Wiener-Hopf Technique for Electromagnetic Scattering Problems Containing Cylindrical Finite Domain Regions

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Abstract

The investigation of the electromagnetic scattering problems containing cylindrical finite domain regions through spectral and asymptotic techniques has been always a challenging problem. In fact, the formulation of the problem in spectral domain involves entire unknowns with exponential phase factors due to the application of spectral transformations along finite boundaries and regions. This work has two primary scopes: to propose an innovative mathematical approach in the framework of Wiener-Hopf techniques that can effectively deal with entire unknowns and, to present examples of solutions in the reported class of problems using semi-analytical spectral and asymptotic techniques.

1. Introduction

The Wiener-Hopf (WH) method [1]-[6] has recently been extended to systematically analyze electromagnetic scattering problems which consider complex structures constituted of sub-domains with different geometries and arbitrary materials, such as angular and layered regions of infinite and semi-infinite length [7]-[12].
The technique consists of the following steps in order [4],[5],[6],[11]: (1) deduction of functional equations in spectral domain by applying Fourier/Laplace transform to wave-equation in each subregion and using the characteristic Green's function procedure [13], (2) imposition of boundary conditions to get the Generalized Wiener-Hopf equations and (3) solution of the system of the WH equations using exact or semi-analytical approximate techniques of factorization as the Fredholm factorization technique [14]. In the case of angular region, special oblique cartesian coordinate systema are used to represent the unknowns.
In this paper we consider a further extension of the proposed technique to problems containing also cylindrical finite domain regions like thick slots, bricks, generalized truncated structures, see for instance structures in Fig. 1.
This paper introduces a solution procedure for the at least not completely addressed problem in WH technique of handling entire unknowns with exponential phase factors, among which the modified Wiener-Hopf equations are considered [3]-[4]. Such unknowns arise whenever the Wiener-Hopf technique is applied to scattering problems with finite penetrable or impenetrable sub-regions because Fourier/Laplace spectral transformations are applied to finite and staggered domains and boundaries.

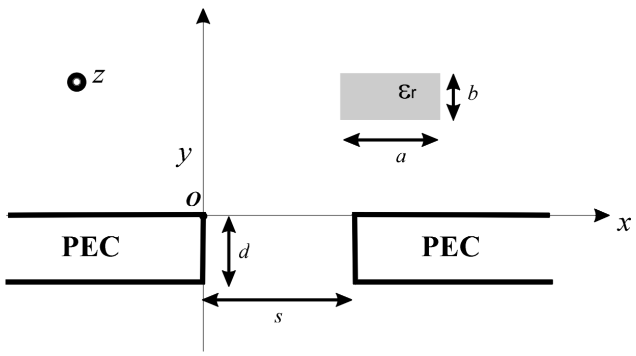

Figure 1. Examples of cylindrical scattering structures considered in the paper made by penetrable and impenetrable materials: brick and thick slot.

The new proposed methodology handles this set of problems with the formulation of incomplete Wiener-Hopf equations in spectral domain subdividing the geometry of the problem into canonical sub-regions of infinite, semi-infinite, finite length. The method exploits the characteristic Green's function procedure [13], the Mittag-Leffler's theorem [15] and a novel version of Fredholm factorization that reduces the Wiener-Hopf problem to the solution of explicit Fredholm integral equations of second kind in spectral domain.
The explicitness of the integral equations obtained with our method avoids classical drawbacks present in alternative specialized methods for modified WH equations such as the availability of explicit pre-factorization and the need of determination of infinite unknown coefficients through systems of infinite equations. These alternative methods are known as Jones-like-methods as they derive from the original work of Jones to handle modified WH equations [16].
Another alternative method to solve WH equations containing entire functions with exponential phase factors is reported in [17] where a sampled representation of entire functions in terms of unknown coefficients is proposed recalling information theory and yielding to system of integral equations for the solution of problems.
All these alternative methods impose the determination of coefficients via infinite system of equations that are unrelated to the physics of the problem and can create convergence limits. On the contrary, our proposed method

solves the incompleteness of WH formulation with Mittag-Leffler's theorem via sample of the physical unknowns. Furthermore, by applying a version of integral Cauchy representation formula [10] we get a representation of sampled unknowns in terms of augmented kernel terms in the Fredholm factorization method, by preserving the properties of the Fredholm integral equation formulation. We demonstrate the importance of Cauchy representations perfectly integrated into Fredholm factorization method also in particular for entire functions with potential exponential phase factor and not just on semi-infinite problems as done in [8],[10].
The convergence characteristics of Fredholm integral equation modeling allow simple approximations to get highly accurate results through the application of reconstruction formulas [18].
As a result, unlike iterative methods, we obtain the true spectra of field components in the overall structure from which we extract by asymptotics physical/engineering phenomena as performed in problems amenable of closed-form spectral analytical solutions.
Finally, the proposed method can be applied systematically in complex problems as both the Wiener-Hopf formulation and the Fredholm integral equation modeling can be interpreted as network relations avoid redundancy per each subregion.
To validate the method, we investigate a well-representative simple problem, i.e. the electromagnetic scattering problem of a slot in a thick metallic screen illuminated by plane-waves, that has been studied with alternative WH approaches with the drawbacks reported in this introduction [19-23].

2. Mathematical Definitions and Formulation

For illustrative purposes, let us consider a representative scattering problem, i.e. the electromagnetic scattering problem of the thick slot illuminated by plane-waves.
The problem is represented in Fig. 1 if the dielectric cylinder is removed.
We first divide the geometry of the problem into three regions of simple canonical shapes: regions 1 and 2 respectively with y>0 and y<-d, and region 3 with 0<x<s, -d<y<0.
For the sake of simplicity, we consider E_z polarization, we assume time harmonic fields with a time dependence specified $e^{+j\omega t}$ (which is omitted) and we assume small losses in the medium which is homogenous to the three regions (the medium can be considered free space in the limit).
We enforce the wave equation with PEC boundary conditions on the two thick half planes, Meixner's condition on the four edges and, the radiation condition.
The source considered in this work is an *Ez*-polarized plane wave with incidence angle φ_o from region 1:

$$E_z^i(x,y) = E_o e^{jk\rho\cos(\varphi-\varphi_o)} = E_o e^{jk(x\cos\varphi_o + y\sin\varphi_o)} \quad (1)$$

where k is the propagation constant. The problem is well-posed and allows unique solution.
The WH equations of the problem are written in terms of Laplace transforms along x direction of the relevant non-null field components at *y=0,-d*.
Eq. (2) reports shifted right unilateral Laplace transforms and left unilateral Laplace transforms, while (3) reports finite Laplace transforms.

$$\begin{aligned} I_{1+}(\eta) &= e^{-j\eta s}\int_s^{\infty} H_x(x,-d)e^{j\eta x}dx, \\ I_{1\pi+}(\eta) &= -\int_{-\infty}^{0} H_x(x,0)e^{-j\eta x}dx, \\ I_{2+}(\eta) &= e^{-j\eta s}\int_0^{\infty} H_x(x,-d)e^{j\eta x}dx, \\ I_{2\pi+}(\eta) &= -e^{j\eta s}\int_{-\infty}^{s} H_x(x,-d)e^{-j\eta x}dx. \end{aligned} \quad (2)$$

$$\begin{aligned} V_{1o}(\eta) &= \int_0^{s} E_z(x,0)e^{j\eta x}dx, \\ V_{2o}(\eta) &= \int_0^{s} E_z(x,-d)e^{j\eta x}dx, \\ I_{1o}(\eta) &= \int_0^{s} H_x(x,0)e^{j\eta x}dx, \\ I_{2o}(\eta) &= \int_o^{s} H_x(x,-d)e^{-j\eta x}dx \end{aligned} \quad (3)$$

Using transverse equations formalism in spectral domain, from (1) we get network relations in spectral domain to model region 1 and region 2, respectively

$$-I_{1\pi+}(-\eta) + I_{1o}(\eta) + e^{j\eta s}I_{1+}(\eta) = Y_c(\eta)[V_{1o}(\eta)] \quad (4)$$

$$I_{2\pi+}(-\eta) - I_{2o}(\eta) - e^{j\eta s}I_{2+}(\eta) = Y_c(\eta)[V_{2o}(\eta)] \quad (5)$$

with $Y_c(\eta) = \xi(\eta)/k\,Z_o, \xi(\eta) = \sqrt{k^2-\eta^2}$.

To model region 3 in the WH formulation we need to start by applying the finite Laplace transform (3) to the wave equations obtaining

$$\left(\frac{d^2}{dy^2} + \tau^2\right)\tilde{E}_{zo}(\eta, y) = f_{o\eta}(y) \quad (6)$$

with

$$f_{o\eta}(y) = -j\omega\mu\, H_y(s,y)e^{j\eta s} + j\omega\mu\, H_y(0,y) \quad (7)$$

By applying the Green’s function procedure, and observing the solution at y=0,-d, we get two-port network model

$$\begin{aligned} &Y_{11}(\eta)V_{1o}(\eta) + Y_{12}(\eta)V_{2o}(\eta) = -e^{j\eta s}q_o(\eta) + p_o(\eta) - I_{1o}(\eta) \\ &-Y_{21}(\eta)V_{1o}(\eta) - Y_{22}(\eta)V_{2o}(\eta) = -e^{j\eta s}s_o(\alpha) + r_o(\eta) - I_{2o}(\eta) \end{aligned} \quad (8)$$

where

$$Y_{11}(\eta)=Y_{22}(\eta)=-jY_c(\eta)\cot[\xi(\eta)d],$$
$$Y_{12}(\eta)=Y_{21}(\eta)=Y_m(\eta)=j\frac{Y_c(\eta)}{\sin[\xi(\eta)d]} \qquad (9)$$

and $q_o(\eta), p_o(\eta), s_o(\eta), r_o(\eta)$ are not explicit terms related to particular integrals originated from (7).
We manipulate $q_o(\eta), p_o(\eta), s_o(\eta), r_o(\eta)$ with the help of Mittag-Leffler theorem: we analyze the poles of the expressions and provide representations in term of samples of WH unknowns in the poles. We note that due to the structure of $q_o(\eta), p_o(\eta), s_o(\eta), r_o(\eta)$ the poles are related to the physics of the structure and sub-regions.
In order to get explicit formulations, we represent the samples of spectral unknowns through a modified version of integral Cauchy representation formula, for instance see

$$V_{io}(-\alpha_n)=\frac{1}{2\pi j}\int_{\gamma_1}\frac{V_{io}(t)}{t+\alpha_n}dt=\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{V_{io}(t)}{t+\alpha_n}dt,\ i=1,2 \qquad (9)$$

with $\eta_n=\sqrt{k^2-(n\frac{\pi}{d})^2}$.
The application of specialized Fredholm factorization method [10,24] provides a system of Fredholm integral equations with augmented kernel for the terms (9) but preserving its convergence properties.
To make solvable and symmetric the system of equations for all the WH unknowns of the problem we perform also a duplication of the equations (4),(5),(8) by substituting η with $-\eta$ and using the auxiliary unknowns (10):

$$V_{1\pi o}(\eta)=e^{j\eta s}V_{1o}(-\eta),\quad V_{2\pi o}(\eta)=e^{j\eta s}V_{2o}(-\eta),$$
$$I_{1\pi o}(\eta)=e^{j\eta s}I_{1o}(-\eta),\quad I_{2\pi o}(\eta)=e^{j\eta s}I_{2o}(-\eta) \qquad (10)$$

3. Solution Procedure

The resulting system of integral equations is of Fredholm 2nd type and explicit in terms of just physical spectral unknowns, thus it can be solved by simple quadrature (as sample and hold) and reconstruction formula [18].
The solution of the problem provides voltage spectra, which allows the computation of diffraction coefficients by asymptotics as in problems with closed form spectral analytical solutions. In fact, the spectral solution contains in its reconstruction formula all physical (source and structure) singularities of the problem in analytical form.
Additional information regarding the formulation, numerical validations, and results will be presented at the conference.
A preliminary numerical demonstration is reported in Fig. 2 where, with reference to Fig. 1 without the dielectric cylinder, the problem is analyzed in free space with dimensions kd=2, ks=7 and illumination constituted by a an Ez polarized plane wave ($\varphi_o=2\pi/9$, Eo=1V/m) (1). Fig. 2 reports the GTD diffraction coefficient for regions 1 and 2.

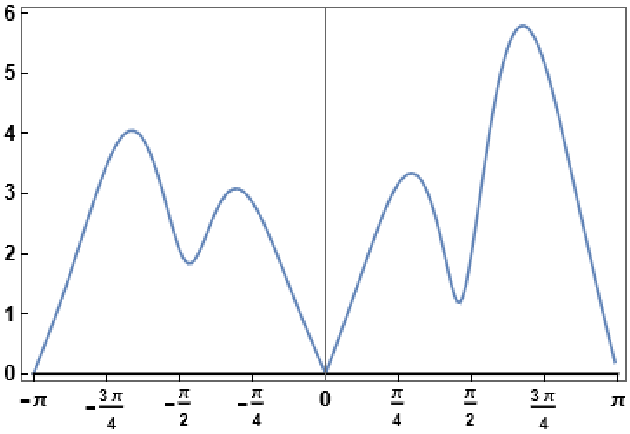

Figure 2. Thick slot in free space with dimensions kd=2, ks=7 illuminate by an Ez polarized plane wave (φo=2π/9, Eo=1V/m). The figure reports GTD diffraction coefficient for regions 1 and 2.

4. Conclusion

In this work, we introduce an effective method for analyzing scattering problems in spectral domain containing cylindrical finite domain regions as reported in Fig. 1. We demonstrate that the combination of the Wiener-Hopf technique, the characteristic Green's function procedure, the Mittag-Leffler theorem, a modified version of integral Cauchy representation formula and, the Fredholm factorization method allow to represent the problem with explicit integral equations of Fredholm type, whose solutions are amenable of asymptotic analysis for the estimation of diffraction.
The paper outlines the procedure for solving Wiener-Hopf problems in presence of entire unknowns and exponential phase factors.
To demonstrate the efficacy of the method, we focused the analysis on the thick slot problem, providing preliminary results.

7. Acknowledgements

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