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On the existence of minimal expansive solutions to the N-body problem

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Abstract

We deal, for the classical N-body problem, with the existence of action minimizing half entire expansive solutions with prescribed asymptotic direction and initial configuration of the bodies. We tackle the cases of hyperbolic, hyperbolic-parabolic and parabolic arcs in a unified manner. Our approach is based on the minimization of a renormalized Lagrangian action on a suitable functional space. With this new strategy, we are able to confirm the already-known results of the existence of both hyperbolic and parabolic solutions, and we prove for the first time the existence of hyperbolic-parabolic solutions for any prescribed asymptotic expansion in a suitable class. Associated with each element of this class we find a viscosity solution of the Hamilton-Jacobi equation as a linear correction of the value function. Besides, we also manage to give a precise description of the growth of parabolic and hyperbolic-parabolic solutions.

1 Introduction and main results

In this paper, we deal with half entire solutions to the N-body problem of Celestial Mechanics in the Euclidean space $\mathbb{R}^d$ of hyperbolic, parabolic or mixed hyperbolic-parabolic type. We first investigate the existence of trajectories to the gravitational N-body problem having prescribed growth at infinity. This classical line of research has recently been re-energized by the injection of new methods of analysis, of perturbative, variational, geometric and/or analytic functional nature. Indeed, in addition to the classical literature on the subject [1, 9, 23, 28, 29], we quote the recent results about the existence of hyperbolic solutions [11, 15, 17, 20], parabolic ones

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[3–5, 18, 19, 30] and hyperbolic-parabolic ones [6], without neglecting those ending in an oscillatory manner [13, 14, 25] and references therein.

To start with, let us consider N point masses $m_1, \ldots, m_N > 0$ moving under the action of the mutual attraction, with the inverse-square law of universal gravitation. We denote the components of the configuration vector $x = (r_1, \ldots, r_N) \in \mathbb{R}^{dN}$ of the positions of the bodies and by $|r_i - r_j|$ the Euclidean distance between two bodies i and j. Newton's equation of motion for the i-th body of the N-body problem reads as

$$m_i \ddot{r}_i = - \sum_{j=1,\ldots,N,\ j\neq i}^{N} m_i m_j \frac{r_i - r_j}{|r_i - r_j|^3}. \tag{1.1}$$

Since these equations are invariant by translation, we can fix the origin of our inertial frame at the center of mass of the system. We can thus define the configuration space of the system as

$$\mathcal{X} = \left\{ x = (r_1, \ldots, r_N) \in \mathbb{R}^{dN},\ \sum_{i=1}^{N} m_i r_i = 0 \right\}$$

and denote by $\Omega = \{x \in \mathcal{X} \mid r_i \neq r_j\ \forall\, i \neq j\} \subset \mathcal{X}$ the set of configurations without collisions, which is open and dense in $\mathcal{X}$, and with Δ its complement, that is the collision set. Now we can write the equations of motion as

$$\mathcal{M}\ddot{x} = \nabla U(x), \tag{1.2}$$

where $\mathcal{M} = \mathrm{diag}(m_1 I_d, \ldots, m_N I_d)$ is the matrix of the masses and the function $U : \Omega \to \mathbb{R} \cup \{+\infty\}$ is the Newtonian potential

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|}. \tag{1.3}$$

Newton's equations define an analytic local flow on $\Omega \times \mathbb{R}^{dN}$ with a first integral given by the mechanical energy:

$$h = \frac{1}{2}\|\dot{x}\|_{\mathcal{M}}^2 - U(x).$$

We will use $\|\cdot\|_{\mathcal{M}}$ to denote the norm induced by the mass scalar product

$$\langle x, y\rangle_{\mathcal{M}} = \sum_{i=1}^{N} m_i \langle r_i, s_i\rangle, \qquad \text{for any } x = (r_1, \ldots, r_N),\ y = (s_1, \ldots, s_N) \in \mathcal{X},$$

where, with a little abuse, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and also in $\mathcal{X}$.

In this paper we will be concerned with the class of expansive motions, which is defined in the following way.

Definition 1.1 A motion $x : [0, +\infty) \to \Omega$ is said to be expansive when all the mutual distances diverge, that is, when $|r_i(t) - r_j(t)| \to +\infty$ as $t \to +\infty$ for all $i < j$. Equivalently, the motion is expansive if $U(x(t)) \to 0$ as $t \to +\infty$.

From the conservation of the energy, we observe that, since $U(x(t)) \to 0$ implies $\|\dot{x}(t)\|^2_{\mathcal{M}} \to 2h$ as $t \to +\infty$, expansive motions can only occur at nonnegative energies.

For a given motion, we introduce the minimum and the maximum separation between the bodies at time t as the two functions

$$r(t) = \min_{i<j} |r_i(t) - r_j(t)| \quad \text{and} \quad R(t) = \max_{i<j} |r_i(t) - r_j(t)|,$$

where we write $|\cdot|$ to denote the standard Euclidean norm in $\mathbb{R}^d$. The next fundamental theorems give us a more accurate description of the system's expansion.

Theorem 1.2 (Pollard, 1967 [27]) *Let x be a motion defined for all $t > t_0$. If r is bounded away from zero, then we have that $R = O(t)$ as $t \to +\infty$. In addition, $R(t)/t \to +\infty$ if and only if $r(t) \to 0$.*

Theorem 1.3 (Marchal-Saari, 1976 [23]) *Let x be a motion defined for all $t > t_0$. Then either $R(t)/t \to +\infty$ and $r(t) \to 0$, or there is a configuration $a \in \mathcal{X}$ such that $x(t) = at + O(t^{2/3})$. In particular, for superhyperbolic motions (i.e. motions such that* $\limsup_{t\to+\infty} R(t)/t = +\infty$*) the quotient $R(t)/t$ diverges.*

Theorem 1.4 (Marchal-Saari, 1976 [23]) *Suppose that $x(t) = at + O(t^{2/3})$ for some $a \in \mathcal{X}$ and that the motion is expansive. Then, for each pair $i < j$ such that $a_i = a_j$, we have $|r_i(t) - r_j(t)| \approx$ [1] $t^{2/3}$.*

Next, let us recall the well-known Chazy classification of the expansive motions for the N-body problem (cfr. [9]), based on the asymptotic order of growth of the distances between the bodies. In light of Theorem 1.2, expansive motions cannot be superhyperbolic, and hence they have the form $x(t) = at + O(t^{2/3})$ for some limit $a \in \mathcal{X}$. Assuming that the center of mass of the system is at rest, Chazy classified these motions in three classes:

- *Hyperbolic*: $a \in \Omega$ and $|r_i(t) - r_j(t)| \approx t$ for all $i < j$;
- *Hyperbolic-parabolic*: $a \in \Delta$ but $a \neq 0$;
- *Completely parabolic*: $a = 0$ and $|r_i(t) - r_j(t)| \approx t^{2/3}$ for all $i < j$.

The following definition is in order.

Definition 1.5 A motion $x(t)$ is said to have limit shape when there is a time-dependent similarity $S(t)$ of the space $\mathbb{R}^d$ such that $S(t)x(t)$ converges to some configuration $a \neq 0$.

[1] Given positive functions f and g, we write $f \approx g$ when there exist two positive constants α and β such that $\alpha \leq \frac{f}{g} \leq \beta$.

In our case, there is a diagonal action of $S(t)$, that is $S(t)x = (S(t)r_1, \ldots, S(t)r_N)$ for $x = (r_1, \ldots, r_N) \in \mathcal{X}$. In particular, the limit shape of a (half) hyperbolic motion is its asymptotic velocity $a = \lim_{t\to+\infty} \frac{x(t)}{t}$ (it is a consequence of Theorem 2.5 below), while the limit shape (if it exists) of a (half) parabolic motion must be a central configuration, that is, a critical point of the potential U constrained on the inertial ellipsoid $\mathcal{E} = \{x \in \mathcal{X} : \|x\|^2_{\mathcal{M}} = 1\}$.

In this paper, we are going to tackle the existence of half entire expansive solutions for the Newtonian N-body problem from a unified perspective, using a global variational approach which involves a suitably renormalized Lagrangian action functional, as the usual Lagrangian can not be integrable on the half line. In particular, referring to Chazy's classification, we will show a proof of existence of motions for each one of the previous three classes of motions. As a first step, we shall revisit recent works by E. Maderna and A. Venturelli about the existence of half hyperbolic and parabolic trajectories from this new angle.

Theorem 1.6 (Maderna and Venturelli 2020, [20]) *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic motion $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = at - \log(t)\nabla U(a) + O(1) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$ and for any collisionless configuration $a \in \Omega$.

As far as the parabolic case is concerned, in addition to providing an alternative proof, we will be able to extend the result of Maderna and Venturelli [19] by improving the estimate of the remainder as follows.

Theorem 1.7 *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a parabolic solution $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty, \tag{1.4}$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$, for any minimal normalized central configuration b_m and for $\beta = \sqrt[3]{\frac{9}{2}U(b_m)}$.

Here, a minimal central configuration is a minimizer of the potential U constrained to the inertia ellipsoid $\mathcal{E} = \{x \in \mathcal{X} : \|x\|^2_{\mathcal{M}} = 1\}$.

As said, the existence of hyperbolic and parabolic solutions for the Newtonian N-body problem has already been proved by Maderna and Venturelli in 2020 and 2009, respectively. In [20], the authors proved the existence of hyperbolic motions for any prescribed limit shape, and initial configuration of the bodies and any positive value of the energy. Their approach is based on the construction of global viscosity solutions for the Hamilton-Jacobi equation $H(x, \nabla u) = h$. In [19], for any starting configuration, they proved the existence of parabolic arcs asymptotic to any prescribed normalized minimal central configuration. More specifically, these solutions were obtained as the limits of solutions of sequences of approximating two-point boundary value problems. To exclude collisions, both proofs in [20] and [19] invoke Marchal's Principle, which ensures the absence of collisions in action-minimizing paths (Theorem 2.1).

Compared to Maderna and Venturelli's articles, in this paper we show alternative and simpler proofs for the existence of hyperbolic and parabolic solutions in a unified framework, which is based on a straightforward application of the Direct Method of the Calculus of Variations to minimize the renormalized Lagrangian actions associated to the problem. This approach has the advantage of allowing us to complement the existence of parabolic arcs with their (almost exact) expansion (1.4).

Finally, after proving Theorems 1.6 and 1.7, we will extend our approach to similarly prove the existence of hyperbolic-parabolic solutions for the N-body problem. In order to state our main result we need to introduce the *a-cluster partition* associated with a collision asymptotic velocity $a \in \Delta \setminus \{0\}$, where clusters are the equivalence classes of the relation $i \sim j \iff a_i - a_j = 0$. Given a cluster K, we consider the associated partial potential U_K, where the sum in (1.3) is restricted to the cluster K. The a-clustered potential U_a is the sum of all the cluster potentials of the partition. Now we can state our main theorem:

Theorem 1.8 *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic-parabolic motion $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = at + \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$, for any collision configuration $a \in \Delta$, for any normalized minimal central configuration $b_m \in \mathcal{X}$ of the a-clustered potential and for any choice of the energy constant $h > 0$ (see Sect. 5 for the exact definition of β and b_m).

Intuitively, hyperbolic-parabolic motions are those expansive motions of the form $x(t) = at + o(t)$, as $t \to +\infty$, when their limit shapes have collisions, that is, $a \in \Delta \setminus \{0\}$. This means that hyperbolic-parabolic motions can be viewed as clusters of bodies moving asymptotically with linear growth, while the distances of the bodies inside each cluster grow with a rate of order $t^{2/3}$ and, referred to its center of mass, the cluster has a limit shape which is a prescribed minimal configuration of the cluster potential U_K. For the Newtonian N-body problem, the existence of hyperbolic-parabolic solutions for any prescribed positive energy and any given initial configuration of the bodies has been tackled by Burgos in [6], where his proof follows from and application of Maderna and Venturelli's Theorem on the existence of hyperbolic motions and a limiting procedure as the limit shape approaches the collision set. With respect to Burgos' result, we can provide a much wider class of such hyperbolic-parabolic trajectories, associated with any asymptotic speed $a \in \Delta$ and minimal a-clustered central configuration b_m. Moreover, our approach provides much more detailed information about the asymptotic behavior of the solution and a better description of the motion of the bodies. Indeed, to prove Theorem 1.8, we partition the set of bodies following the natural cluster partition that was presented by Burgos and Maderna in [7] and is defined as follows: if $x(t) = (r_1(t), \dots, r_N(t))$

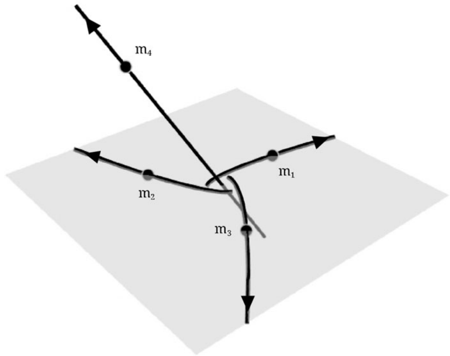

Fig. 1 Example of a cluster partition of four bodies in a hyperbolic-parabolic motion. Supposing $a_1 = a_2 = a_3 \neq a_4$, the index set $\mathcal{N} = \{1, 2, 3, 4\}$ is partitioned into the clusters $K_1 = \{1, 2, 3\}$ and $K_2 = \{4\}$, so $|r_i(t) - r_j(t)| \approx t^{2/3}$, $\forall\ i < j,\ i, j \in K_1$. This means that the point masses m_1, m_2, m_3 move in a triangular parabolic expansion at infinity whose scale is $t^{2/3}$ when is referred to the clusters's barycenter, while the escape of the mass m_4 from the center of mass of the other three bodies, has a linear growth at infinity

and $a = (a_1, \ldots, a_N)$, then $a_i = a_j$ if and only if $|r_i(t) - r_j(t)| = O(t^{2/3})$, and the partition of the set of bodies is defined by this equivalence relation (see Fig. 1 for an example of a cluster partition). Using this particular partition, we are able to decompose the Lagrangian action into two terms: the first is related to the hyperbolic motion of the clusters' barycenters and the second is related to the parabolic motion of the bodies inside the clusters. Through similar proofs to the ones in Theorems 1.6 and 1.7, we can thus apply the Direct Method of the Calculus of Variation and Marchal's Principle also to the case of hyperbolic-parabolic motions.

As a consequence of our variational setting, we also obtain the following corollary, where the absence of collisions is guaranteed by Marchal's Principle and the free-time minimization property (see Definition 1.10 below) is proved in Corollary 6.3.

Corollary 1.9 *The motions $x(t)$ given by Theorems 1.6, 1.7 and 1.8 are continuous at $t = 1$ and collisionless for $t > 1$. Moreover, they are free-time action minimizers at their energy level.*

As already pointed out by Maderna and Venturelli, a family of hyperbolic trajectories that are minimal in free time is associated, via the Busemann function, with a solution of the time-independent Hamilton-Jacobi equation. A further advantage of the approach through the direct minimization of a renormalized action functional is that a value function, dependent on the initial point, is directly defined. As we shall outline in Sect. 7, a linear correction to the value function is, as expected from theory, a solution of the Hamilton-Jacobi equation.

1.1 The renormalized action principle

Our general strategy in the proofs of Theorems 1.6, 1.7 and 1.8 consists in proving the existence of free-time minimizers $x(t)$, having the desired initial configuration and expansion at infinity, for the Lagrangian action associated to Newton's equations (1.2), which is defined as usual (cfr. (2.2)):

$$\mathcal{A}_L(x) = \int_1^{+\infty} L(x(t), \dot{x}(t))\, \mathrm{d}t.$$

Definition 1.10 A curve $\gamma : I \to \mathcal{X}$ is a free-time minimizer for the Lagrangian action at energy h if $\forall\, [a,b], [a',b'] \subset I$ and $\forall\, \sigma : [a',b'] \to \mathcal{X}$ such that $\gamma(a) = \sigma(a')$ and $\gamma(b) = \sigma(b')$, it holds

$$\int_a^b L(\gamma, \dot{\gamma})\, \mathrm{d}t + h(b-a) \leq \int_{a'}^{b'} L(\sigma, \dot{\sigma})\, \mathrm{d}t + h(b'-a').$$

We seek hyperbolic, parabolic or hyperbolic-parabolic trajectories as free-time minimizers having an *a priori* infinite Lagrangian action. This fact calls the introduction of a Renormalized Action Principle as follows. In each of the three cases we fix a background reference curve r_0, taking the form at, $\beta b_m t^{2/3}$ or $at + \beta b_m t^{2/3}$ in accordance with the statement of the corresponding Theorem, and we seek solutions of the form

$$x(t) = r_0(t) + \varphi(t) + x^0 - r_0(1),$$

for some φ belonging to the appropriate Sobolev space (cfr. (2.3)):

$$\mathcal{D} = \{\varphi :\ \varphi(1) = 0 \text{ and } \int_1^{\infty} \|\dot{\varphi}\|^2_{\mathcal{M}} < +\infty\},$$

which ensures $\|\varphi(t)\| = o(t^{1/2})$ as $t \to +\infty$ (see §2 for details). So, x^0 is the starting point and r_0 is definitely the guiding term of the sought curve for large t's, as its minimal growth rate in the three cases is $t^{2/3}$. The correction term φ will be chosen to minimize a Renormalized action.

Definition 1.11 (Renormalized Lagrangian action) Given r_0 and x^0, we define the Renormalized Lagrangian action as

$$\mathcal{A}^{ren}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(\varphi(t) + r_0(t) + x^0 - r_0(1)) - U(r_0(t)) - \langle \mathcal{M}\ddot{r}_0(t), \varphi(t)\rangle\, \mathrm{d}t. \tag{1.5}$$

In contrast to the usual Lagrangian action, the renormalized one is not positive defined. Therefore, a major difficulty will consist in proving its coercivity which, together with weak lower semicontinuity, will yield the actual existence of minimizers. Once done, we will conclude by exploiting the following principle:

Renormalized Action Principle *Given x^0, r_0 and $\mathcal{D}$ as above, if $\varphi^{min} \in \mathcal{D}$ is a minimizer of the renormalized Lagrangian action, then the corresponding expansive motion*

$$x(t) = r_0(t) + \varphi^{min}(t) + x^0 - r_0(1)$$

is a free-time minimizer of the Lagrangian action and, in particular, is a solution of Newton's equations (1.2) *for any* $t \in (1, +\infty)$ (*or for any* $t \in [1, +\infty)$, *if* $x^0 \in \Omega$).

Proof Suppose that a curve $\varphi \in \mathcal{D}$ is a minimizer of the renormalized Lagrangian action and consider the associated expansive motion $x(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$. By Hamilton's Principle of Least Action, since φ is a minimizer, it is a solution of the Euler-Lagrange equations associated with $\mathcal{A}^{ren}$:

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - \mathcal{M}\ddot{r}_0(t) \tag{1.6}$$

at any time $t \in [1, +\infty)$ such that $x(t) \in \Omega$. Moreover, since Corollary 6.3 asserts that $x(t)$ is a free-time minimizer for the Lagrangian action, we can invoke Marchal's Principle (Theorem 2.1), to deduce that $x(t)$ has no collisions for any $t \in (1, +\infty)$. We can thus conclude that φ solves the above systems and, equivalently, that $x(t)$ is a solution of equations (1.2) for any $t \in (1, +\infty)$. □

In order to shorten the notation, throughout the paper we will usually write $\mathcal{A}$ instead of $\mathcal{A}^{ren}$ when there is no ambiguity in interpretation.

2 The variational setting

For the N-body problem, the Hamiltonian H is defined over $\Omega \times \mathbb{R}^{dN}$ as

$$H(x, p) = \frac{1}{2}\|p\|^2_{\mathcal{M}^{-1}} - U(x), \tag{2.1}$$

where the potential U is defined in (1.3), while the Lagrangian is defined over $\Omega \times \mathbb{R}^{dN}$ as

$$L(x, v) = \frac{1}{2}\|v\|^2_{\mathcal{M}} + U(x). \tag{2.2}$$

This means, in particular, that L and H become infinite when x has collisions. Given two configurations $x, y \in \mathcal{X}$ and $T > 0$, we denote by $\mathcal{C}(x, y, T)$ the set of absolutely continuous curves $\gamma : [a, b] \to \mathcal{X}$ going from x to y in time $T = b - a$ and we write $\mathcal{C}(x, y) = \bigcup_{T>0} \mathcal{C}(x, y, T)$. We define the Lagrangian action of a curve $\gamma \in \mathcal{C}(x, y, T)$ as the functional

$$\mathcal{A}_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma})\, \mathrm{d}t = \int_a^b \frac{1}{2}\|\dot{\gamma}\|^2_{\mathcal{M}} + U(\gamma)\, \mathrm{d}t.$$

Hamilton's principle of least action implies that if a curve γ is a minimizer of the Lagrangian action in $\mathcal{C}(x, y, T)$, then γ satisfies Newton's equations at every time $t \in [a, b]$ in which $\gamma(t)$ has no collisions. However, as Poincaré already noticed in [26], there are curves with isolated collisions and finite action, which means that minimizing orbits may not always be true motions. The following theorem, ensuring the absence of collisions for minimal arcs, represents a big step forward in this theory, since it enabled the application of variational techniques to study the Newtonian

N-body problem. The main idea to prove the theorem via averaged variations was introduced by Marchal in [22], while more complete proofs are due to Chenciner in [10] and Ferrario and Terracini in [12].

Theorem 2.1 (Marchal [22], Chenciner [10], Ferrario and Terracini [12]) *Given $x, y \in \mathcal{X}$, if $\gamma \in \mathcal{C}(x, y)$ is defined on some interval $[a, b]$ and satisfies*

$$\mathcal{A}_L(\gamma) = \min\{\mathcal{A}(\sigma) \mid \sigma \in \mathcal{C}(x, y, b-a)\},$$

then $\gamma(t) \in \Omega$ for all $t \in (a, b)$.

Marchal's Theorem will be fundamental in our proofs, since it will guarantee that the minimizers of the action (whose existence is the object of our proofs) are in fact true motions of the N-body problem free of collisions. The Principle of Least Action, jointly with Theorem 2.1, has been widely applied in the search for collisionless periodic solutions to the N-body problem (cfr. e.g. [12, 24]). However, we must now build a suitable variational framework for the search of expansive solutions.

Our minimization will take place on the functional space

$$\begin{aligned}\mathcal{D} = \mathcal{D}_0^{1,2}([1, +\infty), \mathcal{X}) = \{\varphi \in H^1_{loc}([1, +\infty), \mathcal{X}) \ : \ \varphi(1) = 0 \\ \text{and} \int_1^{+\infty} \|\dot\varphi(t)\|^2_{\mathcal{M}}\, \mathrm{d}t < +\infty\},\end{aligned} \tag{2.3}$$

which is endowed with the norm

$$\|\varphi\|_{\mathcal{D}} = \left(\int_1^{+\infty} \|\dot\varphi(t)\|^2_{\mathcal{M}}\, \mathrm{d}t\right)^{1/2}.$$

Remark 2.2 Given a configuration $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{D}_0^{1,2}([1, +\infty), \mathcal{X})$, we will say that its components belong to the space $\mathcal{D}_0^{1,2}([1, +\infty), \mathbb{R}^d)$ and the $\mathcal{D}_0^{1,2}$-norm of each component is

$$\|\varphi_i\|_{\mathcal{D}} = \left(\int_1^{+\infty} |\dot\varphi_i(t)|^2\, \mathrm{d}t\right)^{1/2},$$

for $i = 1, \ldots, N$. We will write $\mathcal{D}_0^{1,2}(1, +\infty)$ to denote both the spaces $\mathcal{D}_0^{1,2}([1, +\infty), \mathcal{X})$ and $\mathcal{D}_0^{1,2}([1, +\infty), \mathbb{R}^d)$, since it will be trivial to distinguish them.

Proposition 2.3 (Cfr. Boscaggin-Dambrosio-Feltrin-Terracini, 2021 [5]) *The space $\mathcal{D}_0^{1,2}(1, +\infty)$ is a Hilbert space containing the set $C_c^\infty(1, +\infty)$ as a dense subspace.*

We recall here the following paramount Hardy-type inequality, which will be used several times in the paper. It states that the space $\mathcal{D}_0^{1,2}(1, +\infty)$ is continuously embedded in a weighted L^2-space with measure $\mathrm{d}t/t^2$.

Proposition 2.4 (*Hardy inequality*, Cfr. Boscaggin-Dambrosio-Feltrin-Terracini, 2021 [5]) *For every* $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$, *it holds that*

$$\int_1^{+\infty} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t \le 4\int_1^{+\infty} \|\dot{\varphi}(t)\|_{\mathcal{M}}^2\,\mathrm{d}t, \tag{2.4}$$

and, *moreover*,

$$\sup_{t\in[1,+\infty)} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t-1} \le \int_1^{+\infty} \|\dot{\varphi}(t)\|_{\mathcal{M}}^2\,\mathrm{d}t. \tag{2.5}$$

In order to prove the existence of minima for the functional $\mathcal{A}$ on $\mathcal{D}_0^{1,2}(1,+\infty)$, we will properly renormalize the Lagrangian action and, after proving its coercivity and weak lower semicontinuity, we will apply the Direct Method of the Calculus of Variations.

To describe the asymptotic expansion of our motions, we will use the following theorem and lemma. The former, applies to the cases of hyperbolic and hyperbolic-parabolic motions, while the latter, which is typically known as *Chazy's Lemma*, states that the set of initial conditions in the phase space that generate hyperbolic motions is an open set and that the map defined on this set that gives the asymptotic velocity in the future is continuous.

Theorem 2.5 (Chazy, 1922 [9]) *Let* $x(t)$ *be a motion with energy constant* $h > 0$ *and defined for all* $t > t_0$.

(i) *The limit*

$$\lim_{t\to+\infty} \frac{R(t)}{r(t)} = \lim_{t\to+\infty} \frac{\max_{i<j} |r_i(t) - r_j(t)|}{\min_{i<j} |r_i(t) - r_j(t)|} = L \in [1,+\infty]$$

always exists.

(ii) *If* $L < +\infty$, *there are a configuration* $a \in \Omega$ *and some function* P, *which is analytic in a neighborhood of* $(0,0)$, *such that for every* t *large enough*, *we have*

$$x(t) = at - \log(t)\nabla U(a) + P(u,v),$$

where $u = 1/t$ *and* $v = \log(t)/t$.

Lemma 2.6 (Maderna-Venturelli, 2020 [20]) *Working on an Euclidean space* E, *which is endowed with an Euclidean norm* $\|\cdot\|$, *let* $U : E^N \to \mathbb{R}\cup\{+\infty\}$ *be a homogeneous potential of degree* -1 *of class* C^2 *on the open set* $\Omega = \{x \in E^N \mid U(x) < +\infty\}$. *Let* $x : [0,+\infty) \to \Omega$ *be a given solution of* $\ddot{x} = \nabla U(x)$ *satisfying* $x(t) = at + o(t)$ *as* $t \to +\infty$ *with* $a \in \Omega$. *Then we have the following*:

1. *The solution* x *has asymptotic velocity* a, *meaning that*

$$\lim_{t\to+\infty} \dot{x}(t) = a.$$

2. (*Chazy's continuity of the limit shape*). *Given* $\varepsilon > 0$, *there are constants* $t_1 > 0$ *and* $\delta > 0$ *such that, for any maximal solution* $y : [0, T) \to \Omega$ *satisfying* $\|y(0) - x(0)\| < \delta$ *and* $\|\dot{y}(0) - \dot{x}(0)\| < \delta$, *we have*
   - $T = +\infty$, $\|y(t) - at\| < t\varepsilon$ *for all* $t > t_1$;
   - *there is* $b \in \Omega$ *with* $\|b - a\| < \varepsilon$ *for which* $y(t) = bt + o(t)$.

3 Existence of minimal half hyperbolic motions

This section is devoted to the proof of Theorem 1.6. The class of hyperbolic motions has the following equivalent definition, also due to Chazy (see [9]).

Definition 3.1 Hyperbolic motions are those motions such that each body has a different limit velocity vector, that is, $\dot{r}_i(t) \to a_i \in \mathbb{R}^d$, as $t \to +\infty$, and $a_i \neq a_j$ whenever $i \neq j$.

We consider the differential system

$$\begin{cases} \mathcal{M}\ddot{x} = \nabla U(x) \\ x(1) = x^0 \\ \lim_{t\to+\infty} \dot{x}(t) = a \end{cases}, \tag{3.1}$$

where $x^0 \in \mathcal{X}$ and $a \in \Omega$.

To prove the existence of hyperbolic motions to Newton's equations (3.1), we will look for solutions having the form $x(t) = \varphi(t) + at + x^0 - a$, where $\varphi : [1, +\infty) \to \mathcal{X}$ belongs to the space $\mathcal{D}_0^{1,2}(1, +\infty)$. We can thus equivalently study the system

$$\begin{cases} \mathcal{M}\ddot{\varphi} = \nabla U(\varphi + x^0 - a + at) \\ \varphi(1) = 0 \\ \lim_{t\to+\infty} \dot{\varphi}(t) = 0 \end{cases}. \tag{3.2}$$

Taking advantage of the problem's variational structure, we would be tempted to prove the existence of hyperbolic motions through the minimization of the Lagrangian action associated to the system (3.2), that is, the functional

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(\varphi(t) + x^0 - a + at)\,\mathrm{d}t, \tag{3.3}$$

where

$$U(\varphi(t) + x^0 - a + at) = \sum_{i<j} \frac{m_i m_j}{|(\varphi_i(t) + x_i^0 - a_i + a_i t) - (\varphi_j(t) + x_j^0 - a_j + a_j t)|}.$$

In attempting to work with the action functional as above, the major problem we encounter is that $U(\varphi(t) + x^0 - a + at)$ needs not to be integrable at infinity. Indeed, when $\varphi \in C_0^\infty([1, +\infty))$, $U(\varphi(t) + x^0 - a + at)$ decays as $\frac{1}{t}$ for $t \to +\infty$.

To overcome this problem, as we can add arbitrary functions to the Lagrangian without changing the associated Euler-Lagrange equations, we can renormalize the action functional in order to have a finite integral in the following way:

$$\mathcal{A}(\varphi) = \mathcal{A}^{ren}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(\varphi(t) + x^0 - a + at) - U(at)\,\mathrm{d}t.$$

3.1 Coercivity

In order to apply the Direct Method of the Calculus of Variations, we start by proving the coercivity of the functional, that is to say, that $\mathcal{A}(\varphi) \to +\infty$ as $\|\varphi\|_{\mathcal{D}} \to +\infty$. From now on, we will use the notations $\varphi_{ij} = \varphi_i - \varphi_j$, $x^0_{ij} = x^0_i - x^0_j$ and $a_{ij} = a_i - a_j$. We observe that the action can be equivalently written as

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2}\sum_{i=1}^N m_i|\dot{\varphi}_i(t)|^2 + U(\varphi(t) + x^0 - a + at) - U(at)\,\mathrm{d}t,$$

where

$$\begin{aligned}
&U(\varphi(t) + x^0 - a + at) - U(at)\\
&\quad= \sum_{i<j}\left(\frac{m_i m_j}{|(\varphi_i(t) + x^0_i - a_i + a_i t) - (\varphi_j(t) + x^0_j - a_j + a_j t)|} - \frac{m_i m_j}{|a_i - a_j|t}\right)\\
&\quad= \sum_{i<j}\left(\frac{m_i m_j}{|\varphi_{ij}(t) + x^0_{ij} - a_{ij} + a_{ij}t|} - \frac{m_i m_j}{|a_{ij}|t}\right).
\end{aligned}$$

Since we are working in the space of configurations whose center of mass is null at every time, we can use Leibniz's formula

$$\sum_{i=1}^N m_i|\dot{\varphi}_i(t)|^2 = \frac{1}{M}\sum_{i<j} m_i m_j|\dot{\varphi}_i(t) - \dot{\varphi}_j(t)|^2, \tag{3.4}$$

where $M = \sum_{i=1}^N m_i$. Indeed, it holds

$$\begin{aligned}
&\sum_{i<j} m_i m_j|\dot{\varphi}_i(t) - \dot{\varphi}_j(t)|^2\\
&\quad= \frac{1}{2}\sum_{i,j} m_i m_j(|\dot{\varphi}_i(t)|^2 + |\dot{\varphi}_j(t)|^2 - 2\langle\dot{\varphi}_i(t), \dot{\varphi}_j(t)\rangle)\\
&\quad= \frac{1}{2}\left(M\sum_{i=1}^N m_i|\dot{\varphi}_i(t)|^2 + M\sum_{j=1}^N m_j|\dot{\varphi}_j(t)|^2 - 2\langle\sum_{i=1}^N m_i\dot{\varphi}_i(t), \sum_{j=1}^N m_j\dot{\varphi}_j(t)\rangle\right)\\
&\quad= \frac{1}{2}\left(M\sum_{i=1}^N m_i|\dot{\varphi}_i(t)|^2 + M\sum_{j=1}^N m_j|\dot{\varphi}_j(t)|^2\right)
\end{aligned}$$

$$= M \sum_{i=1}^{N} m_i |\dot{\varphi}_i(t)|^2.$$

Using (3.4), we can then write the Lagrangian action as

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \sum_{i<j} m_i m_j \left(\frac{|\dot{\varphi}_{ij}(t)|^2}{2M} + \frac{1}{|\varphi_{ij}(t) + x_{ij}^0 - a_{ij} + a_{ij}t|} - \frac{1}{|a_{ij}|t} \right) \mathrm{d}t.$$

Using again Leibniz's formula (3.4), we also notice that $\|\dot{\varphi}\|_{L^2} \to +\infty$ if and only if there is $i<j$ such that $\|\dot{\varphi}_i - \dot{\varphi}_j\|_{L^2} \to +\infty$. Thus, we can prove the coercivity of the action by proving the coercivity of each term $\mathcal{A}_{ij}$, where

$$\mathcal{A}(\varphi) = \sum_{i<j} \mathcal{A}_{ij}(\varphi)$$

and

$$\mathcal{A}_{ij}(\varphi) = \int_1^{+\infty} m_i m_j \left(\frac{|\dot{\varphi}_{ij}(t)|^2}{2M} + \frac{1}{|\varphi_{ij}(t) + x_{ij}^0 - a_{ij} + a_{ij}t|} - \frac{1}{|a_{ij}|t} \right) \mathrm{d}t.$$

Using the inequality

$$|\varphi_i(t)| \leq \|\varphi_i\|_{\mathcal{D}} \sqrt{t}, \qquad \text{for every } i = 1, \ldots, N,\ t \geq 1 \text{ and } \varphi_i \in \mathcal{D}_0^{1,2}, \tag{3.5}$$

which follows from (2.5), we have

$$\begin{aligned} &U(\varphi(t) + x^0 - a + at) - U(at) \\ &\geq \sum_{i<j} \left(\frac{m_i m_j}{\|\varphi_{ij}\|_{\mathcal{D}} \sqrt{t} + |x_{ij}^0 - a_{ij}| + |a_{ij}|t} - \frac{m_i m_j}{|a_{ij}|t} \right); \end{aligned}$$

We can then look for an upper bound for the integral

$$\int_1^{+\infty} \left(\frac{1}{|a_{ij}|t} - \frac{1}{|a_{ij}|t + \|\varphi_{ij}\|_{\mathcal{D}} \sqrt{t} + |x_{ij}^0 - a_{ij}|} \right) \mathrm{d}t.$$

Using the change of variables $t = s^2$, we obtain

$$\frac{2}{|a_{ij}|} \int_1^{+\infty} \left(\frac{1}{s^2} - \frac{1}{s^2 + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{|a_{ij}|} s + \frac{|x_{ij}^0 - a_{ij}|}{|a_{ij}|}} \right) s \, \mathrm{d}s. \tag{3.6}$$

Since

$$\begin{aligned} s^2 + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{|a_{ij}|} s + \frac{|x_{ij}^0 - a_{ij}|}{|a_{ij}|} &= \left(s + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{2|a_{ij}|} \right)^2 - \frac{\|\varphi_{ij}\|_{\mathcal{D}}^2}{4|a_{ij}|^2} + \frac{|x_{ij}^0 - a_{ij}|}{|a_{ij}|} \\ &= \frac{\|\varphi_{ij}\|_{\mathcal{D}}^2}{4|a_{ij}|^2} \left[\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}} + 1 \right)^2 - 1 + \frac{4|x_{ij}^0 - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2} \right], \end{aligned}$$

(3.6) is equal to

$$\frac{2}{|a_{ij}|}\frac{4|a_{ij}|^2}{\|\varphi_{ij}\|_{\mathcal{D}}^2}\int_1^{+\infty}\left[\frac{1}{\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}}\right)^2}-\frac{1}{\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}}+1\right)^2-1+\frac{4|x_{ij}^0-a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2}}\right]s\,\mathrm{d}s. \tag{3.7}$$

Changing variables again with $\tau=\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}}$, we obtain that (3.7) is equal to

$$\frac{2}{|a_{ij}|}\int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\frac{4|x_{ij}^0-a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2}}\right]\tau\,\mathrm{d}\tau.$$

Since we are interested in large values of $\|\varphi_{ij}\|_{\mathcal{D}}$, we can suppose that there is some $\lambda<1$ such that $\frac{4|x_{ij}^0-a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2}\leq\lambda$. We then have

$$\begin{aligned}&\frac{2}{|a_{ij}|}\int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\frac{4|x_{ij}^0-a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2}}\right]\tau\,\mathrm{d}\tau\\&\quad\leq\frac{2}{|a_{ij}|}\int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\lambda}\right]\tau\,\mathrm{d}\tau.\end{aligned}\tag{3.8}$$

The integrand of the last integral is a positive function. We observe that it is asymptotic to $\frac{1}{\tau}$ as $\tau\to 0$ and to $\frac{1}{\tau^2}$ as $\tau\to+\infty$. In particular, the integral exists at infinity, uniformly in λ. Taking $\|\varphi_{ij}\|_{\mathcal{D}}$ large enough, we can equivalently study the integral

$$\int_{\varepsilon}^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\lambda}\right]\tau\,\mathrm{d}\tau,$$

where $\varepsilon=\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}<1$. Since the integrand is asymptotic to $\frac{1}{\tau}$ as $\tau\to 0$, it is equivalent to consider the sum of integrals

$$\int_{\varepsilon}^{1}\frac{1}{\tau}\,\mathrm{d}\tau+\int_1^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\lambda}\right]\tau\,\mathrm{d}\tau,$$

where the second integral is constant (we will call it C_1) and does not depend on ε. We have

$$\int_{\varepsilon}^{1}\frac{1}{\tau}\,\mathrm{d}\tau+\int_1^{+\infty}\left[\frac{1}{\tau^2}-\frac{1}{(\tau+1)^2-1+\lambda}\right]\tau\,\mathrm{d}\tau=\log\tau\Big|_{\varepsilon}^{1}+C_1=-\log\varepsilon+C_1.$$

Then, as $\|\varphi_{ij}\|_{\mathcal{D}}\to+\infty$, we know that the integral on the right-hand side of (3.8) behaves like

$$\frac{2}{|a_{ij}|}\left(-\log\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}+C_1\right)=\frac{2}{|a_{ij}|}\left(\log\|\varphi_{ij}\|_{\mathcal{D}}+C_1-\log 2|a_{ij}|\right)$$

$$= \frac{2}{|a_{ij}|}(\log \|\varphi_{ij}\|_{\mathcal{D}} + C_2),$$

where $C_2 = C_1 - \log 2|a_{ij}|$.

We have thus proved that

$$\int_1^{+\infty} \left(\frac{1}{|a_{ij}|t} - \frac{1}{|a_{ij}|t + \|\varphi_{ij}\|_{\mathcal{D}}\sqrt{t} + |x_{ij}^0 - a_{ij}|} \right) \mathrm{d}t \le \frac{2}{|a_{ij}|}(\log \|\varphi_{ij}\|_{\mathcal{D}} + C_2).$$

This means that given $R > 0$, when $\|\varphi_{ij}\|_{\mathcal{D}} \ge R$ for R large enough, we have

$$\mathcal{A}_{ij}(\varphi) \ge m_i m_j \left[\frac{\|\varphi_{ij}\|_{\mathcal{D}}^2}{2M} - \frac{2}{|a_{ij}|}(\log \|\varphi_{ij}\|_{\mathcal{D}} + C_2) \right]$$

and we can conclude that $\mathcal{A}_{ij}(\varphi) \to +\infty$ as $\|\varphi_{ij}\|_{\mathcal{D}} \to +\infty$.

3.2 Weak lower semicontinuity

Now, we prove that the functional $\mathcal{A}$ is weakly lower semicontinuous. Since the kinetic term $\frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}$ is convex, it is straightforward that the term $\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2\, \mathrm{d}t$ is weakly lower semicontinuous. However, it is worthwhile noticing that Fatou's Lemma cannot be applied to the term $\int_1^{+\infty} U(\varphi(t) + x^0 - a + at) - U(at)\, \mathrm{d}t$, since the integrand is not a positive function, and we must proceed in a different way. We know that there is at least a sequence of functions in $\mathcal{D}_0^{1,2}(1, +\infty)$ that converges uniformly on the compact subsets of $[1, +\infty)$. To show this, consider a bounded sequence $(\varphi^n)_n$ in $\mathcal{D}_0^{1,2}(1, +\infty)$. We also know, by the definition of this space, that $\|\dot{\varphi}^n\|_{L^2([1,+\infty))} < +\infty$ and that $\varphi^n(1) = 0$, for every n. From the inequality

$$\|\varphi(t)\|_{\mathcal{M}} \le \|\dot{\varphi}\|_{L^2}\sqrt{t-1} \le \|\dot{\varphi}\|_{L^2}\sqrt{t} \qquad \text{for every } t \ge 1, \tag{3.9}$$

we have $\|\varphi^n(t)\|_{\mathcal{M}} \le \|\dot{\varphi}^n\|_{L^2}\sqrt{t}$ for every $t \ge 1$ and for every n, which means that the L^∞-norm in $[1, T]$ of φ^n is bounded, for every fixed $T \ge 1$ and for every n. On the other hand, we have

$$\|\varphi^n(t_1) - \varphi^n(t_2)\|_{\mathcal{M}} \le \|\dot{\varphi}^n\|_{L^2}\sqrt{t_1 - t_2},$$

for every $t_1, t_2 \in [1, +\infty)$ and for every n, which implies that the sequence $(\varphi^n)_n$ is equicontinuous on each interval $[1, T]$, for T fixed. Then, by Ascoli-Arzelà's Theorem, we can say that for every fixed $T \ge 1$ there is a subsequence $(\varphi^{n_k})_k$ that converges uniformly on $[1, T]$ (and, consequently, it converges pointwise on each compact). Besides, it can also be proved, through a diagonal procedure, that there is a subsequence converging pointwise on $[1, +\infty)$.

Consider now a sequence $(\varphi^n)_n$ in $\mathcal{D}_0^{1,2}(1, +\infty)$ converging weakly to some limit $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$. By the properties of weak convergence we know that the sequence is bounded on $\mathcal{D}_0^{1,2}(1, +\infty)$ and, from the previous considerations, there is a subsequence $(\varphi^{n_k})_k$ converging uniformly on compact subsets of $[1, +\infty)$ (and hence pointwise in $[1, +\infty)$). We write

$$\begin{aligned} &\frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|} - \frac{1}{|a_{ij}t|} \\ &\quad = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{|a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))|}\right] \mathrm{d}s. \end{aligned} \tag{3.10}$$

However, this inequality holds only when the denominator of the integrand is not zero, which happens for t sufficiently small. In particular, for all $s \in (0, 1)$ we have

$$\begin{aligned} |a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))| &\geq |a_{ij}|t - s(|x_{ij}^0 - a_{ij}| + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}) \\ &> |a_{ij}|t - (|x_{ij}^0 - a_{ij}| + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}), \end{aligned}$$

and, since $|\varphi_{ij}^n(t)| \leq k\sqrt{t}$ for $k \in \mathbb{R}^+$ large enough, we have

$$|a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))| > |a_{ij}|t - (|x_{ij}^0 - a_{ij}| + k\sqrt{t}),$$

where the last term is larger than zero if t is larger than some $\bar{T} = \bar{T}(k)$; it is easy to compute $\bar{T}$ by studying the function $g(t) = |a_{ij}|t - [|x_{ij}^0 - a_{ij}| + k\sqrt{t}]$. For these reasons, it is better to study the potential term separately on the two intervals $[1, \bar{T}]$ and $[\bar{T}, +\infty)$.

We observe that $U(x^0 - a + at + \varphi) \in L^1([1, \bar{T}])$, since

$$\frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|} \leq \frac{1}{|x_{ij}^0 - a_{ij}| - |a_{ij}|t - \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}}.$$

Besides, since U is a positive function, we can use the pointwise convergence of the sequence and Fatou's Lemma to state that

$$\int_1^{\bar{T}} \frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}(t)|}\,\mathrm{d}t \leq \liminf_{n\to+\infty} \int_1^{\bar{T}} \frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|}\,\mathrm{d}t.$$

Now, knowing that the sequence $(\varphi^n)_n$ is bounded, we wish to prove that the term $U(\varphi^n(t) + x^0 - a + at) - U(at)$ converges in $L^1([\bar{T}, +\infty))$. By using (3.10), we can write

$$\begin{aligned} &\int_{\bar{T}}^{+\infty} \frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|} - \frac{1}{|a_{ij}t|}\,\mathrm{d}t \\ &\quad = \int_{\bar{T}}^{+\infty}\left(\int_0^1 -\frac{[a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))](x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))}{|a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))|^3}\,\mathrm{d}s\right)\mathrm{d}t. \end{aligned}$$

Our goal is to find an upper bound for the term

$$\int_{\bar{T}}^{+\infty}\left|\frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|} - \frac{1}{|a_{ij}t|}\right|\mathrm{d}t.$$

To find the upper bound, we will need the inequality

$$\frac{|b+c|^2}{|b|^2-|c|^2} \geq \frac{1}{3}, \qquad \text{for each } b, c \in \mathbb{R}^d \text{ such that } |b| \geq 2|c|, \tag{3.11}$$

which can easily be proved by elementary calculus. By (3.11) and using the fact that $|x_{ij}^0 - a_{ij}| + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t} \leq k'\sqrt{t}$ for $k' \in \mathbb{R}^+$ large enough, we thus have

$$\begin{aligned}
&\int_{\bar{T}}^{+\infty} \left| \int_0^1 - \frac{[a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))](x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))}{|a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))|^3} \, \mathrm{d}s \right| \mathrm{d}t \\
&\quad \leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{|x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t)|}{|a_{ij}t + s(x_{ij}^0 - a_{ij} + \varphi_{ij}^n(t))|^2} \, \mathrm{d}s \right) \mathrm{d}t \\
&\quad \leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 3 \frac{|x_{ij}^0 - a_{ij}| + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}}{|a_{ij}t|^2 - s|x_{ij}^0 - a_{ij} + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}|^2} \, \mathrm{d}s \right) \mathrm{d}t \\
&\quad \leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{3k'\sqrt{t}}{|a_{ij}|^2 t^2 - sk't} \, \mathrm{d}s \right) \mathrm{d}t.
\end{aligned}$$

By choosing $\bar{T}(k) \gg k'/|a_{ij}|^2$ so that $|a_{ij}|^2 t > sk'$ for all $s \in (0,1)$ and for all $t \in [\bar{T}, +\infty)$ (take k large enough), we have that the last integral is finite and we have thus proved that there is a $\hat{T}$ such that, for all $\bar{T} \geq \hat{T}$,

$$\int_{\bar{T}}^{+\infty} \left| \frac{1}{|x_{ij}^0 - a_{ij} + a_{ij}t + \varphi_{ij}^n(t)|} - \frac{1}{|a_{ij}t|} \right| \mathrm{d}t < +\infty.$$

From this result, the L^1 convergence of the term $U(\varphi^n(t) + x^0 - a + at) - U(at)$ follows: by the dominated convergence Theorem we have, in particular,

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\bar{T}}^{+\infty} U(\varphi^n(t) + x^0 - a + at) - U(at) \, \mathrm{d}t \\
&\quad = \int_{\bar{T}}^{+\infty} U(\varphi(t) + x^0 - a + at) - U(at) \, \mathrm{d}t.
\end{aligned}$$

Thus, if we consider any sequence $(\varphi^n)_n$ in $\mathcal{D}_0^{1,2}(1, +\infty)$ converging weakly to some $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, we have

$$\mathcal{A}(\varphi) \leq \liminf_{n\to+\infty} \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}^n(t)\|_{\mathcal{M}}^2 + U(\varphi^n(t) + x^0 - a + at) - U(at) \, \mathrm{d}t,$$

which proves the weak lower semicontinuity of the renormalized Lagrangian action in the space $\mathcal{D}_0^{1,2}(1, +\infty)$.

Remark 3.2 The same reasoning leads to the continuity of the renormalized action with respect to the strong topology, in all elements φ that do not give rise to collisions.

3.3 Absence of collisions and hyperbolicity of the motion

Now, we can apply the Direct Method of the Calculus of Variations, which yields the existence of a minimizer $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ of the renormalized Lagrangian action, and, consequently, the Renormalized Action Principle implies that φ is a solution of equations

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(at + \varphi(t) + x^0 - a).$$

It remains to prove that $\lim_{t\to+\infty}\dot{\varphi}(t) = 0$. We already know that $\dot{\varphi} \in L^2$ and that there is some $k \in \mathbb{R}^+$ such that $\|\varphi(t)\|_{\mathcal{M}} \le k\sqrt{t}$. By this last inequality, we have that

$$\sum_{i<j} m_i m_j \frac{1}{|a_{ij}t + x^0_{ij} - a_{ij} + \varphi_{ij}(t)|} \le \sum_{i<j} m_i m_j \frac{1}{|a_{ij}|t - |x^0_{ij} - a_{ij}| - k\sqrt{t}}$$

and since $|a_{ij}|t - |x^0_{ij} - a_{ij}| - k\sqrt{t} \to +\infty$ as $t \to +\infty$ for all $i, j = 1, \ldots, N$, we obtain that $\lim_{t\to+\infty} U(x(t)) = 0$. Besides, since $\int_1^{+\infty} |\dot{\varphi}_{ij}(t)|^2 \,\mathrm{d}t < +\infty$, we have that

$$\liminf_{t\to+\infty} |\dot{\varphi}_{ij}(t)| = 0. \tag{3.12}$$

Remark 3.3 A solution $x(t) = at + \varphi(t) + x^0 - a$ of the equation $\mathcal{M}\ddot{x} = \nabla U(x)$ has positive energy. Indeed,

$$\frac{1}{2}\|\dot{x}(t)\|^2_{\mathcal{M}} - U(x(t)) = \frac{1}{2}\sum_{i=1}^N m_i|\dot{\varphi}_i(t) + a_i|^2 - U(x(t)) = h,$$

and since by (3.12) there is some $t_k \to +\infty$ such that $\lim_{t_k\to+\infty}\dot{\varphi}_i(t_k) = 0$, we have $h = \frac{1}{2}\|a\|^2_{\mathcal{M}}$.

By Remark 3.3, we can apply Chazy's Lemma (Lemma 2.6), which implies that the limit of $\dot{x}(t)$ exists for $t \to +\infty$. Since, by (3.12), there is at least a sequence $(t_k)_k$ such that $\dot{x}(t_k) \to a$ as $t_k \to +\infty$, we can conclude that

$$\lim_{t\to+\infty} \dot{x}(t) = a.$$

Besides, we can apply Chazy's Theorem (Theorem 2.5) to state that the minimizing motion x has the asymptotic expansion

$$x(t) = at - \log(t)\nabla U(a) + o(1) \quad \text{as } t \to +\infty.$$

We have thus proved that x is a solution of the system

$$\begin{cases} \mathcal{M}\ddot{x} = \nabla U(x) \\ x(1) = x^0 \\ \lim_{t\to+\infty}\dot{x}(t) = a \end{cases},$$

which means that there is a hyperbolic motion for the N-body problem, starting at any initial configuration x^0 and having prescribed asymptotic velocity a without collisions.

4 Existence of minimal half completely parabolic motions

We now focus on the class of completely parabolic motions, that is, those motions that have the form $x(t) = at + O(t^{2/3})$ for $t \to +\infty$, with $a = 0$ and $|r_i(t) - r_j(t)| \approx t^{2/3}$ for $i < j$. Equivalently, we have the following definition.

Definition 4.1 An expansive solution x of the N-body problem is said to be parabolic if the velocity of every body tends to zero.

In this section, we will prove Theorem 1.7. More specifically, we will prove, for the N-body problem, the existence of orbits having the form

$$x(t) = \beta b t^{2/3} + o(t^{1/3^+}), \quad \text{as } t \to +\infty,$$

where $\beta \in \mathbb{R}$ is a proper value and b is a minimal central configuration. The remainder is $o(t^{1/3^+})$ in the sense that it grows less than order γ for every $\gamma > 1/3$.

Definition 4.2 We say that $b \in \mathcal{X}$ is a central configuration if it is a critical point of U when restricted to the inertial ellipsoid

$$\mathcal{E} = \{x \in \mathcal{X} \ : \ \langle \mathcal{M}x, x\rangle = 1\}.$$

A central configuration $b_m \in \mathcal{E}$ is said to be minimal if

$$U(b_m) = \min_{b \in \mathcal{E}} U(b).$$

More precisely, we will work with normalized central configurations, that is, central configurations b such that $\langle \mathcal{M}b, b\rangle = 1$.

Remark 4.3 Obviously, as U is infinite on collisions, a minimal central configuration b_m has no collisions, i.e. $b_m \in \Omega$.

Given a Kepler potential U, we observe that from the definition of central configurations, it follows

$$\nabla U(b) = \lambda \mathcal{M} b,$$

where λ is a Lagrange multiplier. Besides, we have the equality

$$\lambda = \lambda \langle \mathcal{M}b, b\rangle = \langle \nabla U(b), b\rangle = -U(b). \tag{4.1}$$

We first recall that there are self-similar solutions to Newton's equations $\mathcal{M}\ddot{x} = \nabla U(x)$ having the form

$$x(t) = \beta b t^{2/3},$$

for a proper constant β and a central configuration b. Indeed

$$\mathcal{M}\ddot{x} = -\frac{2}{9}\mathcal{M}\beta b t^{-4/3} = \nabla U(x) = \nabla U(\beta b t^{2/3}) = \frac{1}{\beta^2}t^{-4/3}\nabla U(b) = \frac{1}{\beta^2}t^{-4/3}\lambda\mathcal{M}b$$

and, by (4.1), we also have

$$\beta^3 = \frac{9}{2}U(b).$$

This means that, for $\beta = \sqrt[3]{\frac{9}{2}U(b)}$, $x(t) = \beta b t^{2/3}$ is a homothetic solution of Newton's equations.

Now, let us define

$$r_0(t) = \beta b_m t^{2/3},$$

where $b_m \in \Omega$ is a normalized minimal central configuration. We wish to prove the existence of solutions of the system

$$\begin{cases} \mathcal{M}\ddot{x} = \nabla U(x) \\ x(1) = x^0 \\ \lim_{t\to+\infty}\dot{x}(t) = 0 \end{cases},$$

given $x^0 \in \mathcal{X}$. We seek solutions having the form

$$x(t) = r_0(t) + \varphi(t) + x^0 - r_0(1) = r_0(t) + \varphi(t) + \tilde{x}^0, \tag{4.2}$$

where $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$. In this case, we have

$$\nabla U(x(t)) = \mathcal{M}\ddot{x}(t) = \mathcal{M}\ddot{r}_0(t) + \mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t)) + \mathcal{M}\ddot{\varphi}(t),$$

which means that

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t)).$$

We can thus write the renormalized Lagrangian action as

$$\begin{aligned} \mathcal{A}(\varphi) = \int_1^{+\infty} & \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) \\ & - \langle \nabla U(r_0(t)), \varphi(t)\rangle \, \mathrm{d}t. \end{aligned} \tag{4.3}$$

Besides the coercivity and weak lower semicontinuity of the Lagrangian action, we have to verify that:

- $\forall\, \varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that $r_0(t)+\varphi(t)+\tilde{x}^0(t) \neq 0$ for all $t \geq 1$, $\mathcal{A}(\varphi) < +\infty$;
- the action is continuous and C^1 on $\mathcal{D}_0^{1,2} \setminus \{\varphi \in \mathcal{D}_0^{1,2} \,:\, \exists\, t$ such that $r_0(t)+\varphi(t)+\tilde{x}^0(t) = 0\}$.

4.1 Coercivity

To minimize the action on the set $\mathcal{D}_0^{1,2}(1,+\infty)$, we start by proving its coercivity. We do this by reconducting the problem to a Kepler problem, where we denote $U_{min} = \min_{b\in\mathcal{E}} U(b)$. We notice that, for any orbit x,

$$U(x) \geq \frac{U_{min}}{\|x\|},$$

where $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^{dN}$. Indeed, because of the homogeneity of the potential,

$$U(x) = U\left(\|x\|\frac{x}{\|x\|}\right) = \frac{1}{\|x\|}U\left(\frac{x}{\|x\|}\right) \geq \frac{1}{\|x\|}U_{min}. \tag{4.4}$$

Besides,

$$\begin{aligned}\nabla U(r_0) = \nabla U(\beta b_m t^{2/3}) &= \frac{1}{\beta^2 t^{4/3}}\nabla U(b_m) = \frac{1}{\beta^2 t^{4/3}}\lambda\mathcal{M}b_m \\ &= -\frac{U_{min}}{\beta^2 t^{4/3}}\mathcal{M}b_m.\end{aligned} \tag{4.5}$$

Using (4.4) and (4.5), we can then write

$$\begin{aligned}\mathcal{A}(\varphi) \geq \int_1^{+\infty} &\frac{1}{2}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle \\ &+ \frac{U_{min}}{\|r_0(t)+\varphi(t)+\tilde{x}^0\|} - \frac{U_{min}}{\|r_0(t)\|} + \frac{1}{\beta^2 t^{4/3}}\langle U_{min}\mathcal{M}b_m, \varphi(t)\rangle \,\mathrm{d}t \\ = \int_1^{+\infty} &\frac{1}{2}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle + \frac{U_{min}}{\|r_0(t)+\varphi(t)+\tilde{x}^0\|} - \frac{U_{min}}{\|r_0(t)\|} \\ &+ \frac{\langle U_{min}\mathcal{M}r_0(t),\varphi(t)\rangle}{\|r_0(t)\|^3}\,\mathrm{d}t.\end{aligned}$$

We have

$$\begin{aligned}\|r_0(t)+\varphi(t)+\tilde{x}^0\|^2 &= \|r_0(t)\|^2 + 2\langle\mathcal{M}r_0(t),\varphi(t)\rangle + 2\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle \\ &\quad + 2\langle\mathcal{M}r_0(t),\tilde{x}^0\rangle + \|\varphi(t)\|^2 + \|\tilde{x}^0\|^2 = u + v,\end{aligned}$$

where we define

$$\begin{aligned}&u := \|r_0(t)\|^2 \\ &v := 2\langle\mathcal{M}r_0(t),\varphi(t)\rangle + 2\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle + 2\langle\mathcal{M}r_0(t),\tilde{x}^0\rangle + \|\varphi(t)\|^2 + \|\tilde{x}^0\|^2.\end{aligned}$$

Remark 4.4 The following equalities hold:

$$U(b+s)-U(b)=\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}U(b+st)\,\mathrm{d}t=\int_0^1 \langle \nabla U(b+st), s\rangle\,\mathrm{d}t,$$

$$U(b+s)-U(b)-\nabla U(b)s=\int_0^1\int_0^1 \langle \nabla^2 U(b+st_1t_2)s, s\rangle t_2\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

Using Remark 4.4, we then have

$$\|r_0(t)+\varphi(t)+\tilde{x}^0\|^{-1}=(u+v)^{-1/2}=u^{-1/2}-\frac{1}{2}u^{-3/2}v + \frac{3}{4}\int_0^1\int_0^1 \langle (u+stv)^{-5/2}v, v\rangle s\,\mathrm{d}s\,\mathrm{d}t.$$

Since the integral in the last expression is positive, it follows

$$\begin{aligned}\|r_0(t)+\varphi(t)+\tilde{x}^0\|^{-1} &= (u+v)^{-1/2}\\ &\geq u^{-1/2}-\frac{1}{2}u^{-3/2}v\\ &= \|r_0(t)\|^{-1}-\frac{1}{2\|r_0(t)\|^3}[2\langle \mathcal{M}r_0(t),\varphi(t)\rangle+2\langle \mathcal{M}\varphi(t),\tilde{x}^0\rangle\\ &\quad +2\langle \mathcal{M}r_0(t),\tilde{x}^0\rangle+\|\varphi(t)\|^2+\|\tilde{x}^0\|^2]\\ &= \|r_0(t)\|^{-1}-\frac{\langle \mathcal{M}r_0,\varphi(t)\rangle}{\|r_0(t)\|^3}-\frac{\langle \mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\\ &\quad -\frac{\langle \mathcal{M}r_0(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}-\frac{1}{2}\frac{\|\varphi(t)\|^2}{\|r_0(t)\|^3}-\frac{1}{2}\frac{\|\tilde{x}^0\|^2}{\|r_0(t)\|^3}.\end{aligned} \tag{4.6}$$

At this point, we can use (4.6) to obtain

$$\begin{aligned}\mathcal{A}(\varphi) &\geq \int_1^{+\infty} \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle+\frac{U_{min}}{\|r_0(t)+\varphi+\tilde{x}^0\|}-\frac{U_{min}}{\|r_0(t)\|}\\ &\quad +\frac{\langle U_{min}\mathcal{M}r_0(t),\varphi(t)\rangle}{\|r_0(t)\|^3}\,\mathrm{d}t\\ &\geq \int_1^{+\infty}\frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle-\frac{U_{min}}{2}\frac{\|\varphi(t)\|^2}{\|r_0(t)\|^3}-\frac{\langle U_{min}\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\,\mathrm{d}t+C_3,\end{aligned}$$

where C_3 is a constant. By Hardy inequality (2.4) and the fact that, for $\beta=\sqrt[3]{\frac{9}{2}U(b_m)}$,

$$\frac{U_{min}}{\|r_0(t)\|^3}=\frac{U_{min}}{\|\beta b_m t^{2/3}\|^3}=\frac{U_{min}}{\beta^3 t^2\|b_m\|^3}=\frac{2}{9}\frac{1}{t^2}, \tag{4.7}$$

we have

$$\mathcal{A}(\varphi)\geq \int_1^{+\infty}\frac{1}{2}\left[\langle \mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle-\frac{8}{9}\langle \mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle\right]-\frac{U_{min}\langle \mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\,\mathrm{d}t+C_3$$

$$= \int_1^{+\infty} \frac{1}{18}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle - \frac{U_{min}\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle}{\|r_0(t)\|^3}\, \mathrm{d}t + C_3.$$

Using again (4.7), we observe that

$$\frac{U_{min}\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle}{\|r_0(t)\|^3} = \frac{2}{9}\frac{\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle}{t^2}.$$

By Cauchy-Scwartz and Hardy inequalities, it follows

$$\begin{aligned}
\int_1^{+\infty} -\frac{U_{min}\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle}{\|r_0(t)\|^3}\, \mathrm{d}t &\geq -\int_1^{+\infty} \frac{2}{9}\frac{|\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle|}{t^2}\, \mathrm{d}t \\
&\geq -\int_1^{+\infty} \frac{2}{9}\frac{\|\varphi(t)\|_{\mathcal{M}}}{t}\frac{\|\tilde{x}^0\|_{\mathcal{M}}}{t}\, \mathrm{d}t \\
&\geq -\frac{2}{9}\left(\int_1^{+\infty} \frac{\|\varphi(t)\|^2_{\mathcal{M}}}{t^2}\, \mathrm{d}t\right)^{1/2} \\
&\quad \times \left(\int_1^{+\infty} \frac{\|\tilde{x}^0\|^2_{\mathcal{M}}}{t^2}\, \mathrm{d}t\right)^{1/2} \mathrm{d}t \\
&\geq -\frac{4}{9}C_4\|\varphi\|_{\mathcal{D}},
\end{aligned}$$

where C_4 is constant. This means that

$$\mathcal{A}(\varphi) \geq \frac{1}{18}\|\varphi\|^2_{\mathcal{D}} - \frac{4}{9}C_4\|\varphi\|_{\mathcal{D}} + C_3,$$

which proves the coercivity of the action.

4.2 Weak-lower semicontinuity

Now, we can focus on the proof of the weak lower semicontinuity of the action. Consider a sequence of functions $(\varphi^n)_n \subset \mathcal{D}_0^{1,2}(1, +\infty)$ converging weakly in $\mathcal{D}_0^{1,2}(1, +\infty)$ to some φ, for $n \to +\infty$. It trivially follows that, for every n, $\|\varphi\|_{\mathcal{D}} < +\infty$ and $\|\varphi^n\|_{\mathcal{D}} < +\infty$. Let us divide the action in two parts:

$$\mathcal{A}(\varphi) = \mathcal{A}_{[1,\overline{T})}(\varphi) + \mathcal{A}_{[\overline{T},+\infty)}(\varphi),$$

where

$$\begin{aligned}
\mathcal{A}_{[1,\overline{T})}(\varphi) = \int_1^{\overline{T}} &\frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) \\
&- \langle \nabla U(r_0(t)), \varphi(t)\rangle \mathrm{d}t, \\
\mathcal{A}_{[\overline{T},+\infty)}(\varphi) = \int_{\overline{T}}^{+\infty} &\frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) \\
&- \langle \nabla U(r_0(t)), \varphi(t)\rangle\, \mathrm{d}t
\end{aligned}$$

for some $\overline{T} \in (1, +\infty)$. Using Ascoli-Arzelà's Theorem, we can say that $\varphi^n \to \varphi$ uniformly on compact sets, which implies that $\langle \nabla U(r_0), \varphi^n \rangle \to \langle \nabla U(r_0), \varphi \rangle$ uniformly in $[1, \overline{T}]$, as $n \to +\infty$, for every $\overline{T} < +\infty$. Then, using Fatou's Lemma, it easily follows that the term $\mathcal{A}_{[1,\overline{T})}(\varphi)$ is weak lower semicontinuous.

Concerning the term $\mathcal{A}_{[\overline{T},+\infty)}(\varphi)$, we can write:

$$\begin{aligned}\mathcal{A}_{[\overline{T},+\infty)}(\varphi) = \int_{\overline{T}}^{+\infty} & \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \\ & + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ & - \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, \mathrm{d}t.\end{aligned}$$

Claim: The map $\varphi(t) \mapsto \left(\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, \mathrm{d}t \right)^{1/2}$ is an equivalent norm to $\|\cdot\|_{\mathcal{D}}$. Indeed:

- By the homogeneity of the potential, it holds $\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \geq -\frac{2}{9}\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}$ for each $t \in [1, +\infty)$ (see Remark 4.5). Then, by Hardy inequality, we have

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, \mathrm{d}t \geq \frac{1}{2}\left(1 - \frac{8}{9}\right)\|\varphi\|_{\mathcal{D}}^2 = \frac{1}{18}\|\varphi\|_{\mathcal{D}}^2.$$

- Using the fact that, for some constant $C_5 > 0$,

$$\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \leq C_5 \frac{\|\varphi(t)\|_{\mathcal{M}}}{t^2}$$

  and Hardy inequality, we have

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, \mathrm{d}t \leq C_6\|\varphi\|_{\mathcal{D}}^2,$$

  for some constant $C_6 > 0$.

From the equivalence between the two norms, we have that the term $\int_{\overline{T}}^{+\infty} \frac{1}{2} \times \|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, \mathrm{d}t$ is weak lower semicontinuous.

Using Taylor's series expansion, we can write

$$\begin{aligned}&\int_{\overline{T}}^{+\infty} U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ &\quad - \frac{1}{2}\langle \nabla^2 U(r_0(t)), \varphi(t), \varphi(t)\rangle \, \mathrm{d}t \\ &= \int_{\overline{T}}^{+\infty} \int_0^1 \int_0^1 \int_0^1 \langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0))(\varphi^n(t) + \tilde{x}^0), \varphi^n(t) \\ &\quad + \tilde{x}^0, \varphi^n(t) + \tilde{x}^0\rangle \tau_1\tau_2^2 \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_3 \, \mathrm{d}t.\end{aligned}$$

Obviously there is a $\tilde{t} > 1$ such that

$$\|r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0)\|_{\mathcal{M}} > 0$$

for every $t \geq \tilde{t}$. We can then choose $\overline{T} \geq \tilde{t}$ and we have

$$\begin{aligned}
&\langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0))(\varphi^n(t) + \tilde{x}^0), \varphi^n(t) + \tilde{x}^0, \varphi^n(t) + \tilde{x}^0\rangle \\
&\quad \leq C_7 \frac{\|\varphi^n(t) + \tilde{x}^0\|^3_{\mathcal{M}}}{t^{8/3}} \leq C_8 \frac{\|\varphi^n\|^3_{\mathcal{D}} t^{3/2}}{t^{8/3}} \leq \frac{C_9}{t^{7/6}},
\end{aligned}$$

for every $t \geq \overline{T}$ and for proper constants $C_7, C_8, C_9 > 0$. This means that the term $\langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0))(\varphi^n(t) + \tilde{x}^0), \varphi^n(t) + \tilde{x}^0, \varphi^n(t) + \tilde{x}^0\rangle\tau_1\tau_2^2$ is L^1-dominated and the weak lower semicontinuity of $\mathcal{A}_{[\overline{T},+\infty)}$ follows from the Dominated Convergence Theorem.

4.3 The renormalized action is of class C^1 over non-collision sets

Now, we prove that the action $\mathcal{A}$ is C^1 over the set $\mathcal{D}^{1,2}_0([1,+\infty)) \setminus \{\varphi \in \mathcal{D}^{1,2}_0 : \exists\, t \text{ such that } r_0(t) + \varphi(t) + \tilde{x}^0 = 0\}$. The term $\int_1^{+\infty} \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle\, \mathrm{d}t = \frac{1}{2}\|\varphi\|^2_{\mathcal{D}}$ is of course a smooth functional, so we focus on the potential term

$$\mathcal{A}^2(\varphi) := \int_1^{+\infty} K(t, \varphi(t))\, \mathrm{d}t,$$

where

$$K(t, \varphi(t)) := U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle.$$

We have

$$\begin{aligned}
\mathrm{d}\mathcal{A}^2(\varphi)[\psi] &= \int_1^{+\infty} \langle \nabla K(t, \varphi(t)), \psi(t)\rangle\, \mathrm{d}t \\
&= \int_1^{+\infty} \langle \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t)), \psi(t)\rangle\, \mathrm{d}t
\end{aligned}$$

for every $\psi \in \mathcal{D}^{1,2}_0(1,+\infty)$. Given a sequence $(\varphi^n)_n \subset \mathcal{D}^{1,2}_0(1,+\infty)$ we have to prove that if $\varphi^n \to \varphi$ in $\mathcal{D}^{1,2}_0(1,+\infty)$, then

$$\sup_{\|\psi\|_{\mathcal{D}} \leq 1} \left| \int_1^{+\infty} \langle \nabla K(t, \varphi^n(t)) - \nabla K(t, \varphi(t)), \psi(t)\rangle\, \mathrm{d}t \right| \to 0.$$

Since

$$\nabla K(t, \varphi(t)) = \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t)) = \int_0^1 \nabla^2 K(t, s\varphi(t))\varphi(t)\, \mathrm{d}s,$$

we can estimate

$$\|\nabla K(t,\varphi(t))\|_{\mathcal{M}} \le \int_0^1 \|\nabla^2 K(t,s\varphi(t))\|_{\mathcal{M}}\|\varphi(t)\|_{\mathcal{M}}\,\mathrm{d}s \le C_{10}\frac{\|\varphi(t)\|_{\mathcal{M}}}{t^2}, \tag{4.8}$$

where $C_{10} > 0$ is a proper constant. Using the Cauchy-Schwartz inequality we can then compute

$$\begin{aligned}
&\sup_{\|\psi\|_{\mathcal{D}}\le 1}\left|\int_1^{+\infty}\langle \nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t)), \psi(t)\rangle\,\mathrm{d}t\right| \\
&\le \sup_{\|\psi\|_{\mathcal{D}}\le 1}\int_1^{+\infty} t\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}\frac{\|\psi(t)\|_{\mathcal{M}}}{t}\,\mathrm{d}t \\
&\le \sup_{\|\psi\|_{\mathcal{D}}\le 1}\left(\int_1^{+\infty}\frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t\right)^{1/2} \\
&\quad\times\left(\int_1^{+\infty} t^2\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}^2\,\mathrm{d}t\right)^{1/2} \\
&\le 2\left(\int_1^{+\infty} t^2\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}^2\,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Now, using (4.8)

$$\begin{aligned}
&\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}^2 \\
&= \left|\int_0^1 \nabla^2 K(t,\varphi(t)+\sigma(\varphi^n(t)-\varphi))(\varphi^n(t)-\varphi(t))\,\mathrm{d}\sigma\right|^2 \\
&\le \left(\int_0^1 \|\nabla^2 K(t,\varphi(t)+\sigma(\varphi^n(t)-\varphi(t)))(\varphi^n(t)-\varphi(t))\|_{\mathcal{M}}\,\mathrm{d}\sigma\right)^2 \\
&\le \left(\int_0^1 \frac{\|\varphi^n(t)-\varphi(t)\|_{\mathcal{M}}}{t^2}\,\mathrm{d}\sigma\right)^2 \\
&= \frac{\|\varphi^n(t)-\varphi(t)\|_{\mathcal{M}}^2}{t^4}.
\end{aligned}$$

From this last computation, it follows that

$$\begin{aligned}
&\left(\int_1^{+\infty} t^2\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}^2\,\mathrm{d}t\right)^{1/2} \\
&\le \left(\int_1^{+\infty}\frac{\|\varphi^n(t)-\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t\right)^{1/2} \\
&\le 2\left(\int_1^{+\infty}\|\dot\varphi^n(t)-\dot\varphi(t)\|_{\mathcal{M}}^2\,\mathrm{d}t\right)^{1/2} \\
&= 2\|\varphi^n-\varphi\|_{\mathcal{D}}
\end{aligned}$$

and since $\|\varphi^n - \varphi\|_{\mathcal{D}} \to 0$ as $n \to +\infty$, this proves our thesis.

4.4 Absence of collisions and parabolicity of the motion

We can now use an argument similar to that of Sect. 3: since the Direct Method of the Calculus of Variations implies the existence of a minimizer $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$ of the renormalized Lagrangian action, we can apply the Renormalized Action Principle to state that φ is a solution of equations

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(\beta b_m t^{2/3} + \varphi(t) + x^0 - \beta b_m) - \frac{2}{3}\frac{\beta b_m}{t^{1/3}}.$$

To conclude, we observe that given

$$x(t) = \varphi(t) + \beta b_m t^{2/3} + \tilde{x}^0,$$

we have

$$\dot{x}(t) = \dot{\varphi}(t) + \frac{2}{3}\beta b_m t^{-1/3}.$$

To prove that the motion x is indeed parabolic, we still have to prove that

$$\lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \dot{\varphi}(t) = 0.$$

Since $\int_1^{+\infty} |\dot{\varphi}_{ij}(t)|^2 \, \mathrm{d}t < +\infty$, we have

$$\liminf_{t\to+\infty} |\dot{\varphi}_{ij}(t)| = 0.$$

Because of the conservation of the energy along the motion, we have

$$\frac{1}{2}\|\dot{x}(t)\|^2_{\mathcal{M}} - U(x(t)) = \frac{1}{2}\sum_{i=1}^{N} m_i \left|\dot{\varphi}_i(t) + \frac{2}{3}\beta b_{m_i} t^{-1/3}\right|^2 - U(x(t)) = h.$$

Since there is at least a subsequence $(t_k)_k$, with $t_k \to +\infty$, such that $\lim_{t_k\to+\infty} \dot{\varphi}_i(t_k) = 0$, it follows that $h = 0$ and, consequently,

$$\frac{1}{2}\|\dot{x}(t)\|^2_{\mathcal{M}} - U(x(t)) = 0.$$

From this, we have that $\lim_{t\to+\infty} \dot{x}(t) = 0$.

4.5 Asymptotic estimates for half parabolic motions

In order to give a better description of the asymptotic expansion of parabolic motions, we can improve inequality (3.5). In particular, we can show that, for any $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, it holds

$$\|\varphi(t)\|_{\mathcal{M}} \le c t^{\frac{1}{3}+\varepsilon}, \quad \forall \varepsilon > 0, \tag{4.9}$$

for a proper constant $c \in \mathbb{R}$. This section is devoted to the proof of this estimate.

Let us consider a half parabolic motion $x(t)$ having the form (4.2), where $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ is a solution of the equations of motion $\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t))$. We can write:

$$\begin{aligned}
\mathcal{M}\ddot{\varphi}(t) &= \frac{1}{\beta^2 t^{4/3}}\left[\nabla U\left(\frac{x(t)}{\beta t^{2/3}}\right) - \nabla U\left(\frac{r_0(t)}{\beta t^{2/3}}\right)\right] \\
&= \frac{1}{\beta^2 t^{4/3}}\left[\nabla U\left(b_m + \frac{\varphi(t)}{\beta t^{2/3}} + \frac{\tilde{x}^0}{\beta t^{2/3}}\right) - \nabla U(b_m)\right] \\
&= \frac{1}{\beta^3 t^2}\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)(\varphi(t)+\tilde{x}^0)\,\mathrm{d}\theta \\
&= \frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\mathrm{d}\theta\right](\varphi(t)+\tilde{x}^0),
\end{aligned}$$

where we can view the integral term as a matrix.

Fixing a real constant $\delta \in (1,2)$ and a sufficiently big constant $k \in \mathbb{R}$, we define a test function $\psi_k : \mathbb{R} \to \mathcal{X}$ as

$$\psi_k(t) = \eta^2 \min\{k, \|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\}\varphi(t)$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a C^∞-class cut-off function having the form

$$\eta(t) = \begin{cases} 0, & t \in [1,R] \\ 1, & t \in [2R,+\infty) \end{cases},$$

for R big enough, with $0 < \eta(t) < 1$, $\forall t \in (R, 2R)$. We point out that k can be chosen such that $\eta \equiv 1$ when $\|\varphi(t)\|_{\mathcal{M}}^{\delta-1} > k$, so that we have

$$\dot{\psi}_k(t) = \begin{cases} 2\eta\dot{\eta}\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\varphi(t) + \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-2}\langle\varphi(t),\dot{\varphi}(t)\rangle_{\mathcal{M}}, & t \in I_k \\ k\dot{\varphi}(t), & t \in \hat{I}_k \end{cases},$$

where $I_k = \{t \in [1,+\infty) : \|\varphi(t)\|_{\mathcal{M}}^{\delta-1} \le k\}$ and $\hat{I}_k = [1,+\infty) \setminus I_k = \{t \in [1,+\infty) : \|\varphi(t)\|_{\mathcal{M}}^{\delta-1} > k\}$.

Multiplying the equations of motion for $\psi_k(t)$ and integrating, we obtain

$$\begin{aligned}
&\int_R^{+\infty} -\langle\ddot{\varphi}(t),\psi_k(t)\rangle_{\mathcal{M}} \\
&\quad + \left\langle \frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\mathrm{d}\theta\right](\varphi(t)+\tilde{x}^0), \psi_k\right\rangle \mathrm{d}t \\
&= \int_R^{+\infty} \langle\dot{\varphi}(t),\dot{\psi}_k(t)\rangle_{\mathcal{M}} \\
&\quad + \left\langle \frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\mathrm{d}\theta\right](\varphi(t)+\tilde{x}^0), \psi_k\right\rangle \mathrm{d}t.
\end{aligned}$$

Recalling that $\|\nabla^2 U(r_0+\theta(\varphi(t)+\tilde{x}^0))\|_{\mathcal{M}} \le \frac{C_{11}}{t^2}$ for a proper constant C_{11}, for every $t>1$ and for every $\theta\in[0,1]$, we can use Hölder's and Hardy's inequalities to estimate

$$\begin{aligned}
&\int_R^{+\infty} \langle\dot\varphi(t),\dot\psi_k(t)\rangle_{\mathcal{M}} + \left\langle\left[\int_0^1 \nabla^2 U(r_0(t)+\theta(\varphi(t)+\tilde{x}^0))\,\mathrm{d}\theta\right]\varphi(t),\psi_k(t)\right\rangle \mathrm{d}t\\
&= -\int_R^{+\infty}\left\langle\left[\int_0^1 \nabla^2 U(r_0(t)+\theta(\varphi(t)+\tilde{x}^0))\,\mathrm{d}\theta\right]\tilde{x}^0,\psi_k(t)\right\rangle \mathrm{d}t\\
&\le C_{11}\int_R^{+\infty}\frac{\|\psi_k(t)\|_{\mathcal{M}}}{t^2}\,\mathrm{d}t\\
&\le C_{11}\int_R^{+\infty}\frac{\|\varphi(t)\|^{\delta}_{\mathcal{M}}}{t^2}\,\mathrm{d}t\\
&= C_{11}\int_R^{+\infty}\frac{1}{t^{2-\delta}}\frac{\|\varphi(t)\|^{\delta}_{\mathcal{M}}}{t^{\delta}}\,\mathrm{d}t\\
&\le C_{11}\left(\int_R^{+\infty}\frac{1}{t^2}\,\mathrm{d}t\right)^{(2-\delta)/2}\left(\int_R^{+\infty}\frac{\|\varphi(t)\|^{2}_{\mathcal{M}}}{t^2}\,\mathrm{d}t\right)^{\delta/2}\\
&\le C_{12}\|\varphi\|^{\delta}_{\mathcal{D}},
\end{aligned}$$

where C_{12} is a proper constant.

Remark 4.5 We recall that the Keplerian potential U is homogeneous of degree -1. Then,

$$U(r_0(t)) = U\left(\|r_0(t)\|_{\mathcal{M}}\frac{r_0(t)}{\|r_0(t)\|_{\mathcal{M}}}\right) = \frac{U(b_m)}{\|r_0(t)\|_{\mathcal{M}}}.$$

The Hessian matrix of $U(r_0(t))$ can then be written as

$$\begin{aligned}
\nabla^2 U(r_0(t)) = &-\frac{U(b_m)\mathcal{M}}{\|r_0(t)\|^3_{\mathcal{M}}} + 3\frac{U(b_m)}{\|r_0(t)\|^5_{\mathcal{M}}}\mathcal{M}r_0(t)\otimes\mathcal{M}r_0(t)\\
&-2\frac{\nabla_{b_m}U(b_m)\otimes\mathcal{M}r_0(t)}{\|r_0(t)\|^4_{\mathcal{M}}} + \frac{\nabla^2_{b_m}U(b_m)}{\|r_0(t)\|^3_{\mathcal{M}}},
\end{aligned}$$

where $x\otimes x$ denotes the symmetric square matrix with components $(x\otimes x)_{ij} = x_i x_j$ for $i,j\in 1,\ldots,N$, and $\nabla_{b_m}U(b_m)$ and $\nabla^2_{b_m}U(b_m)$ represent the gradient and the Hessian matrix of U with respect to b_m, respectively. Since b_m is the minimum of the restricted potential, we have $\frac{\nabla_{b_m}U(b_m)\otimes\mathcal{M}r_0(t)}{\|r_0(t)\|^4_{\mathcal{M}}}=0$. Besides, since $\mathcal{M}r_0(t)\otimes\mathcal{M}r_0(t)$ and $\nabla^2_{b_m}U(b_m)$ are positive semidefinite quadratic forms, it holds

$$\langle\nabla^2 U(r_0(t))\varphi(t),\psi(t)\rangle \ge -\frac{U(b_m)\|\varphi(t)\|_{\mathcal{M}}\|\psi(t)\|_{\mathcal{M}}}{\|r_0(t)\|^3_{\mathcal{M}}}, \tag{4.10}$$

for $\varphi,\psi\in\mathcal{D}^{1,2}_0(1,+\infty)$.

Using a continuity argument and (4.10), we can also say that for every $\mu > 0$ there is a $\bar{T} > 0$ such that, for every $t > \bar{T}$,

$$\frac{1}{\beta^3 t^2}\nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right) \geq -\frac{2}{9}(1+\mu)\frac{\mathcal{M}}{t^2}$$

in the sense of quadratic forms. It follows

$$\int_R^{+\infty} \langle\dot{\varphi}(t), \dot{\psi}_k(t)\rangle_{\mathcal{M}} + \left\langle\frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right) d\theta\right]\varphi(t), \psi_k(t)\right\rangle dt$$
$$\geq \int_R^{+\infty} \langle\dot{\varphi}(t), \dot{\psi}_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2}, \psi_k(t)\right\rangle_{\mathcal{M}} dt.$$

To estimate the right-hand side of the last inequality, we study the integral separately on the two complementary sets I_k and $\hat{I}_k$. In I_k, we have

$$\int_{I_k} \langle\dot{\varphi}(t), \dot{\psi}_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2}, \psi_k(t)\right\rangle_{\mathcal{M}} dt$$
$$= \int_{I_k} 2\eta\dot{\eta}\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\langle\dot{\varphi}(t), \varphi(t)\rangle_{\mathcal{M}} + \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot{\varphi(t)}\|_{\mathcal{M}}$$
$$-\frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\, dt,$$

which implies

$$\int_{I_k} \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot{\varphi}(t)\|_{\mathcal{M}} - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\, dt$$
$$\leq \int_{I_k} 2\eta\dot{\eta}\|\varphi(t)\|_{\mathcal{M}}^{\delta}\|\dot{\varphi}(t)\|_{\mathcal{M}}\, dt + C_{12}\|\varphi\|_{\mathcal{D}}^{\delta}.$$

where the cut-off function makes sure that the last integral is finite. Besides, we also have

$$\int_{I_k} \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot{\varphi}(t)\|_{\mathcal{M}} - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\, dt$$
$$= \int_{I_k} \frac{4\delta}{(\delta+1)^2}\left(\eta\frac{d}{dt}\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta+1}{2}}\right)^2 - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\, dt.$$

On the other hand, working on the interval $\hat{I}_k$ we obtain

$$\int_{\hat{I}_k} \langle\dot{\varphi}(t), \dot{\psi}_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2}, \psi_k(t)\right\rangle_{\mathcal{M}} dt$$
$$= \int_{\hat{I}_k} k\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\, dt$$

$$\geq \int_{\hat{I}_k} \frac{4\delta}{(\delta+1)^2} k\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t,$$

where we used the fact that $\frac{4\delta}{(\delta+1)^2} < 1$ for every $\delta \in (1,2)$.

Now, we define a function $u_k : \mathbb{R} \to \mathbb{R}$ as

$$u_k(t) = \min\{\eta\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta-1}{2}}, k^{1/2}\}\|\varphi(t)\|_{\mathcal{M}}.$$

Putting everything together, we can use Hardy's inequality to say that

$$\begin{aligned}
&\int_{I_k} \frac{4\delta}{(\delta+1)^2}\left(\eta\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta+1}{2}}\right)^2 - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\,\mathrm{d}t \\
&\quad + \int_{\hat{I}_k} \frac{4\delta}{(\delta+1)^2} k\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t \\
&= \int_1^{+\infty} \frac{4\delta}{(\delta+1)^2}\|\dot{u}_k(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)\frac{\|u_k(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t \\
&\geq \int_1^{+\infty} \left(\frac{4\delta}{(\delta+1)^2} - \frac{8}{9}(1+\mu)\right)\|\dot{u}_k(t)\|_{\mathcal{M}}^2\,\mathrm{d}t.
\end{aligned}$$

In particular, we can choose μ such that $\frac{4\delta}{(\delta+1)^2} - \frac{8}{9}(1+\mu) > 0$, which proves that $u_k \in \mathcal{D}_0^{1,2}(1,+\infty)$.

Since the estimates we obtained do not depend on k, we can take $k \to +\infty$ so that (3.5) leads us to the conclusion of our proof. We have thus shown that for any $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ and for any $\delta \in (1,2)$ there is a constant c, which depends on δ and $\|\varphi\|_{\mathcal{D}}$, such that

$$\|\varphi(t)\|_{\mathcal{M}} \leq c t^{\frac{1}{\delta+1}}, \qquad \forall t \geq 1.$$

5 Existence of minimal half hyperbolic-parabolic motions

This section is devoted to the proof of Theorem 1.8. To prove the existence of hyperbolic-parabolic solutions in the N-body problem, we will use the cluster decomposition that we briefly introduced in Sect. 1 to decompose the Lagrangian action, so that the minimization of the renormalized action over the set $\mathcal{D}_0^{1,2}(1,+\infty)$ can take place.

Definition 5.1 Given a configuration $a \in \mathcal{X}$ and a motion $x(t) = at + O(t^{2/3})$ as $t \to +\infty$, its corresponding natural partition (a-partition) of the index set $\mathcal{N} = \{1,\dots,N\}$ is the one for which $i, j \in \mathcal{N}$ belong to the same class if and only if the mutual distance $|r_i(t) - r_j(t)|$ grows as $O(t^{2/3})$. Equivalently, if $a = (a_1,\dots,a_N)$, then the natural partition is defined by the relation $i \sim j$ if and only if $a_i = a_j$. The partition classes will be called clusters.

We give now some definitions and basic notations related to a given partition $\mathcal{P}$ of the set $\mathcal{N} = \{1, \dots, N\}$.

Definition 5.2 Let $\mathcal{P}$ be a given partition of $\mathcal{N}$ and consider a configuration $x = (r_1, \dots, r_N) \in \mathcal{X}$. For each cluster $K \in \mathcal{P}$ we define the mass of the cluster as

$$M_K = \sum_{i \in K} m_i.$$

Definition 5.3 Let $\mathcal{P}$ be any given partition of $\mathcal{N}$. Then, for every given curve $x(t) = (r_1(t), \dots, r_N(t))$ in $\mathcal{X}$ and for each cluster $K \in \mathcal{P}$, we define the cluster potential

$$U_K(t) = \sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|r_i(t) - r_j(t)|}.$$

which represents the restriction of the potential U to the cluster K.

The system we are studying now is

$$\begin{cases} \mathcal{M}\ddot{x} = \nabla U(x) \\ x(1) = x^0 \\ \lim_{t \to +\infty} \dot{x}(t) = a \end{cases},$$

where $x^0 \in \mathcal{X}$ and $a \in \Delta$. Since we are seeking solutions of the form $x(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$, where r_0 is a proper guiding curve and $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, our problem equivalently reads

$$\begin{cases} \mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - \mathcal{M}\ddot{r}_0(t) \\ \varphi(1) = 0 \\ \lim_{t \to +\infty} \dot{\varphi}(t) = 0 \end{cases}.$$

We can thus apply the Renormalized Action Principle to prove the existence of solutions to the last system.

Partitioning the indexes according to the natural cluster partition, we obtain a partition of $\mathcal{N}$ of the form

$$K_1 := \{1, \dots, k_1\},\ K_2 := \{k_1 + 1, \dots, k_2\},\ K_3 := \{k_2 + 1, \dots, k_3\} \dots$$

For every K_i, we can choose a central configuration b^{K_i} which is minimal for that particular cluster and we can define the configuration

$$b = (b^{K_1}, b^{K_2}, \dots) \in \mathcal{X}.$$

Using this particular definition of b, we can then look for solutions of the form

$$x(t) = at + \beta b t^{2/3} + \varphi(t) + \tilde{x}^0, \tag{5.1}$$

where $\tilde{x}^0 = x^0 - a - b$. Here, β is a real vector with as many components as the number of clusters. Precisely, we have

$$\beta = (\beta_{K_1}, \beta_{K_2}, \ldots),$$

with

$$\beta_{K_1} = \sqrt[3]{\frac{9}{2} U_{min}^{K_1}}$$

and $U_{min}^{K_1}$ denotes the minimum of the potential U restricted to the first cluster;

$$\beta_{K_2} = \sqrt[3]{\frac{9}{2} U_{min}^{K_2}}$$

and $U_{min}^{K_2}$ denotes the minimum of the potential U restricted to the second cluster, and so on... In this section, we write, with an abuse of notation, βb to denote the configuration $(\beta_{K_1} b^{K_1}, \beta_{K_2} b^{K_2}, \ldots) \in \mathcal{X}$.

To apply the Renormalized Action Principle, we need to prove the existence of minimizers of the renormalized Lagrangian action

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2} \|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \ddot{r}_0(t), \varphi(t) \rangle_{\mathcal{M}} \, \mathrm{d}t,$$

where $r_0(t) = at + \beta b t^{2/3}$. In truth, the term $U(r_0)$ will be slightly reworded in order to avoid possible collisions and to facilitate the computations in the proof of coercivity and weak-lower semicontinuity of the functional (of course this will not change the associated Euler-Lagrange equations). To do that, we use the aforementioned cluster partition of the bodies: the main idea is that the renormalized Lagrangian action can be written as the sum of two terms, where the first term refers to the motion of the bodies inside each cluster and the second one refers to the interactions between pairs of bodies that belong to different clusters. Referring to [7], we can see, for example, that the Newtonian potential of a $x = (r_1, \ldots, r_N)$ can be decomposed as

$$U(x) = \sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|r_i - r_j|} \right) + \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \frac{m_i m_j}{|r_i - r_j|} \right).$$

For every $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, we can thus write the renormalized Lagrangian action as

$$\begin{aligned} \mathcal{A}(\varphi) &= \sum_{K \in \mathcal{P}} \mathcal{A}_K(\varphi) + \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \mathcal{A}_{K_1,K_2}(\varphi) \\ &= \sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \mathcal{A}_K^{ij}(\varphi) \right) \\ &\quad + \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \mathcal{A}_{K_1,K_2}^{ij}(\varphi) \right), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned}\mathcal{A}_K^{ij}(\varphi) := \int_1^{+\infty} & \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 \\ & + \frac{m_i m_j}{|\varphi_{ij}(t) + a_{ij}t + \beta_K b_{ij}^K t^{2/3} + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \\ & + \frac{2}{9}\frac{\beta_K}{M} m_i m_j \frac{\langle b_{ij}^K, \varphi_{ij}(t)\rangle}{t^{4/3}}\, \mathrm{d}t,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathcal{A}_{K_1,K_2}^{ij}(\varphi) := \int_1^{+\infty} & \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 \\ & + \frac{m_i m_j}{|\varphi_{ij}(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|a_{ij}t|}\, \mathrm{d}t.\end{aligned} \tag{5.4}$$

Here, we used the notations:

$$b^{K_{1,2}} = (b^{K_1}, b^{K_2})$$

$$\beta_{K_{1,2}} b^{K_{1,2}} = (\beta_{K_1} b^{K_1}, \beta_{K_2} b^{K_2})$$

Remark 5.4 We point out that, in the decomposition above, we made a small change in the renormalization of the Lagrangian action functional. Indeed, if we used $-U(r_0(t))$ as the renormalization term, like we did for the hyperbolic and parabolic case, the cluster decomposition would require us to write this term as

$$\begin{aligned}-U(at + \beta b t^{2/3}) = & -\sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \right) \\ & - \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \frac{m_i m_j}{|a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3}|} \right).\end{aligned}$$

However, we notice that, for small values of t, it may happen that $a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} \times t^{2/3} = 0$ for some indexes $i \in K_1$, $j \in K_2$, with $K_1, K_2 \in \mathcal{P}$, $K_1 \neq K_2$. To avoid this small issue, and also to simplify our computations even more, we use a slight variation of our usual renormalization term, which is given by

$$\begin{aligned}-\tilde{U}(r_0(t)) = & -\sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \right) \\ & - \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \frac{m_i m_j}{|a_{ij}t|} \right).\end{aligned}$$

It is easy to prove that $\int_1^{+\infty} \tilde{U}(r_0(t)) - U(r_0(t))\, \mathrm{d}t < +\infty$ and, since $-\tilde{U}(r_0(t))$ is still a fixed function of t, the minimization of the corresponding renormalized Lagrangian action will still lead us to a solution of Newton's equations.

We also highlight that the term (5.3) is the part of the renormalized Lagrangian action that refer to the (parabolic) motion of the bodies inside each cluster, while the term (5.4) refers to the (linear) motion of the cluster. In the following sections, we will study the two terms separately, in order to apply the Direct Method of the Calculus of Variations and, consequently, the Renormalized Action Principle.

5.1 Coercivity of $\mathcal{A}(\varphi)$

We start with the proof of the coercivity of the Lagrangian action when restricted to a general cluster, where we denote by K the set of indexes related to this cluster. Because of the natural cluster partition of the bodies, we have $a_i = a_j$ for any $i, j \in K$. This means that for any $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$,

$$\mathcal{A}_K(\varphi) = \sum_{i,j\in K,\ i<j} \int_1^{+\infty} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|\varphi_{ij}(t) + \beta_K b_{ij}^K t^{2/3} + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} + \frac{2}{9}\frac{\beta_K}{M} m_i m_j \frac{\langle b_{ij}^K, \varphi_{ij}(t)\rangle}{t^{4/3}}\, \mathrm{d}t.$$

Using the homogeneity of the potential and denoting by U_K the potential U when restricted to the cluster K, we apply the inequality

$$U_K(x) \geq \frac{U_K(b^K)}{\|x\|_{\mathcal{M}}} = \frac{U_{min}}{\|x\|_{\mathcal{M}}}$$

to every configuration x restricted to the cluster K. It follows

$$\mathcal{A}_K(\varphi) \geq \int_1^{+\infty} \sum_{i,j\in K,\ i<j} \left(\frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2\right) + \frac{U_{min}}{\|\varphi(t) + \beta_K b^K t^{2/3} + \tilde{x}^0\|_{\mathcal{M}}} - \frac{U_{min}}{\|\beta_K b^K t^{2/3}\|_{\mathcal{M}}} + \frac{2}{9}\frac{\beta_K}{M}\langle \mathcal{M}_K b^K, \varphi(t)\rangle\, \mathrm{d}t,$$

where $\mathcal{M}_K$ denotes the matrix of the masses of the cluster K. Using the inequality

$$\begin{aligned}
&\frac{1}{\|\varphi(t) + \beta_K b^K t^{2/3} + \tilde{x}^0\|_{\mathcal{M}}} \\
&\geq \frac{1}{\|\beta_K b^K t^{2/3}\|_{\mathcal{M}}} - \frac{1}{2\|\beta_K b^K\|_{\mathcal{M}}^3 t^2}(2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \varphi(t)\rangle \\
&+ 2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle + 2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \tilde{x}^0\rangle \\
&+ \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2),
\end{aligned}$$

which holds because of the convexity of the norm, we obtain

$$\begin{aligned}\mathcal{A}_K(\varphi) \geq \int_1^{+\infty} & \sum_{i,j\in K,\ i<j} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{2}{9}\frac{\beta_K}{M} \langle \mathcal{M}_K b^K, \varphi(t)\rangle \\ & - \frac{U_{min}}{2\beta_K^3 \|b^K\|_{\mathcal{M}}^3 t^2} (2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \varphi(t)\rangle + 2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle \\ & + 2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2)\,\mathrm{d}t \\ = \int_1^{+\infty} & \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 \\ & - \frac{U_{min}}{2\beta_K^3 \|b^K\|_{\mathcal{M}}^3 t^2} (2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle + 2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \tilde{x}^0\rangle \\ & + \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2)\,\mathrm{d}t\end{aligned}$$

We notice that the term

$$C_{13} := \int_1^{+\infty} -\frac{U_{min}}{2\beta_K^3 \|b^K\|_{\mathcal{M}}^3 t^2} (2t^{2/3}\beta_K \langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\tilde{x}^0\|_{\mathcal{M}}^2)\,\mathrm{d}t$$

is constant and finite. Using Hardy and Cauchy-Schwartz inequalities we also have

$$\begin{aligned}& -\int_1^{+\infty} \frac{U_{min}}{\beta_K^3 \|b^K\|_{\mathcal{M}}^3 t^2} \langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle\,\mathrm{d}t \\ & = -\frac{2}{9}\int_1^{+\infty} \frac{1}{t^2}\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle\,\mathrm{d}t \\ & \geq -\frac{2}{9}\left(\int_1^{+\infty} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t\right)^{1/2} \left(\int_1^{+\infty} \frac{\|\tilde{x}^0\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t\right)^{1/2} \\ & \geq -C_{14}\|\varphi\|_{\mathcal{D}},\end{aligned}$$

where $C_{14} := \frac{8}{9}\left(\int_1^{+\infty} \frac{\|\tilde{x}^0\|_{\mathcal{M}}^2}{t^2}\,\mathrm{d}t\right)^{1/2} < +\infty$. Again by Hardy inequality, we obtain

$$\mathcal{A}_K(\varphi) \geq \frac{1}{18}\|\varphi\|_{\mathcal{D}}^2 - C_{14}\|\varphi\|_{\mathcal{D}} + C_{13},$$

which implies that the functional $\mathcal{A}_K$ is coercive.

We now focus on studying the terms

$$\begin{aligned}\mathcal{A}_{K_1,K_2}(\varphi) = \sum_{i\in K_1,\ j\in K_2} \int_1^{+\infty} & \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 \\ & + \frac{m_i m_j}{|\varphi_{ij}(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|a_{ij}t|}\,\mathrm{d}t.\end{aligned}$$

Remark 5.5 We notice that if two bodies of the configuration $b^{K_{1,2}}$ belong to different clusters and have collisions, that is, if there are $i \in K_1$ and $j \in K_2$ such that $b_i^{K_{1,2}} = b_j^{K_{1,2}}$, then the functional reads

$$\mathcal{A}_{K_1,K_2}(\varphi) = \sum_{i \in K_1,\ j \in K_2} \int_1^{+\infty} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|\varphi_{ij}(t) + a_{ij} t + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|a_{ij} t|} \, \mathrm{d}t.$$

Since $a_i \neq a_j$ when $i \in K_1$, $j \in K_2$ and $K_1 \neq K_2$, we have already proved that in this case the action functional $\mathcal{A}$ is coercive.

Assuming $b^{K_{1,2}}$ without collisions, we proceed in the following way. By the triangular inequality, we have

$$\int_1^{+\infty} \frac{1}{|\varphi_{ij}(t) + a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij} t|} \, \mathrm{d}t$$
$$\geq \int_1^{+\infty} \frac{1}{\|\varphi_{ij}\|_{\mathcal{D}} t^{1/2} + |a_{ij}| t + \beta_{K_{1,2}} |b_{ij}^{K_{1,2}}| t^{2/3} + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}| t} \, \mathrm{d}t.$$

Using the changes of variables $s = \|\varphi\|_{\mathcal{D}} u$, we obtain

$$\int_1^{+\infty} \frac{1}{\|\varphi_{ij}\|_{\mathcal{D}} t^{1/2} + |a_{ij}| t + \beta_{K_{1,2}} |b_{ij}^{K_{1,2}}| t^{2/3} + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}| t} \, \mathrm{d}t$$
$$= 2 \int_1^{+\infty} \left(\frac{1}{\|\varphi_{ij}\|_{\mathcal{D}} s + |a_{ij}| s^2 + \beta_{K_{1,2}} |b_{ij}^{K_{1,2}}| s^{4/3} + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}| s^2} \right) s \, \mathrm{d}s$$
$$= \frac{2}{\|\varphi\|_{\mathcal{D}} |a_{ij}|} \int_1^{+\infty} \left(\frac{1}{s^2 + \frac{\beta_{K_{1,2}} |b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{s^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{s}{|a_{ij}| \|\varphi\|_{\mathcal{D}}} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}| \|\varphi\|_{\mathcal{D}}}} - \frac{1}{\frac{s^2}{\|\varphi\|_{\mathcal{D}}^2}} \right) s \, \mathrm{d}s$$
$$= \frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \left(\frac{1}{u^2 + \frac{\beta_{K_{1,2}} |b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{u^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{u}{|a_{ij}|} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}| \|\varphi\|_{\mathcal{D}}}} - \frac{1}{u^2} \right) u \, \mathrm{d}u.$$

We can observe that for $\|\varphi\|_{\mathcal{D}} \to +\infty$, we have $\frac{\beta_{K_{1,2}} |b_{ij}^{K_{1,2}}|}{|a_{ij}| \|\varphi\|_{\mathcal{D}}^{2/3}} \leq 1$ and $\frac{|\tilde{x}_{ij}^0|}{|a_{ij}| \|\varphi\|_{\mathcal{D}}} \leq 1$. So, for $\|\varphi\|_{\mathcal{D}} \to +\infty$, it follows

$$\frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \left(\frac{1}{u^2 + \frac{\beta_{K_{1,2}} |b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{u^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{u}{|a_{ij}|} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}| \|\varphi\|_{\mathcal{D}}}} - \frac{1}{u^2} \right) u \, \mathrm{d}u$$

$$
\begin{aligned}
&\geq \frac{2}{|a_{ij}|}\int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty}\left(\frac{1}{u^2+u^{4/3}+\frac{u}{|a_{ij}|}+1}-\frac{1}{u^2}\right)u\,\mathrm{d}u\\
&=\frac{2}{|a_{ij}|}\int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty}\frac{1}{u}\left(\frac{1}{1+u^{-2/3}+\frac{u^{-1}}{|a_{ij}|}+u^{-1}}-1\right)\mathrm{d}u.
\end{aligned}
$$

Since $1/\|\varphi\|_{\mathcal{D}}\leq 1$ when $\|\varphi\|_{\mathcal{D}}\to+\infty$, we can study the integral separately on the intervals $[1/\|\varphi\|_{\mathcal{D}},1]$ and $[1,+\infty)$. On the second interval, the integral is constant (let us say that it is equal to a constant C_{15}). On the other interval, we have

$$
\frac{2}{|a_{ij}|}\int_{1/\|\varphi\|_{\mathcal{D}}}^{1}\frac{1}{u}\left(\frac{1}{1+u^{-2/3}+\frac{u^{-1}}{|a_{ij}|}+u^{-1}}-1\right)\mathrm{d}u\geq\frac{2}{|a_{ij}|}\int_{1/\|\varphi\|_{\mathcal{D}}}^{1}-\frac{\mathrm{d}u}{u}.
$$

We have thus demonstrated that

$$
\begin{aligned}
&\int_1^{+\infty}\frac{1}{|\varphi_{ij}(t)+a_{ij}t+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\tilde{x}_{ij}^0|}-\frac{1}{|a_{ij}t|}\,\mathrm{d}t\\
&\qquad\geq\frac{2}{|a_{ij}|}\log\frac{1}{\|\varphi\|_{\mathcal{D}}}+C_{15}=-\frac{2}{|a_{ij}|}\log\|\varphi\|_{\mathcal{D}}+C_{15},
\end{aligned}
$$

which concludes the proof of the coercivity of the Lagrangian action.

5.2 Weak lower semicontinuity of $\mathcal{A}(\varphi)$

In order to prove the weak lower semicontinuity of the Lagrangian action, we can use the decomposition (5.2) and study the weak lower semicontinuity of the terms $\mathcal{A}_K$ and $\mathcal{A}_{K_1,K_2}$ separately, given arbitrary clusters $K,K_1,K_2\in\mathcal{P}$.

Concerning the term $\mathcal{A}_K$, we can refer to Sect. 4, since our choice of $\beta_K b^K$ leads us to the same computations.

For the proof of the weak lower semicontinuity of the terms $\mathcal{A}_{K_1,K_2}$, let us consider a sequence $(\varphi^n)_n\subset\mathcal{D}_0^{1,2}(1,+\infty)$ converging weakly in $\mathcal{D}_0^{1,2}(1,+\infty)$ to some φ, as $n\to+\infty$. It follows that there is a constant $k\in\mathbb{R}$ such that $\|\varphi^n\|_{\mathcal{D}}\leq k$ and $\|\varphi\|_{\mathcal{D}}\leq k$ for every $n\in\mathbb{N}$. We would like to use the inequality

$$
\begin{aligned}
&\frac{1}{|\varphi_{ij}^n(t)+a_{ij}t+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\tilde{x}_{ij}^0|}-\frac{1}{|a_{ij}t|}\\
&\quad=\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{|a_{ij}t+s(\varphi_{ij}^n(t)+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\tilde{x}_{ij}^0)|}\right]\mathrm{d}s,
\end{aligned}
\tag{5.5}
$$

which holds true when the denominator of the integrand is not zero. For all $s \in (0,1)$ we have

$$\begin{aligned}
&|a_{ij}t + s(\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)| \\
&\quad \geq |a_{ij}|t - s(\|\varphi_{ij}^n\|_{\mathcal{D}} t^{1/2} + |\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + |\tilde{x}_{ij}^0|) \\
&\quad > |a_{ij}|t - (\|\varphi_{ij}^n\|_{\mathcal{D}} t^{1/2} + |\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + |\tilde{x}_{ij}^0|),
\end{aligned}$$

and since $\|\varphi_{ij}^n\|_{\mathcal{D}} \leq k$, we have

$$|a_{ij}t + s(\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)| > |a_{ij}|t - (kt^{1/2} + |\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + |\tilde{x}_{ij}^0|),$$

where the last term is larger than zero if $t \geq \bar{T} = \bar{T}(k)$, for a proper $\bar{T}$. We can thus study the weak lower semicontinuity of the potential term separately on the two intervals $[1, \bar{T}]$ and $[\bar{T}, +\infty)$.

On $[1, \bar{T}]$, the weak lower semicontinuity easily follows from Fatou's Lemma. On $[\bar{T}, +\infty)$, we can use (5.5):

$$\begin{aligned}
&\int_{\bar{T}}^{+\infty} \frac{1}{|\varphi_{ij}^n(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}t|}\, \mathrm{d}t \\
&= \int_{\bar{T}}^{+\infty} \left(\int_0^1 - \frac{[a_{ij}t + s(\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)](\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)}{|a_{ij}t + s(\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)|^3}\, \mathrm{d}s \right) \mathrm{d}t.
\end{aligned}$$

Using (3.11), we then have

$$\begin{aligned}
&\int_{\bar{T}}^{+\infty} \left| \frac{1}{|\varphi_{ij}^n(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}t|} \right| \mathrm{d}t \\
&\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{|\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|}{|a_{ij}t + s(\varphi_{ij}^n(t) + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0)|^2}\, \mathrm{d}s \right) \mathrm{d}t \\
&\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{3(|kt^{1/2} + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3}| + |\tilde{x}_{ij}^0|)}{|a_{ij}t|^2 - s|kt^{1/2} + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|^2}\, \mathrm{d}s \right) \mathrm{d}t \\
&\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{3k' t^{2/3}}{|a_{ij}|^2 t^2 - s k' t^{4/3}}\, \mathrm{d}s \right) \mathrm{d}t,
\end{aligned}$$

where $k' \in \mathbb{R}$ is big enough so that $|kt^{1/2}| + |\beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0| \leq \sqrt{k'} t^{2/3}$. The denominator of the last integral is positive when

$$t > \left(\frac{|a_{ij}|^2}{k'} \right)^{2/3} =: \hat{T}.$$

If we choose $\bar{T}(k) \gg \hat{T}$, the last integral is finite, which means that

$$\int_{\bar{T}}^{+\infty} \left| \frac{1}{|\varphi_{ij}^n(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}t|} \right| \mathrm{d}t < +\infty.$$

This implies the L^1-convergence of the potential term, which proves its weak lower semicontinuity.

5.3 The action is of class C^1 over non-collision sets

The last thing we have to prove is that the action is of class C^1 over sets of motions that don't undergo collisions. We have already proved this result for the terms $\mathcal{A}_K$, so we can only focus on the terms $\mathcal{A}_{K_1,K_2}$. In particular, denoting by $\mathcal{A}_{K_1,K_2}^2$ the potential term, we wish to prove that the differential

$$\mathrm{d}\mathcal{A}_{K_1,K_2}(\varphi)[\psi] = \int_1^{+\infty} \langle \nabla U(\varphi(t) + at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \tilde{x}^0), \psi(t) \rangle \, \mathrm{d}t$$

is continuous, for every $\varphi, \psi \in \mathcal{D}_0^{1,2}(1, +\infty)$, over the set of non-collisional configurations when the potential U is restricted to the clusters K_1 and K_2.

First of all, we have

$$\begin{aligned} &\|\nabla U(\varphi(t) + at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \tilde{x}^0)\|_{\mathcal{M}} \\ &\quad \le C_{16} \sum_{i \in K_1,\ j \in K_2} \frac{1}{|\varphi_{ij}^n(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0|^2} \end{aligned}$$

for a proper constant C_{16}, where the right-hand side term behaves like $1/t^2$ when $t \to +\infty$. This, together with the Cauchy-Schwartz inequality, proves that the differential is well-defined.

Now, given $(\varphi^n)_n \subset \mathcal{D}_0^{1,2}(1, +\infty)$ such that $\varphi^n \to \varphi$ in $\mathcal{D}_0^{1,2}(1, +\infty)$ for some φ, we wish to prove that

$$\sup_{\|\psi\|_{\mathcal{D}} \le 1} \left| \int_1^{+\infty} \langle \nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t)), \psi(t) \rangle \, \mathrm{d}t \right| \to 0, \qquad \text{as } n \to +\infty,$$

where we write $U(t,\varphi(t)) := U(\varphi(t) + at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \tilde{x}^0)$ to lighten the notation. Using Cauchy-Schwartz and Hardy inequalities, we have

$$\begin{aligned}
&\sup_{\|\psi\|_{\mathcal{D}}\le 1} \left| \int_1^{+\infty} \langle \nabla U(t,\varphi^n(t)) - \nabla U(t,\varphi(t)), \psi(t)\rangle \, \mathrm{d}t \right| \\
&\le \sup_{\|\psi\|_{\mathcal{D}}\le 1} \int_1^{+\infty} t \|\nabla U(t,\varphi^n(t)) - \nabla U(t,\varphi(t))\|_{\mathcal{M}} \frac{\|\psi(t)\|_{\mathcal{M}}}{t} \, \mathrm{d}t \\
&\le \sup_{\|\psi\|_{\mathcal{D}}\le 1} \left(\int_1^{+\infty} \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^2} \, \mathrm{d}t \right)^{1/2} \\
&\quad \times \left(\int_1^{+\infty} t^2 \|\nabla U(t,\varphi^n(t)) - \nabla U(t,\varphi(t))\|_{\mathcal{M}}^2 \, \mathrm{d}t \right)^{1/2} \\
&\le 2 \left(\int_1^{+\infty} t^2 \|\nabla U(t,\varphi^n(t)) - \nabla U(t,\varphi(t))\|_{\mathcal{M}}^2 \, \mathrm{d}t \right)^{1/2}.
\end{aligned}$$

Now, we can write

$$\begin{aligned}
&\int_1^{+\infty} t^2 \|\nabla U(t,\varphi^n(t)) - \nabla U(t,\varphi(t))\|_{\mathcal{M}}^2 \, \mathrm{d}t \\
&= \int_1^{+\infty} t^2 \left| \int_0^1 \nabla^2 U(\varphi(t) + at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \tilde{x}^0 + s(\varphi^n(t) - \varphi(t)))(\varphi^n(t) - \varphi(t)) \, \mathrm{d}s \right|^2 \mathrm{d}t \\
&\le \int_1^{+\infty} t^2 \left(\int_0^1 C_{16} \sum_{i\in K_1,\ j\in K_2} \frac{1}{|\varphi_{ij}(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0 + s(\varphi_{ij}^n(t) - \varphi_{ij}(t))|^3} \|\varphi^n(t) - \varphi(t)\|_{\mathcal{M}} \, \mathrm{d}s \right)^2 \mathrm{d}t \\
&\le \int_1^{+\infty} \left(\int_0^1 C_{16} \sum_{i\in K_1,\ j\in K_2} \frac{1}{|\varphi_{ij}(t) + a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \tilde{x}_{ij}^0 + s(\varphi_{ij}^n(t) - \varphi_{ij}(t))|^3} \|\varphi^n - \varphi\|_{\mathcal{D}} t^{3/2} \, \mathrm{d}s \right)^2 \mathrm{d}t \\
&\le C_{17} \|\varphi^n - \varphi\|_{\mathcal{D}}
\end{aligned}$$

for a proper constant $C_{17} \in \mathbb{R}$, where the last term goes to zero as $n \to +\infty$. This concludes the proof.

5.4 Absence of collisions and hyperbolic-parabolicity of the motion

The existence of a minimizer $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ of the renormalized Lagrangian action follows from the Direct Method of the Calculus of Variations. Once again, the Renormalized Action Principle, which exploits Hamilton's Principle of Least Action and Marchal's Theorem, states that

$$x(t) = at + \beta b t^{2/3} + \varphi(t) + \tilde{x}^0$$

is a solution of Newton's equations (1.2) with $x(1) = x^0$.

Given

$$x(t) = at + \beta b t^{2/3} + \varphi(t) + \tilde{x}^0,$$

we have

$$\dot{x}(t) = a + \frac{2}{3}\beta b t^{-1/3} + \dot{\varphi}(t).$$

In this case, the conservation of the energy implies that the energy of the motion $h = \|a\|^2_{\mathcal{M}}/2 > 0$.

Remark 5.6 We observe that Chazy's Theorem can be applied to the cases of hyperbolic and hyperbolic-parabolic motions, because for completely parabolic motions the energy constant of the internal motion is null. In such cases, the limit shape of $x(t)$ is the shape of the configuration a and, moreover, $L = \lim_{t\to+\infty} \frac{\max_{i<j}|x_{ij}(t)|}{\min_{i<j}|x_{ij}(t)|} < +\infty$ if and only if x is hyperbolic. If the energy $h > 0$ and $L = +\infty$, then either the motion is hyperbolic-parabolic or it is not expansive.

In our case, it is trivial to prove that $L = +\infty$, which implies that the motion is hyperbolic-parabolic.

Remark 5.7 We can observe that if the indexes i, j belong to the same cluster, we have $\dot{x}_{ij}(t) \to 0$ when $t \to +\infty$, while if i, j belong to different clusters, we have $\dot{x}_{ij}(t) \to a_{ij}$ when $t \to +\infty$.

5.5 Hyperbolic-parabolic motions' asymptotic expansion

We have seen that a hyperbolic-parabolic motion x can be written in the form $x(t) = at + \beta b t^{2/3} + \varphi(t) + \tilde{x}^0$, as shown in (5.1), and that the bodies can be divided into subgroups following the natural cluster partition introduced in Definition 5.1. In this section, we will prove that the centers of mass of the clusters follow hyperbolic orbits. Besides, we will show that inside each cluster, the bodies move with respect to the center of mass of the cluster following a parabolic path.

We start with proving that the centers of mass of each cluster have a hyperbolic expansion. For a cluster K, denoting the center of mass of K as

$$c^K(t) = \frac{1}{M_K}\sum_{i\in K} m_i x_i(t),$$

we can compute the equations of motion of the center of mass as

$$\begin{aligned}
M_K \ddot{c}^K(t) &= \sum_{i\in K} m_i \ddot{x}_i(t) \\
&= -\sum_{i\in K}\sum_{j\neq i} m_i m_j \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^3} \\
&= -\sum_{i\in K}\sum_{j\notin K} m_i m_j \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^3}.
\end{aligned}$$

It is easy to see that the right-hand side of the equation is a $O\left(\frac{1}{t^2}\right)$-term for $t \to +\infty$. We also notice that

$$-\sum_{i\in K}\sum_{j\notin K} m_i m_j \frac{x_i(t)-x_j(t)}{|x_i(t)-x_j(t)|^3} \simeq -\frac{1}{t^2}\sum_{i\in K}\sum_{j\notin K} m_i m_j \frac{a_i-a_j}{|a_i-a_j|^3} + O\left(\frac{1}{t^3}\right),$$

for $t \to +\infty$. We can define

$$\tilde{\nabla} U(a^K) = -\sum_{i\in K}\sum_{j\notin K} m_i m_j \frac{a_i-a_j}{|a_i-a_j|^3},$$

which can be seen as a restriction of $\nabla U(a^K)$. Denoting with a^K the restriction of the configuration a to the cluster K, we can thus compute

$$\lim_{t\to+\infty} \frac{M_K c^K(t)}{\log t} = \lim_{t\to+\infty} \frac{M_K \dot{c}^K(t)}{\frac{1}{t}} = -\lim_{t\to+\infty} \frac{M_K \ddot{c}^K(t)}{\frac{1}{t^2}} = -\tilde{\nabla} U(a^K).$$

This implies that the center of mass of the cluster K has the hyperbolic asymptotic expansion

$$c^K(t) = a^K t - \tilde{\nabla} U(a^K)\log t + o(\log t),$$

for $t \to +\infty$.

Now, considering an index $i \in K$, we denote the motion of a body x_i with respect to the center of mass of its cluster as

$$y_i(t) = x_i(t) - c_i^K(t).$$

We are going to show that its asymptotic expansion is a parabolic one.

If the cluster only has one element, we obviously have $y_i \equiv 0$, so we consider the case where K has two or more elements. The equation of motion reads

$$\begin{aligned} m_i\ddot{y}_i(t) &= m_i\ddot{x}_i(t) - m_i\ddot{c}_i^K(t) \\ &= -\sum_{j\in K} m_i m_j \frac{x_i(t)-x_j(t)}{|x_i(t)-x_j(t)|^3} - \sum_{j\notin K} m_i m_j \frac{x_i(t)-x_j(t)}{|x_i(t)-x_j(t)|^3} - m_i\ddot{c}_i^K(t). \end{aligned}$$

Since we already know that $-\sum_{j\notin K} m_i m_j \frac{x_i(t)-x_j(t)}{|x_i(t)-x_j(t)|^3} - m_i\ddot{c}_i^K(t) = O\left(\frac{1}{t^2}\right)$ for $t \to +\infty$, we can then say that

$$m_i\ddot{y}_i(t) = -\sum_{j\in K} m_i m_j \frac{y_i(t)-y_j(t)}{|y_i(t)-y_j(t)|^3} + O\left(\frac{1}{t^2}\right).$$

Using the definition of $x(t)$ and the asymptotic expansion of $c^K(t)$ we found above, we can easily see that

$$y_i(t) = \beta_K b_i^K t^{2/3} + \varphi_i(t) - \log t \sum_{j\notin K} m_i m_j \frac{a_i-a_j}{|a_i-a_j|^3} + o(\log t),$$

for $t \to +\infty$, where $\beta_K = \sqrt[3]{\frac{9}{2}\min_K U}$. Defining $\psi_i(t) := \varphi_i(t) - \sum_{j \notin K} m_i m_j \times \frac{a_i - a_j}{|a_i - a_j|^3} + o(\log t)$, it is easy to prove that $\psi_i \in \mathcal{D}^{1,2}(1, +\infty)$. We can then apply the estimate (4.9) to say that

$$y_i(t) = \beta_K b_i^K t^{2/3} + o(t^{\frac{1}{3}+}),$$

for $t \to +\infty$.

6 Free-time minimization property

"Jacobi's principle brings out vividly the intimate relationship which exists between the motions of conservative holonomic systems and the geometry of curved space" (C. Lanczos, [16, page 138]). Accordingly, trajectories of the N-body problem at energy h are geodesics of the Jacobi-Maupertuis' metric of level h in the configuration space, i.e.,

$$\mathrm{d}\sigma^2 = (U + h)\mathrm{d}s^2_{\mathcal{M}},$$

being $\mathrm{d}s^2_{\mathcal{M}}$ the mass Euclidean metric in the configuration space.

Definition 6.1 A curve $x : [1, +\infty) \to E^N$ is said to be a geodesic ray from $p \in E^N$ if $x(1) = p$ and each restriction to a compact interval is a minimizing geodesic.

In [20], Maderna and Venturelli also proved the following theorem.

Theorem 6.2 (Maderna-Venturelli, 2020 [20]) *Let E be an Euclidean space. For any $h > 0$, $p \in E^N$ and $a \in \Omega$, there is geodesic ray of the Jacobi-Maupertuis' metric of level h with asymptotic direction a and starting at p.*

In facts, geodesic rays turn out to be unbounded free-time minimizer of the Lagrangian action at energy h as in Definition 1.10 (cfr. [2, 16]). Next, we show here that our existence results of expansive motions through the minimization of the renormalized action do indeed agree with Theorem 6.2. More precisely, we prove the following corollary.

Corollary 6.3 *Consider an expansive motion $x : [1, +\infty) \to \mathcal{X}$ of the Newtonian N-body problem of the form*

$$x(t) = r_0(t) + \varphi(t) + \tilde{x}_0,$$

where $\varphi \in \mathcal{D}^{1,2}_0(1, +\infty)$ minimizes the renormalized action (1.5) in any of the settings of Theorems 1.6, 1.7 and 1.8. Then x is actually a free-time minimizer at its energy level. Therefore it is a geodesic ray for the Jacobi-Maupertuis' metric.

Proof We consider a curve $\gamma : [1, +\infty) \to \mathcal{X}$ of the form $\gamma(t) = r_0(t) + \varphi(t) + \tilde{x}^0$ such that φ minimizes the renormalized Lagrangian action on $\mathcal{D}_0^{1,2}(1, +\infty)$.

By contradiction, we suppose that there are T and $\bar{T}$, $\varepsilon > 0$ and there is some curve $\bar{\sigma} : [1, \bar{T}] \to \mathcal{X}$ with $\gamma(T) = \bar{\sigma}(\bar{T})$ such that

$$\int_1^T L(\gamma, \dot{\gamma})\, dt + hT > \int_1^{\bar{T}} L(\bar{\sigma}, \dot{\bar{\sigma}})\, dt + h\bar{T} + \varepsilon. \tag{6.1}$$

By a density and continuity argument, we can then define a compactly supported function $\tilde{\varphi}$ such that $\tilde{\varphi}(t) = \varphi(t)$ on $[1, \hat{T}]$, where $\hat{T} \gg \max\{T, \bar{T}\}$, and $\tilde{\varphi}$ is close enough to φ in the $\mathcal{D}_0^{1,2}$-norm to have

$$\mathcal{A}(\tilde{\varphi}) \leq \mathcal{A}(\varphi) + \varepsilon,$$

where $\mathcal{A}$ is the renormalized Lagrangian action. By the minimizing property of φ we infer

$$\mathcal{A}(\tilde{\varphi}) \leq \mathcal{A}(\psi) + \varepsilon, \quad \forall \psi \in \mathcal{D}_0^{1,2}([1, +\infty)). \tag{6.2}$$

Now, denoting $\tilde{\gamma}(t) = r_0(t) + \tilde{\varphi}(t) + \tilde{x}^0$, we build a curve $\tilde{\sigma} : [1, +\infty) \to \mathcal{X}$ such that

$$\tilde{\sigma}(t) = \begin{cases} \bar{\sigma}(t), & t \in [1, \bar{T}] \\ \tilde{\gamma}(t - \bar{T} + T), & t \in [\bar{T}, +\infty) \end{cases}.$$

Since we supposed that $\gamma(T) = \bar{\sigma}(\bar{T})$, we know for sure that $\tilde{\sigma}$ is continuous. Moreover, we define $\bar{\varphi}(t) = \tilde{\sigma}(t) - r_0(t) - \tilde{x}_0$, so that $\bar{\varphi} \in \mathcal{D}_0^{1,2}(1, +\infty)$ and, by its definition, we have

$$\bar{\varphi}(t) \equiv r_0(t - \bar{T} + T) - r_0(t) = a(T - \bar{T}) + o(1), \quad \forall t \gg \max\{T, \bar{T}\}, \tag{6.3}$$

as $\tilde{\varphi}$ is compactly supported. We notice that we can write

$$\mathcal{A}(\tilde{\varphi}) = \int_1^{+\infty} L(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0(t)\, dt,$$

which easily follows from the fact that also $L(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0(t) \in L^1[1, +\infty)$ and furthermore, since $\tilde{\varphi}$ is compactly supported,

$$\int_1^{+\infty} -\langle \mathcal{M}\ddot{r}_0, \tilde{\varphi}\rangle\, dt = -\langle \mathcal{M}\dot{r}_0, \tilde{\varphi}\rangle\Big|_1^{+\infty} + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\tilde{\varphi}}\rangle\, dt = \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\tilde{\varphi}}\rangle\, dt.$$

On the other hand, from (6.3), using $\dot{r}_0 \simeq t^{-1/3}$, it follows that

$$\begin{aligned}\int_1^{+\infty} -\langle \mathcal{M}\ddot{r}_0, \bar{\varphi}\rangle \,\mathrm{d}t &= -\langle \mathcal{M}\dot{r}_0, \bar{\varphi}\rangle\Big|_1^{+\infty} + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\bar{\varphi}}\rangle \,\mathrm{d}t\\ &= \langle \mathcal{M}a, a\rangle(\bar{T}-T) + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\bar{\varphi}}\rangle \,\mathrm{d}t\\ &= 2h(\bar{T}-T) + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\bar{\varphi}}\rangle \,\mathrm{d}t,\end{aligned}$$

where $h = H(r_0, \dot{r}_0)$ is the energy of r_0, which is positive equal to $\|a\|^2_{\mathcal{M}}/2$ in the hyperbolic and hyperbolic-parabolic case and zero in the completely parabolic case. Consequently, we have

$$\mathcal{A}(\bar{\varphi}) = 2h(\bar{T}-T) + \int_1^{+\infty} L(\bar{\sigma}, \dot{\bar{\sigma}}) - L_0(t)\,\mathrm{d}t.$$

Let us denote $L^h = L - h$ and $L^h_0(t) := L(r_0(t)) - h$. By (6.1), we can say that

$$\begin{aligned}&\int_1^{T} L^h(\gamma,\dot{\gamma})\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L^h_0(t-\bar{T}+T)\,\mathrm{d}t\\ &> \int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}})\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}})\\ &\quad - L^h_0(t-\bar{T}+T)\,\mathrm{d}t + \varepsilon + 2h(\bar{T}-T).\end{aligned} \tag{6.4}$$

Working on left-hand side of equation (6.4), we obtain

$$\begin{aligned}&\int_1^{T} L^h(\gamma,\dot{\gamma})\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L^h_0(t-\bar{T}+T)\,\mathrm{d}t\\ &= \int_1^{T} L^h(\gamma,\dot{\gamma})\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} L^h(\tilde{\gamma}(t-\bar{T}+T),\dot{\tilde{\gamma}}(t-\bar{T}+T)) - L^h_0(t-\bar{T}+T)\,\mathrm{d}t\\ &= \int_1^{T} L^h(\gamma,\dot{\gamma}) - L^h_0(t)\,\mathrm{d}t + \int_{T}^{+\infty} L^h(\tilde{\gamma},\dot{\tilde{\gamma}}) - L^h_0(t)\,\mathrm{d}t + \int_1^{T} L^h_0(t)\,\mathrm{d}t\\ &= \int_1^{+\infty} L^h(\tilde{\gamma},\dot{\tilde{\gamma}}) - L^h_0(t)\,\mathrm{d}t + \int_1^{T} L^h_0(t)\,\mathrm{d}t.\end{aligned}$$

On the other hand, working on right-hand side of (6.4), we have

$$\begin{aligned}&\int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}})\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L^h_0(t-\bar{T}+T)\,\mathrm{d}t + 2h(\bar{T}-T) + \varepsilon\\ &= \int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}}) - L^h_0(t)\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_{\bar T}^{+\infty} L^h(\tilde\sigma,\dot{\tilde\sigma}) - L_0^h(t-\bar T+T) + L_0^h(t) - L_0^h(t)\,\mathrm{d}t +\\
&\qquad\qquad + \int_1^{\bar T} L_0^h(t)\,\mathrm{d}t + 2h(\bar T - T) + \varepsilon\\
&= \int_1^{\bar T} L^h(\bar\sigma,\dot{\bar\sigma}) - L_0^h(t)\,\mathrm{d}t + \int_{\bar T}^{+\infty} L^h(\tilde\sigma,\dot{\tilde\sigma}) - L_0^h(t)\,\mathrm{d}t +\\
&\qquad + \int_1^{\bar T} L_0^h(t)\,\mathrm{d}t + \int_{\bar T}^{+\infty} L_0^h(t) - L_0^h(t-\bar T+T)\,\mathrm{d}t\\
&\qquad\qquad + 2h(\bar T - T) + \varepsilon\\
&= \int_1^{+\infty} L^h(\tilde\sigma,\dot{\tilde\sigma}) - L_0^h(t)\,\mathrm{d}t + \int_1^{\bar T} L_0^h(t)\,\mathrm{d}t + \int_{\bar T}^{+\infty} L_0^h(t) - L_0^h(t-\bar T+T)\,\mathrm{d}t\\
&\qquad\qquad + 2h(\bar T - T) + \varepsilon.
\end{aligned}
$$

It thus follows that

$$
\begin{aligned}
&\int_1^{+\infty} L^h(\tilde\gamma,\dot{\tilde\gamma}) - L_0^h(t)\,\mathrm{d}t\\
&> \int_1^{+\infty} L^h(\tilde\sigma,\dot{\tilde\sigma}) - L_0^h(t)\,\mathrm{d}t + \int_T^{\bar T} L_0^h(t)\,\mathrm{d}t\\
&\quad + \int_{\bar T}^{+\infty} L_0^h(t) - L_0^h(t-\bar T+T)\,\mathrm{d}t + 2h(\bar T - T) + \varepsilon.
\end{aligned}
$$

We recall the following property, which can be demonstrated as a simple exercise.

Proposition 6.4 *Given a function $f \in L^1_{loc}(\mathcal{X})$ such that $f(t) \to 0$ as $t \to \pm\infty$ and such that $f(t) - f(t-\tau) \in L^1$ for some $\tau \in \mathbb{R}$, then*

$$
\int_{-\infty}^{+\infty} f(t) - f(t-\tau)\,\mathrm{d}t = 0.
$$

Since

$$
\begin{aligned}
&\int_T^{\bar T} L_0^h(t)\,\mathrm{d}t + \int_{\bar T}^{+\infty} L_0^h(t) - L_0^h(t-\bar T+T)\,\mathrm{d}t\\
&\quad = \int_{-\infty}^{+\infty} L_0^h(t)\mathcal{X}_{\{t>T\}} - L_0^h(t-\bar T+T)\mathcal{X}_{\{t>\bar T\}}\,\mathrm{d}t,
\end{aligned}
$$

we can apply the Proposition 6.4 to the function $L_0^h(t)\mathcal{X}_{\{t>T\}}$. This eventually yields

$$
\int_1^{+\infty} L^h(\tilde\gamma,\dot{\tilde\gamma}) - L_0^h(t)\,\mathrm{d}t > \int_1^{+\infty} L^h(\tilde\sigma,\dot{\tilde\sigma}) - L_0^h(t)\,\mathrm{d}t + 2h(\bar T - T) + \varepsilon,
$$

and finally

$$\mathcal{A}(\tilde{\varphi}) > \mathcal{A}(\bar{\varphi}) + \varepsilon,$$

in clear contradiction with (6.2). □

7 Hamilton-Jacobi equations

We now emphasize the dependence on the initial point x^0 and define the function

$$\begin{aligned} v(x^0) = \min_{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)} \Bigg\{ \int_1^{+\infty} \frac{1}{2} \|\dot{\varphi}(t)\|_{\mathcal{M}}^2 \,\mathrm{d}t \,+ \\ + \int_1^{+\infty} U(\varphi(t) + r_0(t) + x^0 - r_0(1)) - U(r_0(t)) \\ - \langle \ddot{r}_0(t), \varphi(t) \rangle_{\mathcal{M}} \,\mathrm{d}t \Bigg\} - \langle a, x^0 \rangle_{\mathcal{M}}. \end{aligned} \tag{7.1}$$

We claim that v solves the Hamilton-Jacobi equation

$$H(x, \nabla v(x)) = h \tag{7.2}$$

in the viscosity sense. This can be easily seen by taking a point x^0 of differentiability, and formally differentiate (7.1) with respect to x^0, finding

$$\nabla v(x^0) = -\mathcal{M}\dot{x}(1)$$

where $x(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$ and φ is the minimizer of the renormalized action associated with x^0. Therefore $\mathcal{M}^{-1/2}\nabla v(x^0) = -\mathcal{M}^{1/2}\dot{x}(1)$ and we easily obtain (7.2) from the expression of the Hamiltonian (2.1). Making this argument fully rigorous goes beyond the scope of this paper. The interested reader can retrace step by step the method explained in [20], also taking into account that it is known that the singular set is contained in a locally countable union of smooth hypersurfaces of codimension at least one (cfr. [8]).

Fixing x^0 and $T > 0$, we now consider the boundary value problem

$$\begin{cases} \mathcal{M}\ddot{x} = \nabla U(x) \\ x(1) = x^0 \\ \dot{x}(T) = \dot{r}_0(T) \end{cases}$$

and introduce the associated value function

$$u(T, x^0) = \min_{\gamma \in H^1([1,T]),\ \gamma(1) = x^0} \int_1^T \frac{1}{2} \|\dot{\gamma}(t)\|_{\mathcal{M}}^2 + U(\gamma(t)) \,\mathrm{d}t - \langle \dot{r}_0(T), \gamma(T) \rangle_{\mathcal{M}}.$$

It is a standard result of the theory of Hamilton-Jacobi equations (cfr. [8]) that u is a viscosity solution of

$$-\frac{\partial u}{\partial T} = \frac{1}{2} \|\nabla u\|_{\mathcal{M}^{-1}}^2 - U(x),$$

where the gradient is taken with respect to the second variable.

Remark 7.1 Notice that, compared with [8], we have reversed time orientation.

Now, we define

$$\begin{aligned} v(T,x) &= u(T,x) + \int_1^T \frac{1}{2}\|\dot r_0(t)\|^2_{\mathcal{M}} - U(r_0(t))\,\mathrm{d}t \\ &= u(T,x) + \int_1^T H(r_0(t),\dot r_0(t))\,\mathrm{d}t \end{aligned}$$

and observe that

$$-\frac{\partial v}{\partial T} = \frac{1}{2}\|\nabla v\|_{\mathcal{M}^{-1}} - U(x) - H(r_0,\dot r_0).$$

Assume that $v(T,x)$ converges uniformly to some $v(x)$ as $T\to+\infty$. Then, v is a stationary viscosity solution to the stationary Hamilton-Jacobi equation

$$\frac{1}{2}\|\nabla v\|_{\mathcal{M}^{-1}} - U(x) = \lim_{T\to+\infty} H(r_0,\dot r_0) = \frac{1}{2}\|a\|^2_{\mathcal{M}}.$$

To relate the modified value function v with the minimum of our renormalized action, let us write

$$\gamma(t) = r_0(t) + \varphi(t) + \tilde x^0,$$

with $\tilde x^0 = x^0 - r_0(1)$, and compute

$$\begin{aligned} &\int_1^T \frac{1}{2}\|\dot r_0(t)+\dot\varphi(t)\|^2_{\mathcal{M}} + U(r_0(t)+\varphi(t)+\tilde x^0) + \frac{1}{2}\|\dot r_0(t)\|^2_{\mathcal{M}} \\ &\quad - U(r_0(t))\,\mathrm{d}t - \langle \dot r_0(T), r_0(T)+\varphi(T)+x^0-r_0(1)\rangle_{\mathcal{M}} \\ &= \int_1^T \frac{1}{2}\|\dot\varphi(t)\|^2_{\mathcal{M}} + U(r_0(t)+\varphi(t)+\tilde x^0) - U(r_0(t)) \\ &\quad - \langle \ddot r_0(t),\varphi(t)\rangle_{\mathcal{M}}\,\mathrm{d}t - \langle \dot r_0(T), x^0\rangle_{\mathcal{M}}, \end{aligned}$$

which follows from some integration by parts. Therefore, we have

$$v(T,x^0) = \min_{\varphi\in H^1([1,T]),\ \varphi(1)=0} \mathcal{A}^{ren}_{[1,T]}(\varphi) - \langle \dot r_0(T), x^0\rangle_{\mathcal{M}},$$

where we denoted

$$\begin{aligned} \mathcal{A}^{ren}_{[1,T]}(\varphi) = \int_1^T &\frac{1}{2}\|\dot\varphi(t)\|^2_{\mathcal{M}} + U(r_0(t)+\varphi(t)+\tilde x^0) - U(r_0(t)) \\ &- \langle \ddot r_0(t),\varphi(t)\rangle_{\mathcal{M}}\,\mathrm{d}t. \end{aligned}$$

Then, it becomes natural to let $T \to +\infty$ and define

$$v(x^0) = \min_{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)} \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - U(r_0(t)) - \langle \ddot{r}_0(t), \varphi(t)\rangle_{\mathcal{M}}\, dt - \langle a, x^0\rangle_{\mathcal{M}}.$$

We will prove in a forthcoming paper that

$$v(x) = \lim_{T\to+\infty} v(T, x)$$

uniformly on compact sets of $\mathbb{R}^{dN}$ (actually, in the Hölder norms), so that v solves

$$\frac{1}{2}\|\nabla v\|_{\mathcal{M}^{-1}} - U(x) = \frac{1}{2}\|a\|_{\mathcal{M}}^2$$

in the viscosity sense. This justifies once again our choice for the renormalized action functional.

It is worthwhile noticing that the uniqueness result in [21] ensures that, in the hyperbolic case, our value function v is indeed the Busemann function. Moreover, it may be interesting that the linear correction in (7.1) is itself the Busemann function of the free particle.

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