Design against fatigue failures: lower bound P-S-N curves estimation and influence of runout data

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Abstract
In the present work, three methodologies for the estimation of the fatigue design P-S-N curves are compared. The modified Owen with the staircase method, a method based on the likelihood ratio confidence interval, and an original method based on the bootstrap approach are considered. Three experimental datasets are employed for the experimental validation, showing that the three methods permit to have a reliable estimation of the design curves and that the number of data and, accordingly, the test duration can be significantly reduced with the proper testing strategy.

Keywords: fatigue design; design curves; likelihood ratio confidence interval; Maximum Likelihood; bootstrap approach.

1. INTRODUCTION

Fatigue design is fundamental to guarantee the structural integrity of in-service components, since failures due to repeated cyclic loads represent the most common cause of failures [1]. Fatigue is a complex local phenomenon that involves the formation of cracks and their propagation up to the final fracture, with these phases affected by many different factors, like the material microstructure, the type of load, the specimen geometry, the presence of defects, and the environmental conditions [1]. Therefore, the design against fatigue failures can be complex and must take into account all the factors affecting the fatigue behaviour. The aim of the fatigue design is to avoid the crack initiation or propagation due to loads applied for the number of cycles the component is expected to withstand during its in-service life or before the programmed maintenance. Accordingly, the dependency between the applied stress and the fatigue life or the number of cycles to failure should be determined. Stress-Number of cycles (S-N) curves with different shapes depending on the experimental trend [2-6] are used to model this dependency. Moreover, depending on the failure origin, the models for the fatigue life should also consider the mechanisms of crack nucleation [7-11]. However, due to randomness associated with the fatigue phenomenon, the large experimental scatter should be properly modelled and taken into account when the S-N relationship is assessed [1, 2, 5, 12-14]. Otherwise, a safe design is not guaranteed. For this reason, the analysis of the fatigue results must be carried out in terms of Probabilistic-S-N curves (P-S-N), i.e., of S-N curves at different reliability levels. Ideally, the number of experimental data should be as large as possible to properly assess the experimental variability and scatter [15-19], but this is unpractical due to the high duration of fatigue tests. To take into account that the number of available fatigue data is generally limited, high-reliability P-S-N curves ensuring a safety margin with respect to failures, the so-called design curves, are used for the design of components. Depending on the industrial needs and policy, design curves are assessed in different ways [16, 18] and following different approaches. The research on the testing strategy and on models for the P-S-N curves is therefore of utmost interest in industrial applications.

The experimental data in an S-N plot show different trends [3, 4]. In industrial applications, the trend showing linear decreasing trend ending with a final asymptote, the so-called fatigue limit, is of particular interest. Indeed, even if the occurrence of a fatigue limit has been put in discussion by many papers investigating the Very-High-Cycle Fatigue (VHCF) region [20-22], an asymptotic trend in the High-Cycle Fatigue (HCF) region is generally observed for different materials [6, 23-24] and can model the transition between different failure modes [6]: the surface failure mode in the HCF life range and the internal failure mode in the VHCF life range. Since most of the components are subjected to loads in the HCF life range, a model with a linear trend ending with an asymptote is appropriate in industrial applications and guarantees a safe design.

This paper deals with the estimation of the fatigue design curve from experimental datasets showing a linear decreasing trend with a fatigue limit. A methodology based on the Likelihood ratio Confidence Intervals (LRCI) and an innovative formulation based on the parametric bootstrap approach (bootstrap) have been proposed and compared with a standard method that is widespread among industries and is based on the modified Owen method [16, 18] for the finite life range and on the staircase method for the fatigue limit (O+SC in the following). According to [18], the design curve is assumed as the lower confidence bound of a high-reliability P-S-N curve. The LRCI and the bootstrap methods are based on the application of the Maximum Likelihood (ML) principle and on modelling the distribution of the finite fatigue life together with that of the fatigue limit [4, 25]. In this respect the application of the methods clearly differs from the available literature [26,27] and permits to consider in the analysis both failures and runout data (i.e., unfailed specimens at the end of the test), as recommended by International Standards [19]. An original method based on the parametric bootstrap approach has been also proposed for the estimation of the design curves. The applied bootstrap approach is shown to be of easy implementation and it differs from the bootstrap discussed in [28], where the confidence intervals were stratified by the stress-ratio. The effectiveness of the design curves estimated with LRCI, bootstrap and O+SC methodologies has been verified with experimental datasets obtained by testing, in air environment, a steel, an Alluminium alloy and a composite material. The main features, the strengths and the weaknesses of these methods have been analyzed and compared, focusing also on the influence of the numerousness of runout data. Suggestions and recommendations for optimized testing strategies for the design curves have been finally provided.

2. STATISTICAL METHODS FOR THE DESIGN CURVES

In this Section, the models for the assessment of the design curves are described in detail. In Section 2.1, the modified Owen [16-18] and the staircase methodologies are analyzed. The procedure for the estimation of the design curves through the LRCIs is described in Section 2.2, while Section 2.3 describes an innovative application of the bootstrap method for the assessment of the design curves. Finally, in Section 2.4, details on the implementation of the above-described methods are provided.

In the following, N_f is the number of cycles to failure and s_a is the applied stress amplitude during a fatigue test. In the paper, with design curves we refer to the lower bound of the confidence interval of a high-reliability quantile. According to [18], a design curve can be also indicated with the notation $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$, i.e, the $(1-\mathrm{x_C}/100)$ confidence bound of the $(1-\mathrm{x}_R/100)$ quantile of the P-S-N curve. For example, at a specific N_f, the $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$ fatigue strength, $s_{a,\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}}$, corresponds to the fatigue strength above which the $(1-\mathrm{x}_R/100)$ quantile of the fatigue strength is expected with an $\mathrm{x_C}/100$ confidence level. In the following, design curves and $\mathrm{Rx_R Cx_C}$ curves are interchangeably used. For the sake of clarity, moreover, with fatigue limit, we will refer to the stress amplitude associated to the horizontal asymptote on the S-N plot. If the estimated design curve does not show a clear asymptote, we will generically refer to the fatigue strength s_a at a specific N_f.

2.1. Design curves: Staircase and modified Owen methods.

The staircase (SC in the following) method [29-31] is one of the most widespread methods for assessing the fatigue limit of specimens and its use is suggested in many International Standards in force and books (please refer to [15, 18] and Reference threin). More precisely, the objective of the SC method is the assessment of the fatigue strength at a specific N_f, corresponding to the selected runout number of cycles. In industrial applications, the SC is generally employed for tests at small stress amplitude (e.g., close to the 50% of the ultimate tensile stress of the tested material), with runout number of cycles above 10^6 cycles, to assess the horizontal asymptote of the P-S-N curve, i.e. the fatigue limit. On the other hand, the stress life relationship is generally assessed by assuming a power relationship between s_a and N_f (Basquin's model) and by employing, for example, the modified Owen method [16, 18], as detailed in the following. For this reason, the fatigue strength estimated with the staircase method will be called *fatigue limit* in the following.

Due to its wide diffusion, the SC methodology is only briefly recalled. Details on the formulas that have been considered are not reported here for the sake of brevity and since out of the scope of the present paper, focusing instead on the strengths and the weaknesses of this methodology. In particular, the procedure described in [18] is employed.

The main advantage of the SC is that it allows to concentrate the experimental data close to the mean value of the fatigue limit to be estimated. The fatigue limit is assumed to follow a Normal distribution, with constant mean, μ_{x_l}, and standard deviation, σ_{x_l}. The first test should be carried out at a stress amplitude equal to $s_{a,i=1}$, being i the test number, close to the expected value of the fatigue limit. If the specimen does not fail (runout specimen), the applied stress amplitude in the subsequent test, $s_{a,i=2}$, is increased by a step d selected by the experimenter ($s_{a,i=2} = s_{a,i=1} + d$). Otherwise, the stress amplitude of the second test is decreased by a quantity equal to the step d, $s_{a,i=2} = s_{a,i=1} - d$. This procedure should be repeated to have a sufficient number of data to reliably estimate the mean and the standard deviation of the fatigue limit distribution. According to [15, 18] and to the industrial practice, fifteen valid data are sufficient for a proper application of this methodology. The computation of the fatigue strength and of the standard deviation is based on the least frequent event (failure or runout). Moreover, for a proper application of the methodology, an almost equal number of runout and failures data should be experimentally obtained. The formula developed for the estimation of the fatigue limit and the standard deviation have been originally obtained in [32] by applying the ML principle.

The procedure is of very simple implementation. On the other hand, it does not permit to perform more tests simultaneously, since the applied stress amplitude for the tests i depends on the result of the test number $i-1$. Therefore, tests on the same material cannot be performed "in parallel", but they must be necessarily performed sequentially, and this could be a disadvantage for testing labs where multiple testing machines are available. The choice of the step is moreover fundamental: indeed, the step d should be as close as possible to the standard deviation of the fatigue limit that has to be estimated. The initial guess of $s_{a,i=1}$ is also very important: indeed, the valid data are only those after the first stress inversion, i.e., after the first failure if the SC starts with a runout or after the first runout if the SC starts with a failure. Therefore, if the initial stress amplitude $s_{a,i=1}$ is too small or not properly chosen, many useless runout data may be found before the first failure, with a high experimental cost.

In the following and according to [18], the $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$ in the fatigue limit region, $s_{l,\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}}$, is computed according to Equation 1:

$$s_{l,\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}} = \mu_{x_l} - K \cdot \sigma_{x_l}, \tag{1}$$

where K is a coefficient that permits the estimation of the one side lower bound tolerance limit for a normal distribution. In section 3, K has been retrieved from Tables in [18, 33].

For the finite fatigue life range, i.e., for the life range where the stress amplitude shows a decreasing trend with respect to N_f in the S-N plot, the $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$ curves have been estimated by applying the *approximate Owen one-side tolerance limit* [16, 18]. This approach permits to model the uncertainty associated with the regression analysis. Indeed, the power law (Basquin's model) between s_a and N_f in a bilogarithmic S-N plot is modeled with a linear function, with parameters estimated through the application of the least square method [17-19]. The estimated parameters permit to assess the median (R50) P-S-N curve. The $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$ curves is estimated by shifting the median curve along the horizontal direction by a factor equal to $K_{Owen} \cdot \sigma_s$, being σ_s the sample standard deviation and K_{Owen} a multiplicative coefficient that permits to assess the Owen one side tolerance limit [18]. In other words, the median curve is shifted horizontally by a factor equal to $K_{Owen} \cdot \sigma_s$. In [18], an expression for computing the K_{Owen} is provided. The K_{Owen} factors tabulated in [34] for different sample sizes, reliability and confidence levels have been employed in the following.

2.2. Design curves: Likelihood ratio Confidence Intervals

In this Section, the methodology developed for estimating the LRCI is described. In order to model the continuous decreasing trend in the finite fatigue life region and an asymptotic trend in the infinite fatigue life region, the logarithm of fatigue life and the logarithm of the fatigue limit, respectively, are assumed to follow a Normal distribution, according to Equation 2:

$$F_{Y|x} = \Phi\left(\frac{y-(a+b\cdot x)}{\sigma_Y}\right)\Phi\left(\frac{x-\mu_{X_l}}{\sigma_{X_l}}\right), \tag{2}$$

being $\Phi(\cdot)$ the cumulative distribution function (cdf) of a standardized normal distribution, y the logarithm of the number of cycles to failure N_f, a and b constant coefficients, $x = \log_{10}[s_a]$, μ_{X_l} the mean of the statistical distribution of the fatigue limit and σ_Y and σ_{X_l} the standard deviation of the finite fatigue life and of the fatigue limit distributions, respectively. The mean of the fatigue life distribution is linearly dependent on the applied stress amplitude, whereas the mean of the fatigue limit distribution is assumed to be constant. The constant coefficients a and b, μ_{X_l}, σ_Y and σ_{X_l} must be estimated from the experimental data. The parameter estimation is carried out by applying the ML principle, i.e., by maximizing the Likelihood function, $L[\boldsymbol{\theta}]$:

$$L[\boldsymbol{\theta}] = \prod_{i_f=1}^{n_f} f_{Y|X=x}\left[y_{i_f}; x_{i_f}; \theta\right] \cdot \prod_{j=1}^{n_r}\left(1 - F_{Y|X=x}\left[y_i^*; x_j; \theta\right]\right), \tag{3}$$

being $f_{Y|X=x}$ the probability density function (pdf) of the fatigue life distribution, $i_f = 1..n_s$ the subscript referring to the i-th failed specimen, with n_s the number of failures, $j = 1..n_r$ the subscript referring to the j-th specimen, with n_r the number of runout specimens, y_i^* the logarithm of the runout number of cycles and $\boldsymbol{\theta} = \left(a, b, \sigma_Y, \mu_{X_l}, \sigma_{X_l}\right)$ the set of parameters to be estimated. For the sake of clarity, y_{i_f} and x_{i_f} are the logarithm of N_f and the logarithm of the s_a for the i_f-th failed specimen, whereas x_j is the logarithm of the s_a for the j-th runout specimen. The set of parameters that maximize the $L[\boldsymbol{\theta}]$ in Equation 3 corresponds to the ML estimate, $\widetilde{\boldsymbol{\theta}}$.

The α_{th} quantile of the fatigue life, $y_{\alpha_{th}}$, can be easily estimated from Equation 2, by substituting $F_{Y|x}$ with α_{th} and by solving Equation 2 with respect to x for a selected y. With the same procedure, the α_{th} quantile of the fatigue strength, $x_{\alpha_{th}}$, can be obtained. The objective of this Section is to estimate the lower bound of a specific quantile of the P-S-N curve with the LRCI. An approximate $(1 - \beta_{th})$% LRCI for $x_{\alpha_{th}}$ is given by:

$$PL[\theta_1] = \frac{\max_{\boldsymbol{\theta_2}}[L[\theta_1, \boldsymbol{\theta_2}]]}{L[\widetilde{\boldsymbol{\theta}}]} \geq e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}, \tag{4}$$

being $PL[\theta_1]$ the Profile Likelihood function, θ_1 the investigated parameter (i.e., $\theta_1 = x_{\alpha_{th}}$), $\boldsymbol{\theta_2}$ the set of the other parameters in the model, $\chi^2(1; 1 - \beta_{th})$ the $(1 - \beta_{th})$-th quantile of a Chi-square distribution with 1 degree of freedom.Indeed, the function $-2 \cdot \log(PL[\theta_1])$ is asymptotically Chi-squared with 1 degree of freedom. The notation $\max_{\boldsymbol{\theta_2}}\left[L[\theta_1, \boldsymbol{\theta_2}]\right]$ indicates the Likelihood function $L[\theta_1, \boldsymbol{\theta_2}]$ computed by considering the set of parameters in the $\boldsymbol{\theta_2}$ vector the maximizes $L[\theta_1, \boldsymbol{\theta_2}]$ for a given value of θ_1. It must be noted that the general formulation in Equation 4, which allows for estimating the approximate Confidence intervals for a parameter of interest, has been retrieved from the literature [27, 35, 36]. According to Equation 4, the stress amplitude at which the $PL[\theta_1]$ crosses the $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}$ value corresponds to the lower bound of $x_{\alpha_{th}}$ for a specific N_f. If this procedure is repeated for the investigated life range (i.e., for different values of N_f), the lower bound of the P-S-N curve can be attained. For the sake of clarity, Equation 4 permits to estimate

the $\mathrm{Rx}_R\mathrm{Cx}_\mathrm{C}$ curves: the x_R value corresponds to the $(1-\alpha_{th})\%$ quantile of the P-S-N curve, whereas $(1-\beta_{th})$ is equal to $(2\cdot \mathrm{x}_\mathrm{C}-1)$ (one side confidence interval).
A method for the design curves based on the *LRCI* has been also developed in [26, 27]. In [37] the LRCIs have been computed starting from a model called "*random fatigue limit model*". The diffusion of methodologies based on the LRCI is prevented mainly by the implementation complexity. However, according to [37], "*they have actual coverage probabilities that are closer to the nominal values than do confidence intervals based on normal theory. The likelihood ratio based confidence bounds perform well even in small samples*."
More in details, according to Equation 4, the $PL[\theta_1]$ must be a function of $x_{\alpha_{th}}$. From Equation 2, the α_{th} quantile of the fatigue strength is:

$$\alpha_{th} = \Phi\left(\frac{y_{\alpha_{th}}-\left(a+b\cdot x_{\alpha_{th}}\right)}{\sigma_Y}\right)\Phi\left(\frac{x_{\alpha_{th}}-\mu_{X_l}}{\sigma_{X_l}}\right). \tag{5}$$

From Equation 5, the constant coefficient a can be easily obtained:

$$a = y_{\alpha_{th}} - \Phi^{-1}\left(\frac{\alpha_{th}}{\Phi\left(\frac{x_{\alpha_{th}}-\mu_{X_l}}{\sigma_{X_l}}\right)}\right)\cdot\sigma_Y - b\cdot x_{\alpha_{th}}. \tag{6}$$

By substituting a in Equation 6 with a in Equation 2, the Profile Likelihood function becomes a function of $x_{\alpha_{th}}$ (i.e., $\theta_1 = x_{\alpha_{th}}$ in Equation 4) and the lower bound of $x_{\alpha_{th}}$ can be easily obtained from Equation 4, with $\boldsymbol{\theta_2} = \left(b, \sigma_Y, \mu_{X_l}, \sigma_{X_l}\right)$. Equation 6 may yield infinite values when $\frac{\alpha_{th}}{\Phi\left(\frac{x_{\alpha_{th}}-\mu_{X_l}}{\sigma_{X_l}}\right)} \geq 1$. To avoid this criticality, an expression for μ_{X_l} as a function of $x_{\alpha_{th}}$ can be alternatively obtained from Equation 5:

$$\mu_{X_l} = x_{\alpha_{th}} + \Phi^{-1}\left(\frac{\alpha_{th}}{\Phi\left(\frac{y_\alpha-\left(a+b\cdot x_{\alpha_{th}}\right)}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l}. \tag{7}$$

By replacing μ_{X_l} in Equation 7 with μ_{X_l} in Equation 2, the Profile Likelihood function in Equation 4 becomes a function of $x_{\alpha_{th}}$ and, by considering $\boldsymbol{\theta_2} = \left(a, b, \sigma_Y, \sigma_{X_l}\right)$, the lower bound of $x_{\alpha_{th}}$ can be easily obtained. Details on the procedure implemented for estimating the design curves with the LRCI are provided in Section 2.4.

2.3. Design curves: bootstrap method

The third investigated methodology originally implements the parametric bootstrap approach [38]. With this technique, starting from a known statistical model, a large number of datasets is randomly simulated. For each generated dataset, the parameter estimation is repeated, in order to get information on the statistical variability of the parameters of interest [38]. The model in Equation 2, with the parameters $\boldsymbol{\theta}$ substituted by their ML estimates $\widetilde{\boldsymbol{\theta}}$, is assumed as the model for the population. Starting from this model, n_s random datasets are simulated. The simulation of the datasets is the critical step in the application of the bootstrap approach. The simulated datasets must replicate as much as possible the information carried by the experimental dataset, to eliminate or limit any spurious effect in the simulation phase. At present, to the authors' best knowledge, a general method for the bootstrap simulation of the fatigue datasets has not been proposed in the literature yet. In the present paper, the simulated datasets have been generated by replicating the original experimental dataset in terms of runout number of cycles, number of stress levels, and number of replications for each stress level. This should ensure a correct bootstrap replication of the original experimental dataset. The number of datasets that must be simulated for a reliable estimation is discussed in Sections 2.4 and 3. For each simulated dataset, the parameters are estimated by applying the ML principle and the $\alpha_{th} = (1 - x_R/100)$ quantile P-S-N curve is obtained, i.e., the $x_{(1-\mathrm{x_R}/100)}$ values are estimated for different values of N_f. Finally, for each N_f, the n_S values of $x_{(1-\mathrm{x_R}/100)}$ are sorted in ascending order. The $(1 - \mathrm{x_C}/100)$ confidence bound for the $(1 - \mathrm{x_R}/100)$ P-S-N curve, the $\mathrm{Rx}_R\mathrm{Cx_C}$ curve, is obtained by considering, for each N_f, the $n_S \cdot (1 - \mathrm{x_C}/100)$ ordered value of the sorted $x_{(1-\mathrm{x_R}/100)}$. By repeating this operation for each investigated N_f, the design curve can be finally built. The main weakness of this approach is that ML estimates $\widetilde{\boldsymbol{\theta}}$ are assumed as the actual parameters of the population, but this is not ensured and it may lead to large errors if $\widetilde{\boldsymbol{\theta}}$ is far from $\boldsymbol{\theta}$. Furthermore, the estimation time can be large. Indeed, as the number of simulated datasets increases, the computation time increases, too. A tradeoff between the number of simulated datasets and the estimation effectiveness is necessary, in order to conclude the estimation process in a reasonable time.

2.4. Implementation of the investigated methods

The estimation of the design curves, especially with the LRCI and with the bootstrap method, must necessarily be carried out through a proper programming strategy. In this Section, details on the implementation of the three investigated methods are provided.

The estimation of the design curves with the SC and the modified Owen methods is quite straightforward and it has been carried out by following the guideline reported in [18]. A Matlab script has been created for automatizing this procedure.

On the other hand, the estimation of the design curves with the LRCI is more complicated. A Matlab script has been created. The ML estimates of the unknown parameters, $\widetilde{\boldsymbol{\theta}}$, has been carried out with the function *fminsearch* implemented in Matlab and based on the *Nelder–Mead* simplex algorithm [39]. For each N_f in the range of interest, the solution to the Equation 4 must be found. From a graphical point of view, the solution to Equation 4 corresponds to the points for which the $PL[\theta_1]$ function is equal to the value $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}$. To find this solution, the following procedure has been implemented:

1. For each investigated N_f, the fatigue strength at the quantile of interest is computed by solving Equation 2 with $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}$. For $\tilde{x}_{\alpha_{th}} = x_{\alpha_{th}}(\tilde{\boldsymbol{\theta}})$, the $PL[\tilde{x}_{\alpha_{th}}]$ function in Equation 4 must be equal to 1.
2. $PL[x_{\alpha_{th}}]$ in Equation 4 is computed starting from $x_{\alpha_{th}} = \tilde{x}_{\alpha_{th}}$ and by decreasing $x_{\alpha_{th}}$ with steps equal or smaller than 2 MPa, depending on the investigated material. The parameter estimation is computed with the function *fminsearch*. Equation 6 is employed for the estimation of the $PL[x_{\alpha_{th}}]$. However, if Equation 6 provides an infinite value or cannot be solved, Equation 7 is automatically considered by the implemented algorithm.
3. This procedure is stopped when $PL[x_{\alpha_{th}}]$ is below the value $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}$ and close to 0 (below a threshold of $9 \cdot 10^{-3}$ value). The estimated $PL[x_{\alpha_{th}}]$ points are finally interpolated with a Piecewise Cubic Hermite Interpolating Polynomial (PCHIP).
4. The $x_{\alpha_{th}}$ value that satisfies Equation 6 ($x_{\alpha_{th,LRCI}}$) is the value for which the difference between the estimated interpolating PCHIP and the $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}$ value is minimum. This procedure is automatically repeated to find the $x_{\alpha_{th,LRCI}}$ for each investigated N_f.

Figure 1 shows an example of the $PL[x_{\alpha_{th}}]$ function and helps to understand the steps followed for the estimation of the lower bound of the fatigue strength with the LRCI method.

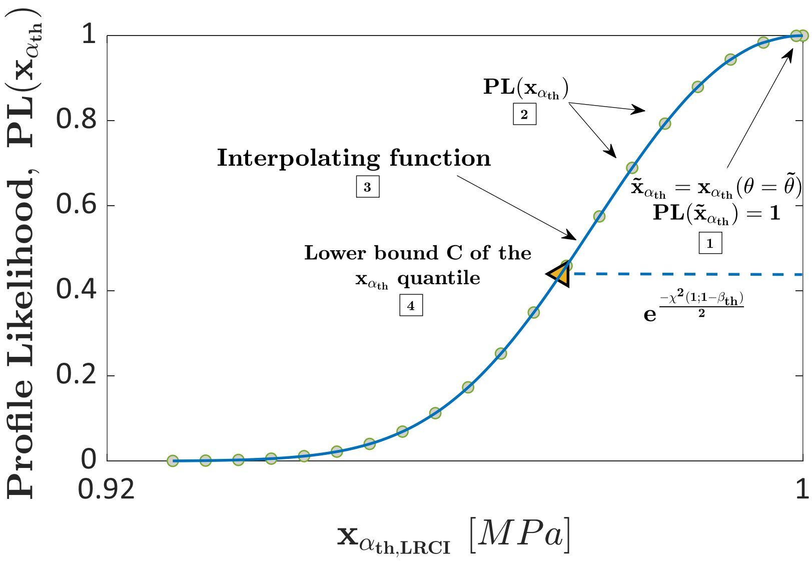


Figure 1: procedure followed to find the lower bound of the fatigue strength with the LRCI method.

A script in Matlab has been also developed to estimate the design curves with the bootstrap method. The core of the bootstrap method is the simulation of the random datasets, starting from the initial statistical model. The ML estimates $\widetilde{\boldsymbol{\theta}}$ of the parameters $\boldsymbol{\theta}$ is firstly estimated. Then, n_b (with n_b equal to the number of actual experimental tests) random values of $\alpha_{th,B}$ (in the range from 0 to 1) are generated. Thereafter, for each experimental stress amplitude x_i (with i ranging from 1 to n_b), a random value of the logarithm of N_f, $y_{i,B}$, is estimated by solving Equation 2 with respect to $y_{i,B}$ for $x = x_i$ and for $\alpha_{th} = \alpha_{th,B}$. If the simulated $y_{i,B}$ is infinite or it is larger than the experimental runout number of cycles, y_r, then $y_{i,B}$ is set equal to y_r. By repeating these steps n_b times, a simulated dataset that mimics the original experimental dataset in terms of runout number of cycles, number of stress levels, and number of replications for each stress level is finally obtained. This simulation procedure is automatically repeated $n_s = 1000$ times in Matlab to get n_s simulated datasets and, for each of them, the $(1 - \mathrm{x_R}/100)$ P-S-N curve is estimated by applying the ML principle. The design curves are finally obtained as described in Section 2.3.

3. EXPERIMENTAL VALIDATION

In this Section, the three investigated methodologies are compared and validated with three datasets. In Section 3.1, the experimental activity and the materials considered for the validation are described. In Section 3.2, the median P-S-N curves are estimated to validate the implemented estimation procedures. In Section 3.3, the design curves are compared, to highlight the differences between the three methodologies. In Section 3.4 the influence of runout specimens is investigated, by gradually reducing the number of runout specimens from the original datasets. In Section 3.5, finally, the capability of the three investigated methodologies to model the fatigue response is discussed.

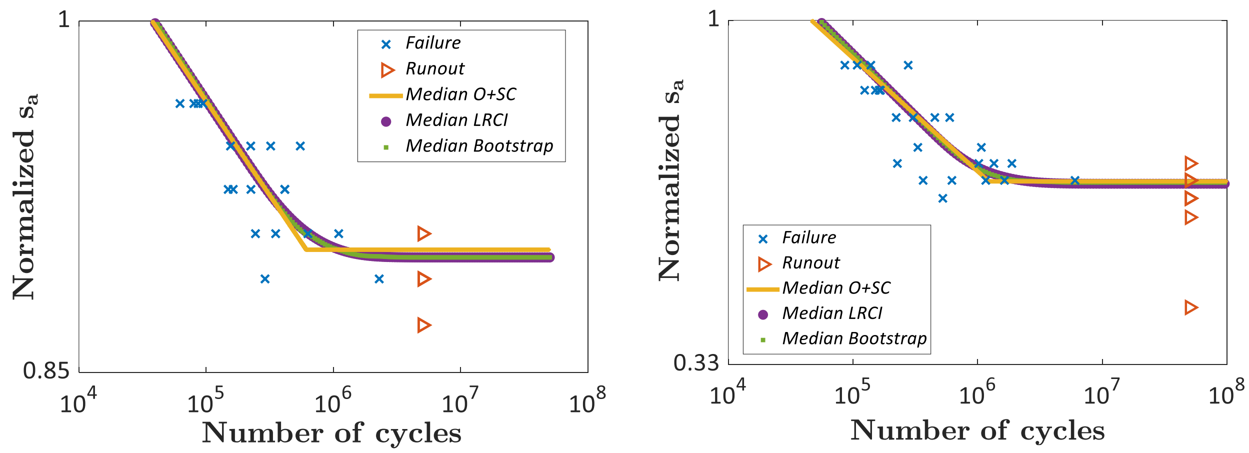
3.1. Experimental validation: material datasets

The strengths and the weaknesses of the investigated methodologies for the estimation of the S-N curves have been verified and compared with three experimental datasets obtained at the CRF lab. All the datasets have been obtained with tests at three or four stress levels for estimating the finite fatigue life range and by applying the SC for the infinite fatigue life range. For each stress level, at least four specimens have been tested.

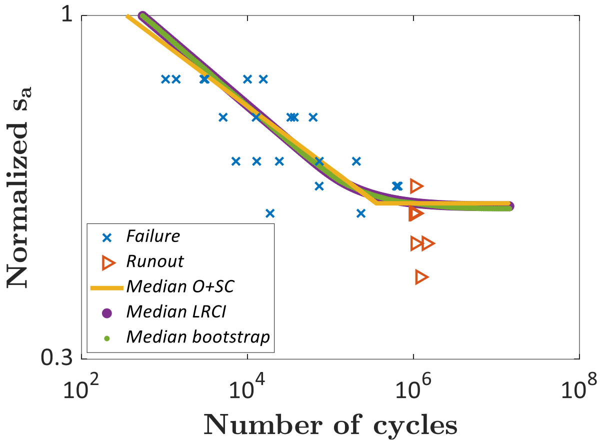
The first tested material is a *TRIP assisted bainitic steel TBC600Y980T*. Fully reversed tension-compression tests in air enviroment at a loading frequency of $50\ \mathrm{Hz}$ were carried out on hourglass specimens, with runout set at $5 \cdot 10^6$ cycles. Fifteen specimens were used for the staircase and seventeen specimens for the finite life range. The second analyzed dataset was obtained with fully reversed tension-compression tests in air enviroment (loading frequency of $150\ \mathrm{Hz}$) on hourglass specimens made of the Aluminum alloy G-AS7C3,5GM (runout N_f at $5 \cdot 10^7$ cycles). For this material, nineteen specimens were used for the staircase and twenty-four for the finite life range. Finally, tension-compression fatigue tests in air enviroment at a stress ratio equal to 0.1 were carried out on the composite material *Polynt SMC LP 2512 R33* (runout N_f at $1 \cdot 10^6$ cycles and loading frequency in the range $[3 - 10]$ Hz). Fifteen data were used for the staircase, whereas twenty data were used for the finite life range. It is worth noting that finite and infinite life data have been considered together for the estimation of the design curves with the LRCI and with the bootstrap method. When the modified Owen and the SC methods have been employed, the P-S-N curves have been obtained by truncating the linear function for the finite life range at $N_{f,t}$, being $N_{f,t}$ the intersection between the linear function for the finite fatigue life and the horizontal line for the fatigue limit at the same quantile. In the following, the ordinate axis of the S-N plots has been normalized for confidential reasons. Moreover, it must be noted that a linear decreasing trend with fatigue limit is valid and can be reliably considered for modelling the stress-life relationship of experimental datasets obtained through tests in air environment. Otherwise, the assumption of a fatigue limit can be non-conservative, with the experimental data showing a continuously decreasing trend without an asymptotic behavior in the HCF region.

3.2. Experimental validation: Median P-S-N curves

Fig. 1 shows the median curves, R50C50 curve, estimated with the three investigated methodologies for the three tested materials. In particular, Fig. 2a is for the *TBC600Y980T* steel ("*steel*" in the following), Fig. 2b is for the *G-AS7C3,5GM* Aluminium alloy (*Al alloy* in the following), and Fig. 2c is for the composite material *Polynt SMC LP 2512 R33* (*Polynt composite* in the following). In the following figures, "*O+SC*" refers to the curves estimated with the Modified Owen (O) method for the finite fatigue life range and with the staircase (SC) for the infinite life range; LRCI refers to curves estimated with the LRCI methods; Bootstrap refers to the curves estimated with the Bootstrap method.



(a) (b)



(c)

Figure 2: comparison of the median curve to verify the effectiveness of the implemented procedures for parameter estimation: a) TRIP assisted bainitic steel TBC600Y980T; b) Aluminum alloy G-AS7C3,5GM; c) Polynt SMC LP 2512 R33.

According to Fig. 2, the differences between the estimated median curves are limited, regardless of the material. The curves tend to overlap, with a similar slope in the finite life region and the same median fatigue limit value. As expected, the transition between the finite fatigue life region is smooth for the curves obtained through approaches based on the ML principle, whereas it shows a stepped trend for the curve obtained with the O+SC. The median P-S-N curves estimated with the bootstrap method overlap the median curve estimated with the ML approach, thus highlighting that the number of simulated datasets, i.e. 1000 datasets, is appropriate. This has been further verified in Fig. 3: Fig. 3a shows the median curve for the Polynt material estimated with the bootstrap method by considering 1000 datasets and 10000 datasets and the median curve estimated with the ML approach.

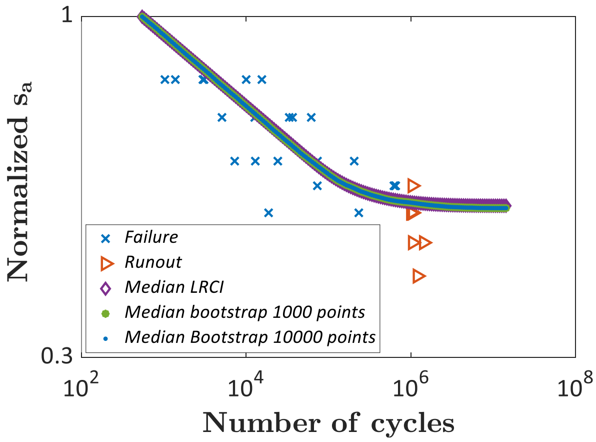


Figure 3: influence of the number of simulated datasets with the bootstrap method (median P-S-N curve for the Polynt material).

According to Fig. 3, 1000 simulated datasets permit to properly estimate the median P-S-N curves, with no differences with the curve estimated with 10000 datasets and that estimated with the *ML*. Accordingly, in the following analysis, the estimation of the design curves with the bootstrap method has been carried out by simulating 1000 datasets, since 10000 simulated datasets do not improve the results, but significantly increase the computation time.

3.3. Experimental validation: Design curves

In this Section, the design curves are compared. In particular, the R90C90 curves, of interest for industrial applications, have been estimated with the three investigated methodologies. Fig. 4 compares the R90C90 curves for the three analyzed materials: Fig. 4a is for the TBC600Y980T steel, Fig. 4b is for the G-AS7C3,5GM Aluminium alloy, and Fig. 4c is for the Polynt SMC LP 2512 R33 composite material.

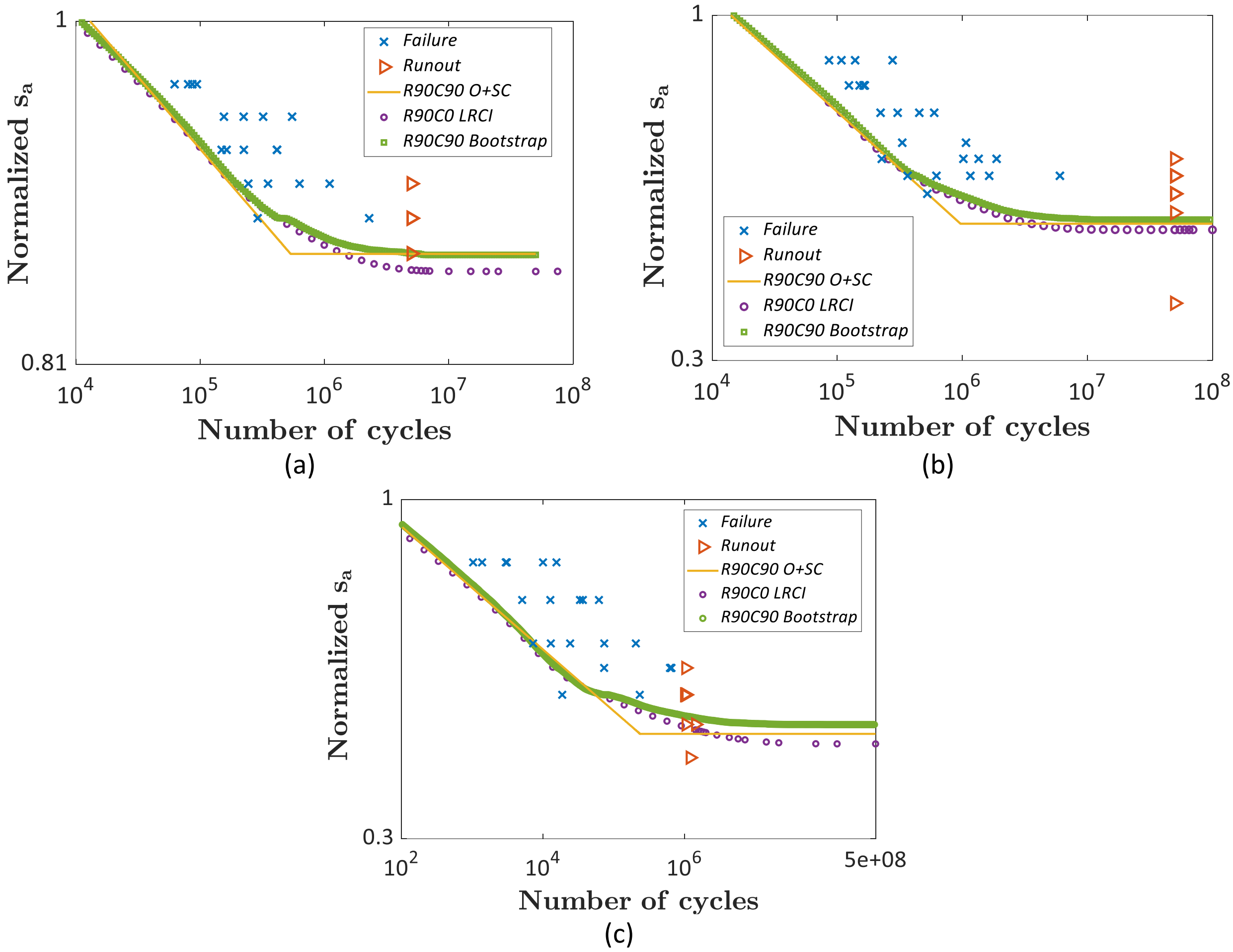
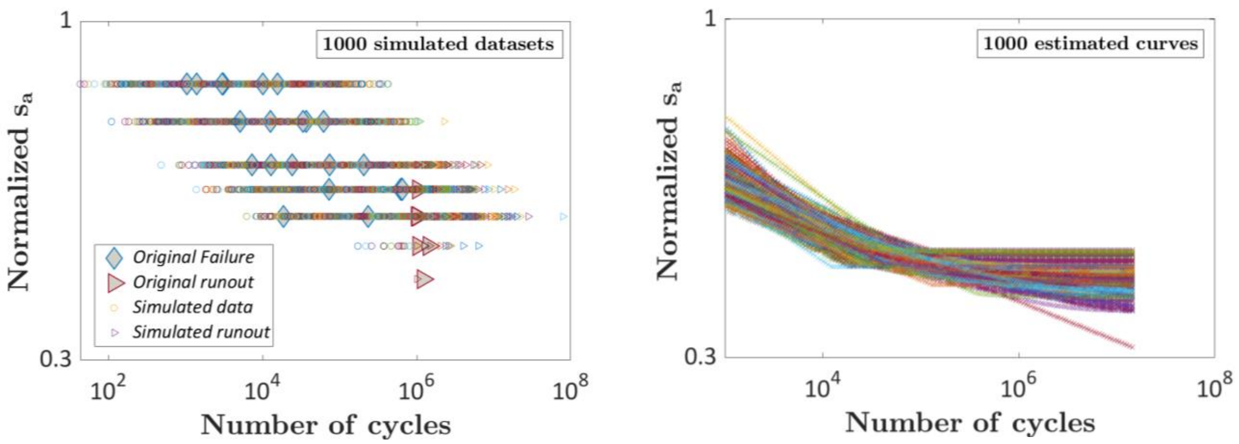


Figure 4: comparison of the R90C90 curve to verify the effectiveness of the parameter estimation: a) TRIP assisted bainitic steel TBC600Y980T; b) Aluminum alloy G-AS7C3,5GM; c) Polynt SMC LP 2512 R33.

According to Fig. 4, the design curves estimated with the three investigated methods are below the experimental failures, apart from 1 datapoint for the steel and the Polynt material and two datapoints for the Aluminum alloy. A more conservative choice of the R or C values would ensure a larger safety margin with respect to failures for the three methodologies. In the finite life region, the design curves are close and have similar trends and slopes, with the LRCI design curves slightly below the other design curves. Close to the transition point, i.e., the ideal point that discriminates between the finite fatigue life region and the infinite fatigue life region, the curve estimated with the O+SC approach is below the other curves. This can be justified by considering that this point is estimated by combining two approaches and two separate estimations for the finite and the infinite fatigue life range, whereas with the other methods the transition between these two life ranges is smoother since a unique model for both regions is considered. The difference between the three approaches increases in the infinite life range. The fatigue limits computed with the SC+O and the bootstrap methods overlap for the steel and the Aluminium alloy. On the other hand, for the Polynt material, the fatigue limit estimated with the Bootstrap method is the largest. In general, the $\mathrm{R90C90}$ fatigue limit estimated with the LRCI is the smallest for the three investigated materials, with differences ranging from 0.9% for the steel up to about 3.4% for the composite Polynt. It is worth noting that it is not possible to assess which methodology is better or permits for a more reliable estimation, since the real values of the population are not known. However, general conclusions can be drawn. In the finite life region, the three investigated methodologies provide similar trends with limited differences, whereas the difference tends to increase in the infinite life range, with the fatigue limit estimated with the LRCI generally below those estimated with the other two methodologies.

A further analysis on the design curves for the Polynt material has been carried out. Indeed, according to Fig. 4, the design curve estimated with the Bootstrap method shows a stepped transition between the finite and the infinite fatigue life. This trend can be justified by analyzing the 1000 simulated datasets (Fig. 5a) and the corresponding estimated $\mathrm{R90}$ curves (Fig. 5b).



(a) (b)

Figure 5: analysis of the simulated datasets for the estimation of the design P-S-N curves of the composite Polynt with the bootstrap method: a) 1000 simulated datasets; b) 1000 estimated P-S-N curves.

According to Fig. 5b, depending on the simulated dataset (Fig. 5a), the trend of the estimated curve changes. Indeed, the simulated curves can show a smooth transition between the finite and the infinite, a stepped transition or a continuous decreasing trend. Accordingly, by considering that the curves with a stepped transition are the curves close to the lower bound of the confidence interval, the transition between the two life ranges for the $R90C90$ curve can be more stepped than that of the median curve (Fig. 2). The number of simulated datasets has been also increased to verify if it influences the trend of the curve, but no differences between the curves estimated with 1000 and 10000 datasets have been found.

3.4. Experimental validation: influence of runout specimens on the design curves

In the present Section, the influence of runout data on the design curves estimated with the investigated methodologies is verified. Runout specimens are important data in the analysis of the fatigue response. Indeed, they are the most time-consuming experimental tests. If the loading frequency is not high, as for composite materials, the time needed to obtain a runout can be high. Depending on the model and on the estimation method, they can be discarded or not considered in the analysis. For example, the least square method does not permit to consider runout data, missing the information they contain. With the SC method, at least $6-8$ runout data must be experimentally found, and this could significantly increase the test duration. These considerations highlight the importance of exploiting and maximizing the information contained in runout data and, at the same time, of minimizing their numerousness, thus increasing the test efficiency without affecting the design curve reliability. Accordingly, a tradeoff between the number of runout specimens to be considered for a safe estimation of the design curves and for reducing the test duration must be found.

Two analyses have been carried out in the following: in the first analysis, the number of runout specimens has been gradually reduced by removing, from the original datasets, all the runout specimens above a defined threshold (Section 3.4.1). In Section 3.4.2, a subset of the data used for the SC has been considered and the Modified staircase method in [17] has been used for the computation of the fatigue limit.

3.4.1. Influence of runout specimens: removal of runout data below a threshold

In the first analysis, the number of runout specimens is gradually reduced. In particular, all the runout data below a defined threshold are eliminated from the SC datasets. Figure 6 helps to understand the procedure followed.

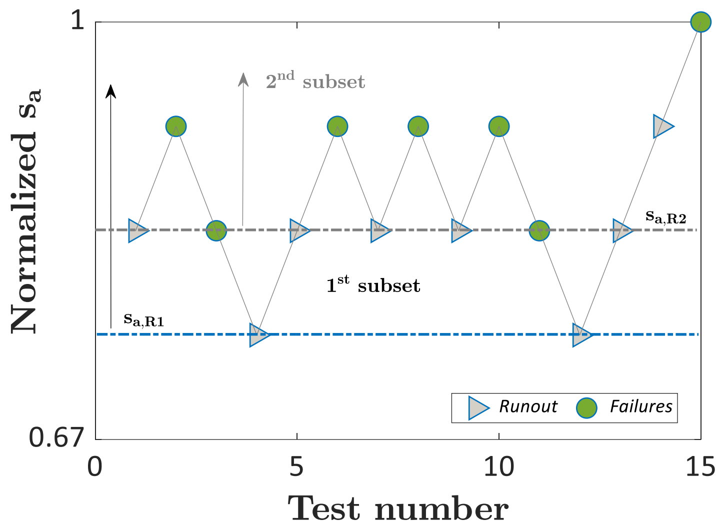
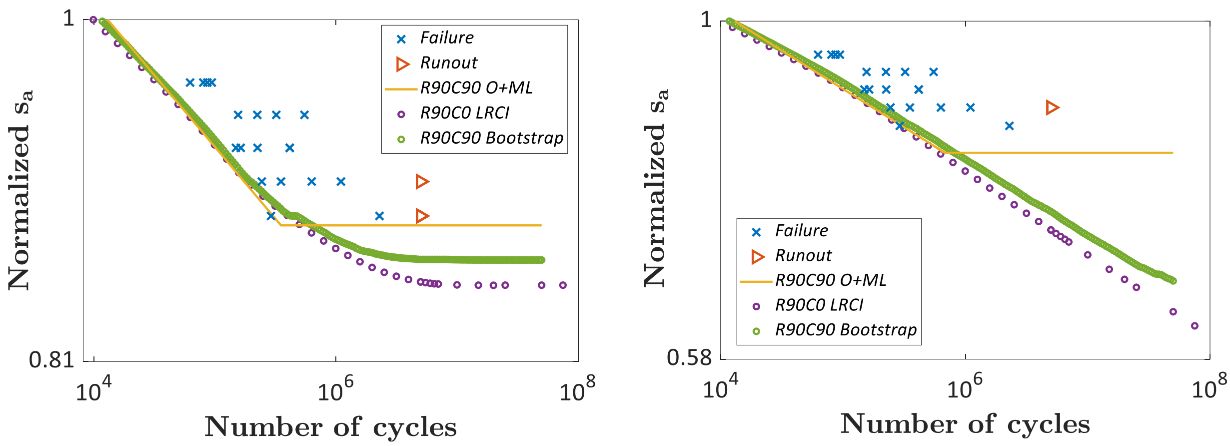


Figure 6: analysis of the influence of the runout specimens: removal of the runout data below a defined threshold from the SC dataset.

According to Fig. 6, the runout stress amplitudes, s_{aR}, are arranged in ascending order. Thereafter, starting from the smallest runout stress amplitude, $s_{aR,1}$, all the runout data below or equal to $s_{aR,1}$ are removed from the SC dataset. Thereafter, according to Figure 6, the second subset is obtained by removing from the SC dataset all the runout data below or equal to $s_{aR,2}$. This procedure is repeated until at least one runout data is available in the subset. If no runout data are available, the procedure is stopped. The design curves with the LRCI and the bootstrap methods are estimated by considering the reduced datasets and following the procedures described in Section 2. On the other hand, since the staircase method cannot be applied if the number of valid data is below 15 (or if the data have not been collected according to the prescribed sequential procedure [18, 31]), the parameters of the distribution of the fatigue limit have been estimated by applying the ML Principle [31]. It must be noted that in the analyzed subsets, the remaining failures are concentrated near the fatigue limit, thus facilitating the estimation. This situation is unlike in real tests, since only with the SC approach runout and failure data can be concentrated near the fatigue limit. However, this analysis permits to estimate a fatigue limit that can be compared with those estimated with the other two methods.

Fig. 7 shows the design curves estimated by considering the failures and the runout data above the 0.87 (Fig. 7a) and the 0.88 (Fig. 7b) values for the investigated steel material. In the following, O+ML refers to the design curves obtained with the Modified Owen method for the finite life range and with the ML approach or the infinite life range [31].



(a) (b)

Figure 7: influence of the numerousness of runout data on the design curves (R90C90) estimated with the three investigated methodologies for the steel material: a) dataset without runout data below the 0.87 value; b) dataset without runout data below the 0.88 value.

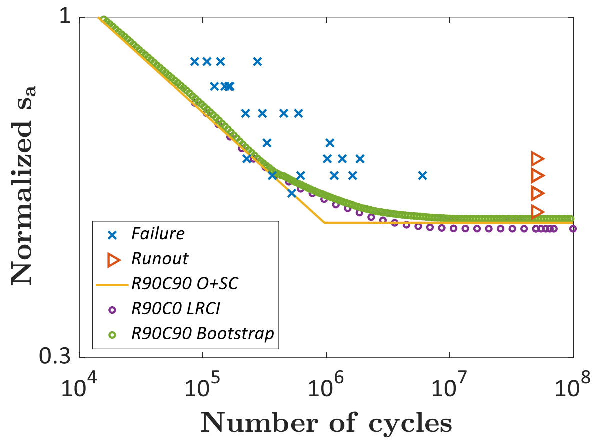
Table 1 summarizes the results of this analysis. If a clear asymptotic trend is not found, the fatigue strength at the runout number of cycles has been considered.

Table 1: summary of the analysis on the influence of runouts on the design curves of the investigated steel material.

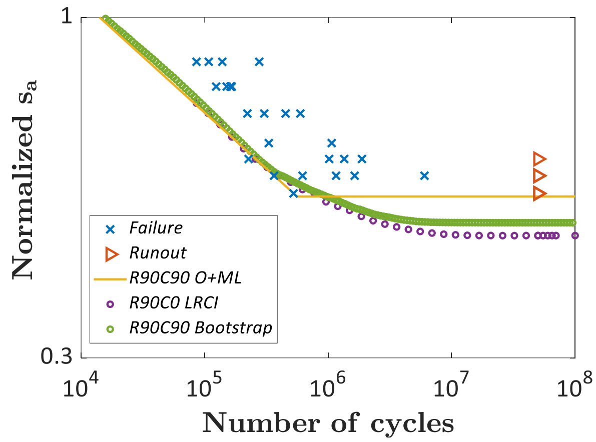
	All data	**1st subset (0.87 threshold)**	**2nd subset (0.88 threshold)**
Saved runout specimens	0	2	6
Left runout specimens	8	6	2
Time saved	0	2 days 8 hours	6 days 23 hours
Fatigue limit O+ML	0.86	0.88	0.86
Fatigue limit LRCI	0.86	0.85	0.78
Fatigue limit Bootstrap	0.86	0.86	0.80

According to Fig. 7 and Table 1, by considering the 0.87 threshold, the fatigue limit slightly increases if computed with the O+ML approach, it slightly decreases by applying the LRCI approach, whereas it does not change if computed with the *bootstrap* method. The design curves computed with the LRCI approach are still below the other curves. Two runout specimens are removed from the analysis, with a saved testing time larger than two days. On the other hand, when only the runout data above the 0.88 threshold are considered, the design curves estimated with the LRCI approach and the *bootstrap* methods do not show an asymptotic trend, whereas they show a continuous decreasing trend with a slope similar to that of the design curve computed with the O+SC approach. With the O+ML method, the presence of an asymptote is assumed, even if the experimental data do not show a clear fatigue limit. On the other hand, with the LRCI and the bootstrap methods, the trend of the curves is not assumed, but depends on the experimental data. Or better, an asymptotic trend cannot fit the experimental data with the selected confidence level. In Table 1 the fatigue strength at the runout number of cycles is compared with the fatigue limit estimated with the O+ML method. This explains the large differences between the lower bound of the fatigue limit estimated with the three methods. It must be noted that the trend found with the LRCI and the bootstrap methods (Fig. 6) does not demonstrate that the tested material is not characterized by an asymptotic trend, but, on the contrary, that the available data do not permit to assess if the curve shows an asymptotic trend with the selected confidence level.

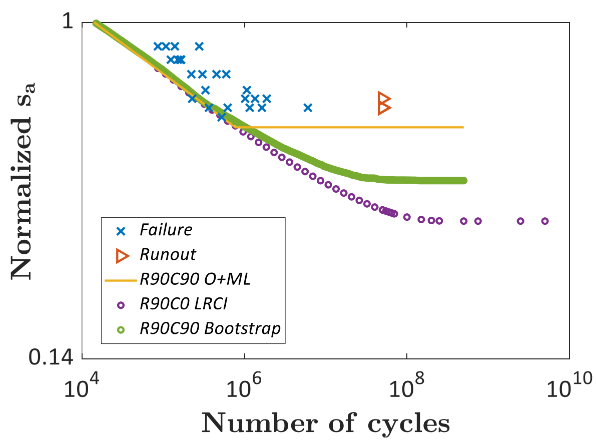
Fig. 8 shows the design curves estimated by considering all failures and runout data above the 0.4 (Fig. 8a), the 0.53 (Fig. 8b), the 0.57 (Fig. 8c) and 0.60 (Fig. 8d) thresholds for the investigated Al material.



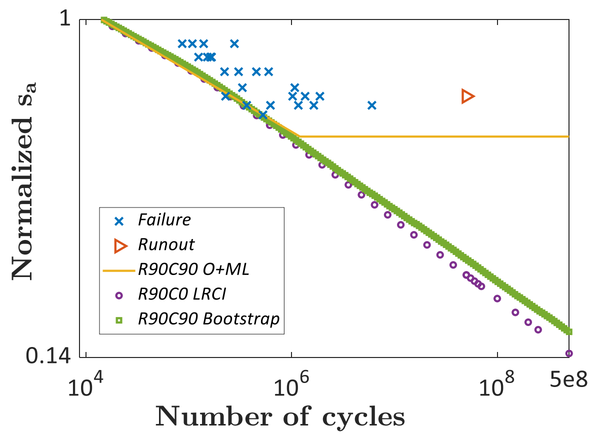
(a)



(b)



(c)



(d)

Figure 8: influence of the runout data numerosity on the design curves (R90C90) estimated with the three investigated methodologies for the Al material: a) runout specimens above 0.4; b) runout specimens above 0.53; c) runout specimens above the 0.57; d) runout specimens above 0.60.

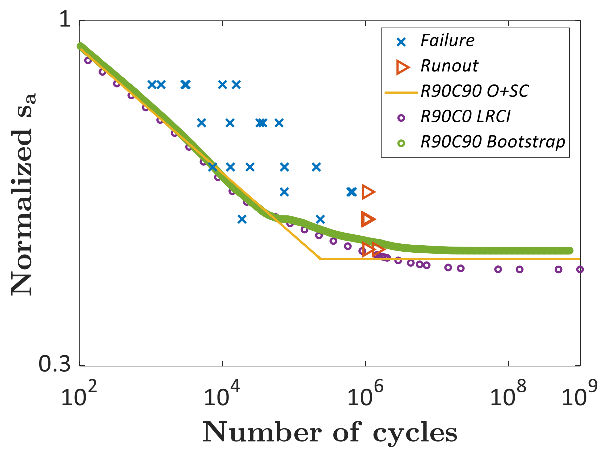
Table 2 summarizes the results of the analysis on the influence of runout specimens on the design curves.

Table 2: summary of the analysis on the influence of runout data on the design curves of the investigated Al material.

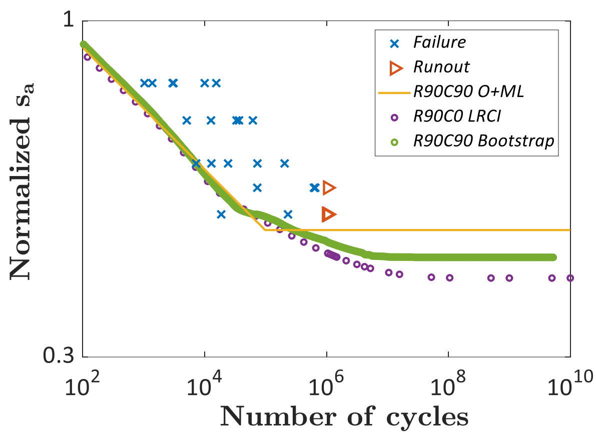
	All data	**1st subset (0.40 threshold)**	**2nd subset (0.53 threshold)**	**3rd subset (0.57 threshold)**	**4th subset (0.60 threshold)**
Saved runout specimens	0	1	2	6	9
Left runout specimens	10	9	8	4	1
Time saved	0	3 days 20 hours	7 days 17 hours	23 days 3 hours	43 days 17 hours
Fatigue limit O+ML	0.51	0.51	0.56	0.53	0.50
Fatigue limit LRCI	0.51	0.51	0.49	0.30	0.22
Fatigue limit Bootstrap	0.52	0.52	0.52	0.39	0.24

According to Figs. 8a and 8b and to Table 2, by removing less than two runout specimens at small stress amplitude, the lower bound of the fatigue limit computed with the LRCI and the Bootstrap is not strongly influenced, with a null or a limited variation. However, due to the high runout number of cycles ($5 \cdot 10^7$), the testing time is significantly reduced, with the test duration reduced by almost 8 days. On the other hand, the fatigue limit computed with the O+ML method significantly increases (about 10% increment). By considering the runout above the 0.57 threshold, the LRCI and the bootstrap methods still fit the data with a model that includes an asymptote, but the estimated fatigue limits are significantly below the fatigue limit computed with the O+ML approach. On the other hand, for the fourth subset, only one runout is available and the dataset does not contain enough information to assess if the curve shows an asymptotic behaviour with high confidence. The fatigue limit estimated with the O+ML is less affected by the runout numerousness. This could be justified by considering that failures are concentrated near the fatigue limit. In real tests, it is unlikely or rather hard to concentrate failures close to the fatigue limit, without obtaining runout data. However, this proves that the fatigue limit can be properly estimated with the ML by considering datasets with a number of runout data smaller than that required by the SC approach.

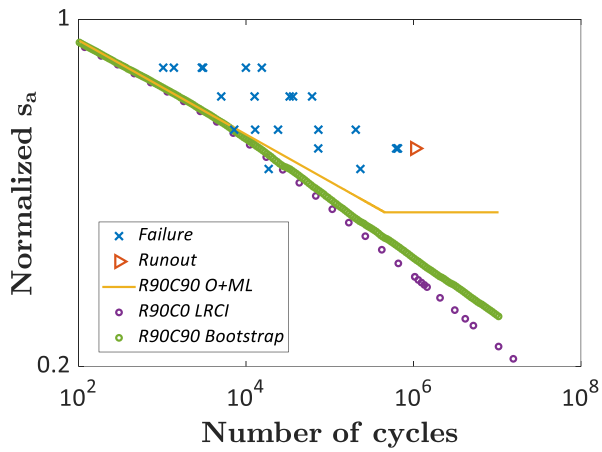
Fig. 9 shows the design curves estimated by considering all the failures and the runout specimens above the 0.4 (Fig. 9a), the 0.45 (Fig. 9b) and the 0.5 (Fig. 9c) value for the investigated composite Polyint materials.



(a)



(b)



(c)

Figure 9: influence of runout specimens on the design curves (R90C90) estimated with the three investigated methodologies for the composite Polynt material: a) runout specimens above 0.4; b) runout specimens above 0.45; c) runout specimens above the 0.50.

Table 3 summarizes the results of the analysis on the influence of runout data on the design curves of the investigated composite Polynt material.

Table 3: summary of the analysis on the influence of runouts on the design curves of the investigated Polynt material.

	All data	**1st subset (0.40 threshold)**	**2nd subset (0.45 threshold)**	**3rd subset (0.50 threshold)**
Saved runout specimens	0	1	3	8
Left runout specimens	9	8	6	1
Time saved	0	1 day 16 hours	4 days 23 hours	13 days 6 hours
Fatigue limit O+ML	0.44	0.44	0.47	0.41
Fatigue limit LRCI	0.42	0.42	0.40	0.30
Fatigue limit Bootstrap	0.45	0.45	0.43	0.33

According to Fig. 9a and to Table 3, as for the steel and the Al material, the removal of one runout at small stress amplitude has no influence on the lower bound of the fatigue limit, but the testing time can be significantly reduced (about 2 days). On the other hand, as the number of runout data is reduced, the lower bound of the fatigue limit tends to decrease (Fig. 9b) and it is not found in Fig. 9c with the LRCI and the bootstrap methods. Accordingly, the fatigue strength computed at the runout number of cycles is significantly smaller than the fatigue limit computed with the O+ML method, whose variation is limited.

The analysis carried out in this Section proved that the number of runout specimens can be reduced even below seven, as required by the SC approach. However, for the datasets characterized by a limited number of runout data (less than five, depending on the dataset), the design curves estimated with the LRCI and the Bootstrap approaches do not show a clear asymptotic trend. This means that there are no sufficient proofs to model an asymptotic trend with high confidence. By separately estimating the finite fatigue life trend with the modified Owen method and trend in the infinite life range with the ML, an asymptotic trend is assumed and thus modeled. The estimated lower bound of the fatigue limit is slightly affected by the runout numerousness and it does not vary too much with respect to that estimated with the complete SC dataset, provided that the experimental failures are concentrated close to the asymptote of the curve (improbable experimental occurrence).

3.4.2 Influence of runout specimens: fatigue limit estimated with the "Modified staircase method"

In this Section, a second analysis on the influence of runout specimens is carried out. Fig. 10 shows the procedure followed to reduce the number of runout specimens in the SC dataset.

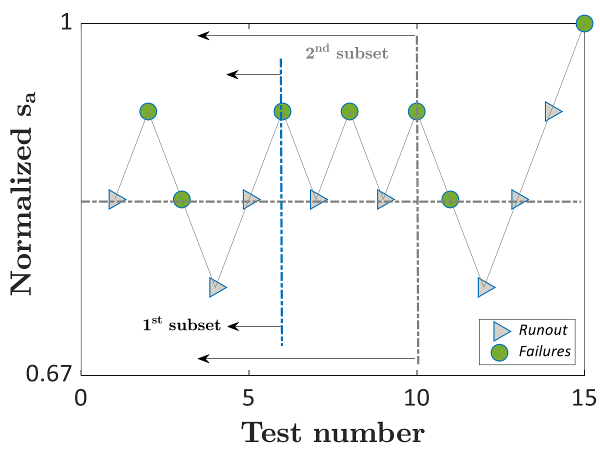
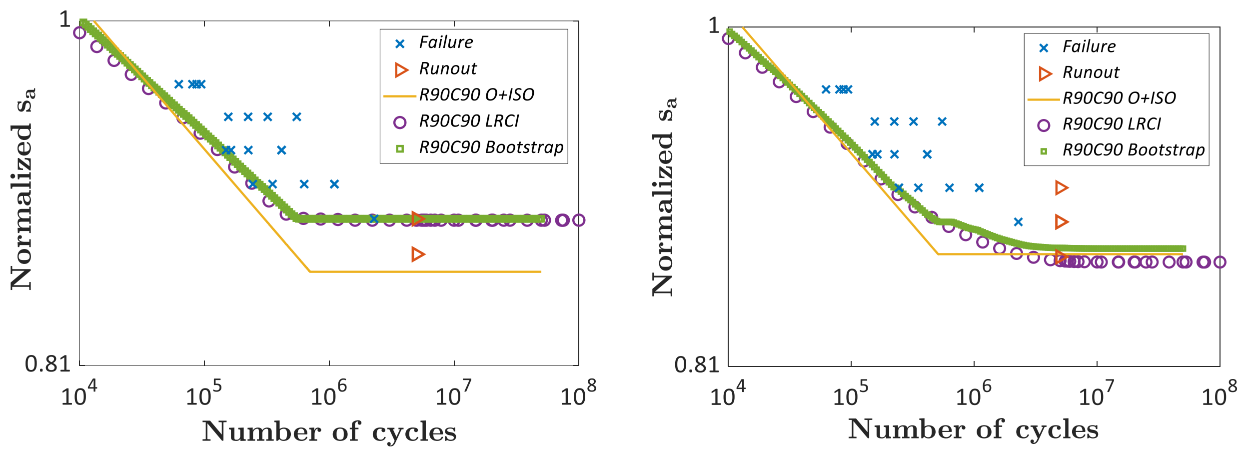


Figure 10: procedure for reducing the number of runout specimens in the SC datasets, with the fatigue limit estimated with the Modified staircase method [4].

According to Fig. 10, two subsets are considered. The first subset considers the first six data obtained with the SC approach, whereas with the second subset the first ten data in the SC dataset are included. The modified staircase method defined in the ISO standard [17] has been applied to compute the lower bound of the fatigue limit from the data contained in these subsets. According to [17], this procedure can be applied on datasets collected following the SC sequential procedure. The datasets can contain less than fifteen specimens required by the SC approach, provided that the standard deviation of the fatigue limit to be estimated is known. Therefore, with the procedure followed in Fig. 10, the modified staircase method can be applied. In the following, the standard deviation of the fatigue limit is assumed equal to the standard deviation estimated with the SC approach by considering the whole dataset. This ensures the application of this method. Moreover, the lower bound of the fatigue limit is estimated according to Equation 1: the multiplicative k coefficient has been retrieved from the "Statistical Tables" in [17].

Fig. 11 shows the design curves estimated with the investigated approaches for the steel material: Fig. 11a is for the first subset and Fig. 11b is for the second subset.



(a) (b)

Figure 11: analysis of the influence of runout specimens on the design curves of the steel material, by considering two reduced subsets and estimating the fatigue limit with the [17]; a) first subset (6 data); b) second subset (10 data).

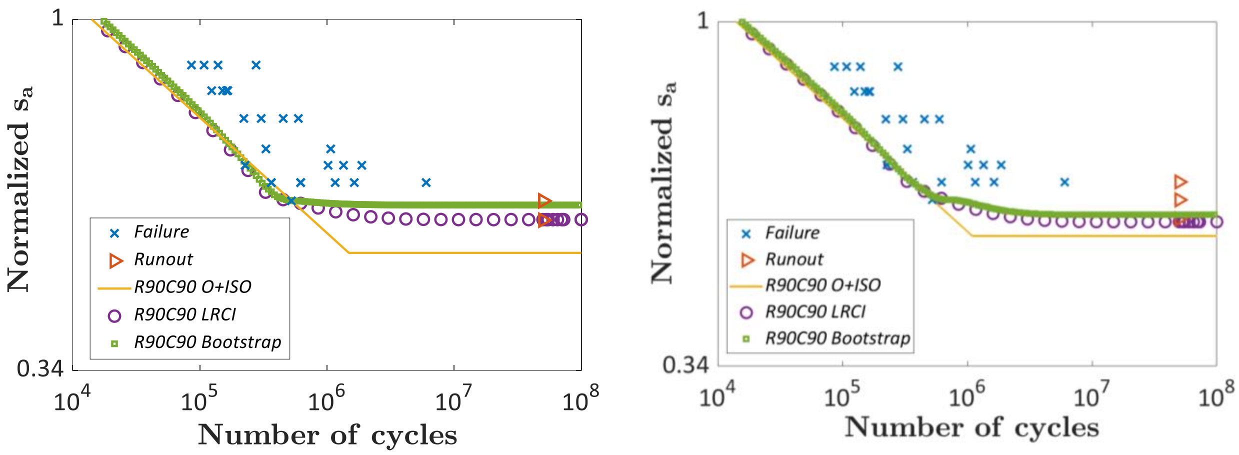
Table 4 summarizes the results of the second analysis on the influence of runout specimens on the design curves of the investigated steel material. In the following, the "*Saved time*" has been computed by considering both the failures and runout removed.

Table 4: summary of the second analysis on the influence of runout data on the design curves of the investigated steel material.

	All data	6 data	10 data
Saved runout specimens	0	4	2
Left runout specimens	9	4	6
Saved time	0	5 days 1 hours	2 days 15 hours
Fatigue limit O+ML	0.86	0.86	0.87
Fatigue limit LRCI	0.86	0.88	0.86
Fatigue limit Bootstrap	0.86	0.88	0.87

According to Fig. 11 and to Table 4, for both the investigated subsets, the three models show a decreasing trend ending with an asymptote. The variation of the fatigue limit is limited and close to that obtained with the whole dataset. Even with a number of data smaller than that required for a proper application of the staircase method, it is thus possible to assess an asymptotic trend with high confidence, with more than five days of tests saved.

Fig. 12 shows the design curves estimated with the investigated approaches for the Al material: Fig. 12a is for the first subset and Fig. 12b is for the second subset.



(a) (b)

Figure 12: analysis of the influence of runout specimens on the design curves of the Al material, by considering two reduced subsets and estimating the fatigue limit with the [17]; a) first subset (6 data); b) second subset (10 data).

Table 5 summarizes the results of the second analysis on the influence of runout specimens on the design curves of the investigated Al material.

Table 5: summary of the second analysis on the influence of runouts on the design curves of the investigated Al material.

	All data	1st subset	2nd subset
Saved runout specimens	0	7	5
Left runout specimens	10	3	5
Saved time	0	24 days 2 hours	16 days 3 hours
Fatigue limit O+ML	0.51	0.48	0.51
Fatigue limit LRCI	0.51	0.53	0.53
Fatigue limit Bootstrap	0.52	0.56	0.54

According to Fig. 12 and Table 5, the reduction of the dataset numerousness does not affect the shape of the design curves, with the three models showing a linear decreasing trend ending with an asymptote. The LRCI and the Bootstrap approaches provide similar values for the fatigue limit, with the latter providing the highest values. The differences between the lower bound of the fatigue limit computed with the three investigated methodologies tend to reduce as the number of data increase (2nd subset). The saved time is significant. For the first subset, i.e., by considering 6 data, about 24 days of tests are saved.

Finally, Fig. 13 shows the design curves estimated with the investigated approaches for the composite Polynt material: Fig. 13a is for the first subset and Fig. 13b is for the second subset.

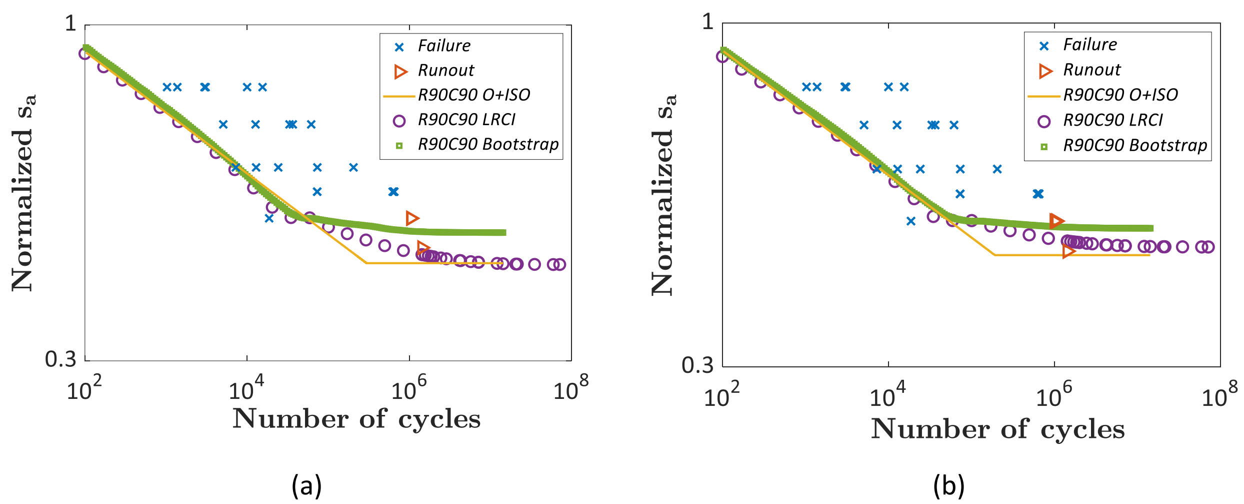


Figure 13: analysis of the influence of runout specimens on the design curves of the composite Polynt material, by considering two reduced subsets and estimating the fatigue limit with the [17]; a) first subset (6 data); b) second subset (10 data).

Table 6 summarizes the results of the second analysis on the influence of runout specimens on the design curves of the investigated composite Polynt material.

Table 6: summary of the second analysis on the influence of runouts on the design curves of the investigated composite Polyint material.

	All data	**6 data**	**8 data**
Saved runout specimens	0	5	3
Left runout specimens	9	3	5
Saved time	0	7 days 7 hours	3 days 24 hours
Fatigue limit O+ML	0.44	0.43	0.44
Fatigue limit LRCI	0.42	0.42	0.46
Fatigue limit Bootstrap	0.45	0.48	0.49

According to Fig. 13 and Table 6, the trend already found for the other tested material is confirmed. Indeed, by considering the two datasets, a model with an asymptotic trend is found to be appropriate for the experimental data. The time saved is up to about eight days with the first subset. The bootstrap method provides the largest values of the lower bound of the fatigue limit, whereas the lower bound of the fatigue limit estimated with the LRCI approach is the smallest by considering the first subset and above the value computed with the O+ML by considering the second subset.

To conclude, a general trend cannot be inferred from this analysis. However, in four out of six cases, the lower bound of the fatigue limit estimated with the bootstrap method tends to be largest, but this depends on the dataset and on the distribution of the data in the S-N plot. Moreover, differently from the analysis carried out in Section 3.4.1, an asymptotic trend at the end of the finite life range has been always found. In this second analysis, a set of data is concentrated close to the median fatigue limit and at least two runout specimens for each subset of data have been considered, highlighting the importance of an appropriate testing design and of runout data to discriminate if the experimental dataset shows an asymptotic or a continuous decreasing trend. Moreover, this analysis confirms that the runout numerousness can be limited, or better, optimized with a proper test design, without affecting the results.

3.5 Discussion

The results of the analysis carried out in previous Sections are here summarized and discussed. In general, the O+SC method is of simple implementation and the design curves can be easily assessed through the application of the least square method for the finite fatigue life range and by applying simple formulas for the infinite life range. On the other hand, the LRCI and the Bootstrap approaches for the estimation of the lower bound of the P-S-N curves require a more difficult implementation. For the LRCI approaches, an optimization algorithm should be used and an automated iterative procedure should be implemented. Moreover, optimization algorithms implemented in commercial software, like the *fminsearch* algorithm in Matlab®, required an initial guess of the parameters to be estimated. A not-appropriate choice of these initial values can affect the final estimation, i.e., a local minimum in place of the global minimum of the Likelihood function can be found. This complexity concerning the implementation of the LRCI method has limited its use in the analysis of fatigue data [37]. Moreover, the optimization process should be repeated iteratively in order to properly estimate the $PL[\theta_1]$ function (Equation 4), and this affects the computational time. However, with the proper programming strategy, the estimation time can be limited (e.g., less than five minutes for more than fifty points in the considered life range). On the other hand, one of the major weaknesses associated with the bootstrap method is the "computational time" for the estimation of the design curves. Indeed, according to Section 3.2, at least 1000 datasets should be simulated and, for each of them, the P-S-N curves are estimated through an optimization process and by numerically solving Equation 2 (e.g., with the *fzero* function in Matlab), thus affecting the computational time. Accordingly, even if the implementation of the methodology is not that complicated, i.e., the procedure for the parameter estimation with the ML principle is implemented one time and then repeated iteratively and automatically, the time for the estimation of the design curve can be larger than fifteen minutes, depending on the computational power.

The investigated methods have proved to be effective in the estimation of the linear decreasing trend in the finite life region and of an asymptotic trend in the infinite fatigue life range. In the finite fatigue life range, the trend has been found to be similar, with the slope of the curve being very close. On the other hand, in the infinite life range, the LRCI method has proved to be the most conservative, since the lower bound of the fatigue limit has been found to be always below that computed with the other methodologies when the whole datasets have been considered. However, since the real value of the fatigue limit is not known, i.e., the constant coefficients have been estimated by considering the experimental sample and not the entire population, it cannot be concluded that the LRCI method provides the most conservative estimation, but only that the estimated LRCI design curves are generally below those estimated with the other investigated methods. It is must be noted that the objective of the SC method is to assess the fatigue strength at the runout number of cycles [18, 32], but, in industrial applications, it is used in combination with the Owen method to assess the fatigue limit, i.e., of the asymptotic trend of the curve.

Another important aspect to be discussed is the capability of the models to assess the real trend of the experimental data in the S-N plot. The three investigated approaches can model a linear decreasing trend ending with an asymptote. However, with the O+SC approach, this trend is always found, since the parameters of the distribution of the fatigue life in the finite and infinite region are estimated separately and a fatigue limit is forcibly assumed. On the other hand, with the LRCI and the Bootstrap approaches, based on the model in Equation 2, the parameter estimation is carried out by considering the whole dataset, without separating it in subsets for the finite and the infinite fatigue life region. These approaches, based on the ML principle, are more flexible and can adapt to the experimental datasets. For example, as shown in Section 3.4.1, the estimated design curves can show a linear decreasing trend with or without a fatigue limit, depending on the analyzed dataset. This does not mean that the investigated material has not a fatigue limit, but that a linear decreasing trend is more appropriate for the experimental data with the selected high reliability and confidence. This represents an important strength of the methods based on the ML principle that do not force the design curves to end with an asymptotic trend.

The above-analyzed results provide interesting indications on the testing strategy to be followed and on the runout numerousness. As discussed above, with the O+SC an asymptote at the end of the curve is assumed, even if a continuous decreasing trend is more appropriate for the experimental data. In general, if experimental failures occur at a number of cycles to failure significantly smaller than the runout number of cycles, the design curves estimated with the LRCI and the Bootstrap show an asymptotic trend. In order to discriminate if a linear trend ending with an asymptote properly fits the experimental data, the experimental campaign should be properly designed. Indeed, Section 3.4.2 has proved that the sequential approach proposed in the SC permits to verify the occurrence of a fatigue limit and that less than fifteen data provide reliable results. For example, with six data, among which at least two runout data at two stress levels, the lower bound of the fatigue limit close to that obtained with the whole dataset has been found, but the testing time has been significantly reduced. For the application of the modified SC method [17] with less than fifteen data, the step should be as close as possible to the standard deviation of the fatigue limit and should be known before the tests. These parameters are generally not known before the tests and should be verified after the tests. On the other hand, with the LRCI and the Bootstrap methods, there are no restrictions or rules to be followed and the experimental tests should be carried out without the need of assuming material parameters before the tests. For example, an optimized and suggested testing strategy would involve collecting less than ten data following a sequential approach to verify the occurrence of a fatigue limit and thereafter performing tests in the finite life range according to the International Standards [17].

One of the weaknesses of the LRCI and the Bootstrap approaches is that an analytical solution for the design curves is not obtained, and a closed-form equation for the design curve cannot be estimated, as for the O+SC method. This would prevent the use of the design curves estimated with the LRCI and the Bootstrap approaches in finite element code that have implemented a Basquin's model for the finite fatigue life range. However, this criticality can be easily overcome, for example, by interpolating the data points for the lower bound curves estimated with the LRCI and the Bootstrap approaches in order to find the best linear fit for the finite fatigue life range.

Table 7 summarizes the results of the analyses carried out in the paper, ranking the properties and the characteristics of the investigated methodologies with the symbol +.

Table 7: Strengths and weaknesses of the investigated methodologies.

	Ease of estimation	Computational time	Indications on the real trend	Limitations/a posteriori verifications	Use for FE code
O+SC	++++	++++	+	+	++++
LRCI	+	++	++++	++++	+++
Bootstrap	++	+	++++	++++	+++

In particular, "++++" indicates a high level for the investigated property (or a strength of the methodology), whereas "+" indicates a low level (or a weakness).

4. CONCUSIONS AND OBSERVATIONS

In the paper, three methodologies for the estimation of the fatigue design curves from experimental data showing a linear decreasing trend ending with an asymptote, i.e., the so-called fatigue limit, have been compared. The design curves showing this trend are of particular interest in industrial applications for components subjected to loads in the High-Cycle Fatigue (HCF) life range. In the paper, the design curve is the lower confidence bound of a high-reliability quantile P-S-N curve. The design curves estimated with the modified Owen (finite life range) and the staircase method in the infinite life range (O+SC), by considering the likelihood ratio confidence interval of a specific quantile P-S-N curve (LRCI) and with a method based on the bootstrap approach (bootstrap in the following) were compared. The investigated methodologies and their effectiveness were validated with experimental datasets obtained by testing three different materials: a steel, an Alluminium alloy and a composite material. It is worth noting that the results obtained in the paper provide useful indications for the design of fatigue tests and on the strength and the weaknesses of the three investigated methodologies, but a larger number of datasets, experimental or simulated, are necessary to reliably generalize the main findings of this work.

The following conclusions and observations can be drawn:

a) The three methods have proved effective in modelling the P-S-N curves, with the median curves found to be in agreement with the experimental data. The O+SC methodology is of easy implementation and the design curves can be easily obtained according to well-established procedures. The implementation of the LRCI approach is rather complicated and a proper iterative algorithm must be developed and implemented. The implementation of Bootstrap method is simpler, but the random simulation of at least 1000 datasets requires a computational time significantly larger than that required by the other investigated methods.

b) In the finite fatigue life region, the three investigated methods have provided similar trends and slopes of the curve. On the other hand, the lower bound of the fatigue limit estimated with the LRCI has been found in general below that estimated with the other investigated methodologies. It is not possible to establish which methodology permits a more reliable estimation of the P-S-N curve, since the real design curve is not known.

c) Generally, when the O+SC methodologies have been employed, an asymptotic trend of the design curves at small stress amplitude was assumed and modeled. Otherwise, the modified Owen method was enough to assess the stress-life relationship, without a time-consuming staircase. However, depending on the dataset, this assumption can be risky, since the experimental data can show a continuously decreasing trend without a fatigue limit. On the other hand, with the LRCI and the Bootstrap methodologies, the employed general model can adapt to the experimental data and model the linear decreasing trend both with or without a final asymptote, without any a-priori assumptions on the data trend.

d) The influence of the runout numerousness has been also investigated. Runout specimens are the most time-consuming data and significantly affect the testing time. Accordingly, the information they contain should be fully exploited and maximized. Runout data have been found to be fundamental to assess if a model with a fatigue limit is appropriate for the investigated dataset. In particular, at least one runout data at two different stress amplitudes should be experimentally found to verify the occurrence of a fatigue limit with high reliability or confidence. However, it has been also shown that the number of runout data can be reduced with respect to that required for the application of the SC method, significantly limiting the testing time.

e) With the LRCI and the Bootstrap methodologies, there is no need to verify the testing parameters after the tests (e.g., with the SC approach the employed step is to be verified after the tests) and all the experimental results can be considered together, without the need of differentiating the datasets for the finite and the infinite life range. This ensures more freedom in the design of the experimental activity. With the LRCI and the Bootstrap methodologies, a procedure that requires less than ten specimens (six can be appropriate) tested with a sequential approach, to verify the occurrence of the fatigue limit, and tests in the finite life range according to the indications provided by International Standards would permit to estimate design curves with a limited testing time.

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