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Original
Small-noise asymptotics of exit times for a class of nonlinear autoregressive processes / Aliev, A., Dzhalilov, A., Fontana, R.. - In: UZBEK MATHEMATICAL JOURNAL. - ISSN 2010-7269. - 70:3(2026), pp. 5-15. [10.29229/uzmj.2026-3-1]

Availability:
This version is available at: 11583/3017512 since: 2026-11-04T10:13:50Z

Publisher:
UZBEKISTAN ACADEMY OF SCIENCES V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

Published
DOI:10.29229/uzmj.2026-3-1

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21 November 2026

Uzbek Mathematical Journal
2026, Volume 70, Issue 3, pp. 5–15
DOI: 10.29229/uzmj.2026-3-1

Small-noise asymptotics of exit times for a class of nonlinear autoregressive processes

Aliev A., Dzhalilov A., Fontana R.

Abstract. We study exit times from the interval $[-1, 1]$ for the nonlinear autoregressive process given, for $n \geq 0$, by

$$X_{n+1}^{(\varepsilon)}(\varphi_{f,g}) = \varphi_{f,g}\big(X_n^{(\varepsilon)}\big) + \varepsilon\xi_{n+1},$$

where $\varepsilon > 0$ is a small noise parameter, $\{\xi_n\}$ is a sequence of independent standard Gaussian random variables, and $\varphi_{f,g}$ is constructed from two continuous nondecreasing maps $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = g(0) = 0$. We consider two classes of such maps. In the first class, the expansive behavior persists up to the endpoints of the interval. In the second class, the maps are expansive on inner intervals and linear and contractive on the corresponding outer intervals. For the first class, we prove that the logarithmic growth rate of the expected exit time vanishes as $\varepsilon \to 0$. For the second class, we determine the exact logarithmic rate of the finite-horizon exit probability and derive an explicit upper bound for the expected exit time. More precisely, our analysis shows that the inner expansive nonlinear part of the drift has zero variational cost, while in the contractive case the outer linear part determines the explicit exponential rate. This extends earlier estimates obtained for particular piecewise-linear autoregressive models to broader classes of monotone nonlinear processes.

Keywords: Exit time, nonlinear autoregressive process, small-noise asymptotics, Markov chain, large deviation principle
MSC (2020): 60F10, 60J05, 37H10

1. Introduction

The study of exit times plays an important role in understanding the behavior of stochastic processes and has applications in population dynamics (see [1], [2], [3]), financial analysis [4], and surveillance analysis [5], among other areas. In this paper, we investigate exit times for nonlinear autoregressive processes driven by Gaussian noise. Throughout, the drift map $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and nondecreasing. Such processes generalize classical linear autoregressive models and may exhibit regime-dependent dynamics, making their first-passage behavior particularly interesting.

Let $\{\xi_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $\xi_1 \sim N(0, 1)$. Consider the autoregressive process

$$X_{n+1}^{(\varepsilon)}(f) = f(X_n^{(\varepsilon)}) + \varepsilon\xi_{n+1},\ n \geq 0,\ X_0^{(\varepsilon)} = x_0 \in (-1, 1). \tag{1.1}$$

We define the exit time from the interval $[-1, 1]$ by (see [6], [7], [8], [9])

$$\tau^{(\varepsilon)}(f) := \min\{k \geq 1 :\ |X_k^{(\varepsilon)}(f)| \geq 1\}, \tag{1.2}$$

where $X_k^{(\varepsilon)}(f)$ is defined by (1.1). Several techniques have been used to study exit times for autoregressive models, including martingale methods and large deviation principles (LDPs). In particular, martingale methods have been used to obtain analytical approximations for the distribution and expectation of exit times in AR(1) processes (see [10], [11], [12], [13]). These methods have also proved effective in the analysis of Ornstein–Uhlenbeck processes in continuous time. In [6], Klebaner and Liptser established an LDP for a general nonlinear autoregressive process of the form

$$X_{n+1}^{(\varepsilon)}(g) := g(X_n^{(\varepsilon)}, \ldots, X_{n-m+1}^{(\varepsilon)}, \varepsilon\xi_{n+1}), \tag{1.3}$$

where $\varepsilon > 0$, $m \geq 1$, and $g(x_1, \ldots, x_m, y)$ is continuous. For comparison, consider a linear autoregressive AR(1) sequence $\mathbb{X}(\lambda) := \{X_n(\lambda),\ n = 1, 2, \ldots\}$ defined by

$$X_{n+1}^{(\varepsilon)}(\lambda) = \lambda X_n^{(\varepsilon)}(\lambda) + \varepsilon\xi_{n+1},\ n \geq 0,\ X_0^{(\varepsilon)} = x_0, \tag{1.4}$$

where $\{\xi_n\}$ is a sequence of i.i.d. random variables with $\xi_1 \sim N(0,1)$ and λ is a constant. Applying the LDP to the linear AR(1) process $\mathbb{X}(\lambda)$, Klebaner and Liptser [6] obtained the following upper bound for the expected exit time.

Theorem 1.1 (see [6]). *Let $\tau^{(\varepsilon)}(\lambda)$ be the exit time from the interval $[-1,1]$ of the random process $\mathbb{X}(\lambda)$. If $|\lambda| < 1$, then*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\lambda) \le \frac{1}{2}(1-\lambda^2).$$

A corresponding lower bound can be obtained by a different method [9]. The multivariate case was studied in [8], and the results were later extended to ARMA processes in [14]. Various aspects of exit-time problems for linear autoregressive processes have been investigated by many authors (see, for example, [15], [16], [17]). A natural problem is to extend these results from linear contractive maps to nonlinear maps. Allen et al. [1] considered examples involving nonlinear contractive maps. Piecewise-linear maps provide a natural intermediate setting between linear and general nonlinear models. Högnäs and Jung [7] studied AR(1) processes driven by continuous increasing contractive maps with a fixed point at $x = 0$. In particular, obtaining an upper bound for $\mathbb{E}\tau^{(\varepsilon)}(f)$ can be reduced to minimizing sums of the form

$$S_N(y_0, y_1, \dots, y_N) := \frac{1}{2}\sum_{i=1}^{N}(y_i - f(y_{i-1}))^2.$$

For details, see the next section. Högnäs and Jung [7] established several useful properties of these sums. Applying those properties to particular piecewise-linear maps, they obtained upper bounds for $\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{E}\tau^{(\varepsilon)}(f)$ and computed the corresponding minimum values of S_N. In the present work, we extend their analysis beyond particular piecewise-linear examples. We consider two-sided drift maps $\varphi_{f,g}$ constructed from continuous nondecreasing one-sided maps f and g that are expansive on inner intervals. The expansive behavior either persists up to the endpoints or is followed by linear contractive behavior on the outer intervals. The precise classes of maps are introduced below.

The contribution of the present paper lies in the explicit solution of the resulting variational problem for these classes of monotone nonlinear drift maps. Unlike the particular piecewise-linear examples treated previously, the maps considered here may exhibit general continuous nondecreasing nonlinear behavior on the inner interval. We show that this inner expansive part has zero variational cost for reaching the interface point. Consequently, in the contractive case, the logarithmic exit cost is determined entirely by the outer linear part of the drift. This leads to an explicit formula for the finite-horizon exit probability starting from the origin and to an upper bound for the expected exit time.

Let $\mathcal{M}$ denote the class of all continuous nondecreasing functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $h(0) = 0$. We introduce the following subclasses of $\mathcal{M}$ (see Fig. 1).

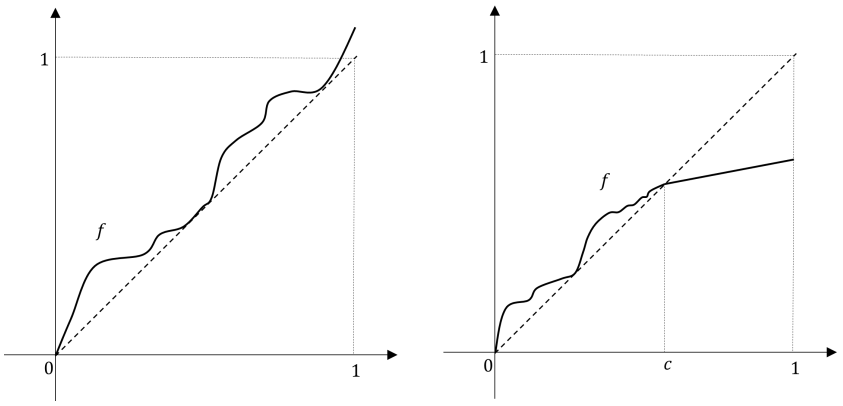


FIGURE 1. Typical graphs of functions from the classes $\mathcal{C}_{\text{up}}$ (left) and $\mathcal{C}_{\text{int}}$ (right).

- The class $\mathcal{C}_{\rm up}$ consists of all functions $h \in \mathcal{M}$ such that $h(x) \geq x$ for every $x \in [0,1]$.
- The class $\mathcal{C}_{\rm int}$ consists of all functions $h \in \mathcal{M}$ for which there exist constants $c_h \in (0,1)$ and $\alpha_h \in (0,1)$ such that
$$h(x) \geq x \quad \text{for every } x \in [0, c_h],$$
and
$$h(x) = \alpha_h x + (1-\alpha_h)c_h \quad \text{for every } x \in [c_h, 1].$$
In particular, $h(c_h) = c_h$ and $h(x) < x$ for every $x \in (c_h, 1]$.

Let $\mathcal{M}^{(c)}$ denote the class of two-sided maps $\varphi_{f,g} : \mathbb{R} \to \mathbb{R}$ associated with pairs $f, g \in \mathcal{M}$ and defined by
$$\varphi_{f,g}(x) = \begin{cases} -f(-x), & \text{if } x \leq 0, \\ g(x), & \text{if } x > 0. \end{cases}$$

The main results are as follows.

Theorem 1.2. *Let $\varphi_{f,g} \in \mathcal{M}^{(c)}$, and let $\mathbb{X}(\varphi_{f,g})$ be the process defined by* (1.1) *with drift map $\varphi_{f,g}$. Then the following assertions hold.*
(i) *If $f, g \in \mathcal{C}_{\rm up}$, then, for every $x_0 \in [-1,1]$,*
$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) = 0.$$
(ii) *If $f, g \in \mathcal{C}_{\rm int}$, then, for the initial point $x_0 = 0$,*
$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = 0 \right) = -\min\{K(f), K(g)\} \tag{1.5}$$
where, for $h \in \{f, g\}$,
$$K(h) := \frac{1+\alpha_h}{1-\alpha_h} \cdot \frac{(1-h(1))^2}{2}.$$

Theorem 1.3. *Let $\varphi_{f,g} \in \mathcal{M}^{(c)}$, and let $\mathbb{X}(\varphi_{f,g})$ be the process defined by* (1.1) *with drift map $\varphi_{f,g}$. Then the following statements hold.*
(i) *If $f, g \in \mathcal{C}_{\rm up}$, then, for every $x_0 \in [-1,1]$,*
$$\lim_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) = 0.$$
(ii) *If $f, g \in \mathcal{C}_{\rm int}$, then*
$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \leq \max\{K(f), K(g)\}. \tag{1.6}$$
In the symmetric case $K(f) = K(g)$ this bound coincides with the constant in (1.5).
Here $K(h)$ is defined as in Theorem 1.2.

Remark 1.1. The upper bound in the second assertion is attained in a simple symmetric case. Indeed, let $f = g = h$, where, for some $c \in (0,1)$ and $\alpha \in (0,1)$,
$$h(x) = \begin{cases} x, & 0 \leq x \leq c, \\ \alpha x + (1-\alpha)c, & c \leq x \leq 1. \end{cases}$$
For this example, a direct computation gives
$$\lim_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{h,h}) = \frac{1-\alpha^2}{2}(1-c)^2 = K(h),$$
so equality holds in the upper bound for this example.

2. The Large Deviation Principle and Past-Dependent Processes

We recall the large deviation framework used by Klebaner and Liptser [6] for past-dependent recursions. Let $\Omega = (\mathbb{R}^\infty, \rho)$ be the complete separable metric space equipped with the metric

$$\rho(\underline{x}, \underline{z}) := \sum_{j\geq 1} 2^{-j} \frac{|x_j - z_j|}{1 + |x_j - z_j|}.$$

Klebaner and Liptser proved an LDP for nonlinear recursions with small noise. In the Markov case, they derived finite-horizon asymptotics for probabilities of the form

$$\mathbb{P}\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1\right).$$

We use this LDP reduction below.

Definition 2.1. *Let $q(\varepsilon) > 0$ be a normalizing function (the speed of the principle) such that $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. We say that the family of Borel probability measures $\{P_\varepsilon,\ \varepsilon > 0\}$ satisfies a large deviation principle (LDP) in Ω with speed $q(\varepsilon)$ and rate function $I : \Omega \to [0, \infty]$ if the following conditions hold:*
I) $0 \leq I(\underline{x}) \leq \infty$ for all $\underline{x} \in \Omega$, I is lower semicontinuous, and the set $\{\underline{x} : I(\underline{x}) \leq r\}$ is compact in Ω for all $r < \infty$;
II) For any $C \subset \Omega$,

$$-\inf_{\underline{x}\in \mathrm{int}(C)} I(\underline{x}) \leq \liminf_{\varepsilon\to 0} q(\varepsilon) \log P_\varepsilon(C) \leq \limsup_{\varepsilon\to 0} q(\varepsilon) \log P_\varepsilon(C) \leq -\inf_{\underline{x}\in \overline{C}} I(\underline{x}), \tag{2.1}$$

where $\mathrm{int}(C)$ and $\overline{C}$ denote the interior and closure of the set C, respectively.

In the present paper we apply this result to the one-dimensional Markov recursion

$$X_{n+1}^{(\varepsilon)} = h\big(X_n^{(\varepsilon)}\big) + \varepsilon\xi_{n+1}, \qquad X_0^{(\varepsilon)} = x_0,$$

where h is a continuous drift map and $\xi_1 \sim N(0, 1)$. In this case the speed is $q(\varepsilon) = \varepsilon^2$, and the corresponding rate function is

$$I_h(\underline{y}) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty} \big(y_n - h(y_{n-1})\big)^2, & y_0 = x_0, \\ \infty, & y_0 \neq x_0. \end{cases}$$

For each $M \geq 1$ put

$$\mathbb{B}_M^{\geq}(x_0) := \big\{(y_0, y_1, \ldots, y_M) \in \mathbb{R}^{M+1} : y_0 = x_0,\ \max_{1\leq i\leq M} |y_i| \geq 1\big\},$$

$$\mathbb{B}_M^{>}(x_0) := \big\{(y_0, y_1, \ldots, y_M) \in \mathbb{R}^{M+1} : y_0 = x_0,\ \max_{1\leq i\leq M} |y_i| > 1\big\}.$$

Applying the LDP to the closed set $\mathbb{B}_M^{\geq}(x_0)$ and to the open set $\mathbb{B}_M^{>}(x_0)$ gives

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0\right) \leq -\inf_{\mathbb{B}_M^{\geq}(x_0)} S_M^h,$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| > 1 \mid X_0^{(\varepsilon)} = x_0\right) \geq -\inf_{\mathbb{B}_M^{>}(x_0)} S_M^h,$$

where

$$S_M^h(y_0, \ldots, y_M) := \frac{1}{2}\sum_{n=1}^{M} \big(y_n - h(y_{n-1})\big)^2.$$

The two variational infima are equal. Indeed, $\mathbb{B}_M^{>}(x_0) \subset \mathbb{B}_M^{\geq}(x_0)$, and the reverse inequality follows by a small perturbation of one coordinate for which $|y_i| = 1$, using the continuity of S_M^h. Hence the above upper and lower bounds give the finite-horizon logarithmic asymptotic

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0\right) = -\inf_{\mathbb{B}_M^{\geq}(x_0)} S_M^h. \tag{2.2}$$

Equivalently, if

$$V_M^h(x_0) := \min_{1\leq N\leq M} \inf_{\substack{y_0=x_0,\ |y_N|\geq 1,\\ |y_k|<1,\ 1\leq k\leq N-1}} \frac{1}{2}\sum_{n=1}^{N} \big(y_n - h(y_{n-1})\big)^2,$$

then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\leq k\leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0\right) = -V_M^h(x_0). \tag{2.3}$$

The reduction from S_M^h to the first-exit sum follows by choosing $y_n = h(y_{n-1})$ after the first hitting time; all subsequent terms then have zero cost.

Thus the finite-horizon exit probability is governed by a deterministic variational problem. This is the same mechanism used by Klebaner and Liptser in the linear Markov case and by Högnäs and Jung for nonlinear autoregressive maps.

For later use, when the drift is clear from the context, we write

$$S_N(y_0, y_1, \ldots, y_N) := \frac{1}{2}\sum_{n=1}^{N} \big(y_n - h(y_{n-1})\big)^2. \tag{2.4}$$

Högnäs and Jung [7] studied variational problems of this type for several classes of nonlinear autoregressive maps.

The following form of their sign-reduction lemma is the version needed below.

Lemma 2.1. *Assume that h is continuous and nondecreasing on $[-1,1]$ and that $h(0) = 0$. Then, for every $N \geq 1$,*

$$\inf_{y_0=0,\ |y_N|\geq 1} S_N(y_0, y_1, \ldots, y_N) = \min\left\{ \inf_{y_0=0,\ y_N=1,\ y_1,\ldots,y_{N-1}\geq 0} S_N(y_0, y_1, \ldots, y_N), \right.$$
$$\left. \inf_{y_0=0,\ y_N=-1,\ y_1,\ldots,y_{N-1}\leq 0} S_N(y_0, y_1, \ldots, y_N) \right\}.$$

In particular, a minimizing path to the right endpoint may be chosen nonnegative, while a minimizing path to the left endpoint may be chosen nonpositive.

For the two-sided drift $h = \varphi_{f,g}$ and the initial point $x_0 = 0$, Lemma 2.1 allows us to treat the positive and negative exits separately. The positive exit is governed by the one-sided map g on $[0,1]$, and the negative exit is governed, after the change of variables $x \mapsto -x$, by the one-sided map f on $[0,1]$.

3. Proof of Theorems 1.2–1.3

For the proof of Theorem 1.2, we first formulate several lemmas, whose proofs are given in the next section.

For a continuous drift map h and $u \in [-1,1]$, let $V_M^h(u)$ be the finite-horizon variational quantity defined in (2.3).

Lemma 3.1. *Let $h = \varphi_{f,g}$.*

- *If $f, g \in \mathcal{C}_{\mathrm{up}}$, then, for every $u \in [-1,1]$,*

$$\lim_{M\to\infty} V_M^h(u) = 0.$$

- *If $f, g \in \mathcal{C}_{\text{int}}$, then*
$$\lim_{M\to\infty} V_M^h(0) = \min\{K(f), K(g)\}.$$
Moreover, for every $u \in [-1, 1]$,
$$\limsup_{M\to\infty} V_M^h(u) \le \max\{K(f), K(g)\}.$$

Equation (2.3) and Lemma 3.1 are the main tools in the proof of the theorems.

For $a, b \in \mathbb{R}$ we write $\vec{y}_N(a, b) := (a, y_1, \dots, y_{N-1}, b)$, so that $y_0 = a$ and $y_N = b$, and, consistently with (2.4),
$$S_N(\vec{y}_N(a, b)) := \frac{1}{2}\sum_{n=1}^{N}(y_n - f(y_{n-1}))^2. \tag{3.1}$$

Before proving Theorem 1.2, we state the following two lemmas; their proofs will be given in the next section.

Lemma 3.2. *Let $[a, b] \subseteq [0, 1]$ and let $f : [a, b] \to [a, \infty)$ be a continuous, nondecreasing function such that*
$$f(a) = a, \qquad f(b) \ge b, \qquad f(x) \ge x \text{ for all } x \in [a, b].$$
Then
$$\inf_{N\in\mathbb{N},\, \vec{y}_N(a,b)} S_N(\vec{y}_N(a, b)) = 0. \tag{3.2}$$

Lemma 3.3. *Let $f(x) = \alpha x + \beta$ be defined on the interval $[c, 1]$, where $\alpha, c \in (0, 1)$ and $f(c) = c$ (equivalently, $\beta = (1 - \alpha)c$). Then $f(x) < x$ for all $x \in (c, 1]$, and*
$$\inf_{N\in\mathbb{N},\, \vec{y}_N(c,1)} S_N(\vec{y}_N(c, 1)) = \frac{(1 + \alpha)(1 - f(1))^2}{2(1 - \alpha)}.$$

Proof of Theorem 1.2. By the finite-horizon asymptotic (2.3), for every fixed M and every initial point $x_0 \in [-1, 1]$ we have
$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\max_{1\le k\le M} |X_k^{(\varepsilon)}| \ge 1 \mid X_0^{(\varepsilon)} = x_0\right) = -V_M^{\varphi_{f,g}}(x_0).$$
If $f, g \in \mathcal{C}_{\text{up}}$, Lemma 3.1 gives $\lim_{M\to\infty} V_M^{\varphi_{f,g}}(x_0) = 0$ for every $x_0 \in [-1, 1]$. This proves the first assertion of Theorem 1.2.

Assume now that $f, g \in \mathcal{C}_{\text{int}}$ and take $x_0 = 0$. By Lemma 2.1, a path from 0 to the right endpoint can be chosen nonnegative and is governed by the one-sided map g, while a path from 0 to the left endpoint is transformed, by $x \mapsto -x$, into a nonnegative path governed by f. Hence Lemmas 3.2 and 3.3 give
$$\lim_{M\to\infty} V_M^{\varphi_{f,g}}(0) = \min\{K(f), K(g)\}.$$
Combining this with (2.3) proves (1.5). Theorem 1.2 is proved.

Proof of Theorem 1.3. Since $\tau^{(\varepsilon)}(\varphi_{f,g})$ takes natural values, $\log E\tau^{(\varepsilon)}(\varphi_{f,g}) \ge 0$. We use the standard block estimate of Klebaner–Liptser and Jung,
$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \le -\lim_{M\to\infty}\lim_{\varepsilon\to 0}\inf_{|x_0|\le 1} \varepsilon^2 \log P\left(\max_{1\le k\le M} |X_k^{(\varepsilon)}| \ge 1 \mid X_0^{(\varepsilon)} = x_0\right),$$
whenever the right-hand side is estimated by the finite-horizon variational problem.

If $f, g \in \mathcal{C}_{\text{up}}$, Lemma 3.1 gives a zero limiting cost uniformly over initial points in $[-1, 1]$, and therefore
$$\lim_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) = 0.$$
If $f, g \in \mathcal{C}_{\text{int}}$, Lemma 3.1 gives the uniform variational upper bound $\max\{K(f), K(g)\}$ for the cost of exiting from $[-1, 1]$. Hence
$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \le \max\{K(f), K(g)\}.$$
This proves Theorem 1.3.

4. Proofs of Lemmas 3.1–3.3

We first prove the variational Lemma 3.1.

Proof of Lemma 3.1.

We first prove the corresponding one-sided assertions. Let $r : [0,1] \to [0,\infty)$ be a continuous and nondecreasing function, with $r(0) = 0$.

Assume first that $r \in \mathcal{C}_{\mathrm{up}}$. Fix $u \in [0,1]$. If some deterministic iterate of u under r reaches or crosses 1, then the path following these iterates has zero cost up to the exit. Otherwise, the sequence of iterates is nondecreasing and bounded above by 1, hence it converges to a fixed point $a \in [u,1]$. Following the deterministic orbit from u to a point arbitrarily close to a costs zero, and the remaining cost from a to 1 is zero by Lemma 3.2, applied on $[a,1]$. Therefore

$$\lim_{M\to\infty} V_M^r(u) = 0, \qquad u \in [0,1].$$

Assume now that $r \in \mathcal{C}_{\mathrm{int}}$. Let $c = c_r$ and $\alpha = \alpha_r$. For every $u \in [0,c]$, the cost of reaching c can be made arbitrarily small by the same deterministic-orbit argument, followed by Lemma 3.2 on the interval whose left endpoint is the limiting fixed point. On the linear interval $[c,1]$, Lemma 3.3 gives the limiting cost $K(r)$ from c to 1. Hence the limiting one-sided cost from any $u \in [0,c]$ to 1 is at most $K(r)$.

If $u \in [c,1]$, the same quadratic minimization as in Lemma 3.3, but with $y_0 = u$ and $y_N = 1$, gives

$$\inf_{\vec{y}_N(u,1)} S_N(\vec{y}_N(u,1)) = \frac{(1-\alpha^2)\big(1-c-\alpha^N(u-c)\big)^2}{2(1-\alpha^{2N})}.$$

Letting $N \to \infty$ yields the same limiting value $K(r)$. Consequently the limiting cost from any $u \in [0,1]$ to 1 is at most $K(r)$.

It remains to justify the equality for the starting point $u = 0$. Let $(0 = y_0, y_1, \dots, y_N = 1)$ be any admissible path. Let $k := \max\{n : y_n \le c\}$; then $y_k \le c < y_{k+1}$, unless c itself is already a node of the path. Since r is nondecreasing and $r(c) = c$, we have $r(y_k) \le c$. Inserting the point c between y_k and y_{k+1} does not increase the cost, because

$$(y_{k+1} - r(y_k))^2 \ge (c - r(y_k))^2 + (y_{k+1} - c)^2.$$

Thus every path from 0 to 1 has cost at least the sum of the optimal costs from 0 to c and from c to 1. By Lemmas 3.2 and 3.3, this lower bound is $K(r)$. The opposite inequality was proved above, hence the limiting one-sided cost from 0 to 1 is exactly $K(r)$.

We now return to the two-sided drift $h = \varphi_{f,g}$. If $u \ge 0$, a positive exit is governed by the one-sided map g. If $u \le 0$, a negative exit is transformed by $x \mapsto -x$ into the corresponding one-sided problem for f. This proves the first assertion when $f, g \in \mathcal{C}_{\mathrm{up}}$.

If $f, g \in \mathcal{C}_{\mathrm{int}}$ and $u = 0$, Lemma 2.1 reduces the centered two-sided problem to the minimum of the two one-sided costs, and therefore

$$\lim_{M\to\infty} V_M^h(0) = \min\{K(f), K(g)\}.$$

For arbitrary $u \in [-1,1]$, choosing the exit on the same side as u gives a cost not exceeding $K(g)$ for $u \ge 0$ and not exceeding $K(f)$ for $u \le 0$. Hence

$$\limsup_{M\to\infty} V_M^h(u) \le \max\{K(f), K(g)\}.$$

The lemma is proved. □

To prove Lemma 3.2 we first establish three auxiliary lemmas.

Lemma 4.1. *Let $f : [p,q] \to [p,q]$, $0 \le p < q \le 1$, be a continuous, nondecreasing function with $f(p) = p$, $f(q) = q$ and $f(x) > x$ for all $x \in (p,q)$. Then*

$$\inf_{N\in\mathbb{N},\, \vec{y}_N(p,q)} S_N(\vec{y}_N(p,q)) = 0.$$

Proof of Lemma 4.1. Fix $\varepsilon \in (0, q-p)$. Set $y_1 = p+\varepsilon$, $y_n = f(y_{n-1})$ for $2 \le n \le N-1$, and $y_N = q$. Since $y_n = f(y_{n-1})$ for $2 \le n \le N-1$, all the middle terms vanish and

$$S_N(\vec{y}_N(p,q)) = \frac{1}{2}\big(\varepsilon^2 + (q - f^{N-1}(p+\varepsilon))^2\big).$$

The iterates $f^n(p+\varepsilon)$ form a nondecreasing sequence (since $f(x) \ge x$) bounded above by q, and hence converge to a fixed point of f in $(p,q]$. As $f(x) > x$ on (p,q), this fixed point is q, so $f^{N-2}(p+\varepsilon) \to q$ as $N \to \infty$. Therefore $\limsup_N S_N(\vec{y}_N(p,q)) \le \frac{1}{2}\varepsilon^2$, and letting $\varepsilon \to 0$ gives the claim. □

Lemma 4.2. *Let $[p,q] \subseteq [0,1]$ and suppose $f(x) = x$ for all $x \in [p,q]$. Then*

$$\inf_{N \in \mathbb{N},\, \vec{y}_N(p,q)} S_N(\vec{y}_N(p,q)) = 0.$$

Proof of Lemma 4.2. For $N \in \mathbb{N}$ take the uniform partition $y_j = p + \frac{j}{N}(q-p)$, $0 \le j \le N$. Then $f(y_{j-1}) = y_{j-1}$ and

$$S_N(\vec{y}_N(p,q)) = \frac{1}{2}\sum_{j=1}^{N}(y_j - y_{j-1})^2 = \frac{(q-p)^2}{2N} \to 0, \quad N \to \infty.$$

□

Lemma 4.3. *Let $f : [p,b] \to [p,\infty)$, $0 \le p < b \le 1$, be a continuous, nondecreasing function with $f(p) = p$, $f(b) \ge b$ and $f(x) > x$ for all $x \in (p,b)$. Then*

$$\inf_{N \in \mathbb{N},\, \vec{y}_N(p,b)} S_N(\vec{y}_N(p,b)) = 0.$$

Proof of Lemma 4.3. We construct a finite backward chain of preimages of b. Set $v_0 = b$. Given $v_k \in (p,b]$, we have $f(p) = p < v_k \le f(v_k)$ (indeed $f(v_k) > v_k$ if $v_k \in (p,b)$, and $f(b) \ge b$ if $v_k = b$); since f is continuous, by the intermediate value theorem the set $\{x \in [p, v_k] : f(x) = v_k\}$ is nonempty, and we put

$$v_{k+1} := \min\{x \in [p, v_k] : f(x) = v_k\}.$$

Then $v_{k+1} > p$ (because $f(p) = p < v_k$) and $v_{k+1} \in (p,b)$, so $v_k = f(v_{k+1}) > v_{k+1}$; thus (v_k) is strictly decreasing and bounded below by p, hence $v_k \downarrow L$ with $f(L) = L$. Since $f(x) > x$ on (p,b), necessarily $L = p$, i.e. $v_k \downarrow p$.

Given $\varepsilon > 0$, choose K with $v_K - p < \varepsilon$ and set

$$y_0 = p, \quad y_1 = v_K, \quad y_j = v_{K-j+1} = f(y_{j-1}) \quad (2 \le j \le K+1),$$

so that $y_{K+1} = v_0 = b$. Every step except the first costs zero, hence

$$S_{K+1}(\vec{y}_{K+1}(p,b)) = \frac{1}{2}(v_K - f(p))^2 = \frac{1}{2}(v_K - p)^2 < \frac{1}{2}\varepsilon^2.$$

Letting $\varepsilon \to 0$ proves the claim. □

Proof of Lemma 3.2. For $a \le u \le v \le b$ write $\mathcal{C}(u,v) := \inf_{N \in \mathbb{N},\, \vec{y}_N(u,v)} S_N(\vec{y}_N(u,v)) \ge 0$. Concatenating admissible sequences at a common endpoint gives, for every finite partition $a = u_0 < u_1 < \cdots < u_M = b$,

$$\mathcal{C}(a,b) \le \sum_{i=1}^{M} \mathcal{C}(u_{i-1}, u_i). \tag{4.1}$$

It suffices to prove $\mathcal{C}(a,b) = 0$.

Let $E := \{x \in [a,b] : f(x) = x\}$, a closed set containing a. Each component (α,β) of $[a,b] \setminus E$ satisfies $f(x) > x$, with $\alpha \in E$ and $\beta \in E$ or $\beta = b$; hence $\mathcal{C}(\alpha,\beta) = 0$ by Lemma 4.1 (if $\beta \in E$, since then f maps $[\alpha,\beta]$ into itself and fixes its endpoints) or by Lemma 4.3 (if $\beta = b$). Moreover $f \equiv \mathrm{id}$ on any interval contained in E, so $\mathcal{C} = 0$ there by Lemma 4.2.

Fix $\delta > 0$. Only finitely many components of $[a,b] \setminus E$ have length $\geq \delta$; each has cost 0, and removing them leaves finitely many closed intervals $I_0, \dots, I_p$ (one of which may abut a or b) inside each of which every component of $[a,b] \setminus E$ is shorter than δ; equivalently, E is δ-dense in each I_m. Fix $I_m = [u,v]$; its endpoints lie in E, with the only possible exception $v = b$. Put $w := v$ if $v \in E$, and $w := \max\big(E \cap [u,v)\big)$ otherwise (then $v - w < \delta$). Setting $e_0 := u$ and, while $e_i < w$,

$$e_{i+1} := \sup\{x \in E \cap [u,w] : \, x \leq e_i + \delta\} \in E,$$

δ-density yields a strictly increasing sequence with steps $< \delta$ which, being unable to accumulate before w, reaches w after finitely many steps $u = e_0 < \cdots < e_P = w$. Since $f(e_i) = e_i$, the vector $(e_0, \dots, e_P)$ is admissible and

$$\mathcal{C}(u,w) \leq \sum_{i=0}^{P-1} \frac{1}{2}(e_{i+1} - e_i)^2 \leq \frac{\delta}{2} \sum_{i=0}^{P-1} (e_{i+1} - e_i) = \frac{\delta}{2}(w - u).$$

If $w = v$ this bounds $\mathcal{C}(I_m)$; if $w < v = b$, the single step $w \to b$ adds at most $\frac{1}{2}(b-w)^2 < \frac{1}{2}\delta^2$. Summing over m via (4.1) and recalling that the removed components have cost 0,

$$\mathcal{C}(a,b) \leq \sum_m \mathcal{C}(I_m) \leq \frac{\delta}{2}(b-a) + \frac{1}{2}\delta^2 \xrightarrow{\delta \to 0} 0.$$

Hence $\mathcal{C}(a,b) = 0$, which completes the proof. □

Proof of Lemma 3.3. On $[c,1]$ we have $f(y_{n-1}) = \alpha y_{n-1} + \beta$, so for a fixed $N \geq 1$

$$S_N(\vec{y}_N(c,1)) = \frac{1}{2} \sum_{n=1}^{N} \big(y_n - \alpha y_{n-1} - \beta\big)^2, \qquad y_0 = c, \quad y_N = 1.$$

We first determine its minimum over the free variables $(y_1, \dots, y_{N-1}) \in \mathbb{R}^{N-1}$. Put

$$d_n := y_n - \alpha y_{n-1} - \beta = y_n - f(y_{n-1}) \qquad (1 \leq n \leq N),$$

so that $S_N = \frac{1}{2}\sum_{n=1}^{N} d_n^2$. Each d_n is an affine function of $(y_1, \dots, y_{N-1})$, and the linear map $(y_1, \dots, y_{N-1}) \mapsto (d_1, \dots, d_N)$ is injective, since y_k enters d_k with coefficient 1. Hence S_N is a strictly convex quadratic function, and its unique global minimum is attained at the (unique) critical point.

Each variable y_k $(1 \leq k \leq N-1)$ occurs in exactly two terms, namely in d_k (with coefficient 1) and in $d_{k+1} = y_{k+1} - \alpha y_k - \beta$ (with coefficient $-\alpha$); therefore

$$\frac{\partial S_N}{\partial y_k} = d_k - \alpha\, d_{k+1} = 0, \qquad 1 \leq k \leq N-1.$$

This recurrence $d_k = \alpha\, d_{k+1}$ gives, with $\lambda := d_N$,

$$d_n = \lambda\, \alpha^{N-n}, \qquad 1 \leq n \leq N.$$

To find λ, multiply $d_n = y_n - \alpha y_{n-1} - \beta$ by α^{N-n} and sum; the y-terms telescope,

$$\sum_{n=1}^{N} \alpha^{N-n}\big(y_n - \alpha y_{n-1}\big) = y_N - \alpha^N y_0 = 1 - \alpha^N c,$$

so, using $\beta = (1-\alpha)c$ (which is equivalent to $f(c) = c$),

$$\sum_{n=1}^{N} \alpha^{N-n} d_n = \big(1 - \alpha^N c\big) - \beta \frac{1-\alpha^N}{1-\alpha} = \big(1 - \alpha^N c\big) - c\big(1 - \alpha^N\big) = 1 - c.$$

On the other hand, $\sum_{n=1}^{N} \alpha^{N-n} d_n = \lambda \sum_{n=1}^{N} \alpha^{2(N-n)} = \lambda\, \dfrac{1-\alpha^{2N}}{1-\alpha^2}$, whence

$$\lambda = \frac{(1-\alpha^2)(1-c)}{1-\alpha^{2N}}.$$

Consequently the minimum for fixed N equals

$$\min_{\vec{y}_N(c,1)} S_N = \frac{1}{2}\sum_{n=1}^{N} d_n^2 = \frac{1}{2}\,\lambda^2 \sum_{n=1}^{N} \alpha^{2(N-n)} = \frac{1}{2}\,\lambda^2\,\frac{1-\alpha^{2N}}{1-\alpha^2} = \frac{(1-\alpha^2)(1-c)^2}{2\,(1-\alpha^{2N})}.$$

It remains to check that this minimizer is admissible, i.e. lies in $[c,1]$. Since $\beta = (1-\alpha)c$, the recurrence $y_n = \alpha y_{n-1} + \beta + d_n$ may be written as

$$y_n - c = \alpha(y_{n-1} - c) + \lambda\alpha^{N-n}, \qquad y_0 - c = 0.$$

Solving this recursion gives

$$y_n - c = \lambda\alpha^{N-n}\frac{1-\alpha^{2n}}{1-\alpha^2}, \qquad 1 \le n \le N.$$

This expression is positive. Furthermore,

$$(y_{n+1} - c) - (y_n - c) = \lambda\alpha^{N-n-1}\frac{(1-\alpha)(1+\alpha^{2n+1})}{1-\alpha^2} > 0,$$

so $c = y_0 < y_1 < \cdots < y_N = 1$. Thus the minimizing vector belongs to $[c,1]^{N+1}$.

Finally, since $\alpha \in (0,1)$, the quantity $1-\alpha^{2N}$ increases to 1, so the above minima decrease in N, and

$$\inf_{N\in\mathbb{N},\,\vec{y}_N(c,1)} S_N(\vec{y}_N(c,1)) = \lim_{N\to\infty} \frac{(1-\alpha^2)(1-c)^2}{2\,(1-\alpha^{2N})} = \frac{(1-\alpha^2)(1-c)^2}{2} = \frac{(1+\alpha)(1-f(1))^2}{2(1-\alpha)}.$$

□

Acknowledgements

This work was partially carried out during a visit of the second author to Great Bay University. The second author is deeply grateful for the warm hospitality extended to him during the visit.

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