Controlling a Social Network of Individuals With Coevolving Actions and Opinions

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***Abstract*—We consider a population of individuals who have actions and opinions, which coevolve, mutually influencing one another on a complex social network. In particular, we formulate a control problem in which we assume that we can inject into the network a committed minority —a set of stubborn nodes— with the objective of steering the population, initially at a consensus, to a different consensus state. Our study focuses on two main objectives: i) determining the conditions under which the committed minority succeeds in its goal, and ii) identifying the optimal placement for such a committed minority. After deriving general monotone convergence result for the controlled dynamics, we leverage these results to build a computationally-efficient algorithm to solve the first problem and an effective heuristic for the second problem, which we prove to be NP-complete. For both algorithms, we establish theoretical guarantees and we demonstrate them though academic and real-world case studies.**

***Index Terms*—evolutionary game theory, opinion dynamics, social networks**

I. Introduction

OVER the past decades, the systems and control community witnessed a growing interest in mathematical modeling of complex social phenomena and collective human behavior [1]–[8]. Particular interest has been devoted to collective decision-making, whereby a population of individuals have to repeatedly make decisions on a specific action to take (often binary). For instance, this scenario often arises in different contexts of social change problems: individuals may decide whether to use a disposable a reusable cup to have a coffee, or whether to use inclusive language or not when writing an email. In these contexts, empirical evidence and social psychology theories suggest that decision-making is deeply intertwined with opinion formation processes [9], [10]. This calls for the development of model paradigms able to integrate these two dynamics in a coevolutionary fashion.

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A. Literature Review

A first step in the direction of integrating these two dynamics is the continuous-opinion discrete-action model [11], and in its main extensions [12], [13]. This model is built on classical opinion dynamics models [2], and relies on the assumption that the opinion formation process entirely shapes the decision-making, whereby actions are a quantization of opinions. Despite relevance for many applications, this assumption limits the possibility to capture the presence of a misalignment between individuals' opinions and their actions. This is the case, e.g., for the phenomenon of unpopular norms [14], [15], whereby a community keeps exhibiting a collective behavior that is disapproved by the most of its members.

This limitation was addressed in [16], [17], with the proposal of a coevolutionary model of actions and opinions, using a game-theoretic framework to jointly model an opinion formation process and a decision-making process [18], [19]. This allows individuals to simultaneously revise their (binary) actions and share their opinions (their support for the action) on a network, accounting for social pressure, opinion influence, and self-consistency. The analysis of this model has shown its ability to reproduce several real-life phenomena, including the emergence and persistence of unpopular norms and polarization. Here, we take a further step, moving from the analysis of social systems to their control.

In the literature, the two most common approaches for controlling social systems are to assume that a policymaker can either i) provide monetary or societal incentives to favor a specific opinion/action [20]–[24] or ii) directly control a committed minority of stubborn agents to exert influence on normal agents. We focus on the latter approach, which has been extensively studied in the contexts of opinion dynamics and decision-making models separately. In the former context, determining the set of agents with maximal influence is often an NP-hard problem [25], and so different heuristics have been proposed, e.g., leveraging a submodularity property [25]–[27]. In the latter context, the problem is also typically NP-hard [28]. Efforts have thus focused on determining graph-theoretic conditions under which the stubborn agents are able to steer the whole population [18], experimentally evaluating the critical mass of stubborn agents needed [29], [30], and establishing effective heuristics to determine their optimal placement [28], [31]. However, distinct models for actions and opinions are used, meaning the same limitations discussed above are still present.

To the best of our knowledge, Ref. [32] is the first (preliminary) effort to address the control problem in the coevolutionary model of actions and opinions, whereby a committed minority is introduced with the objective of steering a population, from one initial consensus state to a different one. Ref. [32] proposed an algorithm to determine whether the committed minority will achieve their goal, but no theoretical guarantees were established, and the problem of determining the optimal placement of the committed minority is still unexplored. Here, we build on this preliminary effort along several major lines, as detailed in the following.

B. Contribution

Building on the model proposed in [17], we focus on the control problem of unlocking a paradigm shift. Specifically, we consider a population initially at a consensus in which all individuals select and support the same action. Then, we introduce a committed minority of stubborn agents with the goal of steering the entire population to a consensus on the opposite action. Stubborn agents may consistently select the opposite action, share opinions supporting it, or both, generalizing the approach from [32], where committed agents necessarily perform both activities. This problem is relevant to many real-life applications. Unlocking a paradigm shift is key for social change, e.g., to favor the collective transition towards more sustainable practices. From the opposite perspective, understanding how malicious agents can steer an entire population is critical to guarantee safety and robustness of our social systems [33].

By leveraging systems and control theoretic tools, we study the controlled model, establishing key properties and theoretical results, including monotonic convergence to an equilibrium point. Building on these theoretical findings, we establish necessary and sufficient conditions for the set of stubborn nodes to unlock a paradigm shift, depending on the model parameters and on the network structure. Then, we deal with the problem of identifying the minimal committed minority needed to control a generic network; Ref. [32] only studied specific network topologies by levering their symmetry. After proving that the problem is NP-complete, we draw inspiration from [31] to design and demonstrate a novel iterative algorithm for its solution.

In detail, the main contribution of this paper is four-fold. First, we incorporate a control action in the coevolutionary model [17] and we formulate two control problems: determining if a control action is sufficient to steer the population to the desired consensus (*effectiveness guarantee problem*), and identifying the minimal set of nodes to be controlled to achieve such a goal (*minimal control set identification problem*). Second, we prove an array of general properties for the controlled dynamics, including convergence, and we characterize the complexity of our research problems, demonstrating that the minimal control set identification problem is NP-complete, which limits the possibility to adopt classical heuristics to approximate its solution. Third, we propose an algorithm to solve the effectiveness guarantee problem, prove its validity and characterize its computational complexity, and use it to evaluate the impact of the model parameters on a synthetic case study, where a closed-form solution of the problem can be derived. Fourth, we propose an iterative algorithm for the minimal control set identification problem with probabilistic convergence properties, and we demonstrate its efficiency on a synthetic case study and a real-world network, whose network topology is reconstructed from face-to-face contact data [34].

The rest of the paper is organized as follows. In Section II, we introduce the controlled coevolutionary dynamics and we formulate our two research problems. In Section III, we present some general results on the uncontrolled and controlled coevolutionary dynamics. Sections IV and V are devoted to the solution of the effectiveness guarantees and the minimal control set problems, respectively, with related case studies. Section VI concludes the paper.

II. Model and Problem Statement

Notation. We denote a vector $\boldsymbol{x}$ with bold lowercase font, with x_i its ith entry; and a matrix $\boldsymbol{A}$ with bold capital font, and a_{ij} the jth entry of its ith row. The all-1 column vector is denoted as $\mathbf{1}$, with appropriate dimension depending on the context. Given two vectors $\boldsymbol{x}, \boldsymbol{y}$ with same dimension, we use $\boldsymbol{x} \leq \boldsymbol{y}$ to denote $x_i \leq y_i$, for all entries i.

A. (Uncontrolled) Coevolutionary Model

We consider a population $\mathcal{V} = \{1, \dots, n\}$ of n individuals. Each $i \in \mathcal{V}$ is associated with a two-dimensional state variable $(x_i(t), y_i(t)) \in \{-1, +1\} \times [-1, +1]$, with discrete time t: $x_i(t) \in \{-1, +1\}$ is the *action* of individual i at time t, $y_i(t) \in [-1, +1]$ their *opinion* on the action ($y_i(t) = -1$ means that i is totally in favor of action -1, $y_i(t) = +1$ that i fully supports action $+1$). Actions and opinions are gathered in vectors $\boldsymbol{x}(t) \in \{-1, 1\}^n$ and $\boldsymbol{y}(t) \in [-1, 1]^n$, and the state of the system is represented by the joint $2n$-dimensional vector $\boldsymbol{z}(t) := (\boldsymbol{x}(t), \boldsymbol{y}(t)) \in \{-1, 1\}^n \times [-1, 1]^n$. Given $i \in \mathcal{V}$, we define as $\boldsymbol{z_{-i}} := (\boldsymbol{x_{-i}}, \boldsymbol{y_{-i}}) \in \{-1, 1\}^{n-1} \times [-1, 1]^{n-1}$ the $(2n-2)$-dimensional vector with the state of all other individuals. At each time step t, we define a set $\mathcal{R}(t) \subseteq \mathcal{V}$ of individuals who simultaneously revise their state at time t.

Assumption 1 (Revision sequence)**.** *There exists a constant $T < \infty$ such that $\cup_{s=0}^{T-1} \mathcal{R}(t+s) = \mathcal{V}$, for any $t \geq 0$.*

Remark 1. *Assumption 1 encompasses many synchronous and asynchronous update rules: for synchronous update rules, $\mathcal{R}(t) = \mathcal{V}$ for all t; for asynchronous update rules, $\mathcal{R}(t)$ comprises a single individual. The requirement $\cup_{s=0}^{T-1} \mathcal{R}(t+s) = \mathcal{V}$ closely mirrors typical assumptions in opinion dynamics with time-varying networks [3, Section 3], as it ensures that over T consecutive time steps, every agent activates at least once. Note that no restrictions are imposed on the ordering of the agent activations, allowing for both deterministic or stochastic mechanisms to determine the activation sequence.*

At time t, each individual $i \in \mathcal{R}(t)$ updates their state, aiming to maximize the utility function defined in [17], that accounts for three contributions: i) individuals' tendency to coordinate actions; ii) opinions exchanged with peers; and iii)

TABLE I: Models variables and parameters.

$x_i(t) \in \{-1,+1\}$	action of individual i at time t
$y_i(t) \in [-1,+1]$	opinion of individual i at time t
$a_{ij} \in [0,1]$	influence of j's action on i
$w_{ij} \in [0,1]$	influence of j's opinion on i
$\lambda_i \in (0,1]$	weight of actions
$\beta_i \in (0,1]$	weight of opinions

an individual's tendency to have consistency between their action and opinion. Following [17], we define the utility that i receives for selecting an action and opinion pair $\boldsymbol{z_i} = (x_i, y_i)$ when the state of the others is $\boldsymbol{z_{-i}}$ as

$$u_i(\boldsymbol{z_i}, \boldsymbol{z_{-i}}) = \tfrac{\lambda_i(1-\beta_i)}{2} \sum_{j\in\mathcal{V}} a_{ij}\big[(1-x_j)(1-x_i) + (1+x_j)\cdot (1+x_i)\big] - \beta_i(1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij}(y_i-y_j)^2 - \lambda_i\beta_i(x_i-y_i)^2, \quad (1)$$

where $a_{ij} \in [0,1]$ and $w_{ij} \in [0,1]$ are the influence of individual j's action and opinion, respectively; and $\lambda_i \in (0,1]$ and $\beta_i \in (0,1]$ the weights given to actions observed and opinions exchanged, respectively. The quantities a_{ij} and w_{ij} are gathered into two matrices $\boldsymbol{A}$ and $\boldsymbol{W}$, which we assume to be stochastic (i.e., $\boldsymbol{A}\mathbf{1} = \boldsymbol{W}\mathbf{1} = \mathbf{1}$). Such a structure induce a two-layer network $\mathcal{G} = (\mathcal{V}, \mathcal{E}_A, \boldsymbol{A}, \mathcal{E}_W, \boldsymbol{W})$, where $\mathcal{E}_A$ are the edges on the influence layer on which individuals see others' actions and $\mathcal{E}_W$ are the edges on the communication layer, on which individuals discuss about their opinions. All parameters are summarized in Table I. Before explicitly presenting the uncontrolled coevolutionary dynamics, we make some observations on Eq. (1), with proof reported in Appendix A.

Proposition 1. *A game with the utility function in Eq. (1) is supermodular.*

In Eq. (1), we enforce $\lambda_i > 0$ and $\beta_i > 0$ to guarantee a nontrivial coupling between the two variables. In the limit case in which one of these parameters is equal to 0, the coevolutionary model would reduce to a simpler (and well-known) dynamics, as commented in the following.

Remark 2. *The utility in Eq. (1) generalizes classical network majority (and coordination) games [18], [19] (obtained for $\lambda_i = 1$) and the French-DeGroot opinion dynamics model [2], [35] (for $\beta_i = 1$). See, [17] for more details.*

We are now ready to present the coevolutionary dynamics, in which agents who activate seek to maximize their utility function in Eq. (1). Consequently, for each $i \in \mathcal{V}$, the action and opinion are revised as follows:

$$x_i(t+1), y_i(t+1) = \begin{cases} \underset{\boldsymbol{z_i}\in\{-1,1\}\times[-1,1]}{\operatorname{argmax}} \; u_i(\boldsymbol{z_i}, \boldsymbol{z_{-i}}) & i \in \mathcal{R}(t), \\ x_i(t), y_i(t) & i \notin \mathcal{R}(t), \end{cases} \quad (2)$$

with the convention that, when the $\operatorname{argmax} u_i(\boldsymbol{z_i}, \boldsymbol{z_{-i}})$ comprises multiple elements, we set $x_i(t+1) = x_i(t)$. In other words, each individual $i \in \mathcal{R}(t)$ performs a joint best-response with respect to Eq. (1), as illustrated in Fig. 1.

In Proposition 1, we proved that the game is supermodular, which is key to prove convergence of Eq. (2). However, to characterize its equilibria, we need to derive a closed-form expression for Eq. (2), following [17], [32].

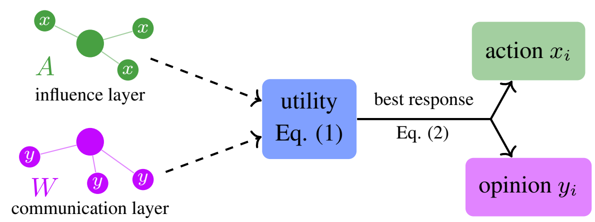


Fig. 1: Schematic of the update rule for a generic individual $i \in \mathcal{R}(t)$.

Proposition 2. *Individual $i \in \mathcal{R}(t)$ updates their state as:*

$$x_i(t+1) = s(\boldsymbol{z}(t)), \quad (3a)$$
$$y_i(t+1) = (1-\lambda_i)\sum\nolimits_{j\in\mathcal{V}} w_{ij}y_j(t) + \lambda_i s(\boldsymbol{z}(t)), \quad (3b)$$

where

$$s(\boldsymbol{z}(t)) = \begin{cases} +1 & \text{if } \delta_i(\boldsymbol{z}(t)) > 0, \\ -1 & \text{if } \delta_i(\boldsymbol{z}(t)) < 0, \\ x_i(t) & \text{if } \delta_i(\boldsymbol{z}(t)) = 0, \end{cases}$$

with

$$\delta_i(\boldsymbol{z}(t)) = 2\beta_i(1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij}y_j(t) + (1-\beta_i)\sum_{j\in\mathcal{V}} a_{ij}x_j(t).$$

From Proposition 2, we derive the following observation.

Proposition 3 (Proposition 3 from [32])**.** *The (uncontrolled) coevolutionary dynamics in Eq. (2) has at least two equilibria: $\boldsymbol{x} = \boldsymbol{y} = -\mathbf{1}$ and $\boldsymbol{x} = \boldsymbol{y} = \mathbf{1}$, being the unique equilibria in which the action vector is at a consensus ($x_i = x_j$, $\forall\, i, j \in \mathcal{V}$).*

B. Controlled Dynamics and Problem Statement

We study a scenario in which, at time $t = 0$, the population is at one consensus equilibrium and we want to steer it to the opposite one. Without loss of generality, assume the starting consensus is $\boldsymbol{x}(0) = \boldsymbol{y}(0) = -\mathbf{1}$, and thus our goal is to reach $\boldsymbol{x} = \boldsymbol{y} = +\mathbf{1}$. We consider this problem from the perspective of a policymaker/designer, and assume that our control lever is in the form of directly acting on the state of a subset of agents by setting their opinion and/or action to $+1$ for all $t \geq 1$, yielding the following assumption (illustrated in Fig. 2).

Assumption 2 (Controlled dynamics)**.** *Consider a two-layer network $\mathcal{G} = (\mathcal{V}, \mathcal{E}_A, \boldsymbol{A}, \mathcal{E}_W, \boldsymbol{W})$ with $\boldsymbol{A}$ and $\boldsymbol{W}$ stochastic and irreducible. Given $\mathcal{C}^X$ the set of controlled actions, and $\mathcal{C}^Y$ the set of controlled opinions, there holds*

$$\begin{cases} x_i(t) = +1 & \forall i \in \mathcal{C}^X, \forall t \geq 1, \\ y_j(t) = +1 & \forall j \in \mathcal{C}^Y, \forall t \geq 1, \\ x_i(0) = -1 & \forall i \in \mathcal{V}\backslash\mathcal{C}^X, \\ y_j(0) = -1 & \forall j \in \mathcal{V}\backslash\mathcal{C}^Y. \end{cases} \quad (4)$$

For $i \notin \mathcal{C}^X$, if $i \in \mathcal{R}(t)$, then $x_i(t+1)$ follows Eq. (3a); for $i \notin \mathcal{C}^Y$, if $i \in \mathcal{R}(t)$, then $y_i(t+1)$ follows Eq. (3b).

Remark 3. *We identify three scenarios of particular interest:*

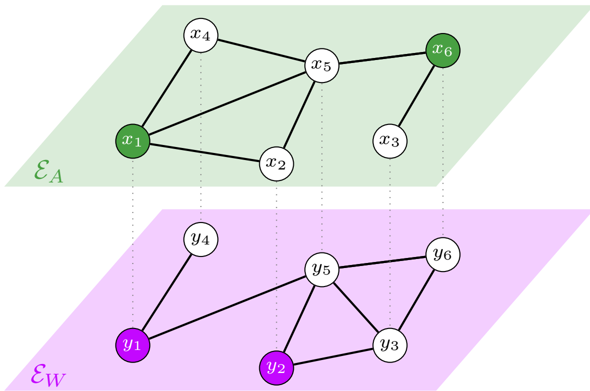


Fig. 2: Example of a setting that satisfies Assumption 2. The top layer represents individuals' action (white for -1, green for $+1$), the bottom layer represents individuals' opinion (shades from white for -1 to violet for $+1$). Control sets are $\mathcal{C}^X = \{1, 6\}$ and $\mathcal{C}^Y = \{1, 2\}$.

1) **opinion control**, *in which one controls only individuals' opinions,* $\mathcal{C}^X = \emptyset$;
2) **action control**, *in which one controls only individuals' actions,* $\mathcal{C}^Y = \emptyset$;
3) **joint control**, *in which one controls both variables* ($\mathcal{C}^X = \mathcal{C}^Y$), *which is the case considered in [32].*

These three scenarios reflect different possible real-world interventions, which act only on opinions, on actions, or on both, capturing technical limitations or constraints which may prevent a policymaker from controlling both layers.

Hereafter, we will refer to a *controlled coevolutionary dynamics* as a coevolutionary dynamics with utility function in Eq. (1), under Assumptions 1 and 2. The goal of the controller, i.e., to lead all the agents to the desired consensus, can be formalized by first defining the objective function

$$\phi(\mathcal{C}^X, \mathcal{C}^Y) := \mathbb{P}[\exists T < \infty : \boldsymbol{x}(t) = \mathbf{1}, \forall t \geq T], \quad (5)$$

i.e., the probability (over the probability space generated by the revision sequence, which might be stochastic, as observed in Remark 1) that all individuals definitively switch their action to $+1$ in finite time when the control sets are $(\mathcal{C}^X, \mathcal{C}^Y)$. The controller's goal is achieved if and only if (iff) $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$. Hence, we formalize the following research problem.

Problem 1 (Effectiveness guarantees)**.** *Given a network $\mathcal{G}$, consider a controlled evolutionary dynamics on the network under Assumptions 1 and 2 with specified parameters. For given control sets $(\mathcal{C}^X, \mathcal{C}^Y)$, compute $\phi(\mathcal{C}^X, \mathcal{C}^Y)$.*

Solving Problem 1 would allow us to determine whether controlling the opinion and/or action of some nodes is sufficient to guarantee convergence to the desired consensus state. At this stage, a second question naturally follows: what is the minimal set of individuals that one should control in order to guarantee that the goal is achieved? Clearly, when controlling an individual, technical limitations may prevent from controlling both actions and opinions, as discussed in Remark 3. Hence, when formulating the problem, we introduce two additional constraints to denote the set of nodes whose action and opinion can be controlled as $\mathcal{V}^X$ and $\mathcal{V}^Y$, respectively, allowing us to define the problem as follows.

Problem 2 (Minimal control set)**.** *Given a network $\mathcal{G}$, consider a controlled evolutionary dynamics on the network under Assumptions 1 and 2 with specified model parameters. Determine the solution to the following optimization problem*

$$\begin{array}{rl} \arg\min_{\mathcal{C}^X \subseteq \mathcal{V}, \mathcal{C}^Y \subseteq \mathcal{V}} & |\mathcal{C}^X \cup \mathcal{C}^Y| \\ \textit{s.t.} & \phi(\mathcal{C}^X, \mathcal{C}^Y) = 1, \\ & \mathcal{C}^X \subseteq \mathcal{V}^X,\ \mathcal{C}^Y \subseteq \mathcal{V}^Y, \end{array} \quad (6)$$

where $\mathcal{V}^X \subseteq \mathcal{V}$ and $\mathcal{V}^Y \subseteq \mathcal{V}$ are constraints on the nodes whose action and opinion can be controlled, respectively.

Remark 4. *By setting $\mathcal{V}^X$ and $\mathcal{V}^Y$, one can enforce a specific form for the solution. In fact, by setting $\mathcal{V}^X = \emptyset$ or $\mathcal{V}^Y = \emptyset$, we obtain solutions of Problem 2 with opinion or action control, respectively, i.e., the first two scenarios discussed in Remark 3. On the contrary, if the same constraints are imposed on the two sets ($\mathcal{V}^X = \mathcal{V}^Y$), if a solution to Eq. (6) exists, then there is necessarily a solution with joint control ($\mathcal{C}^X = \mathcal{C}^Y$), as it will be clear in the next section, after Corollary 1.*

III. Main Properties of the Controlled Dynamics

In general, the uncontrolled dynamics requires restrictive assumptions for convergence, such as homogeneous parameters, symmetric layers, and self-loops [17] or asynchronous updates (see Proposition 1). For the controlled dynamics, instead, only mild conditions on the activation sequence (Assumption 1) and on the weight matrices (Assumption 2) are needed.

Theorem 1. *Consider a controlled coevolutionary dynamics under Assumptions 1–2. Then, there exists an equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ such that the action vector $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}^*$ in finite time, and the opinion vector $\boldsymbol{y}(t)$ converges to $\boldsymbol{y}^*$ asymptotically. Moreover, both the opinion and action vectors are monotonically nondecreasing functions of time, i.e., $\boldsymbol{x}(t+1) \geq \boldsymbol{x}(t)$ and $\boldsymbol{y}(t+1) \geq \boldsymbol{y}(t)$, for all $t \geq 0$.*

Since the proof, reported in Appendix B, is based on Proposition 1, the irreducibility of $\boldsymbol{W}$ and $\boldsymbol{A}$ (equivalently the strong connectivity of each layer of $\mathcal{G}$) are not strictly required for convergence. However, some of our later results rely on the irreducibility property. If a network is not strongly connected, one can partition it into strongly connected components and control each separately.

Remark 5. *Supermodularity of the game implies that all trajectories of the controlled dynamics are lower-bounded by the trajectory with initial condition that satisfies Assumption 2. Hence, $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$ is a sufficient condition to reach the desired consensus from any initial condition, not just from the one that satisfies Assumption 2, for which the condition is also necessary. Consequently, a solution of Problem 2, which is optimal (in the sense of the solution being a minimal control set) under Assumption 2, will also yield a feasible solution for any initial condition, but optimality might be lost.*

Theorem 1 guarantees that under the hypotheses of Assumption 2 the controlled coevolutionary dynamics converges and that actions converge in finite time. Moreover, it also guarantees monotonicity of the trajectory of the state vector $\boldsymbol{z}(t)$. As a consequence, if $i \notin \mathcal{C}^X$ switches to action $+1$ at a certain time, then i will never flip back. This observation will be fundamental to development of a systematic approach to addressing our research problems, as presented in Sections IV and V. Finally, it is worth noticing that the hypothesis in Assumption 2 on the uncontrolled node dynamics are key for obtaining monotonicity (which then yields convergence). In fact, from a general initial condition, one may observe non-monotone trajectories (see, e.g., [17]). Besides monotonicity, we bring attention to the following property of the controlled dynamics, proved in Appendix C.

Lemma 1. *Let us consider control sets* $(\mathcal{C}^X, \mathcal{C}^Y)$. *If* $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$, *then all control sets* $(\bar{\mathcal{C}}^{\mathcal{X}}, \bar{\mathcal{C}}^{\mathcal{Y}})$ *such that* $\mathcal{C}^{\mathcal{X}} \subseteq \bar{\mathcal{C}}^{\mathcal{X}}$ *and* $\mathcal{C}^{\mathcal{Y}} \subseteq \bar{\mathcal{C}}^{\mathcal{Y}}$ *satisfy* $\phi(\bar{\mathcal{C}}^{\mathcal{X}}, \bar{\mathcal{C}}^{\mathcal{Y}}) = 1$. *If* $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 0$, *then all control sets* $(\bar{\mathcal{C}}^{\mathcal{X}}, \bar{\mathcal{C}}^{\mathcal{Y}})$ *such that* $\bar{\mathcal{C}}^{\mathcal{X}} \subseteq \mathcal{C}^{\mathcal{X}}$ *and* $\bar{\mathcal{C}}^{\mathcal{Y}} \subseteq \mathcal{C}^{\mathcal{Y}}$ *satisfy* $\phi(\bar{\mathcal{C}}^{\mathcal{X}}, \bar{\mathcal{C}}^{\mathcal{Y}}) = 0$.

An immediate consequence of Lemma 1 is the following, whose proof is reported in Appendix D.

Corollary 1. *If Problem 2 admits a solution, then there is always a solution such that, letting* $\mathcal{C} := \mathcal{C}^X \cup \mathcal{C}^Y$, *then* $\mathcal{C}^X = \mathcal{C} \cap \mathcal{V}^X$ *and* $\mathcal{C}^Y = \mathcal{C} \cap \mathcal{V}^Y$.

In plain words, Corollary 1 states that, if it is possible to control both the action and the opinion of a given individual, then it is always optimal to either control both variables or to control none. As a straightforward consequence of Lemma 1 and Corollary 1, Problem 2 can be reduced to an optimization problem over a single control set, as stated in the following.

Corollary 2. *Let*

$$\begin{aligned} \mathcal{C}^* = \arg\min\nolimits_{\mathcal{C} \subseteq \mathcal{V}} \quad & |\mathcal{C}| \\ \text{s.t.} \quad & \phi(\mathcal{C} \cap \mathcal{V}^X, \mathcal{C} \cap \mathcal{V}^Y) = 1. \end{aligned} \tag{7}$$

Then, an optimal solution of Problem 2 is given by $C^X = \mathcal{C}^* \cap \mathcal{V}^X$ *and* $C^Y = \mathcal{C}^* \cap \mathcal{V}^Y$.

It is worth noticing that, despite the useful properties of the controlled coevolutionary dynamics demonstrated in the above, the problem of controlling the dynamics and determining the minimal control sets is inherently complex, as stated in the following result, with the proof in Appendix E.

Theorem 2. *Problem 2 is NP-complete.*

Moreover, we can show that the objective function in Eq. (5) is not submodular, hindering the possibility to easily derive sub-optimal solutions via greedy algorithms [36], as is done for related control problems on social networks [28]. The proof is reported in Appendix F.

Proposition 4. *The function* $\phi(\mathcal{C}^X, \mathcal{C}^Y)$ *in Eq. (5) is not submodular with respect to any of its two variables.*

IV. Effectiveness Guarantees Problem

The results from the previous section call for the development of algorithms able to solve our research problem in an efficient way. We start from Problem 1. In our preliminary work [32], we proposed an algorithm and conjectured that this algorithm can solve Problem 1 in the simplified setting of joint control, $\mathcal{C}^X = \mathcal{C}^Y$ (see Remark 3). Here, we propose a refined version of the algorithm that accounts for the more general setting in Assumption 2, while also reducing the total computational complexity. Then, we rigorously prove the conjecture in the general scenario, demonstrating that the proposed algorithm solves Problem 1 in polynomial time. Our procedure is based on the following iterative scheme.

We consider the general case in which we control the action for nodes in $\mathcal{C}^X$ and the opinion for nodes in $\mathcal{C}^Y$ as described in Assumption 2. At iteration $k = 1$, we initialize the algorithm by defining $\mathcal{A}(1) = \mathcal{C}^X$, which involve only the nodes whose action is controlled. At each step of the algorithm k, we construct a candidate equilibrium with action vector $\hat{\boldsymbol{x}}$ with

$$\hat{x}_i = \begin{cases} +1 & \text{if } i \in \mathcal{A}(k), \\ -1 & \text{if } i \notin \mathcal{A}(k), \end{cases} \tag{8}$$

and opinion vector $\hat{\boldsymbol{y}}$, computed by solving the linear system

$$\hat{y}_i = \begin{cases} (1-\lambda_i)\sum_{j \in \mathcal{V}} w_{ij}\hat{y}_j + \lambda_i \hat{x}_i & \text{if } i \notin \mathcal{C}^Y, \\ +1 & \text{if } i \in \mathcal{C}^Y, \end{cases} \tag{9}$$

which has a unique solution, as we will prove later. Observe that solving Eq. (9) requires inverting a matrix that is independent of the variables. This operation can be optimized by computing it in advance, before running the iterations.

Then, we will demonstrate in Theorem 3 below the following properties. First, we show that given an action vector $\hat{\boldsymbol{x}}$, there exists a unique $\hat{\boldsymbol{y}}$ such that $\hat{\boldsymbol{z}} = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is a candidate equilibrium of the controlled coevolutionary dynamics. To check whether $\hat{\boldsymbol{z}}$ is an actual equilibrium, we check if any individual i who plays action -1 at $\hat{\boldsymbol{z}}$ would switch to $+1$. According to Proposition 2, this can be checked by computing the sign of $\delta_i(\hat{\boldsymbol{z}})$ for all $i \notin \mathcal{A}(k)$. If all $\delta_i(\hat{\boldsymbol{z}}) \leq 0$, then no individual will switch action, and the candidate $\hat{\boldsymbol{z}}$ is indeed the equilibrium reached by the system. Otherwise, we will prove that all individuals with $\delta_i(\hat{\boldsymbol{z}}) > 0$ will eventually switch to $+1$. Hence, $\hat{\boldsymbol{z}}$ is not an equilibrium of the controlled coevolutionary dynamics, and we need to consider other potential equilibria where also those individuals with $\delta_i(\hat{\boldsymbol{z}}) > 0$ switch to action $+1$. To this aim, we increase the iteration index k by 1, and we enlarge the set $\mathcal{A}(k)$ by incorporating these individuals into $\mathcal{A}(k-1)$, and we iterate the procedure, until the termination criterion $\mathcal{A}(k) = \mathcal{A}(k-1)$, which implies that no more individuals would change action. According to this procedure, we get a non-decreasing sequence of sets. When the termination criterion is met, the algorithm returns $\mathcal{A}_f$.

This algorithm, for which a computationally-improved pseudo-code is reported in Algorithm 1, offers a tool to solve Problem 1 in a polynomial time, as summarized in the following statement, whose proof is reported in Appendix G.

Theorem 3. *Under Assumptions 1 and 2, Algorithm 1 solves Problem 1 in time* $O(n^3)$. *In fact, given control sets* $(\mathcal{C}^X, \mathcal{C}^Y)$ *and output* $\mathcal{A}_f$ *of Algorithm 1, then*

$$\phi(\mathcal{C}^X, \mathcal{C}^Y) = \begin{cases} 1 & \text{if } \mathcal{A}_f = \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the equilibrium reached by a controlled coevolutionary dynamics that satisfies Assumptions 1 and 2 with control sets $(\mathcal{C}^X, \mathcal{C}^Y)$ *is* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, *with* $\boldsymbol{x}^*$ *defined as in Eq. (8) with* $\mathcal{A}(k) = \mathcal{A}_f$ *and* $\boldsymbol{y}^*$ *the solution of Eq. (9) given* $\boldsymbol{x}^*$.

Algorithm 1: Equilibrium computation

Data: $\boldsymbol{A}, \boldsymbol{W}, \mathcal{C}^X, \mathcal{C}^Y, \lambda_i$ and β_i, for all $i \in \mathcal{U}$
Result: $\mathcal{A}_f := \mathcal{A}(k)$, i.e., individuals with $x^* = +1$
$k \leftarrow 1$; $\mathcal{A}(0) \leftarrow \emptyset$; $\mathcal{A}(1) \leftarrow \mathcal{C}^X$; $\hat{y}_i \leftarrow +1\ \forall i \in \mathcal{C}^Y$;
$\boldsymbol{M} \leftarrow (\boldsymbol{I} - (\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\lambda}))\boldsymbol{W})^{-1}$;
while $\mathcal{A}(k) \neq \mathcal{A}(k-1)$ **do**
    Define $\hat{\boldsymbol{x}}$ using Eq. (8);
    $\hat{y}_i \leftarrow (\boldsymbol{M}\mathrm{diag}(\boldsymbol{\lambda})\hat{\boldsymbol{x}})_i$ for all $i \notin \mathcal{C}^Y$;
    $k \leftarrow k+1$; $\mathcal{A}(k) \leftarrow \mathcal{A}(k-1)$;
    check **for** $i \in \mathcal{V}$ & $i \notin \mathcal{A}(k)$ **do**
        if $\delta_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) > 0$ **then**
            $\mathcal{A}(k) \leftarrow \mathcal{A}(k) \cup \{i\}$;
        end
    end
end

In most practical scenarios, only a few iterations are needed for convergence, since $\mathcal{A}(k)$ often increases by more than one individual at each iteration, further reducing the computational effort needed. Moreover, the heaviest operation is the computation of matrix $\boldsymbol{M} = (\boldsymbol{I} - [\boldsymbol{1} - \mathrm{diag}(\boldsymbol{\lambda})]\boldsymbol{W})^{-1}$, which is used to solve Eq. (9). However, $\boldsymbol{M}$ is independent of the control sets. Hence, it can be precomputed and used for multiple instances of Algorithm 1. Finally, one can leverage symmetry properties of the network structure to reduce the dimension of the system, as illustrated in the following example.

A. Case Study I: Complete Graph

We consider the case study of a complete graph (including self-loops) with homogeneous parameters and weights.

Assumption 3 (Homogeneous complete graph)**.** *Let $\mathcal{G}$ be a two-layer network with $a_{ij} = w_{ij} = \frac{1}{n-1}$ and $a_{ii} = w_{ii} = 0$, $\forall\, i \neq j \in \mathcal{V}$. Moreover, let $\lambda_i = \lambda$ and $\beta_i = \beta$, $\forall\, i \in \mathcal{V}$.*

In [32], we have analyzed the scenario of a complete graph with joint control (see Remark 3). Here, we focus on the other two scenarios of interest discussed in Remark 3, viz. opinion and action control, assuming that we are able to control a certain number of individuals (whose position is irrelevant, due to the network symmetry). Using Algorithm 1 and Theorem 3, we solve Problem 1, determining whether a target control set $\mathcal{C}$ is sufficient to guarantee convergence to the desired equilibrium, in the three different cases described in Remark 3.

Proposition 5. *Consider a coevolutionary dynamics that satisfies Assumptions 1–2 on a complete graph $\mathcal{G}$ with n nodes that satisfies Assumption 3. Given $\mathcal{C} \subseteq \mathcal{V}$, let $\gamma = \frac{|\mathcal{C}|}{n-1}$. Then,*

i) *for opinion control, $\phi(\emptyset, \mathcal{C}) = 1$ iff it holds*
$$\frac{\beta[3(1-\lambda)\gamma + 2\lambda\gamma(1-\lambda) + \lambda(2\lambda-1)]}{\gamma + \lambda - \lambda\gamma} > 1; \quad (10)$$
ii) *for action control, $\phi(\mathcal{C}, \emptyset) = 1$ iff $\gamma > 1/2$;*
iii) *for joint control, $\phi(\mathcal{C}, \mathcal{C}) = 1$ iff it holds*
$$2\beta(1-\lambda)\Big(\frac{\gamma - \lambda + \lambda\gamma}{\gamma + \lambda - \lambda\gamma}\Big) + (1-\beta)(2\gamma - 1) > 0. \quad (11)$$

Proof. In all cases, we apply Algorithm 1 and Theorem 3.

i) We start with $\mathcal{A}(1) = \emptyset$. The candidate equilibrium $\hat{\boldsymbol{z}}$ has $\hat{\boldsymbol{x}} = -\boldsymbol{1}$ and $\hat{\boldsymbol{y}}$ with $\hat{y}_i = +1$ for all $i \in \mathcal{C}$, and by solving Eq. (9) for all $i \notin \mathcal{C}$. By symmetry, all $i \notin \mathcal{C}$ have necessarily $\hat{y}_i = +1$, solution of $\hat{y}_i = (1-\lambda)(\gamma + (1-\gamma)\hat{y}_i) - \lambda$, which yields $\hat{y}_i = \frac{(1-\lambda)\gamma - \lambda}{1-(1-\lambda)(1-\gamma)}$. Substituting this in $\delta_i(\hat{\boldsymbol{z}})$, we get that $\delta_i(\hat{\boldsymbol{z}}) > 0$ iff the condition in Eq. (10) is satisfied. In this scenario, $A(2) = \mathcal{V}$; otherwise, $A(2) = A(1)$. In both cases the algorithm terminates, and Theorem 3 yields claim i).

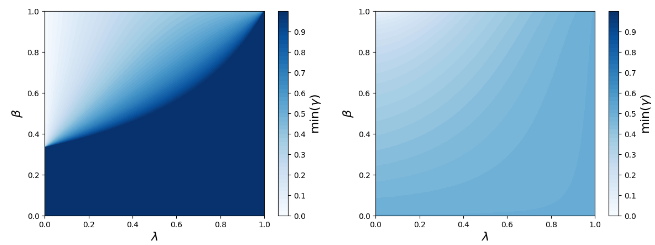
ii) We start with $\mathcal{A}(1) = \mathcal{C}$. The corresponding candidate equilibrium $\hat{\boldsymbol{z}}$ has $\hat{\boldsymbol{x}}$ defined using Eq. (8). Conversely, for the opinion vector $\hat{\boldsymbol{y}}$ we have to distinguish two different cases: $\hat{y}_C$ indicates the equilibrium value for those nodes whose action is controlled, $\hat{y}_U$ the one for those nodes which are not controlled at all. The equilibrium vector is so defined as follows:
$$\begin{cases} \hat{y}_C &= (1-\lambda)[\gamma\hat{y}_C + (1-\gamma)\hat{y}_U] + \lambda \\ \hat{y}_U &= (1-\lambda)[\gamma\hat{y}_C + (1-\gamma)\hat{y}_U] - \lambda. \end{cases} \quad (12)$$
Solving Eq. (12) and substituting the solution in Eq. (15) for any $i \notin \mathcal{C}$, we obtain $\delta_i(\hat{\boldsymbol{z}}) = (2\gamma-1)((1-2\lambda)\beta\lambda + 1)$. It is easy to verify that $((1-2\lambda)\beta\lambda + 1) > 0$ for any choice of λ and β, yielding the condition $\gamma > 1/2$. If $\gamma > 1/2$, then $\delta_i(\hat{\boldsymbol{z}}) > 0$, and $\mathcal{A}(2) = \mathcal{V}$; otherwise, $\mathcal{A}(2) = \mathcal{A}(1) = \mathcal{C}$. In both cases, the algorithm terminates, and Theorem 3 yields ii).

iii) We apply Theorem 3 to [32, Proposition 3]. □

Remark 6. *Observe that Eq. (10) and Eq. (11) are never satisfied for $\gamma = 0$, and their left-hand sides are monotonically increasing functions of γ in $[0, 1]$. We can thus re-write the expressions as inequalities of the form $\gamma > \bar{\gamma}$, where $\bar{\gamma}$ is a function of β and λ (e.g., Eq. (10) reads $\gamma > \bar{\gamma} = \frac{2\lambda}{3\beta+2\lambda-1}$). Moreover, observe that given γ, the choice of the control nodes is irrelevant for Proposition 5. Hence, any control set $\mathcal{C}$ with $|\mathcal{C}| = \lfloor\bar{\gamma}(n-1) + 1\rfloor$ solves Problem 2 for a complete graph.*

From Proposition 5 and Remark 6, we can draw some interesting conclusions. First, from item ii), we observe that for action control we need more than 50% of nodes being controlled. This is consistent with results on game-theoretic models that account only for decision making [18], [19]. On the contrary, when considering opinion control and joint control, the conditions become dependent on the model parameters. This indicates that the intertwining between actions and opinion nontrivially shapes the committed minority in social networks. Predicatively, we can show that Eq. (10) and $\gamma > 1/2$ are always more restrictive than Eq. (11), implying that joint control (if possible) is always preferable. Then, further observations can be made by plotting the minimal value of γ that satisfies Eq. (10) and Eq. (11), as detailed in Remark 6.

These plots, reported in Fig. 3 using color-code intensity, illustrate how the model parameters strongly impact the cardinality of the controlled set $\mathcal{C}$, especially for opinion control. In fact, for small values of β and large λ, effective policies should entail opinion control of a large majority of the population. A similar dependency is also present for joint control, but it is less pronounced. Interestingly, when β is large and λ is small-to-moderate (top left corner in Fig. 3a), opinion control is more effective than action control. In fact, the required fraction of



(a) Opinion Control (b) Joint Control

Fig. 3: Results for a complete graph. The color intensity represents the cardinality of the minimal control set $\bar{\gamma} = |\mathcal{C}|/(n-1)$ that solves Problem 2.

population to control can be less than 50%, which is instead the case for action control. In this vein, Proposition 5 can be used to help assist decision makers in designing policies to favor social change or increase the network robustness against malicious attacks. In fact, our model-informed insights can indicate, e.g., whether it is more effective for a committed minority to share content to influence others' opinion or to promote the visibility of their actions.

V. Minimal Control Set Identification

In the previous section, we have shown how Algorithm 1 can be used to solve our research problems for simple network structures such as a complete graph (Section IV-A) or a star graph (see [32]). However, in more realistic scenarios, the network does not have such a level of symmetry, and Algorithm 1 is not efficient to solve Problem 2. In fact, in order to find the optimal control sets $\mathcal{C}^X$ and $\mathcal{C}^Y$, one should look for all pairs of subsets of $\mathcal{V}$ to find those that guarantee that the constraint $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$ is satisfied. First, we observe that Corollary 2 allows us to simplify the problem (and the notation), since we simply need to find an optimal control set $\mathcal{C}$, and then define the solution of Problem 2 as $(\mathcal{C}^X, \mathcal{C}^Y)$ with

$$\mathcal{C}^X = \mathcal{C} \cap \mathcal{V}^X \text{ and } \mathcal{C}^Y = \mathcal{C} \cap \mathcal{V}^Y . \quad (13)$$

However, even with this simplification, the determination of the minimal control set remains computationally challenging.

With an aim to reduce the computational cost to perform such a task, we now build on the optimal targeting algorithm proposed in [31] in order to design an algorithm able to solve Problem 2. Intuitively, the rationale behind the proposed methodology consists of starting from controlling all the possible individuals and moving backwards, removing individuals from the control set, until we find the minimal control set that allows us to reach the goal in Problem 2. However, such a naive approach might result in being stuck in a local minimum, not being able to reach the global optimum. For this reason, we employ a stochastic approach, which consists of defining a discrete-time Markov chain [37] that explores the space of control sets that are feasible solutions of Problem 2 in such a way that its invariant distribution will provably concentrate about the global optimal solutions of Eq. (7), allowing us to define an effective heuristic to solve Problem 2.

More precisely, given Problem 2 with constraints $\mathcal{C}^X \subseteq \mathcal{V}^X$ and $\mathcal{C}^Y \subseteq \mathcal{V}^Y$, we define the set of all *controllable nodes* as $\mathcal{V}^* := \mathcal{V}^X \cup \mathcal{V}^Y$, and we let $n^* := |\mathcal{V}^*|$ to be the number of controllable nodes. Then, we can define the space of all *potential control sets*, which is nothing but the power set of $\mathcal{V}^*$, i.e., $\mathscr{C} := \{\mathcal{C} : \mathcal{C} \subseteq \mathcal{V}^*\}$. Moreover, we say that a control set $\mathcal{C} \in \mathscr{C}$ is *admissible* iff $\phi(\mathcal{C} \cap \mathcal{V}^X, \mathcal{C} \cap \mathcal{V}^Y) = 1$. In other words, an admissible control set is a feasible solution of Eq. (7). We denote the space of all admissible control sets as $\bar{\mathscr{C}} := \{\mathcal{C} \subseteq \mathscr{C} : \phi(\mathcal{C} \cap \mathcal{V}^X, \mathcal{C} \cap \mathcal{V}^Y) = 1\}$, which is clearly the set of all feasible solutions of Problem 2.

Our algorithm starts from the worst case in which we control all the controllable nodes, i.e., we set $\mathcal{C} = \mathcal{V}^*$. First, we need to check whether $\mathcal{C}$ is admissible, i.e., we define $\mathcal{C}^X$ and $\mathcal{C}^Y$ as in Eq. (13), and check whether $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$. This check is done by employing Algorithm 1. If $\mathcal{C}$ is admissible, then it follows that $(\mathcal{C}^X, \mathcal{C}^Y)$ is a feasible solution to Problem 2. Conversely, if $\mathcal{C}$ is not admissible, then Problem 2 is unfeasible and there is no need to proceed. Trivially, we observe that if $\mathcal{V}^X = \mathcal{V}$ then $\mathcal{C}$ is always admissible, being $\mathcal{C}^X = \mathcal{V}$, and $\phi(\mathcal{V}, \mathcal{C}^Y) = 1$ for any choice of $\mathcal{C}^Y$.

If Problem 2 is feasible, we then adopt the following iterative procedure, which is detailed in Algorithm 2. At the kth iteration we start with the control set $\mathcal{C}$. We select a node r, uniformly at random among the controllable nodes, i.e., $r \in \mathcal{V}^*$. Then, two cases are possible:

1) The node belongs to the control set ($r \in \mathcal{C}$). In this case, if the set $\mathcal{C} \setminus \{r\}$ is admissible (which is checked using Algorithm 1), then the control set is updated to $\mathcal{C} \setminus \{r\}$; otherwise it remains $\mathcal{C}$; or
2) The node does not belong to the control set ($r \notin \mathcal{C}$). In this case, we introduce a probability $\varepsilon \in (0, 1]$, and the node r is added to the control set with probability ε, i.e, the control set is updated to $\mathcal{C} \cup \{r\}$; otherwise, $\mathcal{C}$ remains unchanged.

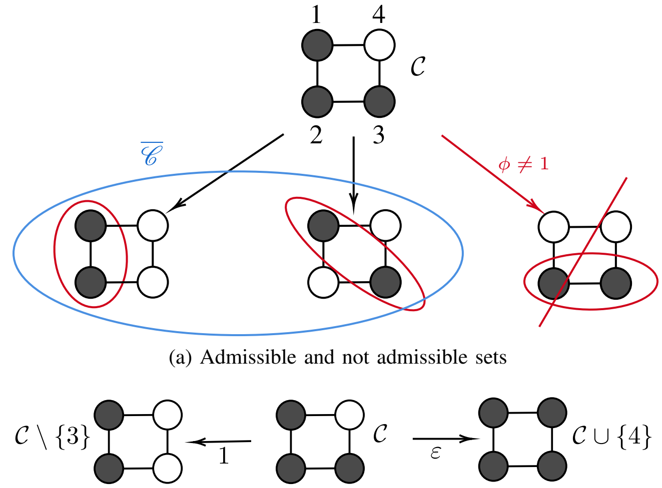
The iteration counter is thus increased to $k+1$ and the process is repeated. As a design choice, we will consider a maximum number of iterations for the algorithm equal to T.

Before formally presenting the Markov chain induced by this iterative process and illustrating how this can be used to solve Problem 2, we offer here a simple example to elucidate the procedure described above. We consider a network with $n = 4$ nodes and with $\mathcal{V}^* = \mathcal{V}$. We start, at an arbitrary iteration step, from a control set $\mathcal{C} = \{1, 2, 3\}$. Figure 4a illustrates the three elements of $\mathscr{C}$ that can be reached from $\mathcal{C}$, depending on which node r is selected. Assume that Algorithm 1 prescribes that only the two sets on the left belong to $\bar{\mathscr{C}}$ and are thus admissible, while the third one is not admissible, and it is thus barred in Fig. 4a. If nodes $r = 2$ or $r = 3$ are selected, then we are in step 1) and the chain has a transition to $\mathcal{C} \setminus \{2\}$ or $\mathcal{C} \setminus \{3\}$, respectively. If instead node $r = 1$ is selected, we are in step 1) but the chain remains in $\mathcal{C}$, since $\mathcal{C} \setminus \{1\}$ is not admissible. Finally, if node $r = 4$ is selected, we are in step 2) and the chain transitions to $\mathcal{C} \cup \{4\}$ with probability ε, as illustrated in Fig. 4b.

Proposition 6. *Algorithm 2 induces a discrete-time Markov chain $Z_\varepsilon(t)$, which is defined on the space of admissible control sets $\bar{\mathscr{C}}$, has initial state $Z_\varepsilon(t) = \mathcal{V}^*$, and, given any pair $\mathcal{A}, \mathcal{B} \in \bar{\mathscr{C}}$, its transition probabilities are defined as*

Algorithm 2: Optimal control set identification

Data: $\boldsymbol{A}, \boldsymbol{W}, \lambda, \beta, \varepsilon, n, T, \mathcal{V}^X$, and $\mathcal{V}^Y$
Result: $\hat{\mathcal{C}}$ such that $(\hat{\mathcal{C}}^X, \hat{\mathcal{C}}^Y)$ solves Problem 2
$k \leftarrow 1$; $\mathcal{C} \leftarrow \mathcal{V}^X \cup \mathcal{V}^Y$; $\hat{\mathcal{C}} \leftarrow \emptyset$;
if $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$ **then**
  $\hat{\mathcal{C}} \leftarrow \mathcal{C}$;
  while $k < T$ **do**
    $k \leftarrow k+1$; Choose at random a node $r \in \mathcal{V}^X$;
    if $r \in \mathcal{C}$ **then**
      if *Algorithm 1 yields* $\mathcal{A}_f = \mathcal{V}$ **then**
        $\mathcal{C} \leftarrow \mathcal{C} \setminus \{r\}$;
        if $|\mathcal{C}| < |\hat{\mathcal{C}}|$ **then**
          $\hat{\mathcal{C}} \leftarrow \mathcal{C}$;
        end
      end
    end
    else
      $\mathcal{C} \leftarrow \mathcal{C} \cup \{r\}$ with probability ε;
    end
  end
end



(a) Admissible and not admissible sets

(b) Transitions of the chain

Fig. 4: Example of one iteration of the Markov chain defined in Algorithm 2, starting from set $\bar{\mathcal{C}}$. In (a), we highlight with a blue circle the set of admissible control sets. The red circles highlight all the candidate control sets. The last control set is not admissible and so it is not considered. In (b), we illustrate the transitions of the chain, described in Eq. (14). Nodes in the control set are denoted in black, nodes not in the control set in white.

$\mathcal{P}[Z_\varepsilon(t+1) = \mathcal{B} | Z_\varepsilon(t) = \mathcal{A}] = P_{\mathcal{A},\mathcal{B},\varepsilon}$, *where*

$$P_{\mathcal{A},\mathcal{B},\varepsilon} = \begin{cases} 1/n^* & \text{if } \mathcal{B} \subset \mathcal{A} \text{ and } |\mathcal{B}| = |\mathcal{A}| - 1, \\ \varepsilon/n^* & \text{if } \mathcal{B} \supset \mathcal{A} \text{ and } |\mathcal{B}| = |\mathcal{A}| + 1, \\ 1 - \alpha_\varepsilon(\mathcal{A}) & \text{if } \mathcal{B} = \mathcal{A}, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

with $\alpha_\varepsilon(\mathcal{A}) = \frac{\varepsilon(n^* - |\mathcal{A}|) + n_c(\mathcal{A})}{n^*}$, *where* $n_c(\mathcal{A})$ *is the number of admissible configuration that can be reached by removing a node from* $\mathcal{A}$, *i.e.,* $n_c(\mathcal{A}) := |\{\mathcal{C} \in \bar{\mathscr{C}} : \mathcal{C} = \mathcal{A} \setminus \{r\}, r \in \mathcal{A}\}|$.

Proof. First, we observe that the iterative procedure described in Algorithm 2 explores only admissible control set. We proceed by induction. If Problem 2 is feasible, then we start from $\mathcal{C} = \mathcal{V}^*$, which is feasible. Then, if at the kth iteration the set $\mathcal{C}$ is an admissible control set, we demonstrate that this holds true also at iteration $k+1$. In fact, if 1) occurs, then either $\mathcal{C} \setminus \{r\}$ is admissible by construction or the control set remains unchanged; if 2) occurs, then the control set is updated to a superset of $\mathcal{C}$ (possibly coinciding with $\mathcal{C}$ with probability $1 - \varepsilon$), which is admissible due to Lemma 1, yielding that Algorithm 2 induces a stochastic process with state space $\bar{\mathscr{C}}$.

Second, let us denote by $\mathcal{A}$ and $\mathcal{B}$ the control set a the kth and $(k+1)$th iteration of Algorithm 2, respectively. Preliminary, we observe that, given $\mathcal{A}$, $\mathcal{B}$ is is independent of the previous history of the process; hence, it is a Markov chain [37]. Then, if node r (selected uniformly at random among n^* nodes) is such that $r \in \mathcal{A}$, then $\mathcal{B} = \mathcal{A} \setminus \{r\}$. Hence, a generic set $\mathcal{B} \subset A$ with $|\mathcal{B}| = |\mathcal{A}| - 1$ is reached with probability $1/n^*$, yielding the first line in Eq. (14). If the node $r \notin \mathcal{A}$, then $\mathcal{B} = \mathcal{A} \cup \{r\}$ with probability ε. Hence, a generic set $\mathcal{B} \supset A$ with $|\mathcal{B}| = |\mathcal{A}| + 1$ is reached with probability ε/n^*, yielding the second line in Eq. (14). No other state can be reached according to the algorithm. Therefore, we conclude that these probabilities match exactly those in Eq. (14), where the third line is simply obtained as the probability of the complementary event, yielding the claim. □

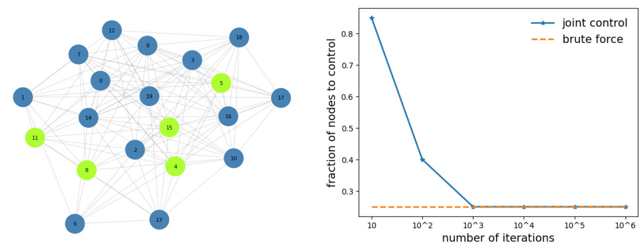
The Markov chain defined in Proposition 6 plays a key role in solving Problem 2, as claimed in the following statement.

Theorem 4. *Algorithm 2 induces a Markov chain whose invariant distribution* $\mu_\varepsilon \in [0,1]^{\bar{\mathscr{C}}}$ *is such that* $\lim_{\varepsilon \searrow 0} \mu_\varepsilon = \mu$ *where* μ *is the uniform probability distribution on the set of solutions of Eq. (7) and, consequently, of Problem 2.*

Proof. We observe that the Markov chain $Z_\varepsilon(t)$ with transition probabilities in Eq. (14) is ergodic, since every admissible configuration $\mathcal{B} \in \bar{\mathscr{C}}$ can be reached from any other one $\mathcal{A} \in \bar{\mathscr{C}}$ following a path of non-zero probability (trivially, first by adding nodes to $\mathcal{A}$ until reaching $\mathcal{V}^*$, and then by removing nodes until reaching $\mathcal{B}$) [37]. This path passes only through admissible control sets, due to Lemma 1. Hence, the chain converges to an invariant distribution.

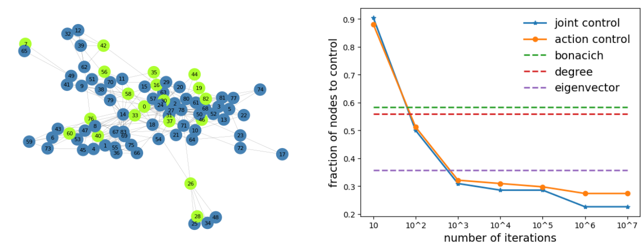
To compute its invariant distribution, we follow the arguments from [31, Theorem 2], to conclude that $\mathcal{Z}_\varepsilon(t)$ has invariant distribution $\mu_\varepsilon \in [0,1]^{\bar{\mathscr{C}}}$, such that its generic component associated with $\mathcal{C} \in \bar{\mathscr{C}}$ is equal to $[\mu_\varepsilon]_\mathcal{C} = \frac{1}{K_\varepsilon} \varepsilon^{|\mathcal{C}|}$, where K_ε is a normalizing coefficient. Hence, for $\varepsilon \searrow 0$, it holds that the only non-zero components of μ_ε are all equal and are those associated with the admissible control sets $\mathcal{C}$ of minimal cardinality, yielding the claim. □

Remark 7. *The convergence result proved in Theorem 4 guarantees that, in the limit* $t \to \infty$, *the Markov chain defined in Proposition 6 concentrates about the solution(s) of Problem 2. From a practical point of view, in Algorithm 2 we incorporate a variable* $\hat{\mathcal{C}}$ *to keep track of the minimal control set reached so far, so that it could be used also as an heuristic for admissible configurations space exploration.*



(a) Network (b) Performance

Fig. 5: Case Study II. In (a), the network structure, with the optimal (joint) control set identified by Algorithm 2 as green nodes. In (b), comparison of Algorithm 2 and the brute-force solution for joint control.



(a) Network (b) Performance

Fig. 6: Case Study III. In (a), the network, with the optimal (joint) control set identified by Algorithm 2 as green nodes. In (b), the blue and orange curves represent the fraction of agents to be controlled with joint and action control identified using Algorithm 2, respectively; dashed lines represent greedy solutions obtained by selecting nodes with largest centrality measures.

Remark 8. *The computations required to solve Algorithm 2 may be reduced. First, the most computationally complex operation is associated with the solution of Eq. (9) in Algorithm 1. However, this operation, which is performed by using matrix $\boldsymbol{M}$, is independent of the iteration. Hence, Algorithm 2 can be optimized by computing the matrix $\boldsymbol{M}$ before starting the iterations and providing it as an input to Algorithm 1, so that the inversion of a matrix needs to be performed only once. Second, if $\mathcal{V}^Y = \emptyset$, i.e., we enforce action control, then in order to verify that a control set obtained by removing a node r from an admissible control set is admissible, we need not apply Algorithm 1 thoroughly. In fact, it is enough to stop the iteration in Algorithm 1 as soon as $\delta_r(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) > 0$, due to Lemma 1 and the monotonicity property of actions $\boldsymbol{x}$.*

A. Case Study II: Watts–Strogatz Network

We validate our approach on a small case study, consisting of a Watts–Strogatz small-world network with 20 nodes. The relatively small size of the network allows us to compute the exact solution using a brute-force algorithm, and compare it with the one obtained using Algorithm 2. The results, reported in Fig. 5, suggests that our algorithm reaches a plateau coinciding with the global optimum in a small number of iterations, computed in 0.2 seconds. Similar results on larger synthetic networks supports scalability [38]. However, when dealing with large-scale networks, it may be more convenient to implement targeted interventions at the level of clusters.

B. Case Study III: Real-world Malawi Village Network

We further demonstrate our approach on a real-world case study of a network of social contacts in a village in rural Malawi, whose dataset is available on Sociopatterns [34]. We use the largest connected component of the network, consisting of 84 individuals and 346 weighted undirected edges, illustrated in Fig. 6a. We apply Algorithm 2 with the exploration parameter set to $\varepsilon = 0.2$, and we keep track of how the quality of the optimal solution (in terms of fraction of nodes to be controlled) evolves as the number of iterations increases. The code is available at [38].

The results (Fig. 6b) allows us to make several observations. First, Algorithm 2 quickly outperforms greedy heuristics — e.g., with joint control, after 10^6 iterations, the cardinality of the minimum control set is reduced to 22.6%, outperforming the 35.7% obtained with the best greedy algorithm (violet dashed line in Fig. 6b). Second, increasing the number of iterations yields an initial reduction in the fraction of nodes that need to be controlled. However, this effect quickly reaches a limit, suggesting that the algorithm is able to reach the optimal solution (or a good proxy of it) in a reasonable time ($100,000$ iterations are performed in 70 seconds on a standard PC). Since each run of Algorithm 1 takes approximately $0.8e^{-3}$ seconds, a brute-force algorithm is unfeasible.

Third, while joint control outperforms action control, the performance difference is not substantial. Hence, controlling only actions, which one expects would require a weaker enforcement policy, could be a valid choice to reduce costs. On the contrary, opinion control is not sufficient to steer the system to the desired consensus. These results underline how our approach can effectively determine the most influential individuals in a complex social network, offering a tool that policymakers can leverage to promote social change (e.g., identifying who could act as a "testimonial") or guarantee social safety (e.g., determining where malicious attacks could be more dangerous). Finally, it is worth noticing that the fraction of nodes needed to be controlled in this real-world case study is comparable with the critical mass of innovators observed in experimental studies [29], [30], providing further empirical support to the coevolutionary model. Additional simulations, performed to demonstrate robustness of our results with respect to the value of the parameters are reported in [38].

VI. Discussion and Conclusion

We formalized a novel control problem for social networks by incorporating a committed minority in a coevolutionary model of actions and opinions. We established a general convergence result and we leverage it to tackle two research problems: i) determine whether the committed minority are able to steer the population to the desired state and ii) identify the minimal control set needed to achieve the goal. By developing algorithms to address these two questions, we offer a novel set of effective tools to assess the robustness of social systems against malicious attacks and assist policy makers in designing policies to promote social change.

The results presented in this paper are not exempt from limitations, outlining several lines of future research. First, this paper proposes a static control policy. Future research should focus on designing dynamic control policies, towards minimizing the total control effort. Second, we focused on the problem of steering a population to a consensus, which is relevant, e.g., to promote social change. However, different applications may require different goals, e.g., favoring diversity or reaching a clustered consensus. While some of our results may directly extend to these scenarios (e.g., Algorithm 1), different control actions might be designed (e.g., if one needs to enforce non-monotonic trajectories), for which a different analysis is needed, revealing a key research direction. Third, this paper focused on minimizing the number of nodes to control, whereas real-world scenarios can mean that different nodes may be associated different costs to control, calling for the generalization of Problem 1 to minimize a non-trivial cost function of the controlled nodes. In this case, while the theoretical tools developed in Section IV remain valid, one needs to build on Algorithm 2 and re-design an ad-hoc Markov chain so that its invariant distribution concentrates about the minima of the designed cost function. Fourth, the improved performance of Algorithm 2 in identifying the optimal control set compared to classical heuristics suggests that our approach can be possibly extended to similar control problems for other multi-dimensional supermodular games.

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Appendix

A. Proof of Proposition 1

Being the domain compact and the utility function upper-semicontinuous, to prove supermodularity we need to check that Eq. (1) has increasing differences [39], i.e., that given $\boldsymbol{z_i}' \geq \boldsymbol{z_i}$ and $\boldsymbol{z_{-i}}' \geq \boldsymbol{z_{-i}}$, it holds that $\Delta_i(\boldsymbol{z_i}', \boldsymbol{z_i}, \boldsymbol{z}') := u_i(\boldsymbol{z_i}', \boldsymbol{z}') - u_i(\boldsymbol{z_i}, \boldsymbol{z}') \geq \Delta_i(\boldsymbol{z_i}', \boldsymbol{z_i}, \boldsymbol{z_{-i}})$. Eq. (1) yields

$$\begin{aligned}\Delta_i(\boldsymbol{z_i}', \boldsymbol{z_i}, \boldsymbol{z_{-i}}) &= \lambda_i(1-\beta_i)\textstyle\sum_{j\in\mathcal{V}} a_{ij}(x_i'-x_i)x_j \\ &+2\beta_i(1-\lambda_i)\textstyle\sum_{j\in\mathcal{V}} w_{ij}y_j(y_i'-y_i) + \psi(\boldsymbol{z_i}', \boldsymbol{z_i}),\end{aligned} \tag{15}$$

where $\psi(\boldsymbol{z_i}', \boldsymbol{z_i})$ is a function that depends only on $\boldsymbol{z_i}'$ and $\boldsymbol{z_i}$. Hence, being $x_i' \geq x_i$ and $y_i' \geq y_i$, Eq. (15) is monotonic nondecreasing in x_j and y_j, $j \in \mathcal{V}$. Hence, $\Delta_i(\boldsymbol{z_i}', \boldsymbol{z_i}, \boldsymbol{z_{-i}}') \geq \Delta_i(\boldsymbol{z_i}', \boldsymbol{z_i}, \boldsymbol{z_{-i}})$ for any $\boldsymbol{z_{-i}}' \geq \boldsymbol{z_{-i}}$, yielding the claim. □

B. Proof of Theorem 1

Consider the controlled game under Assumption 2. Proposition 1 guarantees that the game is supermodular. Moreover, we observe that under Assumption 2, Eq. (2), as previously introduced, can be equivalently characterized as the minimal best response $(x_i(t+1), y_i(t+1)) = \min(\operatorname{argmax} u_i(\boldsymbol{z_i}, \boldsymbol{z_{-i}}))$ [40]. It is known from [40] that the trajectory of a minimal best response with smallest initial conditions is monotonically non-decreasing [40]. Hence, since Assumption 2 fixes the initial condition for all the uncontrolled nodes to $(-1,-1)$, then for all trajectories, it holds $\boldsymbol{x}(t) \geq \boldsymbol{x}(t-1)$ and $\boldsymbol{y}(t) \geq \boldsymbol{y}(t-1)$ for all $t \geq 0$. Note that the same monotonicity results can be obtained by using the explicit dynamics in Proposition 2, following the arguments used for the special case of joint control in [32, Lemmas 1–2]. Finally, monotonicity implies convergence due to the monotone convergence theorem [41]. □

C. Proof of Lemma 1

To prove the first claim, we observe from Eq. (15) that $\delta_i(\boldsymbol{z})$ is a monotonic nondecreasing function of $\boldsymbol{z}$. Consequently, from Eq. (3), $\boldsymbol{x}(t+1)$ and $\boldsymbol{y}(t+1)$ are monotonic nondecreasing functions of $\boldsymbol{z}(t)$. Let us define the controlled coevolutionary dynamics $\boldsymbol{z}(t)$, with initial condition $\boldsymbol{z}(0)$ according to Eq. (4). Let us consider control sets $\bar{\mathcal{C}}^X \supset \mathcal{C}^X$ and $\bar{\mathcal{C}}^Y \supset \mathcal{C}^Y$, and corresponding dynamics $\bar{\boldsymbol{z}}(t)$ with initial conditions defined according to according to Eq. (4). Then, fixed any common activation sequence for $\boldsymbol{z}(t)$ and $\bar{\boldsymbol{z}}(t)$, the monotonicity properties described in the above yields $\bar{\boldsymbol{z}}(t) \geq \boldsymbol{z}(t)$. Finally, for almost every activation sequence we have that $\boldsymbol{x}(t) \to \mathbf{1}$, being $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$. Consequently, also $\bar{\boldsymbol{x}}(t) \to \mathbf{1}$ for all configurations (except for a zero-measure set), yielding the claim. The second claim is proved following a similar (symmetric) argument. □

D. Proof of Corollary 1

Let $\mathcal{C}^X$, $\mathcal{C}^Y$ be an optimal solution of Eq. (6). Then, we define $\mathcal{C} := \mathcal{C}^X \cup \mathcal{C}^Y$, $\hat{\mathcal{C}}^X := \mathcal{C} \cap \mathcal{V}^X$, and $\hat{\mathcal{C}}^Y := \mathcal{C} \cap \mathcal{V}^Y$. Clearly, $\hat{\mathcal{C}}^X \supseteq \mathcal{C}^X$ and $\hat{\mathcal{C}}^Y \supseteq \mathcal{C}^Y$. Hence, by Lemma 1, since $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$, also $\phi(\hat{\mathcal{C}}^X, \hat{\mathcal{C}}^Y) = 1$. Moreover, $\mathcal{C}^X \subseteq \mathcal{V}^X$ and $\mathcal{C}^Y \subseteq \mathcal{V}^Y$ by construction, so $(\hat{\mathcal{C}}^X, \hat{\mathcal{C}}^Y)$ is a feasible solution of Eq. (6). Finally, we observe that $\hat{\mathcal{C}}^X \cup \hat{\mathcal{C}}^Y = \mathcal{C}$. Hence, $|\hat{\mathcal{C}}^X \cup \hat{\mathcal{C}}^Y| = |\mathcal{C}^X \cup \mathcal{C}^Y| = |\mathcal{C}|$, and $(\hat{\mathcal{C}}^X, \hat{\mathcal{C}}^Y)$ is an optimal solution of Eq. (6), yielding the claim. □

E. Proof of Theorem 2

First, we recall that when $\lambda_i = 1$, for all $i \in \mathcal{V}$, our model reduces to a majority game, for which it is well-known that finding the minimal control set is NP-hard [31]. Hence, also Problem 2 is NP-hard. We now show that it belongs to the NP-class, which yields NP-completeness. To prove that Problem 2 is NP, we demonstrate that, given an instance of the coevolutionary dynamics and a control set $(\mathcal{C}^X, \mathcal{C}^Y)$, we can check whether $(\mathcal{C}^X, \mathcal{C}^Y)$ is a feasible solution of Eq. (6) in a polynomial time. In other words, if there exists an algorithm able to solve Problem 1 in polynomial time, then clearly one can determine whether $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$, while checking whether $\mathcal{C}^X$ is a subset of $\mathcal{V}^X$ and $\mathcal{C}^Y$ of $\mathcal{V}^Y$ can be checked in time $O(n)$ using, e.g., an hash function. To prove that there exists an algorithm to solve Problem 1 in polynomial time, we refer to Theorem 3, which proves that Algorithm 1 solves Problem 1 in time $O(n^3)$, yielding the claim. □

F. Proof of Proposition 4

We build a counterexample. Consider 2 nodes connected by a link with $w_{12} = a_{12} = w_{21} = a_{21} = 1/3$ and $w_{11} = a_{11} = w_{22} = a_{22} = 2/3$, and $\mathcal{R}(t) = \mathcal{V}$, for all t. We consider $\mathcal{C}_1^X = \{1\}$, $\mathcal{C}_2^X = \{2\}$, and $\mathcal{C}^Y = \mathcal{V}$. Clearly, it holds $\mathcal{C}_1^X \cup \mathcal{C}_2^X = \mathcal{V}$, which implies that $\phi(\mathcal{C}_1^X \cup \mathcal{C}_2^X, \mathcal{C}^Y) = \phi(\mathcal{V}, \mathcal{V}) = 1$. On the other hand, for control sets $\mathcal{C}_1^X, \mathcal{C}^Y)$, we immediately observe that the only state that can change is $x_2(t)$. At the first time instant t at which $2 \in \mathcal{R}(t)$, which occurs in finite time due to Assumption 1, individual 2 switches to $+1$ iff $\delta_2(\boldsymbol{z}(t)) = 2\beta_2(1-\lambda_2) - (1-\beta_2) > 0$, according to Proposition 2. By symmetry, a similar condition holds for individual 1 when $\mathcal{C}_2^X$, exchanging the role of individuals 1 and 2. This implies that

$$\phi(\mathcal{C}_i^X, \mathcal{C}^Y) = \begin{cases} 1 & \text{if } \lambda_{3-i} \leq \frac{3\beta_{3-i}-1}{2\beta_{3-i}}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we set $\beta_1 = \beta_2 = 1/2$ and $\lambda = 2/3$, then $\phi(\mathcal{C}_1^X, \mathcal{C}^Y) = \phi(\mathcal{C}_2^X, \mathcal{C}^Y) = 0$, which implies that the condition $\phi(\mathcal{C}_1^X, \mathcal{C}^Y) + \phi(\mathcal{C}_2^X, \mathcal{C}^Y) \geq \phi(\mathcal{S} \cup \mathcal{T}, \mathcal{C}^Y) + \phi(\mathcal{S} \cap \mathcal{T}, \mathcal{C}^Y)$ required by submodularity [39] is not satisfied by the first variable. Similar, we can build a counterexample to prove that the function is not submodular also with respect to the second variable, yielding the claim. □

G. Proof of Theorem 3

We start by proving the following result.

Lemma 2. *Given control sets $(\mathcal{C}^X, \mathcal{C}^Y)$ and a vector $\hat{\boldsymbol{x}} \in \{-1, 1\}^n$, let $\hat{\boldsymbol{y}}$ be the solution of Eq. (9) given $\hat{\boldsymbol{x}}$. Then, configuration $\hat{\boldsymbol{z}} = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is an equilibrium for the controlled coevolutionary dynamics under Assumptions 1–2 iff $\hat{x}_i\delta_i(\hat{\boldsymbol{z}}) \geq 0$, $\forall\, i \notin \mathcal{C}^X$. If there exists at least an individual $i \notin \mathcal{C}^X$ with $\hat{x}_i\delta_i(\hat{\boldsymbol{z}}) < 0$, there exist no equilibria with $\boldsymbol{x} = \hat{\boldsymbol{x}}$.*

Proof. Fixed the action vector $\hat{\boldsymbol{x}}$ and the opinion of $j \in \mathcal{C}^Y$ to $\hat{y}_j = 1$ (because of Assumption 2), the dynamics in Eq. (3b) for a generic individual $i \notin \mathcal{C}^Y$ reduces to

$$y_i(t+1) = (1-\lambda_i)\Big[\sum_{j\notin\mathcal{C}^Y} w_{ij}y_j(t) + \sum_{j\in\mathcal{C}^Y} w_{ij}\hat{y}_i\Big] + \lambda_i\hat{x}_i$$
$$= (1-\tau_i)\sum\nolimits_{j\notin\mathcal{C}^Y} \bar{w}_{ij}y_j(t) + \tau_i u_i,$$

with $\tau_i = 1-(1-\lambda_i)\sum_{j\notin\mathcal{C}^Y} w_{ij}$, $\bar{w}_{ij} = \frac{w_{ij}}{\sum_{k\notin\mathcal{C}^Y} w_{ik}}$, and

$$u_i = \frac{(1-\lambda_i)(1-\sum_{j\notin\mathcal{C}^Y} w_{ij}) + \lambda_i\hat{x}_i}{1-(1-\lambda_i)\sum_{j\notin\mathcal{C}^Y} w_{ij}},$$

with the convention that, $\bar{w}_{ij} = 0$ if $w_{ij} = 0$. This can be seen as the update rule of a Friedkin–Johnsen opinion dynamics model [2], which is known to converge under Assumption 1 to the unique solution of $\hat{y}_i = (1-\tau_i)\sum_{j\notin\mathcal{C}^Y} \bar{w}_{ij}\hat{y}_j + \tau_i u_i$, which coincides with Eq. (9), see [3]). Hence, $\hat{\boldsymbol{z}} = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is the only admissible equilibrium with action vector equal to $\hat{\boldsymbol{x}}$.

Now, we observe that $\hat{\boldsymbol{z}}$ is an equilibrium iff there are no individuals that would change their action according to Eq. (3a) when the system is at $\hat{\boldsymbol{z}}$. This corresponds to verify that all individuals $i \notin \mathcal{C}^X$ with $\hat{x}_i = -1$ have $\delta_i(\hat{\boldsymbol{z}}) \leq 0$, and all those with $\hat{x}_i = 1$ have $\delta_i(\hat{\boldsymbol{z}}) \geq 0$. In fact, if there exists $i \notin \mathcal{C}^X$ with $\hat{x}_i = -1$ and $\delta_i(\hat{\boldsymbol{z}}) > 0$, then Assumption 1 guarantees that within a finite time-window, i activates and flips action to $+1$ (being $\delta_i(\hat{\boldsymbol{z}}) > 0$). □

Now, we use Lemma 2 to prove the following result.

Lemma 3. *The equilibrium reached by a controlled coevolutionary dynamics that satisfies Assumptions 1 and 2 with control sets $(\mathcal{C}^X, \mathcal{C}^Y)$ is $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, with $\boldsymbol{x}^*$ defined as in Eq. (8) with $\mathcal{A}(k) = \mathcal{A}_f$ (output of Algorithm 1) and $\boldsymbol{y}^*$ solution of Eq. (9) given $\boldsymbol{x}^*$.*

Proof. In the first iteration of the algorithm ($k = 1$), Lemma 2 establishes that state $\hat{\boldsymbol{z}}$ defined using Eq. (8) with set $\mathcal{A}(1)$ and Eq. (9) is an equilibrium iff $\mathcal{A}(2) = \mathcal{A}(1)$. Otherwise, we will now prove that individuals in $\mathcal{A}(2) \setminus \mathcal{A}(1)$ will eventually switch action to $+1$. In fact, as long as $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}$, then $y_j(t)$ converges asymptotically to $\hat{y}_j$ for all $j \notin \mathcal{C}^Y$ (due to the observations made in the proof of Lemma 2). Hence, $\delta_i(\boldsymbol{z}(t))$ converges asymptotically to $\delta_i(\hat{\boldsymbol{z}}) > 0$. By continuity, $\exists\, \tilde{t}$ such that $\delta_i(\boldsymbol{z}(t)) > 0$ for all $t \geq \tilde{t}$, as long as $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}$. Moreover, since $\delta_i(\boldsymbol{z}(t))$ is monotonically increasing in $\boldsymbol{z}$ and $\boldsymbol{z}(t)$ is monotonically increasing in t, then $\delta_i(\boldsymbol{z}(t))$ is a monotonically increasing function of time. This implies that $\delta_i(\boldsymbol{z}(t)) > 0$ for all $t \geq \tilde{t}$. This, together with Assumption 1, guarantees that i switches to $+1$ (Proposition 2) and cannot switch back (Theorem 1), then $x_i(t) = +1$ for all $t \geq \tilde{t} + T$.

If $\mathcal{A}(2) = \mathcal{A}(1)$, then the system necessarily converges to the equilibrium $\hat{\boldsymbol{z}}$, yielding the claim. Otherwise, $\hat{\boldsymbol{z}}$ is not an equilibrium. In this case, all individuals in $\mathcal{A}(2) \setminus \mathcal{A}(1)$ will necessarily switch action to $+1$ in finite time. Hence, we re-iterate considering the set $\mathcal{A}(2)$ and computing the corresponding $\hat{\boldsymbol{z}}$, observing that, if i has switched to $+1$, Theorem 1 guarantees that i will never switch back, so we just need to check whether all $i \in \mathcal{V} \setminus \mathcal{A}(2)$ have $\delta_i(\hat{z}) \leq 0$ to get the terminal condition $\mathcal{A}(3) = \mathcal{A}(2)$, for which the system necessarily converges to the equilibrium $\hat{\boldsymbol{z}}$. Otherwise, we re-iterate the process. Finally, in each iteration k in which the terminal condition is not met, the size of $\mathcal{A}(k)$ increases by at least 1, implying that within at most $k = n - |\mathcal{C}^X|$ iterations we would get $\mathcal{A}(k) = \mathcal{V}$, for which $\hat{\boldsymbol{z}} = (\mathbf{1}, \mathbf{1})$ is a trivial equilibrium, terminating the algorithm. □

Theorem 1 implies that a controlled coevolutionary dynamics always converge to an equilibrium. Lemma 3 implies that the equilibrium is independent of the activation sequence, but depends only on the model parameters and on the initial condition, which are determined by $\mathcal{C}^X$ and $\mathcal{C}^Y$. Hence, fixed the parameters and given the $\mathcal{C}^X$ and $\mathcal{C}^Y$ either the system converges to $\boldsymbol{x} = \mathbf{1}$, implying $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 1$ or to any other equilibrium, implying $\phi(\mathcal{C}^X, \mathcal{C}^Y) = 0$.

Finally, observe that Eq. (9) can be rewritten as $\hat{\boldsymbol{y}} = (\boldsymbol{I} - (\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\lambda}))\boldsymbol{W})^{-1}\mathrm{diag}(\boldsymbol{\lambda})\hat{\boldsymbol{x}}$. The matrix $\boldsymbol{M} := (\boldsymbol{I} - [(\mathbf{1} - \boldsymbol{\lambda})]\boldsymbol{W})^{-1}$ does not depend on the iteration step, thus it can be computed once at the beginning of the iterations (such procedure requires $O(n^3)$ operations). Then, at each iteration of Algorithm 1, the dominant operation is the computation of $\hat{\boldsymbol{y}}$ which requires $O(n^2)$ operations. Since the number of iterations is at most $n - |\mathcal{C}^X|$, the total computational complexity of Algorithm 1 is $O(n^3)$, yielding the claim. □

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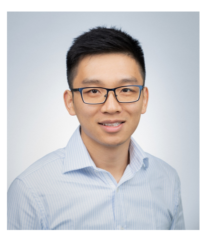

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