flat) \cap \ker(LV^\flat)$

$$R^B(V, X, Y, Z) = -\nabla^B_X dV^\flat(Y, Z), \quad R^B(LV, X, Y, Z) = -\nabla^B_X dLV^\flat(Y, Z) = -\nabla^B_X \eta_L(Y, Z). \tag{4.A.1}$$

1. We prove the first statement for $L = I$, since the others follow analogously. If we take $X = IV$, and Y, Z horizontal vector fields, we have that X, Y, Z lie for instance in $\ker(V^\flat) \cap \ker(JV^\flat)$, so we may apply the first identity in (4.A.1).

$$\begin{aligned} R^B(V, IV, Y, Z) =& -\nabla^B_{IV} dV^\flat(Y, Z) \\ =& dV^\flat((\iota_Y dIV^\flat)^\sharp, Z) + dV^\flat(Y, (\iota_Z dIV^\flat)^\sharp) \\ =& dV^\flat((\iota_Y \beta_I)^\sharp, Z) + dV^\flat(Y, (\iota_Z \beta_I)^\sharp) \\ =& dV^\flat((\iota_Y \eta_I)^\sharp, Z) + dV^\flat(Y, (\iota_Z \eta_I)^\sharp), \end{aligned}$$

where we used that $dV^\flat((\iota_Y \omega_I^T)^\sharp, Z) + dV^\flat(Y, (\iota_Z \omega_I^T)^\sharp) = 0$ (observe that we already did this computation in details in the proof of Theorem 4.5.2). Therefore for $Y = e_i$ and $Z = e_j$ we get that

$$R^B(V, IV, e_i, e_j) = dV^\flat((\iota_{e_i}\eta_I)^\sharp, e_j) + dV^\flat(e_i, (\iota_{e_j}\eta_I)^\sharp).$$

2. Let X, Y, Z be horizontal vector fields, which ie in $\ker(V^\flat) \cap \ker(LV^\flat)$, for any L. Then, by the first identity in (4.A.1),

$$R^B(V,X,Y,Z) = -\nabla^B_X dV^\flat(Y,Z) = -\nabla^{LC}_X dV^\flat(Y,Z),$$

as $H(X,Y,Z)=0$. By choosing $X=e_i$, $Y=e_j$ and $Z=e_k$ the statement follows.

3. As in the proof of the first statement we deal with the case $L=I$, as the other computations are analogues. As before, let X, Y, Z be horizontal vector fields, which ie in $\ker(V^\flat) \cap \ker(IV^\flat)$. Then by the second identity of (4.A.1) we have that

$$R^B(IV,X,Y,Z) = -\nabla^B_X dIV^\flat(Y,Z) = -\nabla^B_X \eta_I(Y,Z) = -\nabla^{LC}_X \eta_I(Y,Z).$$

Also in this case we chose $X=e_i$, $Y=e_j$ and $Z=e_k$.

It only remains to verify the identities for the Levi-Civita curvature. We will use the [IP, Formula 3.19] which relates the Levi-Civita curvature with the Bismut one. In particular, for any vector fields X, Y, Z, U we have that

$$\begin{aligned} R^{LC}(X,Y,Z,U) =& R^B(X,Y,Z,U) - \frac{1}{2}\nabla^B_X H(Y,Z,U) \\ &+ \frac{1}{2}\nabla^B_Y H(X,Z,U) - \frac{1}{2} g(H(X,Y),H(Z,U)) \\ &- \frac{1}{4} g(H(Y,Z),H(X,U)) + \frac{1}{4} g(H(X,Z),H(Y,U)), \end{aligned} \tag{4.A.2}$$

where the expression of the torsion H is given in (4.5.1).

4. Also in this case we deal with $L=I$. Applying (4.A.2):

$$\begin{aligned} R^{LC}(V,IV,e_i,e_j) =& R^B(V,IV,e_i,e_j) - \frac{1}{2}\nabla^B_V H(IV,e_i,e_j) + \frac{1}{2}\nabla^B_{IV} H(V,e_i,e_j) \\ &- \frac{1}{4} g(H(IV,e_i),H(V,e_j)) + \frac{1}{4} g(H(V,e_i),H(IV,e_j)) \\ =& dV^\flat((\iota_{e_i}\eta_I)^\sharp, e_j) + dV^\flat(e_i,(\iota_{e_j}\eta_I)^\sharp) - \frac{1}{2}\nabla^B_V \eta_I(e_i,e_j) \\ &+ \frac{1}{2}\nabla^B_{IV} dV^\flat(e_i,e_j) - \frac{1}{4} g((\iota_{e_i}\beta_I)^\sharp,(\iota_{e_j}dV^\flat)^\sharp) + \frac{1}{4} g((\iota_{e_j}\beta_I)^\sharp,(\iota_{e_i}dV^\flat)^\sharp) \\ =& dV^\flat((\iota_{e_i}\eta_I)^\sharp, e_j) + dV^\flat(e_i,(\iota_{e_j}\eta_I)^\sharp) - \frac{1}{2} dV^\flat((\iota_{e_i}\eta_I)^\sharp, e_j) \\ &- \frac{1}{2} dV^\flat(e_i,(\iota_{e_j}\eta_I)^\sharp) + \frac{1}{2}\eta_I((\iota_{e_i}dV^\flat)^\sharp, e_j) + \frac{1}{2}\eta_I(e_i,(\iota_{e_j}dV^\flat)^\sharp) \\ &- \frac{1}{4} g((\iota_{e_i}\eta_I)^\sharp,(\iota_{e_j}dV^\flat)^\sharp) + \frac{1}{4} g((\iota_{e_j}\eta_I)^\sharp,(\iota_{e_i}dV^\flat)^\sharp) \\ =& \frac{1}{4}(dV^\flat)_{ik}(\eta_I)_{jk} - \frac{1}{4}(dV^\flat)_{jk}(\eta_I)_{ik} \\ =& \frac{1}{4} R^B(V,IV,e_i,e_j). \end{aligned}$$

5. Exploiting the formula (4.A.2),

$$\begin{aligned} R^{LC}(V,e_i,e_j,e_k) =& R^B(V,e_i,e_j,e_k) - \frac{1}{2}\nabla^B_V H(e_i,e_j,e_k) \\ &+\frac{1}{2}\nabla^B_{e_i} H(V,e_j,e_k) - \frac{1}{2}g(H(V,e_i),H(e_j,e_k)) \\ &-\frac{1}{4}g(H(e_i,e_j),H(V,e_k)) + \frac{1}{4}g(H(V,e_j),H(e_i,e_k)) \\ =& -(\nabla^{LC}_{e_i} dV^\flat)(e_j,e_k) + \frac{1}{2}(\nabla^B_{e_i} dV^\flat)(e_j,e_k) \\ =& -\frac{1}{2}(\nabla^{LC}_{e_i} dV^\flat)(e_j,e_k) \\ =& \frac{1}{2}R^B(V,e_i,e_j,e_k). \end{aligned}$$

6. As in the previous cases we work with $L = I$ since the other cases follows similarly. By applying (4.A.2),

$$\begin{aligned} R^{LC}(IV,e_i,e_j,e_k) =& R^B(IV,e_i,e_j,e_k) - \frac{1}{2}\nabla^B_{IV} H(e_i,e_j,e_k) \\ &+\frac{1}{2}\nabla^B_{e_i} H(IV,e_j,e_k) - \frac{1}{2}g(H(IV,e_i),H(e_j,e_k)) \\ &-\frac{1}{4}g(H(e_i,e_j),H(IV,e_k)) + \frac{1}{4}g(H(IV,e_j),H(e_i,e_k)) \\ =& -(\nabla^{LC}_{e_i} \eta_I)(e_j,e_k) + \frac{1}{2}(\nabla^B_{e_i} \eta_I)(e_j,e_k) \\ =& -\frac{1}{2}(\nabla^{LC}_{e_i} \eta_I)(e_j,e_k) \\ =& \frac{1}{2}R^B(IV,e_i,e_j,e_k). \end{aligned}$$

7. We set again $L = I$. Without loss of generality, we may assume that $U \in \{V, IV, JV, KV\}$. In such a way, $\nabla^B U = 0$, $R^B(V, IV, U, X) = 0$ and $[U, V] = 0$. Using (4.A.2),

$$\begin{aligned} R^{LC}(V,IV,U,e_j) =& -\frac{1}{2}\nabla^B_V H(IV,U,e_j) + \frac{1}{2}\nabla^B_{IV} H(V,U,e_j) - \frac{1}{4}g(H(IV,U),H(V,e_j)) \\ =& -\frac{1}{2}\nabla^B_V \eta_I(U,e_j) + \frac{1}{2}\nabla^B_{IV} dV^\flat(U,e_j). \\ =& 0. \end{aligned}$$

q.e.d.

Chapter 5

The holonomy of the Obata connection on Joyce hypercomplex manifolds

In this chapter we investigate the holonomy of the Obata connection on Joyce hypercomplex manifolds.

We recall that in quaternionic dimension one, the only Joyce hypercomplex manifold is the Hopf surface $S^1 \times \mathrm{SU}(2)$. In this case, the Obata connection is flat with holonomy group $\mathbb{Z}$ [SV].

In quaternionic dimension two, the situation is significantly richer. Soldatenkov showed in [Sol] that the holonomy group of the Obata connection on $\mathrm{SU}(3)$ is *equal* to $\mathrm{GL}(2, \mathbb{H})$. The proof first involves showing that the Obata holonomy group acts irreducibly on the tangent space; the argument relies crucially on properties specific to dimension 8. Secondly, one appeals to the classification of irreducible holonomy groups of torsion-free affine connections in [MS2]. In the latter classification, only three candidates appear as possible subgroups of $\mathrm{GL}(n, \mathbb{H})$: $\mathrm{Sp}(n)$, $\mathrm{SL}(n, \mathbb{H})$ and $\mathrm{GL}(n, \mathbb{H})$, and it is simply a matter of eliminating the former two.

It was subsequently conjectured (see e.g. [SV]) that for any Joyce hypercomplex manifold in the list (1.4.4) (with the exception of the Hopf surface) the holonomy group of the Obata connection is always equal to the full group $\mathrm{GL}(n, \mathbb{H})$. We should point out that even if M is a product manifold in (1.4.4), the Joyce hypercomplex structure is not a product one and as such the Obata holonomy does not necessarily split.

Our first main result is the following theorem, which, in particular, disproves the conjectural picture recalled in the paper of Soldatenkov and Verbitsky [SV]:

Theorem 5.0.1. *Let M be a compact Lie group from the list* (1.4.4)*, except from* $\mathrm{SU}(2n+1)$*, and let ∇^{Ob} denote the Obata connection of a left invariant hypercomplex structure on M. Then the Obata holonomy is a proper subgroup of* $\mathrm{GL}(n, \mathbb{H})$*, i.e., it is strictly contained in* $\mathrm{GL}(n, \mathbb{H})$.

Recall that any left-invariant hypercomplex structure on M arises via Joyce's construction. Furthermore, on each manifold M, there exist in general infinitely many non-equivalent Joyce hypercomplex structures, and Theorem 5.0.1 holds for the Obata connection associated to *each* of these structures. The key result to proving the theorem is Lemma 5.1.6, which provides a sufficient condition for the existence of a ∇^{Ob}-parallel subbundle in terms of the Joyce decomposition of M. This condition depends on geometric properties of the simple Lie group G only and can be verified directly from its Dynkin diagram. Determining the holonomy, however, is still a rather difficult problem. In Section 5.2.2, we compute explicitly the holonomy algebra for $\mathbb{T}^2 \times \mathrm{Sp}(2)$. We show that it is an 11-dimensional subalgebra of the Lie algebra of quaternionic lower triangular matrices (Theorem 5.2.6). The argument relies on a left invariant version of the Ambrose-Singer [Al], together with Theorem 5.0.2 which guarantees that the holonomy algebra of the Obata connection of any left invariant hypercomplex structure lies in $\mathfrak{sl}(3, \mathbb{H})$.

The case of $\mathrm{SU}(2n+1)$ turns out to be rather peculiar and we treat it separately. We show that for $n \geq 2$ there always exist many Joyce hypercomplex structures on $\mathrm{SU}(2n+1)$ for which the Obata holonomy is *strictly* contained in $\mathrm{GL}(n(n+1), \mathbb{H})$ (Theorem 5.2.8). However, there appear to also exist hypercomplex structures for which the holonomy group is full; we provide such an example on $\mathrm{SU}(5)$ (Theorem 5.2.10). We conjecture that there exist many more such examples, in contrast to those in Theorem 5.2.8. The above dichotomy does not occur on $\mathrm{SU}(3)$ since the whole 1-parameter family of Joyce hypercomplex structures have full holonomy.

It was shown in [BDV, SV] that no Joyce hypercomplex manifold has holonomy group contained in $\mathrm{SL}(n, \mathbb{H})$. However, there are many examples for which the *restricted* holonomy lies in $\mathrm{SL}(n, \mathbb{H})$. We show that this occurs whenever the abelian summand $\mathfrak{b}$ in the Joyce decomposition of G vanishes (see Section 1.4.1). In that case, M belongs to the list:

$$\begin{aligned} &\mathbb{T}^k \times \mathrm{SO}(2k+1), \quad \mathbb{T}^{2k} \times \mathrm{SO}(4k), \quad \mathbb{T}^k \times \mathrm{Sp}(k), \\ &\mathbb{T}^7 \times \mathrm{E}_7, \quad \mathbb{T}^8 \times \mathrm{E}_8, \quad \mathbb{T}^4 \times \mathrm{F}_4, \quad \mathbb{T}^2 \times \mathrm{G}_2. \end{aligned} \tag{5.0.1}$$

We show that the corresponding Joyce hypercomplex manifold $\mathbb{T}^{2m-r} \times G$ admits left-invariant solutions to the twisted Calabi–Yau system introduced by [GRST]. We recall that a *twisted Calabi-Yau structure* on a complex manifold (M, I) is a Hermitian metric g together with a complex volume form Ψ of constant norm such that

$$\partial\bar{\partial}\omega = 0\,, \quad d\Psi = \theta \wedge \Psi\,, \quad d\theta = 0\,, \tag{5.0.2}$$

where ω is the fundamental form of g and $\theta = Id^*\omega$ is the Lee form of g. Twisted Calabi-Yau manifolds give solutions to the *twisted Hull-Strominger system* [GRST].

Known examples of twisted Calabi-Yau manifolds are given by Calabi-Yau manifolds (which can be thought of as trivial examples), quaternionic Hopf surfaces, $\mathrm{SU}(3)$-structures on $\mathbb{T}^3 \times S^3$, and $\mathrm{SU}(3)$-structures on $M \times \mathbb{R}$, where M is a Sasakian 5-manifold.

We prove the following:

Theorem 5.0.2 (Theorem 5.3.4)**.** *Let M belong to the list (5.0.1). Then any left invariant hypercomplex structure (I, J, K) on M has restricted Obata holonomy contained in $\mathrm{SL}(n, \mathbb{H})$, and (M, I) admits a left-invariant solution to the twisted Calabi-Yau system (5.0.2).*

The result in this chapter are contained in a preprint joint with U. Fowdar, G. Gentili and L. Vezzoni [BFGV2].

5.1 Technical Lemmas

In this section, we prove some technical lemmas, based on explicit computations. These lemmas will be crucial in the subsequent sections.

Consider the Lie group $\mathbb{T}^\ell \times G$, where G is a compact semisimple Lie group of rank r. Here, as in the Prelimaries, we write $\ell = 2m - r$, where m is the number of layers in the Joyce decomposition of $\mathfrak{g}$ and r the rank of G.

Lemma 5.1.1. *Let G be a compact semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. With respect to the decomposition (1.4.1) we have*

1. $[\mathfrak{b}, \mathfrak{f}_j] \subseteq \mathfrak{f}_j$ *for all* $j = 1, \ldots, m$;
2. $[\mathfrak{d}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_j$ *for* $i > j$;
3. $[\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_i$ *for* $i < j$;
4. $[\mathfrak{f}_i, \mathfrak{f}_i] \subseteq \mathfrak{b} \oplus \bigoplus_{k \geq i} \mathfrak{d}_k \oplus \bigoplus_{k \geq i} \mathfrak{f}_k$.

Proof. Let (I, J, K) be a Joyce hypercomplex structure on the Lie algebra $\ell\mathfrak{u}(1) \oplus \mathfrak{g}$ (see Section 1.4.2).

Set $h := g_E \oplus B$, where g_E is the Euclidean metric on $\ell\mathfrak{u}(1)$ and B is the negative of the Killing–Cartan form of $\mathfrak{g}$. Then h is a bi-invariant metric on $\ell\mathfrak{u}(1) \oplus \mathfrak{g}$ such that the decomposition

$$\ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{i=1}^{m} \mathfrak{d}_i \oplus \bigoplus_{i=1}^{m} \mathfrak{f}_i, \tag{5.1.1}$$

is h-orthogonal [GP]. Observe that we are not assuming h to be compatible with the hypercomplex structure (I, J, K).

The proof of the lemma is a direct computation, using the bi-invariant metric h on $\ell\mathfrak{u}(1) \oplus \mathfrak{g}$, together with repeated application of the definition of the hypercomplex structure (I, J, K) (see Section 1.4.2) and the properties (J1)–(J4) (see Section 1.4.1).

We now prove statement **1**. Fix $b \in \mathfrak{b}$ and $f_j \in \mathfrak{f}_j$. Then

$$h([b, f_j], X) = -h(f_j, [b, X]) = 0$$

for all $X \in \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{k=1}^{m} \mathfrak{d}_k$ using (J1). On the other hand, if $X \in \mathfrak{f}_k$ with $k < j$, we compute:

$$\begin{aligned} h([b, f_j], X) &= -h([b, f_j], I^2 X) \\ &= -h([b, f_j], [e_2^k, IX]) \\ &= -h([[b, f_j], e_2^k], IX) \\ &= -h([[b, e_2^k], f_j], IX) - h([b, [f_j, e_2^k]], IX) = 0, \end{aligned}$$

where we used (J1) and (J3). Similarly, if $X \in \mathfrak{f}_k$ with $k > j$, we have:

$$\begin{aligned} h([b, f_j], X) &= -h([b, I^2 f_j], X) \\ &= -h([b, [e_2^j, I f_j]], X) \\ &= -h([[b, e_2^j], I f_j], X) - h([e_2^j, [b, I f_j]], X) \\ &= -h([e_2^j, [b, I f_j]], X) \\ &= \quad h([b, I f_j], [e_2^j, X]) = 0, \end{aligned}$$

again by (J1) and (J3). This concludes the proof of statement **1**.

We now prove statement **3**. Fix $f_i \in \mathfrak{f}_i$ and $f_j \in \mathfrak{f}_j$. Let $X \in \mathfrak{g}$. Then:

$$\begin{aligned} h([f_i, f_j], X) &= -h([I^2 f_i, f_j], X) \\ &= -h([[e_2^i, I f_i], f_j], X) \\ &= -h([[e_2^i, f_j], I f_i], X) - h([e_2^i, [I f_i, f_j]], X) \\ &= \quad h([[I f_i, f_j], e_2^i], X) \\ &= \quad h([I f_i, f_j], [e_2^i, X]), \end{aligned} \tag{5.1.2}$$

where we used (J3) in the fourth line. It follows from (5.1.2) that $h([f_i, f_j], X) = 0$ whenever $X \in \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{k \neq i} \mathfrak{d}_k \oplus \bigoplus_{l > i} \mathfrak{f}_l$. If instead $X \in \mathfrak{d}_i$, then

$$h([f_i, f_j], X) = h(f_i, [f_j, X]) = 0,$$

again by (J3). Now suppose $X \in \mathfrak{f}_l$ with $l < i$. Then:

$$\begin{aligned} h([f_i, f_j], X) &= -h([f_i, f_j], I^2 X) \\ &= -h([f_i, f_j], [e_2^l, IX]) \\ &= -h([[f_i, f_j], e_2^l], IX) \\ &= -h([[f_i, e_2^l], f_j], IX) - h([f_i, [f_j, e_2^l]], IX) = 0, \end{aligned}$$

where the final equality follows from (J3) and the fact that $l < i < j$. Hence, $[f_i, f_j]$ can only have non-zero components in $\mathfrak{f}_i$, completing the proof of statement **3**.

We now prove statement **2** using statement **3**. Let $d_i \in \mathfrak{d}_i$ and $f_j \in \mathfrak{f}_j$. Then, for all $X \in \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{k=1}^m \mathfrak{d}_k$, we have:

$$h([d_i, f_j], X) = -h(f_j, [d_i, X]) = 0,$$

where we used (J1), (J2), the inclusion $[\mathfrak{d}_i, \mathfrak{d}_i] \subset \mathfrak{d}_i$, and the h-orthogonality of the decomposition (5.1.1). On the other hand, by statement **3**, we also get:

$$h([d_i, f_j], X) = h(d_i, [f_j, X]) = 0$$

for all $X \in \mathfrak{f}_l$ with $l \neq j$, where we used again the h-orthogonality of the decomposition (5.1.1). This proves statement **2**.

Finally, to prove statement **4**, let $f_i, f_i' \in \mathfrak{f}_i$. Then:

$$h([f_i, f_i'], X) = h(f_i, [f_i', X]) = 0$$

if $X \in \ell\mathfrak{u}(1)$ or $X \in \mathfrak{d}_k$ with $k < i$. On the other hand, if $X \in \mathfrak{f}_l$ with $l < i$, then by statement **3**, $[f_i', X] \in \mathfrak{f}_l$, and hence:

$$h([f_i, f_i'], X) = h(f_i, [f_i', X]) = 0,$$

where the last equality follows by the h-orthogonality of the decomposition (5.1.1).
q.e.d.

Lemma 5.1.2. *Let* $(\mathbb{T}^\ell \times G, I, J, K)$ *be a Joyce hypercomplex manifold. Every vector* $b \in \ell\mathfrak{u}(1) \oplus \mathfrak{b}$ *is hyper-holomorphic.*

Proof. The claim is obvious for $b \in \ell\mathfrak{u}(1)$ because it lies in the center of the Lie algebra. Thus, we only need to prove it for $b \in \mathfrak{b}$. We show that b is I-holomorphic, the proof is analogous for J and K. We need to check that

$$0 = (\mathcal{L}_b I)X = [b, IX] - I[b, X]$$

for all $X \in \mathfrak{g}$. The assertion is clear if $X \in \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{j=1}^m \mathfrak{d}_j$, so let us assume $X \in \mathfrak{f}_k$. By assertion **1** of Lemma 5.1.1 we have $[b, X] \in \mathfrak{f}_k$. Therefore

$$[b, IX] - I[b, X] = [b, [e_2^k, X]] - [e_2^k, [b, X]] = [[e_2^k, b], X] = 0\,,$$

concluding the proof. q.e.d.

Remark 5.1.3. Lemma 5.1.2 generalizes the result in [Sol, Proposition 4.1 (1)], where the result was proven for $\mathrm{SU}(3)$.

Using Lemma 5.1.1, we explicitly compute the covariant derivative of e_1^j with respect to the Obata connection for $j = 1, \ldots, m$, which will be useful later.

Lemma 5.1.4. *Let* $(\mathbb{T}^\ell \times G, I, J, K)$ *be a Joyce hypercomplex manifold. Then*

$$\nabla_X^{Ob} e_1^i = \begin{cases} -X & \textit{if } X \in \mathfrak{h}_i \oplus \mathfrak{f}_i\,, \\ 0 & \textit{otherwise}\,, \end{cases}$$

where $\mathfrak{h}_i = \langle e_1^i \rangle_{\mathbb{R}} \oplus \mathfrak{d}_i$.

Proof. We recall the formula for the Obata connection derived in [Sol, (2.5)]:

$$\nabla_X^{Ob} Y = \frac{1}{2}([X, Y] + I[IX, Y] - J[X, JY] + K[IX, JY]). \tag{5.1.3}$$

Using the latter we have

$$\begin{aligned} \nabla_X^{Ob} e_1^i &= \frac{1}{2}([X, e_1^i] + I[IX, e_1^i] - J[X, Je_1^i] + K[IX, Je_1^i]) \\ &= \frac{1}{2}(-J[X, e_3^i] + K[IX, e_3^i])\,, \end{aligned}$$

where the last equality follows from Lemma 5.1.2.

If $X \in \mathfrak{h}_k$, for $k \neq i$, then by (J1) and (J2) we clearly have $\nabla_X^{Ob} e_1^i = 0$. If $X \in \mathfrak{h}_i$ then one can easily verify that

$$\nabla_X^{Ob} e_1^i = \frac{1}{2}(-J[X, e_3^i] + K[IX, e_3^i]) = -X\,.$$

Now, assume $X \in \mathfrak{f}_k$. If $k > i$, then $\nabla_X^{Ob} e_1^i = 0$ by (J3). If $k < i$ using (J2) we get

$$K[IX, e_3^i] = K[[e_2^k, X], e_3^i] = K[[e_2^k, e_3^i], X] + K[e_2^k, [X, e_3^i]] = K[e_2^k, [X, e_3^i]]\,.$$

From statement **2** of Lemma 5.1.1, $[X, e_3^i] \in \mathfrak{f}_k$ and so

$$\nabla_X^{Ob} e_1^i = \frac{1}{2}(-J[X, e_3^i] + K[e_2^k, [X, e_3^i]]) = \frac{1}{2}(-J[X, e_3^i] + KI[X, e_3^i]]) = 0\,.$$

Finally, if $k = i$ then

$$\nabla^{Ob}_X e^i_1 = \frac{1}{2}(J^2X - KJIX) = -X\,,$$

as we wanted to show. q.e.d.

Corollary 5.1.5. *On every Joyce hypercomplex manifold there exists a unique vector field $\mathcal{E}$ such that $\nabla^{Ob}\mathcal{E} = \mathrm{Id}$. In particular, the Obata connection can only preserve tensors of type (k, k).*

Proof. Let $\mathcal{E} := -\sum_{j=1}^m e^j_1$, then clearly $\nabla^{Ob}\mathcal{E} = \mathrm{Id}$ (see Lemma 5.1.4). The uniqueness of $\mathcal{E}$, as well as the fact that the Obata connection can preserve only tensors of type (k, k), has been observed in [Sol, Remark 4.3]. q.e.d.

Lemma 5.1.6. *Let M be a Joyce hypercomplex manifold of quaternionic dimension $n > 1$. If there exists an index $i = 1, \ldots, m$ such that $\mathfrak{f}_i = 0$ in the Joyce decomposition of the semisimple factor of M, then the holonomy of the Obata connection is strictly contained in $\mathrm{GL}(n, \mathbb{H})$.*

Proof. As an immediate consequence of Lemma 5.1.4, we obtain that if there exists an index $i = 1, \ldots, m$ such that $\mathfrak{f}_i = 0$, then the subspace $\mathfrak{h}_i = \langle e^i_1 \rangle_{\mathbb{H}}$ is preserved by the Obata connection.

In particular, since $n > 1$, $\mathrm{Hol}(\nabla^{Ob}) \subsetneq \mathrm{GL}(n, \mathbb{H})$. Indeed, let H_i denote the corresponding left-invariant and parallel* subbundle. Since H_i is I, J, K invariant and parallel, it is preserved under parallel transport. It follows that the holonomy group cannot act transitively on $\mathbb{H}^n \setminus \{0\}$, and hence cannot coincide with $\mathrm{GL}(n, \mathbb{H})$. q.e.d.

5.2 Holonomy reduction

In this section we prove Theorem 5.0.1, compute explicitly the holonomy of the Obata connection on $\mathbb{T}^2 \times \mathrm{Sp}(2)$, and analyse the case of $\mathrm{SU}(2n + 1)$.

5.2.1 Proof of Theorem 5.0.1

Recall that every left invariant hypercomplex structure on a compact Lie group corresponds to a Joyce one. Moreover, from [SV] we also know that the Hopf surface has holonomy group $\mathbb{Z}$. Theorem 5.0.1 is an immediate consequence of these facts together with Theorem 5.2.1 below.

*A subbundle $H \subset \mathrm{T}M$ is called *parallel* if it is preserved by the connection, i.e., $\nabla^{Ob}_X Y \in \Gamma(H)$ for all vector fields X and Y with $Y \in \Gamma(H)$.

Theorem 5.2.1. *Let M be a compact Lie group from the list (1.4.4), except from $S^1 \times \mathrm{SU}(2)$ and $\mathrm{SU}(2n+1)$, and let ∇^{Ob} denote the Obata connection of a Joyce hypercomplex structure on M. Then there exists a left-invariant subbundle of TM preserved by ∇^{Ob}. In particular,*

$$\mathrm{Hol}(\nabla^{Ob}) \subsetneq \mathrm{GL}(n, \mathbb{H}).$$

Proof. Let $M = \mathbb{T}^\ell \times G$ be as in the assumptions of Theorem 5.2.1, and let (I, J, K) be a Joyce hypercomplex structure on M (see Section 1.4.2). As observed in the preliminaries, the Lie algebra of M admits the following decomposition:

$$\ell\mathfrak{u}(1) \oplus \mathfrak{g} = \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{i=1}^{m} \mathfrak{d}_i \oplus \bigoplus_{i=1}^{m} \mathfrak{f}_i = \bigoplus_{i=1}^{m} \mathfrak{h}_i \oplus \bigoplus_{i=1}^{m} \mathfrak{f}_i,$$

where we set

$$\mathfrak{h}_i := \mathbb{R} \oplus \mathfrak{d}_i = \langle e_1^i, e_2^i, e_3^i, e_4^i \rangle_{\mathbb{R}}.$$

We point out that, since $S^1 \times \mathrm{SU}(2)$ has been excluded, any manifold M considered here has quaternionic dimension strictly greater than 1.

We apply Lemma 5.1.6 to prove that, for any M in the theorem, there exists a left invariant subbundle preserved by the Obata connection. Note that the condition established in Lemma 5.1.6 depends only on the Joyce decomposition of the Lie group G and not on the choice of the Joyce hypercomplex structure. Therefore, it suffices to check that the Joyce decomposition of the Lie algebras of the following compact simple Lie groups contains a non trivial $\mathfrak{d}_i$ with corresponding trivial $\mathfrak{f}_i$:

$$\begin{aligned} &\mathrm{SU}(2k)\ (k \geq 2),\ \ \mathrm{SO}(2k+1)\ (k \geq 3),\ \ \mathrm{SO}(4k)\ (k \geq 2),\\ &\mathrm{SO}(4k+2)\ (k \geq 2),\ \ \mathrm{Sp}(k)\ (k \geq 2),\\ &\mathrm{E}_6,\ \ \mathrm{E}_7,\ \ \mathrm{E}_8,\ \ \mathrm{F}_4,\ \ \mathrm{G}_2. \end{aligned}$$

The restrictions on k are imposed to avoid Lie algebra isomorphisms and to exclude the case of $\mathrm{SU}(2)$. To prove this, we describe below two iterative reduction procedures, one using the Wolf spaces (see Section 1.4.1) and the other using Dynkin diagrams, following [OP].

- G_2: $\frac{\mathrm{G}_2}{\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)}$ is a Wolf space of quaternionic dimension 2. By Lemma 1.4.1 the Lie algebra $\mathfrak{g}_2$ has the following decomposition

$$\mathfrak{g}_2 = \mathfrak{d}_1 \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_1.$$

  Therefore, $\mathfrak{d}_2 \neq 0$ but $\mathfrak{f}_2 = 0$.

- $\mathrm{Sp}(k)$: We prove the result by induction on k. For $k = 2$, the space $\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)}$ is a

Wolf space of quaternionic dimension 1. By Lemma 1.4.1, the Lie algebra $\mathfrak{sp}(2)$ admits the decomposition

$$\mathfrak{sp}(2) = \mathfrak{d}_1 \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_1.$$

Therefore, $\mathfrak{d}_2 \neq 0$ but $\mathfrak{f}_2 = 0$. By induction, assume that the Joyce decomposition of $\mathrm{Sp}(k)$ has a trivial summand $\mathfrak{f}_i$ for some i. Since the quotient $\frac{\mathrm{Sp}(k+1)}{\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)}$ is a Wolf space of quaternionic dimension k, we have

$$\mathfrak{sp}(k+1) = \mathfrak{sp}(k) \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1.$$

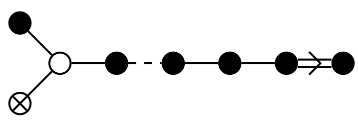
By the inductive hypothesis and Lemma 1.4.1, it follows that there exists an index i such that $\mathfrak{f}_i$ is trivial in the Joyce decomposition of $\mathfrak{sp}(k)$, which implies that $\mathfrak{f}_{i+1}$ is trivial in the Joyce decomposition of $\mathfrak{sp}(k+1)$.

- F_4: $\frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}$ is a Wolf space of quaternionic dimension 7 and, accordingly,

$$\mathrm{Lie}(F_4) = \mathfrak{sp}(3) \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1.$$

Since the Joyce decomposition of $\mathfrak{sp}(3)$ has a trivial $\mathfrak{f}_i$ summand, $\mathfrak{f}_{i+1}$ is trivial in the Joyce decomposition of $\mathrm{Lie}(F_4)$, again by Lemma 1.4.1 †.

- $\mathrm{SO}(2k+1)$: the Joyce decomposition of $\mathfrak{so}(2k+1)$ can be most easily described using extended Dynkin diagrams [OP]. The extended Dynkin diagram of $\mathrm{B}_k = \mathfrak{so}(2k+1,\mathbb{C})$ is given by



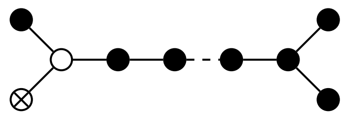
where the coloured and uncoloured dots correspond to the usual simple roots of B_k and the additional crossed dot corresponds to the maximal root.

At the first level of the Joyce decomposition we obtain $\mathfrak{g} = \mathfrak{b}_1 \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1$, where $\mathfrak{b}_1$ is the centraliser of the $\mathfrak{su}(2) = \mathfrak{d}_1$ generated by the maximal root (see Section 1.4.1). The coloured sub-diagram is precisely isomorphic to the Dynkin diagram of $\mathfrak{b}_{1\mathbb{C}}$.

In this case, the sub-diagram is disconnected: one component corresponds to $\mathfrak{so}(2k-3)$ and the isolated black vertex to a $\mathfrak{d}_2$ with trivial $\mathfrak{f}_2$ (see [OP] for the details).

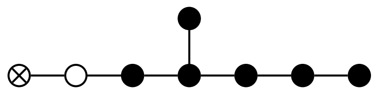
† We write $\mathrm{Lie}(F_4)$ instead of $\mathfrak{f}_4$ to prevent ambiguity with the notation used in the Joyce decomposition.

- $\mathrm{SO}(4k)$, $\mathrm{SO}(4k+2)$: these cases can be treated very similarly to the previous one, and we will treat them together. The extended Dynkin diagram associated to $\mathrm{D}_{2k} = \mathfrak{so}(4k,\mathbb{C})$ ($\mathrm{D}_{2k+1} = \mathfrak{so}(4k+2,\mathbb{C})$) is given by



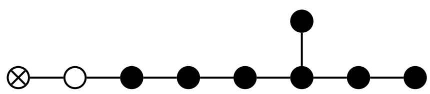
  At the first level of Joyce decomposition we have that the centralizer $\mathfrak{b}_1$ of the $\mathfrak{su}(2)$-copy corresponding to the maximal root is disconnected. One component corresponds to $\mathfrak{so}(4k-4,\mathbb{C})$ ($\mathfrak{so}(4k-2,\mathbb{C})$) and the isolated black vertex to a $\mathfrak{d}_2$ with trivial $\mathfrak{f}_2$.

- E_7: the extended Dynkin diagram of E_7 is the following



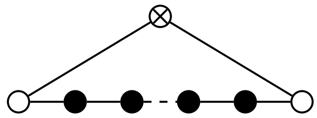
  The colored sub-diagram corresponds to D_6, and, therefore, the centralizer of the $\mathfrak{su}(2)$-copy corresponding to the maximal root is isomorphic to $\mathfrak{so}(12)$. Since the Joyce decomposition of $\mathfrak{so}(12)$ contains a summand $\mathfrak{d}_i$ with corresponding trivial $\mathfrak{f}_i$, the same holds for E_7.

- E_8: this case can be treated similarly to the previous one. The extended Dynkin diagram of E_8 is given by



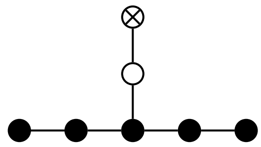
  Since the colored Dynkin diagram of E_8 corresponds to that of E_7, then E_7 is the centralizer of the $\mathfrak{su}(2)$-copy corresponding to the maximal root. As already observed, the Joyce decomposition of E_7 contains a summand $\mathfrak{d}_i$ with trivial $\mathfrak{f}_i$, which implies that the same property holds for E_8.

- $\mathrm{SU}(2k)$: For $\mathrm{SU}(2k)$, the procedure is slightly different. In each of the previous cases, the subalgebra $\mathfrak{b}_1$ was either simple or of the form $\mathfrak{su}(2)\oplus\mathfrak{g}'$, with $\mathfrak{g}'$ simple. In the present case, considering the extended Dynkin diagram of $\mathfrak{sl}(2k,\mathbb{C}) = \mathrm{A}_{2k-1}$,



we have to remove two vertices, ending up with the diagram of A_{2k-3}. The reason is that the centraliser of the maximal root in $\mathfrak{su}(2k)$ is given by $\mathfrak{u}(1) \oplus \mathfrak{su}(2k-2)$; therefore, two simple roots must be removed, rather than just one as in the previous cases. Repeating the process for $\mathfrak{su}(2k-2)$ eventually we will arrive to $\mathfrak{su}(2)$, which yields a trivial $\mathfrak{f}_i$.

- E_6: for E_6, the associated Dynkin diagram is given by



and the colored sub-diagram corresponds to A_5. Therefore, the centralizer of the $\mathfrak{su}(2)$-copy corresponding to the maximal root is $\mathfrak{su}(6)$. By the previous argument, $\mathfrak{su}(6)$ has a $\mathfrak{d}_j$ summand with corresponding trivial $\mathfrak{f}_j$. q.e.d.

In Lemma 5.1.6 we showed that for each index $j = 1, \dots, m$ such that $\mathfrak{f}_j$ is trivial, there exists a left-invariant parallel subbundle $H_j \subset TM$, corresponding to the subalgebra $\mathfrak{h}_j = \langle e_1^j \rangle_{\mathbb{H}}$. Therefore, the number of trivial $\mathfrak{f}_j$ summands in the Joyce decomposition of $\mathfrak{g}$ provides an indication of the extent to which the holonomy of the Obata connection reduces. This count is computed inductively using the procedure described in Theorem 5.2.1, and the results are summarized in Table 5.2.1. Essentially, in Theorem 5.2.1 we proved that for any Joyce hypercomplex manifold $(M = \mathbb{T}^\ell \times G, I, J, K)$ appearing in the list (1.4.4), with the exception of $\mathrm{SU}(2n+1)$, the number of trivial $\mathfrak{f}_j$ in the Joyce decomposition of $\mathfrak{g}$ is at least 1.

Table 5.1: Trivial $\mathfrak{f}_j$ summands in the Joyce decomposition of $\mathfrak{g}$

G	**# trivial $\mathfrak{f}_j$ summand**
G_2	1
F_4	1
E_6	1
E_7	4
E_8	4
$\mathrm{Sp}(k)$	1
$\mathrm{SO}(2k{+}1)$	$\lceil \frac{k}{2} \rceil$
$\mathrm{SO}(4k)$	$k{+}1$
$\mathrm{SO}(4k{+}2)$	k
$\mathrm{SU}(2k)$	1
$\mathrm{SU}(2k{+}1)$	0

As a preliminary to the next corollary, we recall an adaptation of the Ambrose–Singer theorem to the left-invariant setting:

Theorem 5.2.2 (Alekseevskii [Al, Proposition 2.1]). *Let ∇ be a left-invariant linear connection on the Lie group G, and let $\mathfrak{g}$ denote the Lie algebra of G. Then the holonomy algebra $\mathfrak{hol}(\nabla)$, based at the identity element $e \in G$ is the smallest subalgebra of $\mathfrak{gl}(\mathfrak{g})$ containing the curvature endomorphisms $R(x,y)$ for any $x, y \in \mathfrak{g}$, and closed under commutators with the left multiplication operators $\nabla_x : \mathfrak{g} \to \mathfrak{g}$.*

Corollary 5.2.3. *Let $M = \mathbb{T}^\ell \times G$ be in the list (1.4.4), with the exception of $S^1 \times \mathrm{SU}(2)$ and $\mathrm{SU}(2n+1)$. Then, for any Joyce hypercomplex structure (I, J, K) on M the holonomy algebra of the Obata connection satisfies the following inclusion*

$$\mathfrak{hol}(\nabla^{Ob}) \subseteq \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 & a_{1,j+1} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & 0 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{jj} & a_{j,j+1} & \cdots & a_{jn} \\ 0 & 0 & \cdots & 0 & a_{j+1,j+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{H}) \right\}, \qquad (5.2.1)$$

where j is the number of trivial $\mathfrak{f}_i$ summands in the Joyce decomposition of $\mathfrak{g}$.

Proof. The proof follows by Lemma 5.1.6, the proof of Theorem 5.2.1 and by Theorem 5.2.2. q.e.d.

Remark 5.2.4. Theorem 5.2.1 can be slightly generalized to the general case of G semisimple. Let $M = \mathbb{T}^\ell \times G$ with G semisimple and

$$G \neq \prod_i \mathrm{SU}(2k_i + 1).$$

Then for any Joyce hypercomplex structure on M the holonomy of the associated Obata connection is strictly contained in $\mathrm{GL}(n, \mathbb{H})$.

5.2.2 The holonomy of $\mathbb{T}^2 \times \mathrm{Sp}(2)$

In this subsection we compute explicitly the holonomy algebra of the Obata connection on $\mathbb{T}^2 \times \mathrm{Sp}(2)$ for an arbitrary Joyce hypercomplex structure. This is the lowest possible dimensional case covered in Theorem 5.2.1.

We start writing down the Joyce decomposition of $\mathfrak{sp}(2)$. The quaternionic matrices generating $\mathfrak{sp}(2)$ are given by:

$$e_2^1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3^1 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad e_4^1 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$
$$e_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad e_3^2 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad e_4^2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$
$$f_1^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f_2^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad f_3^1 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad f_4^1 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

We denote by φ_j^i, ψ_l^k the associated dual basis i.e. $\varphi_j^i(e_s^r) = \delta_{ir}\delta_{js}$, $\psi_l^k(f_s^r) = \delta_{kr}\delta_{ls}$. The structure equations can be computed as follows:

$$\begin{aligned}
d\varphi_2^1 &= -2\varphi_3^1 \wedge \varphi_4^1 - 2\sigma_1 \\
d\varphi_3^1 &= -2\varphi_4^1 \wedge \varphi_2^1 - 2\sigma_2 \\
d\varphi_4^1 &= -2\varphi_2^1 \wedge \varphi_3^1 - 2\sigma_3 \\
d\varphi_2^2 &= -2\varphi_3^2 \wedge \varphi_4^2 + 2\bar{\sigma}_1 \\
d\varphi_3^2 &= -2\varphi_4^2 \wedge \varphi_2^2 + 2\bar{\sigma}_2 \\
d\varphi_4^2 &= -2\varphi_2^2 \wedge \varphi_3^2 + 2\bar{\sigma}_3 \\
d\psi_1^1 &= \ \ \varphi_2^1 \wedge \psi_2^1 + \varphi_3^1 \wedge \psi_3^1 + \varphi_4^1 \wedge \psi_4^1 - \varphi_2^2 \wedge \psi_2^1 - \varphi_3^2 \wedge \psi_3^1 - \varphi_4^2 \wedge \psi_4^1 \\
d\psi_2^1 &= -\varphi_2^1 \wedge \psi_1^1 - \varphi_3^1 \wedge \psi_4^1 + \varphi_4^1 \wedge \psi_3^1 + \varphi_2^2 \wedge \psi_1^1 + \varphi_4^2 \wedge \psi_3^1 - \varphi_3^2 \wedge \psi_4^1 \\
d\psi_3^1 &= \ \ \varphi_2^1 \wedge \psi_4^1 - \varphi_3^1 \wedge \psi_1^1 - \varphi_4^1 \wedge \psi_2^1 + \varphi_3^2 \wedge \psi_1^1 - \varphi_4^2 \wedge \psi_2^1 + \varphi_2^2 \wedge \psi_4^1 \\
d\psi_4^1 &= -\varphi_2^1 \wedge \psi_3^1 + \varphi_3^1 \wedge \psi_2^1 - \varphi_4^1 \wedge \psi_1^1 + \varphi_4^2 \wedge \psi_1^1 + \varphi_3^2 \wedge \psi_2^1 - \varphi_2^2 \wedge \psi_3^1
\end{aligned} \tag{5.2.2}$$

where

$$\begin{aligned}
\sigma_1 &:= \psi_1^1 \wedge \psi_2^1 + \psi_3^1 \wedge \psi_4^1, \quad \sigma_2 := \psi_1^1 \wedge \psi_3^1 + \psi_4^1 \wedge \psi_2^1, \quad \sigma_3 := \psi_1^1 \wedge \psi_4^1 + \psi_2^1 \wedge \psi_3^1, \\
\bar{\sigma}_1 &:= \psi_1^1 \wedge \psi_2^1 - \psi_3^1 \wedge \psi_4^1, \quad \bar{\sigma}_2 := \psi_1^1 \wedge \psi_3^1 - \psi_4^1 \wedge \psi_2^1, \quad \bar{\sigma}_3 := \psi_1^1 \wedge \psi_4^1 - \psi_2^1 \wedge \psi_3^1.
\end{aligned}$$

We fix any basis $\{e_1^1, e_1^2\}$ for $2\mathfrak{u}(1) \cong \mathbb{R}^2$. The Joyce decomposition is then given by

$$\begin{aligned}
2\mathfrak{u}(1) &= \langle e_1^1, e_1^2 \rangle, \\
\mathfrak{d}_1 &= \langle e_2^1, e_3^1, e_4^1 \rangle, \\
\mathfrak{d}_2 &= \langle e_2^2, e_3^2, e_4^2 \rangle, \\
\mathfrak{f}_1 &= \langle f_1^1, f_2^1, f_3^1, f_4^1 \rangle.
\end{aligned} \tag{5.2.3}$$

Let (I, J, K) be the Joyce hypercomplex structure corresponding to the decomposition (5.2.3) (see Section 1.4.2).

We may rewrite the decomposition above as

$$2\mathfrak{u}(1) \oplus \mathfrak{sp}(2) = \left(\langle e_1^1 \rangle \oplus \mathfrak{d}_1\right) \oplus \mathfrak{f}_1 \oplus \left(\langle e_1^2 \rangle \oplus \mathfrak{d}_2\right) = \mathfrak{h}_1 \oplus \mathfrak{f}_1 \oplus \mathfrak{h}_2, \tag{5.2.4}$$

and by Theorem 5.2.1 the left-invariant subbundle generated by $\mathfrak{h}_2$ via left translations is Obata parallel.

Let us now determine the holonomy algebra of the Obata connection ∇^{Ob} associated to the hypercomplex structure (I, J, K). By remark 1.5.8, for any other Joyce hypercomplex structure $(\tilde{I}, \tilde{J}, \tilde{K})$ with Obata connection $\tilde{\nabla}^{Ob}$ one has that $\mathfrak{hol}(\nabla^{Ob}) \cong \mathfrak{hol}(\tilde{\nabla}^{Ob})$.

We should point out that a classification of irreducible holonomies of torsion-free affine connections is known, see [MS2]. This is for instance used in [Sol] to determine the holonomy of the Obata connection on $\mathrm{SU}(3)$. However, in our case we cannot appeal to this classification since our connection does not act irreducibly: to determine the holonomy algebra we have to compute it directly. With respect to the global co-framing $\{\varphi_j^i, \psi_l^k\}$ we identify ∇^{Ob} with the connection 1-form

$$\Theta = \left(\begin{array}{cccc|cccc|cccc}
-\varphi_1^1 & \varphi_2^1 & \varphi_3^1 & \varphi_4^1 & \psi_1^1 & \psi_2^1 & \psi_3^1 & \psi_4^1 & 0 & 0 & 0 & 0 \\
-\varphi_2^1 & -\varphi_1^1 & -\varphi_4^1 & \varphi_3^1 & -\psi_2^1 & \psi_1^1 & -\psi_4^1 & \psi_3^1 & 0 & 0 & 0 & 0 \\
-\varphi_3^1 & \varphi_4^1 & -\varphi_1^1 & -\varphi_2^1 & -\psi_3^1 & \psi_4^1 & \psi_1^1 & -\psi_2^1 & 0 & 0 & 0 & 0 \\
-\varphi_4^1 & -\varphi_3^1 & \varphi_2^1 & -\varphi_1^1 & -\psi_4^1 & -\psi_3^1 & \psi_2^1 & \psi_1^1 & 0 & 0 & 0 & 0 \\
\hline
-\psi_1^1 & \psi_2^1 & \psi_3^1 & \psi_4^1 & -\varphi_1^1 & \varphi_2^2 & \varphi_3^2 & \varphi_4^2 & 0 & 0 & 0 & 0 \\
-\psi_2^1 & -\psi_1^1 & -\psi_4^1 & \psi_3^1 & -\varphi_2^2 & -\varphi_1^1 & -\varphi_4^2 & \varphi_3^2 & 0 & 0 & 0 & 0 \\
-\psi_3^1 & \psi_4^1 & -\psi_1^1 & -\psi_2^1 & -\varphi_3^2 & \varphi_4^2 & -\varphi_1^1 & -\varphi_2^2 & 0 & 0 & 0 & 0 \\
-\psi_4^1 & -\psi_3^1 & \psi_2^1 & -\psi_1^1 & -\varphi_4^2 & -\varphi_3^2 & \varphi_2^2 & -\varphi_1^1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -3\psi_1^1 & \psi_2^1 & \psi_3^1 & \psi_4^1 & -\varphi_1^2 & \varphi_2^2 & \varphi_3^2 & \varphi_4^2 \\
0 & 0 & 0 & 0 & -\psi_2^1 & -3\psi_1^1 & -\psi_4^1 & \psi_3^1 & -\varphi_2^2 & -\varphi_1^2 & -\varphi_4^2 & \varphi_3^2 \\
0 & 0 & 0 & 0 & -\psi_3^1 & \psi_4^1 & -3\psi_1^1 & -\psi_2^1 & -\varphi_3^2 & \varphi_4^2 & -\varphi_1^2 & -\varphi_2^2 \\
0 & 0 & 0 & 0 & -\psi_4^1 & -\psi_3^1 & \psi_2^1 & -3\psi_1^1 & -\varphi_4^2 & -\varphi_3^2 & \varphi_2^2 & -\varphi_1^2
\end{array}\right) \tag{5.2.5}$$

or equivalently, as the $\mathfrak{gl}(3, \mathbb{H})$-valued 1-form

$$\Theta = \begin{pmatrix}
-\varphi_1^1 + i\varphi_2^1 + j\varphi_3^1 + k\varphi_4^1 & +\psi_1^1 + i\psi_2^1 + j\psi_3^1 + k\psi_4^1 & 0 \\
-\psi_1^1 + i\psi_2^1 + j\psi_3^1 + k\psi_4^1 & -\varphi_1^1 + i\varphi_2^2 + j\varphi_3^2 + k\varphi_4^2 & 0 \\
0 & -3\psi_1^1 + i\psi_2^1 + j\psi_3^1 + k\psi_4^1 & -\varphi_1^2 + i\varphi_2^2 + j\varphi_3^2 + k\varphi_4^2
\end{pmatrix}.$$

We compute the curvature form $F_\Theta := d\Theta + \Theta \wedge \Theta$ as

$$F_\Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_1 & F_2 & F_3 \end{pmatrix}, \tag{5.2.6}$$

where

$$F_1 = -2i\bar{\sigma}_1 - 2j\bar{\sigma}_2 - 2k\bar{\sigma}_3,$$
$$F_3 = +2i\bar{\sigma}_1 + 2j\bar{\sigma}_2 + 2k\bar{\sigma}_3,$$

and

$$\begin{aligned} F_2 = \; & -3\big(\varphi_1^1 \wedge \psi_1^1 + \varphi_2^1 \wedge \psi_2^1 + \varphi_3^1 \wedge \psi_3^1 + \varphi_4^1 \wedge \psi_4^1 \\ & \quad - \varphi_1^2 \wedge \psi_1^1 - \varphi_2^2 \wedge \psi_2^1 - \varphi_3^2 \wedge \psi_3^1 - \varphi_4^2 \wedge \psi_4^1\big) \\ & + i\big(\varphi_1^1 \wedge \psi_2^1 + \varphi_3^2 \wedge \psi_4^1 - \varphi_4^2 \wedge \psi_3^1 - \varphi_2^1 \wedge \psi_1^1 - \varphi_3^1 \wedge \psi_4^1 \\ & \quad + \varphi_4^1 \wedge \psi_3^1 + \varphi_2^2 \wedge \psi_1^1 - \varphi_1^2 \wedge \psi_2^1\big) \\ & + j\big(\varphi_2^1 \wedge \psi_4^1 - \varphi_3^1 \wedge \psi_1^1 - \varphi_4^1 \wedge \psi_2^1 + \varphi_3^2 \wedge \psi_1^1 + \varphi_4^2 \wedge \psi_2^1 \\ & \quad + \varphi_1^1 \wedge \psi_3^1 - \varphi_2^2 \wedge \psi_4^1 - \varphi_1^2 \wedge \psi_3^1\big) \\ & + k\big(- \varphi_3^2 \wedge \psi_2^1 + \varphi_1^1 \wedge \psi_4^1 + \varphi_2^2 \wedge \psi_3^1 - \varphi_2^1 \wedge \psi_3^1 + \varphi_3^1 \wedge \psi_2^1 \\ & \quad - \varphi_4^1 \wedge \psi_1^1 + \varphi_4^2 \wedge \psi_1^1 - \varphi_1^2 \wedge \psi_4^1\big) \end{aligned}$$

Therefore, we see that $F_\Theta(X,Y)$ has 7 generators, which, expressed as tensors, are given by

$$\begin{aligned} \tau_1 &= (\varphi_2^1 - \varphi_2^2) \otimes e_1^2 + (\varphi_1^2 - \varphi_1^1) \otimes e_2^2 + (\varphi_4^2 - \varphi_4^1) \otimes e_3^2 + (\varphi_3^1 - \varphi_3^2) \otimes e_4^2, \\ \tau_2 &= (\varphi_3^1 - \varphi_3^2) \otimes e_1^2 + (\varphi_4^1 - \varphi_4^2) \otimes e_2^2 + (\varphi_1^2 - \varphi_1^1) \otimes e_3^2 + (\varphi_2^2 - \varphi_2^1) \otimes e_4^2, \\ \tau_3 &= (\varphi_4^1 - \varphi_4^2) \otimes e_1^2 + (\varphi_3^2 - \varphi_3^1) \otimes e_2^2 + (\varphi_2^1 - \varphi_2^2) \otimes e_3^2 + (\varphi_1^2 - \varphi_1^1) \otimes e_4^2, \\ \tau_4 &= \psi_1^1 \otimes e_1^2 + \psi_2^1 \otimes e_2^2 + \psi_3^1 \otimes e_3^2 + \psi_4^1 \otimes e_4^2, \\ \tau_5 &= \psi_2^1 \otimes e_1^2 - \psi_1^1 \otimes e_2^2 - \psi_4^1 \otimes e_3^2 + \psi_3^1 \otimes e_4^2, \\ \tau_6 &= \psi_3^1 \otimes e_1^2 + \psi_4^1 \otimes e_2^2 - \psi_1^1 \otimes e_3^2 - \psi_2^1 \otimes e_4^2, \\ \tau_7 &= \psi_4^1 \otimes e_1^2 - \psi_3^1 \otimes e_2^2 + \psi_2^1 \otimes e_3^2 - \psi_1^1 \otimes e_4^2. \end{aligned} \tag{5.2.7}$$

By the Ambrose-Singer theorem (see Theorem 5.2.2), in order to compute the full holonomy algebra of the Obata connection we also need to compute $(\nabla^{Ob}_X F_\Theta)(Y,Z)$ and its higher derivatives. First note that we can write

$$F_\Theta = \alpha_i \otimes \tau_i,$$

where α_i correspond to the 2-forms appearing in F_1, F_2, F_3. If we now compute

$$(\nabla^{Ob}_X F_\Theta)(Y,Z) = (\nabla^{Ob}_X \alpha_i)(Y,Z)\tau_i + \alpha_i(Y,Z)\nabla^{Ob}_X \tau_i,$$

then we see that we recover the endomorphisms in (5.2.7) and obtain possibly new endomorphisms from the terms of the form $\alpha_i(Y,Z)\nabla^{Ob}_X \tau_i$. Thus, it only suffices to consider those.

On the other hand, from Theorem 5.2.1 we see that $\nabla^{Ob}\mathfrak{h}_2$ lies in $\mathfrak{h}_2$. We deduce that the holonomy algebra has to be generated by $\mathbb{H}$-matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 & q_2 & q_3 \end{pmatrix}, \qquad q_i \in \mathbb{H}\,. \tag{5.2.8}$$

This is a Lie algebra of real dimension 12. The crucial point is to see if the holonomy algebra is a strict sub-algebra of the latter. To this end we begin by determining the span of $(\nabla^{Ob}_X F_\Theta)(Y,Z)$.

Lemma 5.2.5. *We have the following*

$$\begin{aligned}
\nu_1 &:= \nabla^{Ob}_{e_1^1}\tau_1 = \varphi_2^1 \otimes e_1^2 - \varphi_1^1 \otimes e_2^2 - \varphi_4^1 \otimes e_3^2 + \varphi_3^1 \otimes e_4^2,\\
\nu_1 &:= \nabla^{Ob}_{e_2^1}\tau_1 = \varphi_1^1 \otimes e_1^2 + \varphi_2^1 \otimes e_2^2 + \varphi_3^1 \otimes e_3^2 + \varphi_4^1 \otimes e_4^2,\\
\nu_3 &:= \nabla^{Ob}_{e_3^1}\tau_1 = -\varphi_4^1 \otimes e_1^2 + \varphi_3^1 \otimes e_2^2 - \varphi_2^1 \otimes e_3^2 + \varphi_1^1 \otimes e_4^2,\\
\nu_4 &:= \nabla^{Ob}_{e_4^1}\tau_1 = \varphi_3^1 \otimes e_1^2 + \varphi_4^1 \otimes e_2^2 - \varphi_1^1 \otimes e_3^2 - \varphi_2^1 \otimes e_4^2.
\end{aligned} \tag{5.2.9}$$

Proof. This follows by a direct computation using the connection form (5.2.5). q.e.d.

Theorem 5.2.6. *The holonomy algebra $\mathfrak{hol}(\nabla^{Ob})$ is the 11-dimensional Lie algebra*

$$\mathfrak{hol}(\nabla^{Ob}) \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 & q_2 & p \end{pmatrix} \in \mathfrak{gl}(3,\mathbb{H}) \mid q_1, q_2 \in \mathbb{H},\ p \in \mathrm{Im}(\mathbb{H}) \right\}. \tag{5.2.10}$$

Proof. From Lemma 5.2.5, the endomorphisms $\nu_1, \ldots, \nu_4$ are linearly independent, and, moreover, they are also linearly independent of $\tau_1, \ldots, \tau_7$.

Thus, this yields a total of 11 generators for the holonomy algebra:

$$\langle \tau_1, \ldots, \tau_7, \nu_1, \ldots \nu_4 \rangle \subset \mathfrak{hol}(\nabla^{Ob}).$$

More explicitly, we have the identification:

$$\langle \tau_1, \ldots, \tau_7, \nu_1, \ldots \nu_4 \rangle = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 & q_2 & p \end{pmatrix} \in \mathfrak{gl}(3,\mathbb{H}) \mid q_1, q_2 \in \mathbb{H},\ p \in \mathrm{Im}(\mathbb{H}) \right\}.$$

Since $11 = \dim(\langle \tau_1, \ldots, \tau_7, \nu_1, \ldots \nu_4 \rangle) \leq \dim(\mathfrak{hol}(\nabla^{Ob})) \leq 12$, to prove the result we only need to show that the holonomy algebra cannot be 12-dimensional. This follows from Theorem 5.3.4 since the holonomy algebra of ∇^{Ob} is necessarily a sub-algebra of $\mathfrak{sl}(3,\mathbb{H})$

i.e. q_3 in (5.2.8) has to be purely imaginary. Therefore, the dimension of $\mathfrak{hol}(\nabla^{Ob})$ is equal to 11 and this concludes the proof. q.e.d.

5.2.3 The case of $\mathrm{SU}(2n+1)$

In this section, we focus on $\mathrm{SU}(2n+1)$ (with $n > 1$), which is the only case excluded by Theorem 5.2.1. This case is particularly interesting: unlike the situations covered by Theorem 5.2.1, where the reduction of the holonomy holds regardless of the chosen Joyce hypercomplex structure, in the $\mathrm{SU}(2n+1)$ case we prove that for *specific choices* of hypercomplex structures, there exists a ∇^{Ob} invariant subbundle of $T\mathrm{SU}(2n+1)$.

The Joyce decomposition of $\mathfrak{su}(2n+1)$ can be constructed iteratively via the relation

$$\mathfrak{su}(2n+1) = \mathfrak{su}(2n-1) \oplus \mathfrak{u}(1) \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1.$$

In particular, for any $k \leq n$, the Lie algebra $\mathfrak{su}(2k+1)$ is embedded as a subalgebra of $\mathfrak{su}(2n+1)$. This leads to the chain of inclusions

$$\mathfrak{su}(3) \subset \mathfrak{su}(5) \subset \cdots \subset \mathfrak{su}(2n+1).$$

Note that, in general, $\mathfrak{su}(2k+1)$ is not invariant under the action of the triple (I, J, K) of $\mathfrak{su}(2n+1)$. When this is the case, i.e. for appropriate choices of (I, J, K), we will be able to show that $\mathfrak{su}(2k+1)$ is Obata invariant.

Proceeding iteratively, we obtain the decomposition

$$\mathfrak{su}(2n+1) = \mathfrak{b} \oplus \bigoplus_{i=1}^{n} \mathfrak{d}_i \oplus \bigoplus_{i=1}^{n} \mathfrak{f}_i,$$

where $\mathfrak{f}_i = (2n-(2i-1))\mathbb{C}^2$. The Lie algebra $\mathfrak{su}(2n+1)$ consists of $(2n+1) \times (2n+1)$ skew-Hermitian, trace-free matrices. Such matrices can be written in block form as

$$\left(\begin{array}{c|c|c|c|c}
D_1 & \multicolumn{4}{c}{F_1} \\ \hline
 & D_2 & \multicolumn{3}{c}{F_2} \\ \cline{2-5}
-\bar{F}_1^t & & \ddots & \multicolumn{2}{c}{\vdots} \\ \cline{4-5}
 & -\bar{F}_2^t & & D_n & F_n \\ \cline{4-5}
 & & \cdots & -\bar{F}_n^t & B
\end{array}\right), \tag{5.2.11}$$

where each $D_i \in \mathfrak{u}(2)$, each F_i is a matrix in $M(2 \times (2n-(2i-1)), \mathbb{C})$, and $B \in \mathbb{C}$ is such that $\sum_i \mathrm{tr}(D_i) + B = 0$. The subspace $\mathfrak{d}_i$ consists of matrices in which only D_i is non-zero, $\mathfrak{f}_i$ consists of those where only F_i is non-zero, and $\mathfrak{b}$ consists of trace-free diagonal matrices that commute with every D_i.

With this block description, it is clear that the copy of $\mathfrak{su}(2k+1)$ inside $\mathfrak{su}(2n+1)$ is embedded in the bottom-right square block of size $2k+1$.

Since $\dim(\mathfrak{b}) = n$, there are up to n^2 parameters of hypercomplex structures determined by the choice of isomorphism $\mathfrak{b} \cong \mathbb{R}^n$. We will show that, for some of these hypercomplex structures, the associated Obata connection preserves a left invariant subbundle.

For the sake of simplicity, let us first consider the case of $\mathrm{SU}(5)$. By the argument given in the proof of Theorem 5.2.1, we have

$$\mathfrak{su}(5) = \mathfrak{u}(1) \oplus \mathfrak{su}(3) \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1,$$

and the Joyce decomposition is obtained by further decomposing

$$\mathfrak{su}(3) = \mathfrak{u}(1) \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_2.$$

In this case, the subspace $\mathfrak{b}$ has dimension 2, and one can choose a basis $\{E_1, E_2\}$ of $\mathfrak{b}$ such that E_2 generates the $\mathfrak{u}(1)$-copy inside $\mathfrak{su}(3)$. More precisely:

$$\mathfrak{b} = \left\langle E_1 := \left(\begin{array}{cc|ccc} 3i & 0 & 0 & 0 & 0 \\ 0 & 3i & 0 & 0 & 0 \\ \hline 0 & 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 & -2i \end{array}\right), \; E_2 := \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -2i \end{array}\right) \right\rangle$$

Joyce hypercomplex structures depend on the choice of 4 parameters, which correspond to the choice of all the possible basis $\{e_1^1, e_1^2\}$ (see Section 1.4.2 for the notation). The possible hypercomplex structures can then be identified with the choice of the change-of-basis matrix $A = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right) \in \mathrm{GL}(4, \mathbb{R})$, where $e_1^1 = a_1 E_1 + a_2 E_2$, $e_1^2 = a_3 E_1 + a_4 E_2$.

Theorem 5.2.7. *Consider* $\mathrm{SU}(5)$ *with the Joyce hypercomplex structure* (I, J, K) *corresponding to the matrix*

$$A = \begin{pmatrix} a_1 & 0 \\ a_2 & a_4 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}),$$

i.e.,

$$\mathfrak{su}(5) = \Big(\langle a_1 E_1 + a_2 E_2 \rangle \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1\Big) \oplus \Big(\langle a_4 E_2 \rangle \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_2\Big).$$

Then the hypercomplex subspace $\mathfrak{su}(3) = \langle a_4 E_2 \rangle \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_2$ *is invariant by the Obata connection* ∇^{Ob} *of* (I, J, K). *In particular, the holonomy group* $\mathrm{Hol}(\nabla^{Ob}) \subsetneq \mathrm{GL}(6, \mathbb{H})$.

Proof. Let $X \in \mathfrak{su}(5)$ and $Y \in \mathfrak{su}(3)$, we want to show that $\nabla^{Ob}_X Y \in \mathfrak{su}(3)$.

By the choice of the basis $\{E_1, E_2\}$, and using that $[\mathfrak{f}_2, \mathfrak{f}_2] \subset \mathfrak{su}(3)$, we immediately get that $\mathfrak{su}(3)$ is a hypercomplex sub-algebra. Consequently, if $X \in \mathfrak{su}(3)$ then the result is clear from the expression of the Obata connection [Sol]:

$$\nabla^{Ob}_X Y = \frac{1}{2}([X,Y] + I[IX,Y] - J[X,JY] + K[IX,JY]. \tag{5.2.12}$$

Suppose now $X = e_1^1 = a_1E_1 + a_2E_2$, then

$$\begin{aligned} \nabla^{Ob}_{e_1^1} Y &= \nabla^{Ob}_Y e_1^1 + [e_1^1, Y] \\ &= [e_1^1, Y] \in \mathfrak{f}_2 \subset \mathfrak{su}(3) \end{aligned}$$

where we used Lemma 5.1.4, (J1) and Lemma 5.1.1.

Suppose now $X \in \mathfrak{d}_1$. Again from Lemma 5.1.4, $\nabla^{Ob}_X e_1^2 = 0$ and so $\nabla^{Ob}_X \mathfrak{d}_2 = 0$. For the remaining case, we have

$$2\nabla^{Ob}_X \mathfrak{f}_2 = [X,\mathfrak{f}_2] + I[IX,\mathfrak{f}_2] - J[X,J\mathfrak{f}_2] + K[IX,J\mathfrak{f}_2] \in \mathfrak{f}_2 \subset \mathfrak{su}(3),$$

since $[X,\mathfrak{f}_2], [X,J\mathfrak{f}_2]$ vanish from (J3) and $[IX,\mathfrak{f}_2], [IX,J\mathfrak{f}_2]$ lie in $\mathfrak{f}_2$ since IX belongs to $\langle e_1^1\rangle \oplus \mathfrak{d}_1$.

Lastly, suppose that $X \in \mathfrak{f}_1$. As before from Lemma 5.1.4 we have $\nabla^{Ob}_X e_1^2 = 0$, and hence $\nabla^{Ob}_X \mathfrak{d}_2 = 0$. It remains to compute $\nabla^{Ob}_X \mathfrak{f}_2$. Let $Z_1 \in \mathfrak{f}_1$ and $Z_2 \in \mathfrak{f}_2$ then

$$\begin{aligned} [Z_1,Z_2] + I[IZ_1,Z_2] &= [Z_1,Z_2] + I[[e_2^1,Z_1],Z_2] \\ &= [Z_1,Z_2] - I[[Z_1,Z_2],e_2^1] - I[[Z_2,e_2^1],Z_1] \\ &= [Z_1,Z_2] - I(-I[Z_1,Z_2]) \\ &= 0 \end{aligned}$$

where we used (J3) and that $[Z_1,Z_2] \in \mathfrak{f}_1$ from Lemma 5.1.1. Comparing with expression (5.2.12), and using that $\mathfrak{f}_2$ is J-invariant, we see that $\nabla^{Ob}_X \mathfrak{f}_2 = 0$. q.e.d.

Let us now consider the Lie algebra $\mathfrak{su}(2n+1)$, $n > 1$, and let us fix a basis $E_1,\dots,E_n$ of the abelian subalgebra $\mathfrak{b} \cong \mathbb{R}^n$ such that each element E_k belongs to the subalgebra $\mathfrak{su}(2n+3-2k) \subset \mathfrak{su}(2n+1)$. For instance, $E_1 \in \mathfrak{su}(2n+1)$, $E_2 \in \mathfrak{su}(2n-1)$, and $E_n \in \mathfrak{su}(3)$.

From the matrix point of view (see (5.2.11)), each $E_k = \mathrm{diag}(a_1,a_2,\dots,a_{2n+1})$ is a diagonal, trace-free matrix that commutes with every block D_j, and satisfies $a_i = 0$ for all $i = 1,\dots,2k-2$.

Joyce hypercomplex structures are parametrized by change-of-basis matrices in $\mathrm{GL}(n,\mathbb{R})$ written with respect to the basis $\{E_1,\dots,E_n\}$. We generalize Theorem 5.2.7 as follows:

Theorem 5.2.8. *Fix $n > 1$ and an integer $k \in \{1,\dots,n-1\}$. Consider the Lie group*

$\mathrm{SU}(2n+1)$ *with the Joyce hypercomplex structure* (I, J, K) *corresponding to the matrix*

$$A = \left(\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n-k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-k,1} & \cdots & a_{n-k,n-k} & 0 & \cdots & 0 \\ \hline a_{n-k+1,1} & \cdots & a_{n-k+1,n-k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{array}\right) \in \mathrm{GL}(n, \mathbb{R}),$$

i.e.,

$$\mathfrak{su}(2n+1) = \bigoplus_{i=1}^{n-k} \left(\langle e_1^i \rangle \oplus \mathfrak{d}_i \oplus \mathfrak{f}_i\right) \oplus \bigoplus_{i=n-k+1}^{n} \left(\langle e_1^i \rangle \oplus \mathfrak{d}_i \oplus \mathfrak{f}_i\right),$$

where for $i = 1, \dots, n-k$*:*

$$e_1^i = \sum_{j=1}^{n} a_{ji} E_j,$$

and for $i = n-k, \dots, n$*:*

$$e_1^i = \sum_{j=n-k+1}^{n} a_{ji} E_j \in \mathfrak{su}(2k+1).$$

Then the hypercomplex subspace

$$\mathfrak{su}(2k+1) = \bigoplus_{i=n-k+1}^{n} \left(\langle e_1^i \rangle \oplus \mathfrak{d}_i \oplus \mathfrak{f}_i\right)$$

is invariant by the Obata connection ∇^{Ob} *of* (I, J, K)*. In particular, the holonomy group* $\mathrm{Hol}(\nabla^{Ob}) \subsetneq \mathrm{GL}(n(n+1), \mathbb{H})$.

Proof. The proof is analogous to that of Theorem 5.2.7, and uses crucially the fact that, with respect to the hypercomplex structure (I, J, K), the subalgebra $\mathfrak{su}(2k+1)$ is hypercomplex. q.e.d.

Combining Theorems 5.2.1, 5.2.8 and the fact that $S^1 \times \mathrm{SU}(2)$ has holonomy $\mathbb{Z}$ [SV], we get the following:

Corollary 5.2.9. *Every manifold in the list* (1.4.4)*, aside from* $\mathrm{SU}(3)$*, admits a Joyce hypercomplex structure with Obata holonomy strictly contained in* $\mathrm{GL}(n, \mathbb{H})$.

We showed that for specific hypercomplex structures, the Obata connection preserves a left invariant subbundle of $T\mathrm{SU}(2n+1)$. However, this does not cover all the possible cases.

Let us consider more closely the case of $\mathrm{SU}(5)$. The Joyce hypercomplex structures on $\mathfrak{su}(5)$ are parametrized by matrices in $\mathrm{GL}(2,\mathbb{R})$ with respect to the basis $\{E_1, E_2\}$ (see Theorem 5.2.7).

In Theorem 5.2.7, we proved that for hypercomplex structures corresponding to matrices of the form

$$\begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix},$$

the holonomy group of the associated Obata connection acts preserves a left invariant subbundle of $T\mathrm{SU}(5)$. In the complementary case where $a_2 \neq 0$ the subalgebra $\mathfrak{su}(3)$ is no longer hypercomplex.

Although a full characterization of this case remains open, our working conjecture is that the Obata connection associated with Joyce hypercomplex structures in this case has holonomy *equal* to $\mathrm{GL}(6,\mathbb{H})$. We expect that a similar behaviour also holds for the generic $\mathrm{SU}(2n+1)$, with $n > 1$. As a partial evidence, we show:

Theorem 5.2.10. *Consider* $\mathrm{SU}(5)$ *with the Joyce hypercomplex structure* (I, J, K) *corresponding to the matrix*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}),$$

i.e.,

$$\mathfrak{su}(5) = \Big(\langle E_2\rangle \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1\Big) \oplus \Big(\langle E_1\rangle \oplus \mathfrak{d}_2 \oplus \mathfrak{f}_2\Big).$$

Then the Obata holonomy group $\mathrm{Hol}(\nabla^{Ob})$ *is isomorphic to* $\mathrm{GL}(6,\mathbb{H})$.

Proof. Denote by R the curvature tensor of the Obata connection (5.1.3) and let $\nabla^{Ob^k}\mathcal{R}$ denote the space of k^{th} covariant derivative of the curvature tensor i.e.

$$\nabla^{Ob^k}\mathcal{R} := \{\nabla^{Ob^k}_{x_1,\cdots,x_k} R_{x,y} \mid \forall\, x, y, x_i \in \mathfrak{su}(5)\}.$$

Observe that R and its derivatives can be computed using only the Lie bracket and the hypercomplex structures I, J, K (which are themselves determined by the Lie bracket after fixing $\mathcal{B}$, see Section 1.4.2).

With the aid of MAPLE software, we have been able to determine the dimension of the latter spaces as follows:

$$\dim(\mathcal{R}) = 52, \quad \dim(\{\mathcal{R}, \nabla^{Ob}\mathcal{R}\}) = 138, \quad \dim(\{\mathcal{R}, \nabla^{Ob}\mathcal{R}, \nabla^{Ob^2}\mathcal{R}\}) = 144. \qquad (5.2.13)$$

Since $\dim(\mathfrak{gl}(6,\mathbb{H})) = 144$ and we know that $\mathfrak{hol}(\nabla^{Ob}) \subset \mathfrak{gl}(6,\mathbb{H})$, the result follows from Theorem 5.2.2, the classification of the holonomy groups of irreducible torsion free connections in [MS2] and the fact that $\mathrm{SU}(5)$ is simply connected. q.e.d.

Note that our proof relies on directly determining the dimension of the holonomy algebra with the help of the computer; this is typically unfeasible in higher dimensions. In [Sol], Soldatenkov was able to prove the irreducibility of the holonomy group in the $\mathrm{SU}(3)$ case without having to determine the rank of the holonomy algebra. Unfortunately, we have been unable to generalise his argument to the $\mathrm{SU}(2n+1)$ case for $n \geq 2$. It is worth mentioning, however, that we were able to check that one needs to compute up to $\nabla^{Ob\,4}\mathcal{R}$ in the $\mathrm{SU}(3)$ case in order to generate the full holonomy algebra of $\mathfrak{gl}(2,\mathbb{H})$. Our above proof remains somewhat unsatisfactory as it relies on heavy computation, so it would be interesting to have a neater argument which is applicable to all the cases excluded by Theorem 5.2.8.

As a consequence of Theorems 5.2.8 and 5.2.10, it follows that the irreducibility of the Obata holonomy group is not preserved under variations of the hypercomplex structure, even within the Joyce parameter space $\mathrm{GL}(2,\mathbb{R})$. For instance, one can deform $A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $A_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ via the curve

$$A_t = \begin{pmatrix} t & 1-t \\ 1+t & -t \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}), \quad t \in (-\varepsilon, 1+\varepsilon).$$

We can summarise the above observation into:

Corollary 5.2.11. *The irreducibility of the holonomy of the Obata connection is not preserved under deformations of the hypercomplex structure.*

5.3 Restricted holonomy in $\mathrm{SL}(n,\mathbb{H})$

In this section, we investigate Joyce hypercomplex manifolds whose Obata connection has restricted holonomy group contained in $\mathrm{SL}(n,\mathbb{H})$. For a hypercomplex manifold, the restricted holonomy lies in $\mathrm{SL}(n,\mathbb{H})$ if and only if the Ricci tensor of the Obata connection vanishes [AM, Theorem 5.6]. In view of this, we begin by determining an explicit formula of the Obata-Ricci tensor on a hypercomplex Lie algebra.

Lemma 5.3.1. *Let $\mathfrak{g}$ be a Lie algebra equipped with a hypercomplex structure (I,J,K). Then the Obata connection 1-form is given by*

$$\eta(X) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_X) - \frac{1}{2}\mathrm{tr}(L\mathrm{ad}_{LX})\,, \qquad X \in \mathfrak{g}\,,$$

for any $L \in \{I,J,K\}$. In particular, the Obata-Ricci tensor is

$$\mathrm{Ric}^{\mathrm{Ob}}(X,Y) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_{[X,Y]}) + \frac{1}{2}\mathrm{tr}(L\mathrm{ad}_{L[X,Y]})\,, \qquad X,Y \in \mathfrak{g}\,.$$

Proof. Since η does not depend on the chosen complex structure it suffices to prove the Lemma for $L = I$.

Let α denote the $(1,0)$ part of η with respect to I. Then, according to [FG2, Lemma 3.3], $\bar{\alpha} = \Lambda_\Omega(\bar{\partial}\Omega)$, where

$$\Omega = \frac{\omega_J + i\omega_K}{2}$$

is the $(2,0)$ form associated to a given hyperhermitian metric g compatible with (I, J, K).

Fix $W \in \mathfrak{g}^{1,0}$ and an orthonormal basis $\{e_1, \ldots, e_{4n}\}$ adapted to (I, J, K). Set $Z_i = e_{2i-1} - iIe_{2i-1} = e_{2i-1} - ie_{2i}$, for $i = 1, \ldots, 2n$. With respect to this basis we have

$$\Omega = \frac{1}{2}\sum_{i=1}^{n} Z^{2i-1} \wedge Z^{2i} .$$

We compute

$$\begin{aligned}
\bar{\alpha}(\bar{W}) &= (\Lambda\bar{\partial}\Omega)(\bar{W}) \\
&= \frac{1}{2}\sum_{i=1}^{n} \bar{\partial}\Omega(\bar{W}, Z_{2i-1}, Z_{2i}) \\
&= \frac{1}{2}\sum_{i=1}^{n} \big(-\Omega([\bar{W}, Z_{2i-1}], Z_{2i}) + \Omega([\bar{W}, Z_{2i}], Z_{2i-1}) - \Omega([Z_{2i-1}, Z_{2i}], \bar{W})\big) .
\end{aligned}$$

Since Ω is of type $(2,0)$ the last term vanishes, yielding

$$\begin{aligned}
\bar{\alpha}(\bar{W}) &= \frac{1}{2}\sum_{i=1}^{n} \big(-\Omega([\bar{W}, Z_{2i-1}], Z_{2i}) + \Omega([\bar{W}, Z_{2i}], Z_{2i-1})\big) \\
&= -\frac{1}{2}\sum_{j=1}^{2n} \Omega([\bar{W}, Z_j], J\bar{Z}_j) \\
&= -\frac{1}{2}\sum_{j=1}^{2n} g([\bar{W}, Z_j], \bar{Z}_j) .
\end{aligned}$$

In terms of the real basis we rewrite this as

$$\begin{aligned}
\bar{\alpha}(\bar{W}) &= -\frac{1}{2}\sum_{j=1}^{2n} g([\bar{W}, e_{2j-1} - ie_{2j}], e_{2j-1} + ie_{2j}) \\
&= -\frac{1}{2}\sum_{j=1}^{2n} \big(g([\bar{W}, e_{2j-1}], e_{2j-1}) + g([\bar{W}, e_{2j}], e_{2j}) - ig([\bar{W}, e_{2j}], e_{2j-1}) + ig([\bar{W}, e_{2j-1}], e_{2j})\big) \\
&= -\frac{1}{2}\sum_{k=1}^{4n} g([\bar{W}, e_k], e_k) - \frac{i}{2}\sum_{k=1}^{4n} g([\bar{W}, e_k], Ie_k) \\
&= -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_{\bar{W}}) + \frac{i}{2}\,\mathrm{tr}(I\mathrm{ad}_{\bar{W}}) .
\end{aligned}$$

We conclude that for every $X \in \mathfrak{g}$

$$\eta(X) = 2\mathrm{Re}\left(\bar{\alpha}(X^{0,1})\right) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_X) - \frac{1}{2}\mathrm{tr}(I\mathrm{ad}_{IX})\,.$$

Finally,

$$\mathrm{Ric}^{\mathrm{Ob}}(X,Y) = d\eta(X,Y) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_{[X,Y]}) + \frac{1}{2}\mathrm{tr}(I\mathrm{ad}_{I[X,Y]})$$

as claimed. q.e.d.

From now on we shall focus on those Joyce hypercomplex structures admitting a (strong) HKT metric obtained by extending the Killing–Cartan form (see Section 1.5.1).

The advantage in the HKT case is that the Obata-Ricci tensor coincides with the differential of the Lee form [IP] (see also [FG2, Proposition 3.4 (d)] for the hyperhermitian setting).

Also, such strong HKT metrics are related to the *twisted Calabi–Yau system* introduced in [GRST]. Indeed, on a hyperhermitian manifold (M, I, J, K, g), the $(2,0)$ form Ω is non-degenerate, namely $\Psi = \Omega^n$ is a complex volume form of constant norm. Furthermore, the hyperhermitian structure is HKT if and only if $\partial\Omega = 0$ [GP] and in this case we have

$$d\Psi = \theta \wedge \Psi\,,$$

which is the second equation of the twisted Calabi-Yau system (5.0.2) (in fact it is enough to have $\partial\Omega^{n-1} = 0$ [FG2]). If the HKT metric is strong, then $\omega_I = g(I\cdot,\cdot)$ satisfies the pluriclosed condition $\partial\bar{\partial}\omega_I = 0$. In conclusion, in order for the Killing–Cartan form to provide a solution to the system we have to further impose that $d\theta = 0$, which is the same condition needed to have restricted holonomy of the Obata connection contained in $\mathrm{SL}(n,\mathbb{H})$. To see when the Lee form is closed let us compute it in terms of the basis chosen in Section 1.5.1 to extend the Killing–Cartan form to a bi-invariant strong HKT metric.

Proposition 5.3.2. *Let G be a compact simple Lie group and (I, J, K) a Joyce hypercomplex structure on $\mathbb{T}^\ell \times G$ such that $(\mathbb{T}^\ell \times G, I, J, K)$ admits a bi-invariant strong HKT metric g constructed as in Section 1.5.1. Then the Lee form of the HKT structure (I, J, K, g) takes the following expression*

$$\theta = 2\sum_{j=1}^{m} \frac{1}{\lambda_j^2}\left(1 + \dim_{\mathbb{H}}(\mathfrak{f}_j)\right)(e_1^j)^\sharp\,,$$

where $(e_1^j)^\sharp$ denotes the 1-form dual to e_1^j with respect to the metric g and $\lambda_j^2 = g(e_1^j, e_1^j)$.

Proof. As mentioned, in the HKT setting the Lee form coincides with the Obata connection 1-form η. From Lemma 5.3.1, using the unimodularity of $\mathbb{T}^\ell \times G$ and the bi-invariance of g, we obtain

$$\theta(X) = \eta(X) = -\frac{1}{2}\mathrm{tr}(J\mathrm{ad}_{JX}) = \frac{1}{2}\sum_{i=1}^{4n} g(J[JX, e_i], e_i) = -\frac{1}{2}\sum_{i=1}^{4n} g(X, J[e_i, Je_i]),$$

where $\{e_1, \ldots, e_{4n}\}$ is an orthonormal basis with respect to g.

For the sake of computations, let us fix the orthogonal basis $\{e_1^1, \ldots, e_4^1, \ldots, e_1^m, \ldots, e_4^m\}$ on $\ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{j=1}^m \mathfrak{d}_j$ that defines the hypercomplex structure (see Section 1.5.1 for the definition of the basis). Let $4d_j := \dim_{\mathbb{R}}(\mathfrak{f}_j)$. On the subspaces $\mathfrak{f}_j$ we pick instead any orthonormal basis $\{f_1^j, \ldots, f_{4d_j}^j\}$ adapted to I, J, K, that is, $f_{4k-2}^j = If_{4k-3}^j$, $f_{4k-1}^j = Jf_{4k-3}^j$, $f_{4k}^j = Kf_{4k-3}^j$, for any $k = 1, \ldots, d_j$.

It is straightforward to check that for all $j = 1, \ldots, m$ we have

$$\sum_{k=1}^{4} J[e_k^j, Je_k^j] = -4e_1^j.$$

Furthermore, it is computed in [FG2] that

$$\sum_{k=1}^{4d_j} J[f_k^j, Jf_k^j] = -4d_j e_1^j.$$

We repeat the calculation for the readers' convenience. First of all we note that

$$\begin{aligned}
&[f_{4k-3}^j, Jf_{4k-3}^j] + [f_{4k-2}^j, Jf_{4k-2}^j] + [f_{4k-1}^j, Jf_{4k-1}^j] + [f_{4k}^j, Jf_{4k}^j] \\
&\quad = [f_{4k-3}^j, f_{4k-1}^j] - [f_{4k-2}^j, f_{4k}^j] - [f_{4k-1}^j, f_{4k-3}^j] + [f_{4k}^j, f_{4k-2}^j] \\
&\quad = 2\left([f_{4k-3}^j, Jf_{4k-3}^j] + [f_{4k-2}^j, Jf_{4k-2}^j]\right).
\end{aligned}$$

By definition of the hypercomplex structure on $\mathfrak{f}_j$ and using the Jacobi's identity, we have:

$$\begin{aligned}
[f_{4k-3}^j, Jf_{4k-3}^j] + [f_{4k-2}^j, Jf_{4k-2}^j] &= [f_{4k-3}^j, Kf_{4k-2}^j] - [f_{4k-2}^j, Kf_{4k-3}^j] \\
&= [f_{4k-3}^j, [e_4^j, f_{4k-2}^j]] - [f_{4k-2}^j, [e_4^j, f_{4k-3}^j]] \\
&= [e_4^j, [f_{4k-3}^j, f_{4k-2}^j]].
\end{aligned}$$

Therefore, summing over k we get:

$$\sum_{i=1}^{4d_j} [f_i^j, Jf_i^j] = 2\sum_{k=1}^{d_j} [e_4^j, [f_{4k-3}^j, f_{4k-2}^j]]. \tag{5.3.1}$$

To conclude, we show that for any fixed $k \in \{1, \ldots, d_j\}$ the bracket $[e_4^j, [f_{4k-3}^j, f_{4k-2}^j]]$ has non-zero component only along e_3^j. To do this we use the bi-invariant metric g and the properties of the Joyce decomposition. First

$$g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],X) = -g([f_{4k-3}^j,f_{4k-2}^j],[e_4^j,X])\,,$$

which clearly vanishes for all $X \in \ell\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \bigoplus_{k\neq j} \mathfrak{d}_k \oplus \bigoplus_{l>j} \mathfrak{f}_l \oplus \langle e_4^j \rangle$ thanks to (J1), (J2), (J3). On the other hand, if $X = e_2^j$ we have

$$\begin{aligned} g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],e_2^j) &= -2g([f_{4k-3}^j,f_{4k-2}^j],e_3^j) \\ &= -2g(f_{4k-3}^j,[f_{4k-2}^j,e_3^j]) \\ &= -2g(f_{4k-3}^j,f_{4k}^j) = 0\,, \end{aligned}$$

whereas if $X = e_3^j$ we get

$$\begin{aligned} g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],e_3^j) &= 2g([f_{4k-3}^j,f_{4k-2}^j],e_2^j) \\ &= 2g(f_{4k-3}^j,[f_{4k-2}^j,e_2^j]) \\ &= 2g(f_{4k-3}^j,f_{4k-3}^j) = 2\,. \end{aligned}$$

Now, assume $X \in \mathfrak{f}_l$ with $l < j$ then

$$\begin{aligned} g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],X) &= -g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],I^2X) \\ &= -g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],[e_2^l,IX]) \\ &= \;\; g([e_2^l,[e_4^j,[f_{4k-3}^j,f_{4k-2}^j]]],IX) = 0 \\ &= \;\; g([[e_2^l,e_4^j],[f_{4k-3}^j,f_{4k-2}^j]],IX) + g([e_4^j,[e_2^l,[f_{4k-3}^j,f_{4k-2}^j]]],IX) = 0\,, \end{aligned}$$

where the last equality holds by using Jacobi's identity and (J2)–(J3). Finally if $X \in \mathfrak{f}_j$

$$\begin{aligned} g([e_4^j,[f_{4k-3}^j,f_{4k-2}^j]],X) &= -g([f_{4k-3}^j,f_{4k-2}^j],[e_4^j,X]) = -g([f_{4k-3}^j,f_{4k-2}^j],KX) \\ &= -g([f_{4k-3}^j,f_{4k-2}^j],[e_2^j,JX]) = g([e_2^j,[f_{4k-3}^j,f_{4k-2}^j]],JX) \\ &= \;\; g([[e_2^j,f_{4k-3}^j],f_{4k-2}^j],JX) + g([f_{4k-3}^j,[e_2^j,f_{4k-2}^j]],JX) \\ &= \;\; g([f_{4k-2}^j,f_{4k-2}^j],JX) - g([f_{4k-3}^j,f_{4k-3}^j],JX) = 0\,. \end{aligned}$$

Putting everything together we have shown that $[e_4^j,[f_{4k-3}^j,f_{4k-2}^j]] = 2\frac{e_3^j}{\lambda_j^2}$, where $\lambda_j = \sqrt{g(e_j^1,e_j^1)}$. In order to compute the Lee form, we now consider the orthonormal basis

$\{e_1, \dots, e_{4n}\} = \{\frac{e_1^j}{\lambda_j}, \dots, \frac{e_4^j}{\lambda_j}, f_i^k\}$. We therefore obtain from (5.3.1):

$$\begin{aligned} \sum_{i=1}^{4n} J[e_i, Je_i] &= \sum_{j=1}^{m} \sum_{k=1}^{4} \frac{1}{\lambda_j^2} J[e_k^j, Je_k^j] + \sum_{j=1}^{m} \sum_{i=1}^{4d_j} J[f_i^j, Jf_i^j] \\ &= -4 \sum_{j=1}^{m} \frac{1}{\lambda_j^2} e_1^j + 2 \sum_{j=1}^{m} \sum_{k=1}^{d_j} J[e_4^j, [f_{4k-3}^j, f_{4k-2}^j] \\ &= -4 \sum_{j=1}^{m} \frac{1}{\lambda_j^2} (1 + d_j)\, e_1^j\,, \end{aligned}$$

implying

$$\theta = 2 \sum_{j=1}^{m} \frac{1}{\lambda_j^2} (1 + d_j)\, (e_1^j)^\sharp\,,$$

as claimed. q.e.d.

Corollary 5.3.3. *Let G be a compact simple Lie group of rank r and (I, J, K) a Joyce hypercomplex structure on $\mathbb{T}^\ell \times G$ such that $(\mathbb{T}^\ell \times G, I, J, K)$ admits a bi-invariant strong HKT metric g constructed as in Section 1.5.1. Then the Lee form of g is closed if and only if $\ell = m = r$ in the Joyce decomposition (1.4.1), i.e. if and only if the abelian summand $\mathfrak{b}$ in the Joyce decomposition is trivial.*

Proof. We use Proposition 5.3.2 to compute the differential of the Lee form associated to (I, J, K, g). Since the Lie algebra $\mathfrak{g}$ is simple, the commutator of $\ell\mathfrak{u}(1)\oplus\mathfrak{g}$ is $\mathfrak{g}$, implying that

$$d\theta = \theta|_{\mathfrak{g}}\,.$$

By Proposition 5.3.2, $d\theta = 0$ if and only if $e_1^j \in \ell\mathfrak{u}(1)$ for all $j = 1, \dots, m$, i.e. if and only if $\ell = 2m - r = m$ ‡. q.e.d.

We are ready to prove Theorem 5.0.2:

Theorem 5.3.4. *Let M belong to the list (5.0.1). Then any left invariant hypercomplex structure (I, J, K) on M has restricted Obata holonomy contained in* $\mathrm{SL}(n, \mathbb{H})$*, and (M, I) admits a left-invariant solution to the twisted Calabi-Yau system (5.0.2).*

Proof. Recall that any left invariant hypercomplex structure on M arises from the Joyce construction in Section 1.4.2. According to [SSTVP, Section 6] (see also [OP, Section 5]) a simple Lie group has trivial $\mathfrak{b}$ in its Joyce decomposition if and only if it is one of the following:

$$\mathrm{SO}(2k+1), \quad \mathrm{SO}(4k), \quad \mathrm{Sp}(k), \quad \mathrm{E}_7, \quad \mathrm{E}_8, \quad \mathrm{F}_4, \quad \mathrm{G}_2,$$

‡Recall that, by Section 1.5.1, the basis $\{e_1^j\}$ is chosen such that the first ℓ vectors are in $\ell\mathfrak{u}(1)$ and the remaining $m - r$ are in $\mathfrak{b}$.

and therefore any Joyce hypercomplex structure on a manifold M in the list (5.0.1) satisfies the hypothesis of Corollary 5.3.3. Indeed, the condition of having trivial summand $\mathfrak{b}$ in the Joyce decomposition has the consequence that every Joyce hypercomplex structure is compatible with an extension g of the Killing–Cartan form (see Section 1.5.1). We conclude that the pair $(g, \Psi = \Omega^n)$ provides a solution to the twisted Calabi-Yau equations (5.0.2). Furthermore, by [IP, Corollary 4.2], the Obata connection has restricted holonomy group in $\mathrm{SL}(n, \mathbb{H})$. q.e.d.

Remark 5.3.5. When $\mathfrak{b} \neq 0$ in the Joyce decomposition of $\mathfrak{g}$, not every Joyce hypercomplex structure is constructed as in Section 1.5.1; however, there are infinitely many that are. For these, the restricted holonomy of the Obata connection is never a subgroup of $\mathrm{SL}(n, \mathbb{H})$, by Corollary 5.3.3.

This occurs in Joyce manifolds constructed starting from the compact simple Lie groups:

$$S^1 \times \mathrm{SU}(2k)\ (k \geq 2), \quad \mathrm{SU}(2k+1), \quad \mathbb{T}^{2k-1} \times \mathrm{SO}(4k+2), \quad \mathbb{T}^2 \times \mathrm{E}_6.$$

We wonder what happens on these manifolds when equipped with a Joyce hypercomplex structure that is not compatible with the bi-invariant metric. The case of $\mathrm{SU}(2k+1)$ is easy because, being simply connected, the restricted holonomy group of the Obata connection coincides with the full holonomy group. Moreover, as shown in [BDV, SV], no Joyce hypercomplex manifold is an $\mathrm{SL}(n, \mathbb{H})$-manifold; in particular, no Joyce hypercomplex structure on $\mathrm{SU}(2k+1)$ has restricted holonomy contained in $\mathrm{SL}(n, \mathbb{H})$.

Chapter 6

Holonomy of the Obata connection on 2-step hypercomplex nilmanifolds

Nilmanifolds have played a fundamental role in differential geometry and topology since the seminal works of Mal'cev and Mostow. They provide a natural class of spaces where algebraic and geometric properties interact tightly, offering a fair balance between algebraic manageability and rich geometric behavior.

Invariant complex structures on nilmanifolds have been particularly investigated, largely due to the foundational work of Salamon [Sal2], where it is shown that the existence of invariant complex structures J on nilmanifolds admits special features: e.g., the existence of a so-called Salamon basis [Sal2, Theorem 1.3], and the fact that the subspace $[\mathfrak{g}, \mathfrak{g}] + J[\mathfrak{g}, \mathfrak{g}]$ is always a proper ideal of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ [Sal2, Corollary 1.4].

In the hypercomplex setting, many of these nice behaviors fail to carry over in a unified way. Each complex structure J_α ($\alpha = 1, 2, 3$) may satisfy suitable compatibility conditions individually, but it is not clear whether these structures collectively induce analogous global properties. For instance, it is natural to ask whether the subspace

$$\mathfrak{g}_1^{\mathbb{H}} := [\mathfrak{g}, \mathfrak{g}] + J_1[\mathfrak{g}, \mathfrak{g}] + J_2[\mathfrak{g}, \mathfrak{g}] + J_3[\mathfrak{g}, \mathfrak{g}]$$

is a proper ideal of $\mathfrak{g}$. This is an open question and has been explicitly conjectured in [Gor1, Gor2]. The difficulty in addressing this problem reflects the more rigid nature of hypercomplex geometry compared to the complex case.

A central geometric object in this setting is the Obata connection: the unique torsion-free connection preserving all three complex structures. Its holonomy group $\mathrm{Hol}(\nabla^{Ob})$ lies in $\mathrm{GL}(n, \mathbb{H})$ and plays a crucial role in understanding the geometry of hypercomplex manifolds [IP, SV]. In general, explicit descriptions of $\mathrm{Hol}(\nabla^{Ob})$ are rare (see, for instance, [Sol]). For nilmanifolds, however, it is known that the holonomy is contained in the commutator subgroup $\mathrm{SL}(n, \mathbb{H})$ [BDV, Corollary 3.3].

In this chapter, we investigate left-invariant hypercomplex structures on 2-step nilpotent Lie groups. Surprisingly, all known examples in this class are Obata-flat.

This naturally leads to the question: *Are all 2-step hypercomplex nilpotent Lie algebras Obata-flat?* We answer this in the negative, and clarify the role of the nilpotency step of each complex structure. In particular, we show that if the Obata connection is not flat, then each J_α must be 3-step nilpotent. No such examples had been previously known.

In Section 6.2, we construct examples of 2-step hypercomplex nilpotent Lie algebras in which one, two, or all three complex structures are 3-step nilpotent. Among these, Example 6.2.3 provides the first known example of a 2-step hypercomplex nilpotent Lie algebra with non-flat Obata connection. We also provide new examples of k-step hypercomplex nilpotent nilmanifolds, with arbitrary k, which are not Obata flat.

In Section 6.4, we prove that the holonomy algebra of the Obata connection is always abelian for 2-step hypercomplex nilpotent Lie groups, and we describe it explicitly (Theorem 6.4.1). This reveals a surprising rigidity: even when the Obata connection is non-flat, the holonomy remains severely restricted. Our approach relies on proving the Gorginyan conjecture (Conjecture 6.3.3) in the 2-step case (Theorem 6.3.4). We further show that the Obata holonomy group of a 2-step hypercomplex nilmanifold is trivial if and only if the nilmanifold is a torus.

In the final part of the chapter, we extend our analysis to the 3-step nilpotent case, showing that, under suitable conditions, the holonomy of the Obata connection remains abelian.

The results contained in this chapter were obtained in collaboration with A. Andrada and M. L. Barberis, and appeared in the paper [ABB], submitted to journal.

6.1 Computation of the Obata curvature

Let $(\mathfrak{g}, \{J_\alpha\})$ be a 2-step hypercomplex nilpotent Lie algebra. As already observed, the commutator ideal $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ lies in the center $\mathfrak{z}$ of $\mathfrak{g}$. This special feature allows for an explicit computation of the curvature tensor of the Obata connection.

Proposition 6.1.1. *Let $(\mathfrak{g}, \{J_\alpha\})$ be a 2-step hypercomplex nilpotent Lie algebra. Then, for* *any $X, Y, Z \in \mathfrak{g}$ the curvature tensor of the Obata connection is given by*

$$\begin{aligned}4R^{Ob}(X,Y)Z = &- J_\beta[X, J_\beta[Y,Z]] + J_\gamma[J_\alpha X, J_\beta[Y,Z]]\\ &- J_\gamma[X, J_\beta[J_\alpha Y, Z]] - J_\beta[J_\alpha X, J_\beta[J_\alpha Y, Z]]\\ &- [X, J_\beta[Y, J_\beta Z]] - J_\alpha[J_\alpha X, J_\beta[Y, J_\beta Z]]\\ &+ J_\alpha[X, J_\beta[J_\alpha Y, J_\beta Z]] - [J_\alpha X, J_\beta[J_\alpha Y, J_\beta Z]]\\ &+ J_\beta[Y, J_\beta[X,Z]] - J_\gamma[J_\alpha Y, J_\beta[X,Z]]\\ &+ J_\gamma[Y, J_\beta[J_\alpha X, Z]] + J_\beta[J_\alpha Y, J_\beta[J_\alpha X, Z]]\\ &+ [Y, J_\beta[X, J_\beta Z]] + J_\alpha[J_\alpha Y, J_\beta[X, J_\beta Z]]\\ &- J_\alpha[Y, J_\beta[J_\alpha X, J_\beta Z]] + [J_\alpha Y, J_\beta[J_\alpha X, J_\beta Z]]\\ &- 2[J_\beta[X,Y], J_\beta Z] + 2J_\gamma[J_\beta[X,Y], J_\alpha Z],\end{aligned} \tag{6.1.1}$$

where (α, β, γ) is any cyclic permutation of $(1,2,3)$.

Proof. The proof is an application of the formula (1.3.2) and the integrability of $\{J_\alpha\}$. Using the formula (1.3.2) twice, the inclusion $\mathfrak{g}^1 \subset \mathfrak{z}$ and that $\nabla^{Ob}_X J_\alpha Y = J_\alpha \nabla^{Ob}_X Y$, we compute:

$$\begin{aligned}4\nabla^{Ob}_X \nabla^{Ob}_Y Z = &-J_\beta[X, J_\beta[Y,Z]] + J_\gamma[J_\alpha X, J_\beta[Y,Z]]\\ &- J_\gamma[X, J_\beta[J_\alpha Y, Z]] - J_\beta[J_\alpha X, J_\beta[J_\alpha Y, Z]]\\ &- [X, J_\beta[Y, J_\beta Z]] - J_\alpha[J_\alpha X, J_\beta[Y, J_\beta Z]]\\ &+ J_\alpha[X, J_\beta[J_\alpha Y, J_\beta Z]] - [J_\alpha X, J_\beta[J_\alpha Y, J_\beta Z]].\end{aligned}$$

Similarly, applying again (1.3.2) and using the inclusion $\mathfrak{g}^1 \subset \mathfrak{z}$, we compute:

$$\begin{aligned}2\nabla^{Ob}_{[X,Y]} Z &= J_\alpha[J_\alpha[X,Y], Z] + J_\gamma[J_\alpha[X,Y], J_\beta Z]\\ &= [J_\alpha[X,Y], J_\alpha Z] + [J_\beta[X,Y], J_\beta Z]\\ &\quad - [J_\alpha[X,Y], J_\alpha Z] - J_\gamma[J_\beta[X,Y], J_\alpha Z]\\ &= [J_\beta[X,Y], J_\beta Z] - J_\gamma[J_\beta[X,Y], J_\alpha Z],\end{aligned}$$

where the second equality follows from the integrability of J_α on the first term and the integrability of J_γ on the second term. q.e.d.

As a consequence of Proposition 6.1.1 we obtain the following result. This is a generalization of [B1, Proposition 5.1], where it was shown that if $\mathfrak{z}$ is J_α invariant for all α then $\{J_\alpha\}$ is Obata flat.

Corollary 6.1.2. *Let $(\mathfrak{g}, \{J_\alpha\})$ be a 2-step hypercomplex nilpotent Lie algebra. If any of the three complex structures J_1, J_2, J_3 is 2-step nilpotent, then $(\mathfrak{g}, \{J_\alpha\})$ is Obata flat.*

Proof. If J_β is 2-step nilpotent, $J_\beta\mathfrak{g}^1 \subset \mathfrak{z}$ (Lemma 1.1.9). The result then follows from Proposition 6.1.1, since each term contains an element of the form $[\cdot, J_\beta\mathfrak{g}^1]$. q.e.d.

Remark 6.1.3. As a consequence of Corollary 6.1.2, we have that if the Obata connection associated to $(\mathfrak{g}, \{J_\alpha\})$ is not flat, then J_α is 3-step for all α. Moreover, any complex structure J_y in the hypercomplex 2-sphere is 3-step.

6.2 Construction of non-flat 2-step hypercomplex nilmanifolds

To the best of our knowledge, in all known examples of 2-step hypercomplex nilpotent Lie algebras $(\mathfrak{g}, \{J_\alpha\})$, each complex structure J_α is 2-step [B2, DF1, DF2, BD, LW, FG2]. In particular, every known example of a 2-step nilpotent Lie group equipped with a left-invariant hypercomplex structure is Obata flat (see Corollary 6.1.2).

According to [DF2, Section 3], in dimension 8 any nilpotent Lie algebra admitting a hypercomplex structure is 2-step nilpotent, and, moreover, each complex structure J_α is 2-step. This naturally raises the question of whether this behavior is general among 2-step hypercomplex nilpotent Lie algebras, or whether there exist examples in which one, two, or all of the complex structures are 3-step.

Pushing this further: if such examples where each J_α is 3-step exist, are the corresponding 2-step hypercomplex nilpotent Lie groups still Obata flat? Looking at Formula (6.1.1), this is not clear.

In this section, we address these questions.

The construction

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra. We consider $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^r$.

Let $\mu \in \bigwedge^2 \mathfrak{n}^* \otimes \mathbb{R}^r$ be a 2-form on $\mathfrak{n}$ with values in $\mathbb{R}^r$ such that $\mathfrak{n}^1 \subset \ker(\mu)$. We use μ to define a bracket $[\cdot,\cdot]'$ on $\mathfrak{g}$ as follows:

$$
\begin{aligned}
&[X, Y]' = ([X, Y], \mu(X, Y)), \ \ X, Y \in \mathfrak{n},\\
&[X, Y]' = 0, \ \ X, Y \in \mathbb{R}^r,\\
&[X, Y]' = 0, \ \ X \in \mathfrak{n} \ \text{ and } \ Y \in \mathbb{R}^r.
\end{aligned}
$$

Then, $(\mathfrak{g}, [\cdot,\cdot]')$ is a 2-step nilpotent Lie algebra.

Let us now assume that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra *endowed with a hypercomplex structure* $\{J_\alpha\}$, and let $(\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^{4k}, [\cdot,\cdot]')$ be the 2-step nilpotent Lie algebra constructed as above. $\mathfrak{g}$ carries a natural almost complex structure $J'_\alpha = J_\alpha \oplus I_\alpha$, where

$\{I_\alpha\}$ is the standard hypercomplex structure on $\mathbb{R}^{4k}$. The following result is a characterization of $N_{J'_\alpha} = 0$.

Lemma 6.2.1. *The almost complex structure J'_α is integrable if and only if*

$$I_\alpha \mu(X,Y) = \mu(J_\alpha X, Y) + \mu(X, J_\alpha Y) + I_\alpha \mu(J_\alpha X, J_\alpha Y), \ \ X, Y \in \mathfrak{n}. \tag{6.2.1}$$

Let us now assume that the 2-step hypercomplex nilpotent Lie algebra $(\mathfrak{n}, \{J_\alpha\})$ satisfies that each J_α is 2-step. If there exists $X \in \mathfrak{n}^1$ such that $\mu(J_\alpha X, \cdot) \neq 0$, then the corresponding complex structure J'_α on $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^{4k}$ is 3-step, according to Lemma 1.1.9. We will use this procedure to construct new examples of 2-step hypercomplex nilpotent Lie algebras where one, two or all three complex structures are 3-step.

Examples

Let $\mathfrak{n}_8$ be the 8-dimensional 2-step nilpotent Lie algebra

$$de^i = 0, \ \ i = 1, \dots, 7, \ \ de^8 = e^{12} - e^{34}. \tag{6.2.2}$$

It has been shown in [DF1, DF2] that $\mathfrak{n}_8$ admits a hypercomplex structure $\{J_\alpha\}$ defined as

$$\begin{aligned} &J_1 e_1 = e_2, \ \ J_1 e_3 = e_4, \ \ J_1 e_5 = e_6, \ \ J_1 e_7 = e_8, \\ &J_2 e_1 = e_3, \ \ J_2 e_2 = -e_4, \ \ J_2 e_5 = e_7, \ \ J_2 e_6 = -e_8, \\ &J_3 e_1 = e_4, \ \ J_3 e_2 = e_3, \ \ J_3 e_5 = e_8, \ \ J_3 e_6 = e_7. \end{aligned} \tag{6.2.3}$$

In particular, since $\mathfrak{n}_8^1 = \text{span}\{e_8\}$ and the center $\mathfrak{z}(\mathfrak{n}_8) = \text{span}\{e_5, e_6, e_7, e_8\}$, J_α is 2-step for any $\alpha = 1, 2, 3$.

Consider $\mathfrak{g} = \mathfrak{n}_8 \oplus \mathbb{R}^4$ and fix the standard basis $\{e_9, e_{10}, e_{11}, e_{12}\}$ of $\mathbb{R}^4$. We point out that $\{I_\alpha\}$ are defined as

$$\begin{aligned} &I_1 e_9 = e_{10}, \ \ I_1 e_{11} = e_{12}, \\ &I_2 e_9 = e_{11}, \ \ I_2 e_{10} = -e_{12}, \\ &I_3 e_9 = e_{12}, \ \ I_3 e_{10} = e_{11}. \end{aligned}$$

Example 6.2.2. We consider the following Lie bracket $[\cdot,\cdot]'$ on $\mathfrak{g}$ given by $[\cdot,\cdot]' = [\cdot,\cdot] + \mu$, where $\mu \in \bigwedge^2 \mathfrak{n}_8^* \otimes \mathbb{R}^4$ is defined as

$$\mu = e^{15} \otimes e_9 + e^{25} \otimes e_{10} + e^{35} \otimes e_{11} + e^{45} \otimes e_{12}.$$

Since $\mu(e_8, \cdot) = 0$, and Equation 6.2.1 is satisfied for any $\alpha = 1, 2, 3$, the 2-step nilpotent Lie algebra $(\mathfrak{g} = \mathfrak{n}_8 \oplus \mathbb{R}^4, [\cdot, \cdot]')$ can be endowed with the hypercomplex structure $\{J'_\alpha = J_\alpha \oplus I_\alpha\}$. In particular, since $\mu(J_3 e_8, \cdot) = \mu(-e_5, \cdot) \neq 0$, J'_3 is 3-step nilpotent, while J'_1, J'_2 are 2-step nilpotent.

To sum up, the 2-step nilpotent Lie algebra $\mathfrak{g}$ defined by the structure equations

$$
\begin{aligned}
&de^i = 0, \quad i = 1, \dots, 7, \quad de^8 = e^{12} - e^{34},\\
&de^9 = -e^{15}, \ \ de^{10} = -e^{25},\\
&de^{11} = -e^{35}, \ \ de^{12} = -e^{45},
\end{aligned}
$$

can be endowed with a hypercomplex structure $\{J'_\alpha\}$, where J'_1, J'_2 are 2-step nilpotent and J'_3 is 3-step nilpotent.

In order to construct a 2-step hypercomplex nilpotent Lie algebra in which two of the complex structures are 3-step nilpotent and one is 2-step nilpotent, we consider the same underlying vector space $\mathfrak{g} = \mathfrak{n}_8 \oplus \mathbb{R}^4$ as above, but we modify the Lie bracket by changing the 2-form μ. Let us define on $\mathfrak{g}$ the Lie bracket $[\cdot, \cdot]' = [\cdot, \cdot] + \mu$, where $\mu \in \bigwedge^2 \mathfrak{n}_8^* \otimes \mathbb{R}^4$ is now defined as

$$
\mu = \left(e^{15} + e^{16}\right) \otimes e_9 + \left(e^{25} + e^{26}\right) \otimes e_{10} + \left(e^{35} + e^{36}\right) \otimes e_{11} + \left(e^{45} + e^{46}\right) \otimes e_{12}.
$$

Since $\mu(e_8, \cdot) = 0$, and Equation 6.2.1 is satisfied for any $\alpha = 1, 2, 3$, the 2-step nilpotent Lie algebra $(\mathfrak{g} = \mathfrak{n}_8 \oplus \mathbb{R}^4, [\cdot, \cdot]')$ can be endowed with the hypercomplex structure $\{J'_\alpha = J_\alpha \oplus I_\alpha\}$. In particular, since $\mu(J_2 e_8, \cdot) = \mu(e_6, \cdot) \neq 0$ and $\mu(J_3 e_8, \cdot) = \mu(-e_5, \cdot) \neq 0$, J'_2 and J'_3 are 3-step nilpotent, while J'_1 is 2-step nilpotent.

In summary, the 2-step nilpotent Lie algebra $\mathfrak{g}$ defined by the structure equations

$$
\begin{aligned}
&de^i = 0, \quad i = 1, \dots, 7, \quad de^8 = e^{12} - e^{34},\\
&de^9 = -e^{15} - e^{16}, \ \ de^{10} = -e^{25} - e^{26},\\
&de^{11} = -e^{35} - e^{36}, \ \ de^{12} = -e^{45} - e^{46},
\end{aligned}
$$

can be endowed with a hypercomplex structure $\{J'_\alpha\}$, where J'_1 is 2-step nilpotent and J'_2 and J'_3 are 3-step nilpotent.

We point out that both of examples constructed above are still Obata flat by Corollary 6.1.2.

Example 6.2.3. In this last example we construct a 2-step hypercomplex nilpotent Lie algebra where each complex structure is 3-step. Let us consider again $\mathfrak{g} = \mathfrak{n}_8 \oplus \mathbb{R}^4$ with bracket $[\cdot, \cdot]'$ defined as $[\cdot, \cdot]' = [\cdot, \cdot] + \mu$ where $\mu \in \bigwedge^2 \mathfrak{n}_8^* \otimes \mathbb{R}^4$ has the following definition:

$$
\mu = e^{56} \otimes e_9 - e^{67} \otimes e_{10} + e^{57} \otimes e_{12}.
$$

Since $\mu(e_8,\cdot)=0$, and Equation 6.2.1 is satisfied for any $\alpha=1,2,3$, the 2-step nilpotent Lie algebra $(\mathfrak{g}=\mathfrak{n}_8\oplus\mathbb{R}^4,[\cdot,\cdot]')$ can be endowed with the hypercomplex structure $\{J'_\alpha = J_\alpha\oplus I_\alpha\}$. Since $\mu(J_1e_8,\cdot)=\mu(-e_7,\cdot)\neq 0$, $\mu(J_2e_8,\cdot)=\mu(e_6,\cdot)\neq 0$ and $\mu(J_3e_8,\cdot)=\mu(-e_5,\cdot)\neq 0$, J'_1, J'_2 and J'_3 are 3-step nilpotent complex structures.

To sum up, the 2-step nilpotent Lie algebra $\mathfrak{g}$ defined by the structure equations

$$\begin{aligned} &de^i=0,\quad i=1,\dots,7,\quad de^8=e^{12}-e^{34},\\ &de^9=-e^{56},\quad de^{10}=0,\quad de^{11}=e^{67},\quad de^{12}=-e^{57},\end{aligned}$$

admits a hypercomplex structure $\{J'_\alpha\}$, where J'_1, J'_2 and J'_3 are 3-step nilpotent. We claim that this hypercomplex Lie algebra is not Obata flat. Using Formula (1.3.1):

$$R'(e_8,e_1)e_1=\nabla^{Ob'}_{e_8}\nabla^{Ob'}_{e_1}e_1=-\frac{1}{2}\nabla^{Ob'}_{e_8}e_7=-\frac{1}{4}e_9,$$

where we used that $\nabla^{Ob'}_{e_8}e_1=0$ and $[e_1,e_8]=0$.

In Section 6.4, we will prove that if we consider the corresponding 2-step hypercomplex nilpotent Lie group, then the holonomy algebra $\mathfrak{hol}(\nabla^{Ob'})$ is abelian.

We point out that the 2-step nilpotent Lie groups corresponding to the three Lie algebras constructed in this section admit lattices [Mal].

Semidirect product construction

We present next a construction which is a slight generalization of [BF, Lemma 3.1].

Let $(\mathfrak{g},\{J_\alpha\})$ be an arbitrary hypercomplex Lie algebra and let $\{I_\alpha\}$ be the standard hypercomplex structure on $\mathbb{R}^{4k}$. Let $\rho:\mathfrak{g}\to\mathfrak{gl}(k,\mathbb{H})$ be a Lie algebra homomorphism, that is,

$$\rho([X,Y])=[\rho(X),\rho(Y)],\quad \rho(X)I_\alpha=I_\alpha\,\rho(X),\quad \text{for all } X,\,Y\in\mathfrak{g},\ \alpha=1,2,3.$$

Consider the Lie algebra $\mathfrak{h}:=\mathfrak{g}\ltimes_\rho\mathbb{R}^{4k}$, that is,

$$[(X,V),(Y,W)]=([X,Y],\rho(X)W-\rho(Y)V),\quad X,Y\in\mathfrak{g},\ V,W\in\mathbb{R}^{4k}.$$

The Jacobi identity on $\mathfrak{h}$ follows from the fact that ρ is a Lie algebra homomorphism. Note that $\{(0,V):V\in\mathbb{R}^{4k}\}$ is an abelian ideal of $\mathfrak{h}$. Let $\widetilde{J}_\alpha$, $\alpha=1,2,3$, be the endomorphisms of $\mathfrak{h}$ defined by $\widetilde{J}_\alpha=J_\alpha\oplus I_\alpha$.

Proposition 6.2.4. *Let $(\mathfrak{g},\{J_\alpha\})$ and $(\mathfrak{h},\{\widetilde{J}_\alpha\})$ be as above, then $\{\widetilde{J}_\alpha\}$ is a hypercomplex structure on $\mathfrak{h}$. Moreover, the Obata connection $\widetilde{\nabla^{Ob}}$ of $\{\widetilde{J}_\alpha\}$ and the curvature tensor $\widetilde{R}$ of*

$\widetilde{\nabla^{Ob}}$ *are given by:*

$$\begin{aligned}\widetilde{\nabla^{Ob}}_{(X,V)}(Y,W) &= (\nabla^{Ob}_X Y, \rho(X)W),\\ \widetilde{R}\left((X,V),(Y,W)\right)(Z,U) &= (R(X,Y)Z,0)\,, \quad X,Y,Z\in\mathfrak{g},\ V,W,U\in\mathbb{R}^{4k},\end{aligned}$$

where ∇^{Ob} *is the Obata connection of* $\{J_\alpha\}$ *and* R *is the curvature tensor of* ∇^{Ob}*. In particular,* $\mathfrak{hol}(\widetilde{\nabla^{Ob}}) \cong \mathfrak{hol}(\nabla^{Ob})$.

Proof. Since $\{J_\alpha\}$ is integrable on $\mathfrak{g}$ and $\rho(X)$ commutes with I_α for all $X\in\mathfrak{g}$, $\alpha = 1,2,3$, it turns out that $\{\widetilde{J}_\alpha\}$ is integrable on $\mathfrak{h}$.

The rest of the proof follows the lines of [BF, Lemma 3.1]. q.e.d.

Lemma 6.2.5.

(1) *The center and commutator ideal of the Lie algebra* $\mathfrak{h}$ *are given by:*

$$\begin{aligned}\mathfrak{z}(\mathfrak{h}) &= \left\{(X,V) : X\in\mathfrak{z}(\mathfrak{g})\cap\ker\rho,\ V\in\bigcap_{Y\in\mathfrak{g}}\ker\rho(Y)\right\},\\ \mathfrak{h}^1 &= \left\{(X,V) : X\in\mathfrak{g}^1,\ V\in\rho(\mathfrak{g})\mathbb{R}^{4k}\right\}.\end{aligned}$$

(2) *If* $\mathfrak{g}$ *is* l*-step nilpotent and*

$$m_\rho = \min\{j : \rho(X_1)\cdots\rho(X_j) = 0,\ \textit{for all}\ \ X_1,\dots,X_j\in\mathfrak{g}\},$$

then $\mathfrak{h}$ *is* k*-step nilpotent, where* $k = \max\{l, m_\rho\}$*. In particular, if* $\mathfrak{g}$ *is* 2*-step nilpotent and* $\rho(X_1)\cdot\rho(X_2) = 0$ *for any* $X_1,X_2\in\mathfrak{g}$*, then* $\mathfrak{h}$ *is* 2*-step nilpotent.*

(3) *If* $\mathfrak{h}$ *is 2-step nilpotent, then* $\widetilde{J}_\alpha$ *is 2-step nilpotent if and only if* $J_\alpha\mathfrak{g}^1\subset\mathfrak{z}(\mathfrak{g})\cap\ker\rho$*. In particular, if* J_α *is 3-step nilpotent, then* $\widetilde{J}_\alpha$ *is 3-step nilpotent.*

Proof. The only non-trivial statement is the second, which follows from the identity

$$\operatorname{ad}^k_{(X,U)} = \left(\begin{array}{c|c}\operatorname{ad}^k_X & 0\\ \hline B^k_U & \rho(X)^k\end{array}\right),$$

where the operator $B^k_U : \mathfrak{g}\to\mathbb{R}^{4k}$ is given by

$$B^k_U Y = \left(\sum_{l=1}^{k}(-1)^l\binom{k}{l}\rho(X)^{k-l}\rho(Y)\rho(X)^{l-1}\right)U.$$

q.e.d.

New examples of Obata non flat 2-step hypercomplex nilpotent Lie algebras can be obtained by applying the semidirect product construction, as we show next.

Example 6.2.6. Let $\mathfrak{g}$ be the Lie algebra from Example 6.2.3 with the hypercomplex structure considered there. We define $\rho : \mathfrak{g} \to \mathfrak{gl}(2, \mathbb{H}) \subset \mathfrak{gl}(8, \mathbb{R})$ by:

$$\rho(e_1) = \begin{bmatrix} 0_4 & 0_4 \\ I_4 & 0_4 \end{bmatrix}, \qquad \rho(e_i) = 0, \quad i \geq 2,$$

where I_4 and 0_4 are the real 4×4 identity and zero matrices, respectively. It follows from Example 6.2.3, Proposition 6.2.4 and Lemma 6.2.5 that $\{\widetilde{J}_\alpha\}$ is not Obata flat. Theorem 6.4.1 in Section 6.4 below implies that the holonomy algebra $\mathfrak{hol}(\widetilde{\nabla^{Ob}})$ of the corresponding 2-step hypercomplex nilpotent Lie group is abelian.

Example 6.2.7. Let $\mathfrak{g}$ be as in Example 6.2.3 and set $\rho : \mathfrak{g} \to \mathfrak{gl}(k, \mathbb{H}) \subset \mathfrak{gl}(4k, \mathbb{R})$, $k \geq 3$, defined by:

$$\rho(e_1) = \begin{bmatrix} 0_4 & 0_4 & \cdots & \cdots & 0_4 \\ I_4 & 0_4 & 0_4 & \cdots & 0_4 \\ 0_4 & I_4 & 0_4 & \cdots & 0_4 \\ \vdots & \vdots & \ddots & \ddots & 0_4 \\ 0_4 & 0_4 & 0_4 & I_4 & 0_4 \end{bmatrix} \in M_{4k}(\mathbb{R}), \qquad \rho(e_i) = 0, \quad i \geq 2.$$

In this case, $\mathfrak{h}$ is k-step nilpotent and $\{\widetilde{J}_\alpha\}$ is not Obata flat. By Proposition 6.2.4, the holonomy algebra coincides with the holonomy algebra of the Example 6.2.3. In particular, it is abelian (see Section 6.4).

We note that the simply connected Lie groups corresponding to the Lie algebras from examples 6.2.6 and 6.2.7 admit lattices, according to [Mal].

6.3 $\mathbb{H}$-solvability of 2-step nilpotent Lie algebras

We recall the following definition.

Definition 6.3.1 ([Gor1])**.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with a hypercomplex structure (J_1, J_2, J_3). Define inductively $\mathbb{H}$-invariant Lie subalgebras:*

$$\mathfrak{g}_k^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] = [\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] + J_1[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] + J_2[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] + J_3[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}],$$

where $\mathfrak{g}_0^{\mathbb{H}} = \mathfrak{g}$. Note that $\mathfrak{g}_1^{\mathbb{H}} := \mathbb{H}\mathfrak{g}^1 = \mathfrak{g}^1 + J_1\mathfrak{g}^1 + J_2\mathfrak{g}^1 + J_3\mathfrak{g}^1$.

A hypercomplex nilpotent Lie algebra $\mathfrak{g}$ is called $\mathbb{H}$-solvable *if the following sequence eventually becomes zero for some $k \in \mathbb{N}$:*

$$\mathfrak{g}_1^{\mathbb{H}} \supset \mathfrak{g}_2^{\mathbb{H}} \supset \cdots \supset \mathfrak{g}_{k-1}^{\mathbb{H}} \supset \mathfrak{g}_k^{\mathbb{H}} = 0. \tag{6.3.1}$$

The smallest such k is called the $\mathbb{H}$-solvability step of $(\mathfrak{g}, \{J_\alpha\})$.

Lemma 6.3.2. *Let $(\mathfrak{g}, \{J_\alpha\})$ be a hypercomplex nilpotent Lie algebra. Then for any $k \in \mathbb{N}$, $\mathfrak{g}_k^{\mathbb{H}}$ is a subalgebra of $\mathfrak{g}$ and it is an ideal in $\mathfrak{g}_{k-1}^{\mathbb{H}}$.*

Proof. Since $\mathfrak{g}^1 \subset \mathfrak{g}_{\mathbb{H}}^1$, $\mathfrak{g}_1^{\mathbb{H}}$ is clearly a subalgebra. By induction, assume that $\mathfrak{g}_{k-1}^{\mathbb{H}}$ is a subalgebra. Then

$$[\mathfrak{g}_k^{\mathbb{H}}, \mathfrak{g}_k^{\mathbb{H}}] = [\mathbb{H}[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}], \mathbb{H}[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}]].$$

Since $\mathfrak{g}_{k-1}^{\mathbb{H}}$ is a subalgebra $[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] \subset \mathfrak{g}_{k-1}^{\mathbb{H}}$. In particular

$$[\mathfrak{g}_k^{\mathbb{H}}, \mathfrak{g}_k^{\mathbb{H}}] \subset [\mathbb{H}\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathbb{H}\mathfrak{g}_{k-1}^{\mathbb{H}}] = [\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] \subset \mathfrak{g}_k^{\mathbb{H}}.$$

Since $\mathfrak{g}_k^{\mathbb{H}}$ is a subalgebra for each k, we have:

$$[\mathfrak{g}_k^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] = [\mathbb{H}[\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}], \mathfrak{g}_{k-1}^{\mathbb{H}}] \subset [\mathbb{H}\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] = [\mathfrak{g}_{k-1}^{\mathbb{H}}, \mathfrak{g}_{k-1}^{\mathbb{H}}] \subset \mathfrak{g}_k^{\mathbb{H}}.$$

q.e.d.

Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra endowed with a complex structure J. It was shown by Salamon [Sal2, Corollary 1.4] that the ideal $\mathfrak{g}^1 + J\mathfrak{g}^1$ is a proper ideal of $\mathfrak{g}$. This naturally raises the question of whether an analogous property holds for nilpotent Lie algebras endowed with a hypercomplex structure.

The condition that $\mathfrak{g}_1^{\mathbb{H}} = \mathfrak{g}^1 + J_1\mathfrak{g}^1 + J_2\mathfrak{g}^1 + J_3\mathfrak{g}^1$ is a proper ideal of $\mathfrak{g}$ is necessary to the $\mathbb{H}$-solvability of $(\mathfrak{g}, \{J_\alpha\})$. This observation motivates the following conjecture:

Conjecture 6.3.3 ([Gor1])**.** Let $(\mathfrak{g}, \{J_\alpha\})$ be a hypercomplex nilpotent Lie algebra. Then it is $\mathbb{H}$-solvable.

When the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is $\mathbb{H}$-solvable, it turns out that, for a generic complex structure J_y in the hypercomplex 2-sphere, the complex nilmanifold $(\Gamma\backslash G, J_y)$ does not admit any complex curves [Gor1].

The conjecture holds true for Lie algebras of hypercomplex nilmanifolds with flat Obata connection [Gor2] and for nilpotent Lie algebras with abelian hypercomplex structure [AV, Proposition 4.5]. Here we prove the conjecture for 2-step nilpotent Lie algebras.

Theorem 6.3.4. *Let $(\mathfrak{g}, \{J_\alpha\})$ be a 2-step hypercomplex nilpotent Lie algebra. Then $(\mathfrak{g}, \{J_\alpha\})$ is $\mathbb{H}$-solvable. In particular, $\mathfrak{g}_1^{\mathbb{H}}$ is a proper ideal of $\mathfrak{g}$ and the $\mathbb{H}$-solvability step is at most 3.*

Proof. To prove Theorem 6.3.4, we show that the $\mathbb{H}$-solvable series (6.3.1) terminates at 0. In particular, this implies that for any $k \in \mathbb{N}_{\geq 1}$, $\mathfrak{g}_k^{\mathbb{H}} \subsetneq \mathfrak{g}$, proving that $\mathfrak{g}_1^{\mathbb{H}}$ is a proper ideal of the Lie algebra $\mathfrak{g}$.

Let us consider

$$\mathfrak{g}_2^{\mathbb{H}} = \mathbb{H}[\mathfrak{g}_1^{\mathbb{H}}, \mathfrak{g}_1^{\mathbb{H}}].$$

By Lemma 1.1.9,

$$\begin{aligned}[\mathfrak{g}_1^{\mathbb{H}}, \mathfrak{g}_1^{\mathbb{H}}] =& [\mathfrak{g}^1 + J_1\mathfrak{g}^1 + J_2\mathfrak{g}^1 + J_3\mathfrak{g}^1, \mathfrak{g}^1 + J_1\mathfrak{g}^1 + J_2\mathfrak{g}^1 + J_3\mathfrak{g}^1] \\ =& [J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] + [J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + [J_2\mathfrak{g}^1, J_3\mathfrak{g}^1].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{H}[\mathfrak{g}_1^{\mathbb{H}}, \mathfrak{g}_1^{\mathbb{H}}] = & [J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] + [J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + [J_2\mathfrak{g}^1, J_3\mathfrak{g}^1] \\ & + J_1([J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] + [J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + [J_2\mathfrak{g}^1, J_3\mathfrak{g}^1]) \\ & + J_2([J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] + [J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + [J_2\mathfrak{g}^1, J_3\mathfrak{g}^1]) \\ & + J_3([J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] + [J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + [J_2\mathfrak{g}^1, J_3\mathfrak{g}^1]).\end{aligned}$$

Since by Equation 1.1.2 $J_\alpha[J_\alpha\mathfrak{g}^1, J_\beta\mathfrak{g}^1] \subset \mathfrak{g}^1$, we may write:

$$\mathfrak{g}_2^{\mathbb{H}} = \mathbb{H}[\mathfrak{g}_1^{\mathbb{H}}, \mathfrak{g}_1^{\mathbb{H}}] \subset \mathfrak{g}^1 + J_1[J_2\mathfrak{g}^1, J_3\mathfrak{g}^1] + J_2[J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] + J_3[J_1\mathfrak{g}^1, J_2\mathfrak{g}^1].$$

We claim that $J_1[J_2\mathfrak{g}^1, J_3\mathfrak{g}^1] = J_2[J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] = J_3[J_1\mathfrak{g}^1, J_2\mathfrak{g}^1]$. Using the quaternionic identities and Equation 1.1.2 with J_1, we have:

$$J_2[J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] = J_3J_1[J_1\mathfrak{g}^1, J_3\mathfrak{g}^1] = J_3[J_1\mathfrak{g}^1, J_2\mathfrak{g}^1],$$

and, similarly, using the integrability of J_2,

$$J_3[J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] = J_1J_2[J_1\mathfrak{g}^1, J_2\mathfrak{g}^1] = J_1[J_3\mathfrak{g}^1, J_2\mathfrak{g}^1].$$

Thus, all three summands coincide and we obtain that $\mathfrak{g}_2^{\mathbb{H}} \subset \mathfrak{g}^1 + J_1[J_2\mathfrak{g}^1, J_3\mathfrak{g}^1] \subset \mathfrak{g}^1 + J_1\mathfrak{g}^1$. Therefore,

$$\mathfrak{g}_3^{\mathbb{H}} = \mathbb{H}[\mathfrak{g}_2^{\mathbb{H}}, \mathfrak{g}_2^{\mathbb{H}}] \subset \mathbb{H}[J_1\mathfrak{g}^1, J_1\mathfrak{g}^1] = 0,$$

since $J_1\mathfrak{g}^1$ is abelian by Lemma 1.1.9. q.e.d.

6.4 Holonomy of 2-step hypercomplex nilmanifolds

We use Theorem 6.3.4 to prove a structural result about the holonomy algebra of the Obata connection on 2-step nilpotent Lie groups endowed with invariant hypercomplex structures.

Theorem 6.4.1. *Let G^{4n} be a 2-step nilpotent Lie group endowed with an invariant hypercomplex structure $\{J_\alpha\}$. Then the holonomy Lie algebra $\mathfrak{hol}(\nabla^{Ob})$ is abelian, where ∇^{Ob} is* *the Obata connection of* $(G, \{J_\alpha\})$.

In particular, it is an abelian subalgebra of $\mathfrak{sl}(n, \mathbb{H})$ *with trivial product, that is, for any* $A, B \in \mathfrak{hol}(\nabla^{Ob}),\ A \cdot B = 0$.

Proof. The proof is divided into 4 steps. We start by proving

- **Step 1**: for any $X, Y, Z \in \mathfrak{g}$, $R(X, Y)Z \in \mathfrak{g}^1 \cap J_2\mathfrak{g}^1 + J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$.

  For $(\alpha, \beta, \gamma) = (1, 2, 3)$, each term appearing in Formula (6.1.1) falls into one of four distinct types:

  1. $J_2[\cdot, J_2[\cdot, \cdot]] = [J_2\cdot, J_2[\cdot, \cdot]] \in \mathfrak{g}^1 \cap J_2\mathfrak{g}^1$, where the equality follows by Equation (1.1.2),

  2. $[\cdot, J_2[\cdot, \cdot]] \in \mathfrak{g}^1 \cap J_2\mathfrak{g}^1$, as $J_2\mathfrak{g}^1$ is an ideal by Lemma 1.1.9,

  3. $J_3[\cdot, J_2[\cdot, \cdot]] = J_1J_2[\cdot, J_2[\cdot, \cdot]] = J_1[J_2\cdot, J_2[\cdot, \cdot]] \in J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$, where the first equality follows by the quaternionic identities and the second equality follows by Equation (1.1.2),

  4. $J_1[\cdot, J_2[\cdot, \cdot]] = -J_3J_2[\cdot, J_2[\cdot, \cdot]] = -J_3[J_2\cdot, J_2[\cdot, \cdot]] \in J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$, where the first equality follows by the quaternionic identities and the second equality follows again by Equation (1.1.2).

  We set $\mathfrak{h} := \mathfrak{g}^1 \cap J_2\mathfrak{g}^1 + J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$. It is not difficult to note that $\mathfrak{h}$ is $\{J_\alpha\}$-invariant and $\mathfrak{h} \subset \mathfrak{g}_1^{\mathbb{H}}$.

- **Step 2**: for any $X \in \mathfrak{h}$, $\nabla^{Ob}X = 0$.

  We use the following:

  Lemma 6.4.2. *For any* $X \in \mathfrak{h}$ *and* $Y \in \mathfrak{g}$,

  $$[Y, X] + J_1[J_1Y, X] = 0 \tag{6.4.1}$$

  Proof. Let $X \in \mathfrak{h}$. Then $X = X_1 + X_2$, where $X_1 \in \mathfrak{g}^1 \cap J_2\mathfrak{g}^1$ and $X_2 \in J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$. Since $X_1 \in \mathfrak{g}^1$,

  $$[Y, X] + J_1[J_1Y, X] = [Y, X_2] + J_1[J_1Y, X_2].$$

  Using that $X_2 \in J_1\mathfrak{g}^1$, $[X_2, Y] = -J_1[X_2, J_1Y]$ by Equation (1.1.2). Therefore,

  $$[Y, X] + J_1[J_1Y, X] = -J_1[J_1Y, X_2] + J_1[J_1Y, X_2] = 0.$$

  q.e.d.

Let $Y \in \mathfrak{g}$ and $X \in \mathfrak{h}$. Applying Formula (1.3.1)

$$\begin{aligned} 2\nabla^{Ob}_Y X &= [Y, X] + J_1[J_1Y, X] - J_2[Y, J_2X] + J_3[J_1Y, J_2X] \\ &= [Y, X] + J_1[J_1Y, X] - J_2([Y, J_2X] + J_1[J_1Y, J_2X]) \\ &= 0, \end{aligned}$$

where we used twice Lemma 6.4.2 and the $\{J_\alpha\}$-invariance of $\mathfrak{h}$.

- **Step 3:** Expression for the curvature endomorphisms and ∇^{Ob}.

By Theorem 6.3.4, the ideal $\mathfrak{g}_1^{\mathbb{H}}$ is a proper ideal of $\mathfrak{g}$ containing $\mathfrak{h}$. Therefore, we may write $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{g}_1^{\mathbb{H}}$, where $\mathfrak{k}_1 \neq 0$ is any $\{J_\alpha\}$-invariant complement of $\mathfrak{g}_1^{\mathbb{H}}$. Similarly, we decompose $\mathfrak{g}_1^{\mathbb{H}} = \mathfrak{h}_1 \oplus \mathfrak{h}$, where $\mathfrak{h}_1$ is any $\{J_\alpha\}$-invariant complement of $\mathfrak{h}$ inside $\mathfrak{g}_1^{\mathbb{H}}$ (such complements always exist, as we can consider the orthogonal complements with respect to a fixed hyperHermitian metric). We note that $\mathfrak{h}_1$ could be trivial; nonetheless, the proof proceeds in exactly the same manner without any differences.

With respect to the decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}$, using Step 1 and Step 2, we get that for any $X, Y \in \mathfrak{g}$,

$$R(X, Y) = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline * & * & 0 \end{array}\right).$$

We have the following two Lemmas:

Lemma 6.4.3. *Let $X, Y \in \mathfrak{g}$. Then $\nabla^{Ob}_X Y \in \mathfrak{g}_1^{\mathbb{H}}$.*

Proof. Straightforward by Formula (1.3.1). q.e.d.

Lemma 6.4.4. *Let $X \in \mathfrak{g}$. Then $\nabla^{Ob}_X \mathfrak{g}_1^{\mathbb{H}} \subset \mathfrak{h}$.*

Proof. Since $\nabla^{Ob}_X J_\alpha = J_\alpha \nabla^{Ob}_X$ and $\mathfrak{h}$ is $\{J_\alpha\}$-invariant, it suffices to prove that $\nabla^{Ob}_X \mathfrak{g}^1 \subset \mathfrak{h}$. Applying Formula (1.3.1), we see that for any $Y \in \mathfrak{g}^1$:

$$\nabla^{Ob}_X Y = \frac{1}{2}\big(- J_2[X, J_2Y] + J_3[J_1X, J_2Y]\big).$$

By Step 1, the first term lies in $\mathfrak{g}^1 \cap J_2\mathfrak{g}^1$, while the second term lies in $J_1\mathfrak{g}^1 \cap J_3\mathfrak{g}^1$. The proof follows by the definition of $\mathfrak{h}$. q.e.d.

By the previous Lemmas we get that with respect to the decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}$:

$$\nabla^{Ob}_X = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline * & 0 & 0 \\ \hline * & * & 0 \end{array}\right). \tag{6.4.2}$$

- **Step 4:** Application of the Ambrose-Singer Theorem.

  Having in mind the left invariant version of the Ambrose-Singer Theorem (see Theorem 5.2.2), let us consider span$\{R(X,Y) \mid X, Y \in \mathfrak{g}\}$. As already observed in Step 3, with respect to the decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}$,

$$\text{span}\{R(X,Y) \mid X, Y \in \mathfrak{g}\} \subset \left\{ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline * & * & 0 \end{array} \right) \in \mathfrak{sl}(n, \mathbb{H}) \right\}.$$

  We set $\mathfrak{a} := \left\{ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline * & * & 0 \end{array} \right) \in \mathfrak{sl}(n, \mathbb{H}) \right\}$. We point out that $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{sl}(n, \mathbb{H})$.

  Let $A \in \mathfrak{a}$ and let $X \in \mathfrak{g}$. Using the expression of ∇^{Ob}_X given in Step 3, the commutator

$$[A, \nabla^{Ob}_X] \in \mathfrak{a}.$$

  Since $\mathfrak{a}$ is closed under commutators with $\nabla^{Ob}_{\mathfrak{g}}$, by Theorem 5.2.2 we get that $\mathfrak{hol}(\nabla^{Ob}) \subset \mathfrak{a}$. In particular, since $\mathfrak{a}$ is abelian with trivial product, Theorem 6.4.1 is proved.

q.e.d.

Example 6.4.5. Let $(\mathfrak{g}, \{J_\alpha\})$ be the 2-step nilmanifold hypercomplex Lie algebra constructed in Example 6.2.3 and let us consider the corresponding 2-step hypercomplex nilpotent Lie group G. As already observed, the Obata connection ∇^{Ob} on G is not flat. A direct computation shows that the holonomy algebra $\mathfrak{hol}(\nabla^{Ob})$ is given by

$$\mathfrak{hol}(\nabla^{Ob}) = \left\{ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline q & p \cdot j & 0 \end{array} \right) \in \mathfrak{sl}(3, \mathbb{H}) \mid q \in \mathbb{H},\ p \in \mathbb{R} \right\},$$

where we are considering $\mathfrak{g} = \text{span}_{\mathbb{H}}\{e_1, e_5, e_9\}$.

This provides an explicit example of a non-trivial, abelian holonomy algebra arising from a non-flat Obata connection on a 2-step nilmanifold.

Let G be a 2-step nilpotent Lie group endowed with an invariant hypercomplex structure $\{J_\alpha\}$. If the Obata connection is flat, then the full holonomy group is discrete. A natural question is whether this could be trivial. To address this question, we prove the following theorem.

Theorem 6.4.6. *Let* $(\Gamma\backslash G, \{J_\alpha\})$ *be a* 2*-step nilmanifold endowed with an invariant hypercomplex structure* $\{J_\alpha\}$ *such that the Obata connection is flat. Then the holonomy group* $\mathrm{Hol}(\nabla^{Ob})$ *is trivial if and only if* $\Gamma\backslash G$ *is a torus.*

Proof. The implication from right to left is clear. Let us now prove the converse.

The Obata connection is complex with respect to each J_α and has vanishing (and hence parallel) torsion. Moreover, as shown in the proof of Theorem 6.4.1, for any $X \in \mathfrak{g}$, the endomorphism $\nabla^{Ob}_X : \mathfrak{g} \to \mathfrak{g}$ is nilpotent (see Equation (6.4.2)). Since the Obata connection ∇^{Ob} is flat and torsion-free, it follows from [Kim, Theorem 2.2 and Proposition 1.2] that ∇^{Ob} is complete.

We apply [ABD, Theorem 4.1] to conclude that $(\Gamma\backslash G, J_\alpha, \nabla^{Ob})$ is affinely biholomorphic to $(\Gamma_0\backslash G_0, J^0_\alpha, \nabla^0)$ for each α, where G_0 is the connected and simply connected Lie group whose Lie algebra is given by

$$P^{\nabla^{Ob}} = \{X \in \mathfrak{X}(\Gamma\backslash G) \mid \nabla^{Ob} X = 0\},$$

Γ_0 is a lattice in G_0, J^0_α is the complex structure induced on $P^{\nabla^{Ob}}$ by J_α and ∇^0 is the $(-)$-connection.

Since ∇^{Ob} is torsion free, for any $X, Y \in P^{\nabla^{Ob}} = \mathfrak{g}_0$, $[X, Y] = 0$. In particular, G_0 is abelian and $\Gamma_0\backslash G_0$ is a torus.

Since $\Gamma\backslash G$ is diffeomorphic to $\Gamma_0\backslash G_0$, then G is isomorphic to G_0 as a Lie group [Mal]. It then follows that G is abelian, and so $\Gamma\backslash G$ is also a torus. q.e.d.

Remark 6.4.7. Theorem 6.4.6 still holds true if instead of considering a 2-step nilmanifold we assume that $\Gamma\backslash G$ is a solvmanifold with G a completely solvable almost abelian Lie group. Indeed, in this case the Obata connection is automatically flat, according to [AB1, Proposition 3.7], and is complete, due to [AB1, Corollary 3.8]. For the last paragraph, we use the Saito rigidity Theorem [Sai], which extends the result of Mal'cev from nilpotent to completely solvable Lie groups.

6.5 3-step case

Let $(\mathfrak{g}, \{J_\alpha\})$ be a Lie algebra endowed with a hypercomplex structure such that $J_\alpha\mathfrak{g}^1 \subset \mathfrak{z}$ for all $\alpha = 1, 2, 3$. We claim that $\mathfrak{g}$ is at most 3-step nilpotent. Indeed, using the integrability of J_α and the assumption $J_\alpha\mathfrak{g}^1 \subset \mathfrak{z}$, one obtains

$$[[X, Y], Z] = -J_\alpha[[X, Y], J_\alpha Z], \tag{6.5.1}$$

which shows that $\mathfrak{g}^2 \subset \mathfrak{z}$, and hence $\mathfrak{g}^3 = [\mathfrak{g}^2, \mathfrak{g}] = 0$. Note that by Equation (6.5.1), $[\mathfrak{g}^1, \mathfrak{g}^1] = 0$.

Since in the 2-step case the corresponding Obata connection is flat (see Corollary 6.1.2), we will focus our attention on the 3-step case. We observe that the hypercomplex structure must be non-abelian. Indeed, if $\{J_\alpha\}$ is abelian and $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$, then

$$\mathfrak{g}^1 \subset J_\alpha \mathfrak{z} = \mathfrak{z},$$

which forces $\mathfrak{g}$ to be 2-step nilpotent.

Examples of 3-step nilpotent Lie algebras with hypercomplex structures have been given in [DF3, BD, AB2]. Among these, the only examples with non-abelian hypercomplex structures are those in [AB2]; however, those are Obata flat and none of them satisfies the condition $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$ for all $\alpha = 1, 2, 3$.

Due to the lack of examples of this kind in the literature, we include here a 3-step nilpotent example satisfying $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$ for any $\alpha = 1, 2, 3$.

Let $\mathfrak{g}$ be the 3-step nilpotent Lie algebra defined by the structure equations

$$\begin{aligned}
&de^i = 0, \;\; i = 1, \ldots, 4, \;\; de^5 = e^{12} - e^{34}, \\
&de^i = 0, \;\; i = 6, 7, 8, \;\; de^9 = e^{13} - e^{42}, \\
&de^i = 0, \;\; i = 10, 11, 12, \;\; de^{13} = -e^{25} + e^{39}, \\
&de^{14} = e^{15} + e^{49}, \;\; de^{15} = e^{45} - e^{19}, \;\; de^{16} = -e^{35} - e^{29}.
\end{aligned}$$

Then $\mathfrak{g}$ can be endowed with the following hypercomplex structure

$$\begin{aligned}
&J_1 e_1 = e_2, \;\; J_1 e_3 = e_4, \;\; J_1 e_5 = e_6, \;\; J_1 e_7 = e_8, \\
&J_1 e_9 = e_{10}, \;\; J_1 e_{11} = e_{12}, \; J_1 e_{13} = e_{14}, \;\; J_1 e_{15} = e_{16}, \\
&J_2 e_1 = e_3, \;\; J_2 e_2 = -e_4, \;\; J_2 e_5 = e_7, \;\; J_2 e_6 = -e_8, \\
&J_2 e_9 = e_{11}, \;\; J_2 e_{10} = -e_{12}, J_2 e_{13} = e_{15}, \;\; J_2 e_{14} = -e_{16}, \\
&J_3 e_1 = e_4, \;\; J_3 e_2 = e_3, \;\; J_3 e_5 = e_8, \;\; J_3 e_6 = e_7, \\
&J_3 e_9 = e_{12}, \;\; J_3 e_{10} = e_{11}, \;\; J_3 e_{13} = e_{16}, \;\; J_3 e_{14} = e_{15},
\end{aligned}$$

satisfying $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$, for each $\alpha = 1, 2, 3$.

Proposition 6.5.1. *Let $(\mathfrak{g}, \{J_\alpha\})$ be a hypercomplex Lie algebra such that $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$ for each $\alpha = 1, 2, 3$. Then*

$$R(X, Y) = -\operatorname{ad}_{[X,Y]},$$

and $(\mathfrak{g}, \{J_\alpha\})$ is non-flat if and only if $\mathfrak{g}$ is 3-step nilpotent.
Let G be the corresponding connected and simply connected 3-step nilpotent Lie group. Then the holonomy algebra of the Obata connection on G is an abelian subalgebra of $\mathfrak{sl}(n, \mathbb{H})$.

Proof. As we pointed out at the beginning of this section, $\mathfrak{g}$ is at most 3-step nilpotent.

Applying Formula (1.3.1) and the hypothesis that $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$ for each α, we get that

$$\begin{aligned}\nabla^{Ob}_X \nabla^{Ob}_Y Z &= \frac{1}{4}([X,[Y,Z]] + J_1[J_1X,[Y,Z]] - J_2[X,[Y,J_2Z]] + J_3[J_1X,[Y,J_2Z]]) \\ &= 0,\end{aligned}$$

where we used Equation (6.5.1). Therefore,

$$R(X,Y)Z = -\nabla^{Ob}_{[X,Y]} Z = -\frac{1}{2}([[X,Y],Z] - J_2[[X,Y],J_2Z]) = -[[X,Y],Z],$$

where the last equality again follows from Equation (6.5.1).

Set $\mathfrak{a} := \operatorname{span}\{\operatorname{ad}_{[X,Y]} \mid X,Y \in \mathfrak{g}\}$. Then, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{sl}(n,\mathbb{H})$. In fact,

$$\operatorname{ad}_{[X,Y]} \operatorname{ad}_{[Z,W]} U = [[X,Y],[[Z,W],U]] = 0,$$

as $\mathfrak{g}$ is 3-step nilpotent.

Furthermore, using that $J_\alpha \mathfrak{g}^1 \subset \mathfrak{z}$, $\mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}^1] \subset \mathfrak{z}$ and $[\mathfrak{g}^1,\mathfrak{g}^1] = 0$,

$$\begin{aligned}2\nabla^{Ob}_U \operatorname{ad}_{[X,Y]} Z = {} & [U,[[X,Y],Z]] + [J_1U,[[X,Y],Z]] \\ & - J_2[U,J_2[[X,Y],Z]] + J_3[J_1U,J_2[[X,Y],Z]] = 0,\end{aligned}$$

and

$$2\operatorname{ad}_{[X,Y]} \nabla^{Ob}_U Z = [[X,Y],[U,Z]] = 0.$$

By Theorem 5.2.2, $\mathfrak{a} = \mathfrak{hol}(\nabla^{Ob})$, concluding the proof. q.e.d.

Theorem 6.4.1 and Proposition 6.5.1 highlight an unexpected rigidity in the holonomy algebra of the Obata connection. A related result is proved in [Ge], where the author shows that if the hypercomplex structure is abelian then the holonomy algebra of the Obata connection is abelian.

In the 3-step nilpotent setting, as recalled above, all previously known examples were either Obata-flat or equipped with an abelian hypercomplex structure (the example constructed in this section is neither Obata-flat nor endowed with an abelian hypercomplex structure). Nevertheless, in every known case, the holonomy algebra of the Obata connection is abelian. This recurring behavior suggests that such rigidity might be a general feature of the nilpotent setting, motivating the following question:

Question 6.5.2. Let G be a nilpotent Lie group endowed with a left-invariant hypercomplex structure. Is it true that the holonomy algebra of the Obata connection is always abelian?

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