Fake supersymmetry versus Hamilton-Jacobi

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Abstract: We explain when the first-order Hamilton-Jacobi equations for black holes (and domain walls) in (gauged) supergravity, reduce to the usual first-order equations derived from a fake superpotential. This turns out to be equivalent to the vanishing of a newly found constant of motion and we illustrate this with various examples. We show that fake supersymmetry is a necessary condition for having physically sensible extremal black hole solutions. We furthermore observe that small black holes become scaling solutions near the horizon. When combined with fake supersymmetry, this leads to a precise extension of the attractor mechanism to small black holes: the attractor solution is such that the scalars move on specific curves, determined by the black hole charges, that are purely geodesic, although there is a non-zero potential.

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Contents

1 Introduction

Supersymmetry has proven to be a very powerful tool in string theory. In the supergravity approximation, supersymmetry can be used as a tool to generate solutions since supersymmetric solutions obey first-order equations that are derived from the preservation of some fraction of supersymmetry. However, the ultimate goal is to perform computations in string theory in non-supersymmetric contexts. For the sake of finding supergravity solutions, such as black holes for instance, the so-called fake supergravity formalism borrows tricks of supersymmetry to find non-supersymmetric solutions. Fake supergravity was first introduced in the context of domain wall solutions in [1]. For supersymmetric domain wall solutions, the first order supersymmetry equations are determined by the superpotential. The idea of fake supergravity (or fake supersymmetry) is to define a fake superpotential, not directly related to the superalgebra, whose first-order gradient flow also satisfies the second-order equations of motion. Apart from being a technical tool, fake supersymmetry has a physical meaning in the sense that it guarantees the stability of the domain wall solution. It thus becomes important to understand when a domain wall solution is fake supersymmetric or not, a question which was raised in [2].

FLRW solutions supported by time-dependent scalars subject to the force derived from some scalar potential are very similar to gravitational domain wall solutions. In fact

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there exists a one to one map between such solutions [3]. This implies that the notion of fake supersymmetry should carry over to time-dependent solutions. This is called pseudo-supersymmetry since time-dependent solutions cannot possess unitary superalgebras.[1]

Domain walls and FLRW cosmologies are solutions that depend on one coordinate. The second-order equations of motion take the form of ODE's that can be derived from an effective Lagrangian that defines a Hamiltonian system. This suggests a close link between fake supergravity and the Hamilton-Jacobi formalism, since the latter also defines first-order equations. Indeed such a link was established in [7] (see also [8–10]). Nonetheless, it has never been precisely formulated what the difference is between fake supersymmetry and the Hamilton-Jacobi formalism. It is the aim of this paper to make this precise. It is important to note that not all domain walls are derived from fake supersymmetry. Reference [11] gave an explicit example for which one can easily proof that there is no fake supersymmetry in the standard sense. We will recall and simplify this solution in this paper and present even simpler examples.

Spherically symmetric stationary black holes are another example of co-homogeneity one solutions in supergravity that effectively define a Hamiltonian system. It is therefore not surprising that the fake supergravity formalism can be applied to such black holes as well. This was initiated in [12] and many other references soon followed that constructed fake superpotentials for non-supersymmetric extremal black holes (see e.g. [13–15] and the review [16] for more references). Similarly to domain walls, the relation with the Hamilton-Jacobi formalism was noted in [17]. Interestingly the existence of first-order gradient flow equations was then also found for non-extremal black holes in Einstein Maxwell-theory [18] and extended to more general models in [19]. The most general form for the first-order equations for non-extremal solutions was found in [20], where it was emphasized that the flow equations do differ from those for extremal black holes in the sense that the black hole warp factor appears in a non-trivial way, different from extremal solutions.

The only physical interpretation of the fake superpotential for black holes so far is as a Liapunov function [21]. A deeper physical meaning of the existence of fake supersymmetry, similar to the assurance of stability for domain walls, has not been understood. One of the purposes of this paper is to fill this gap for extremal black holes.

This paper is organized as follows. We recall the concept of fake supersymmetry in section 2 and the Hamilton-Jacobi (HJ) formalism in section 3. In section 4, we show that the HJ equations reduce to the standard fake supersymmetry equations if a certain, newly-found, constant of motion vanishes. We furthermore give a new physical interpretation of fake supersymmetry in the context of extremal black holes: fake supersymmetry is necessary for having physically acceptable solutions. We use this in section 5 to uncover new general properties of the near horizon regions of small black holes, which can be regarded as an extension of the attractor mechanism for large extremal black holes [22, 23]. We end with a discussion in section 6. Appendix A contains several examples that clarify the statements in the bulk of the paper.

[1]For earlier remarks on the first-order formalism in cosmology we refer to [4, 5] and for applications to bent branes, see [6].

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2 Fake supersymmetry

Fake supersymmetry is a concept that is usually formulated on the level of effective one-dimensional actions for black hole solutions, domain walls and, through the map between domain walls and FLRW cosmologies [7], also for the latter, where it is referred to as pseudo-supersymmetry. We briefly recall these effective actions before we recall the notion of fake supersymmetry.

Effective actions for black holes. A typical ungauged supergravity is described by N real scalar fields ϕ^i that parameterize a Riemannian target space with metric G_{ij} and M Abelian vector fields that couple to these scalars. Spherical and static black hole solutions to such theories are described by the following Ansatz

$$\mathrm{d}s_4^2 = -\mathrm{e}^{2\mathrm{U}(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2\mathrm{U}(\tau)}\big(\mathrm{e}^{4A(\tau)}\mathrm{d}\tau^2 + \mathrm{e}^{2A(\tau)}\mathrm{d}\Omega_2^2\big)\,. \tag{2.1}$$

In the absence of a scalar potential (ungauged supergravity) the function $A(\tau)$ is independent of the matter content:

$$\text{extremal:} \qquad \mathrm{e}^{A(\tau)} = \frac{1}{\tau}\,, \tag{2.2}$$

$$\text{non-extremal:} \qquad \mathrm{e}^{A(\tau)} = \frac{c}{\sinh(c\tau)}\,. \tag{2.3}$$

The constant c is the so-called non-extremality parameter and τ is a reparametrisation of the usual radial coordinate. When $c^2 < 0$, we find non-physical ('over-extremal') solutions (although the metric is still real for some coordinate range).

Due to spherical symmetry the scalars only depend on the radial direction and the vector fields can easily be integrated out in terms of the magnetic and electric charges. The equations of motion are captured by the effective action

$$S = \int \mathrm{d}\tau\big(4\dot{U}^2 + G_{ij}\dot{\phi}^i\dot{\phi}^j - \mathrm{e}^{2U}V_{\mathrm{BH}}(\phi)\big)\,, \tag{2.4}$$

where a dot represents a derivative with respect to τ. The black hole potential $V_{\mathrm{BH}}(\phi)$ is the term generated by integrating out field strengths and is strictly negative. The solutions are subject to the following energy constraint

$$G_{ij}\dot{\phi}^i\dot{\phi}^j + 4\dot{U}^2 + \mathrm{e}^{2U}V_{\mathrm{BH}} = 4c^2. \tag{2.5}$$

Effective actions for domain walls. Domain wall solutions are usually supported by scalar fields that are subject to a non-trivial scalar potential $V_{\mathrm{DW}}(\phi)$. The standard Ansatz for flat domain wall solutions is given by

$$\mathrm{d}s_2^2 = \mathrm{e}^{2\sqrt{3}\mathrm{U}(z)}\mathrm{d}z^2 + e^{\frac{2}{\sqrt{3}}\mathrm{U}(z)}(\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2)\,. \tag{2.6}$$

Consistent with the symmetries one then typically assumes that the scalars depend on z only. The effective action then becomes one-dimensional

$$S = \int \mathrm{d}z\big(4\dot{U}^2 - G_{ij}\dot{\phi}^i\dot{\phi}^j - \mathrm{e}^{2\sqrt{3}U}V_{\mathrm{DW}}(\phi)\big)\,, \tag{2.7}$$

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and is supplemented with a zero energy condition

$$4\dot{U}^2 - G_{ij}\dot{\phi}^i\dot{\phi}^j + \mathrm{e}^{2\sqrt{3}U}V_{\mathrm{DW}}(\phi) = 0\,. \tag{2.8}$$

Up to signs and factors of e^U there is no difference between extremal black holes and domain walls from the point of view of the effective action.

There is a simple one to one map between domain walls and FLRW cosmologies. For Minkowski-sliced domain walls and $k = 0$ FLRW cosmologies, given by the metric,[2]

$$\mathrm{d}s_2^2 = -\mathrm{e}^{2\sqrt{3}\mathrm{U}(t)}\mathrm{d}t^2 + e^{\frac{2}{\sqrt{3}}\mathrm{U}(t)}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{2.9}$$

the map proceeds by flipping the sign of the scalar potential V and replacing $z \to t$ in $\mathrm{U}(z), \phi^i(z)$.

The fake superpotential. Supersymmetric domain walls and black holes are a special subset of solutions that fulfill certain first-order differential equations that follow from the Killing spinor equations. These equations take the form of flow equations, derived from the superpotential function W:

$$\dot{\phi}^i = \epsilon \mathrm{e}^{aU}G^{ij}\partial_j W(\phi)\,, \qquad 4\dot{U} = a\mathrm{e}^{aU}W(\phi)\,, \tag{2.10}$$

where

$$\begin{aligned} &\text{Domain walls:} && \epsilon = -1 && a = \sqrt{3}\,, \\ &\text{Black holes:} && \epsilon = +1 && a = 1\,. \end{aligned}$$

As is well known, supersymmetric black holes are necessarily extremal. Hence the zero energy condition for both domain walls and black holes implies a relation for the superpotential function W

$$\epsilon\, G^{ij}\partial_i W \partial_j W + \frac{a^2}{4}W^2 = -V(\phi)\,. \tag{2.11}$$

For supersymmetric solutions, W is the superpotential derived from the Killing spinor equations. The essence of fake supersymmetry is that *any* function W that obeys the above relation (2.11) defines a first-order flow, through (2.10), that can easily be demonstrated to solve the full second-order equations of motion. Hence solutions that can be found from a flow governed by a fake superpotential, in a certain sense mimic supersymmetric solutions.

3 Hamilton's principal function

To introduce the Hamilton-Jacobi formalism in this context, we first write the above effective actions (2.4), (2.7) in a more formal Hamiltonian system notation. For that we define the configuration space variables $q^a = (U, \phi^i)$ and the corresponding metric

$$G_{ab} = \begin{pmatrix} 4 & 0 \\ 0 & \epsilon G_{ij} \end{pmatrix}. \tag{3.1}$$

[2]We have chosen a similar z parametrization as for domain walls rather than the usual cosmological time.

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We furthermore define

$$\mathcal{V}(q) = \frac{1}{2}\mathrm{e}^{2aU}V(\phi)\,, \tag{3.2}$$

then the effective action is compactly written as

$$S = \int \mathrm{d}\tau \left(\frac{1}{2}G_{ab}\dot{q}^a\dot{q}^b - \mathcal{V}(q)\right). \tag{3.3}$$

This action is scaled with a factor $\frac{1}{2}$ with respect to the effective actions (2.4), (2.7). The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}G_{ab}\dot{q}^a\dot{q}^b + \mathcal{V}(q)\,, \tag{3.4}$$

which is zero for extremal black holes and domain walls and equal to $2c^2$ for non-extremal black holes. Now we follow the steps of [17]. The canonical momenta p^a are defined via

$$p_a = G_{ab}\dot{q}^b. \tag{3.5}$$

Let us assume that there exists a local Hamilton-Jacobi (HJ) formulation. We will later comment on this assumption. HJ implies the existence of new variables P, Q that obey the following equations in terms of the *principal Hamiltonian function* (or Hamilton's principal function) $S(q, P, \tau)$:

$$\frac{\partial S}{\partial q^a} = p_a\,, \qquad \frac{\partial S}{\partial P^a} = Q_a\,, \qquad \frac{\partial S}{\partial \tau} = -\mathcal{H}\,. \tag{3.6}$$

The P^a are constants of motion. If we therefore focus on the appearance of the q's we deduce from the Hamiltonian constraint ($\mathcal{H} = 2c^2$)

$$S(q, \tau) = \mathcal{W}(q) - 2c^2\tau\,, \tag{3.7}$$

and the velocities follow a gradient flow set by $\mathcal{W}$:

$$p_a = \frac{\partial S}{\partial q^a} \qquad \Rightarrow \qquad \dot{q}^a = G^{ab}\frac{\partial \mathcal{W}(q)}{\partial q^b}\,. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\frac{1}{2}G^{ab}\partial_a\mathcal{W}^i\partial_b\mathcal{W} + \mathcal{V}(q) = 2c^2. \tag{3.9}$$

The function $\mathcal{W}$ is often called Hamilton's characteristic function. Note that the Hamilton-Jacobi equations always coincide exactly with a rewriting of the action as a sum and difference of squares, up to a total derivative:

$$S = \int \frac{1}{2}G_{ab}\left(\dot{q}^a - G^{ac}\frac{\partial \mathcal{W}}{\partial q^c}\right)\left(\dot{q}^b - G^{bd}\frac{\partial \mathcal{W}}{\partial q^d}\right). \tag{3.10}$$

It is a basic theorem of analytical mechanics that, locally, one can always define the first-order Hamilton-Jacobi equations. Typical global phenomena are the existence of branch cuts such that S can become multi-valued. Barring these subtleties one can safely claim that there always exists a function S, for any solution.

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4 Fake supersymmetry and Hamilton-Jacobi

In this section we explain the link between Hamilton-Jacobi theory and fake supersymmetry for domain walls and extremal black holes ($\mathcal{H} = 2c^2 = 0$). The condition for fake supersymmetry is that Hamilton's characteristic function factorizes as $\mathcal{W} = \mathrm{e}^{aU} W(\phi)$.

We use this observation to make the statement that for *regular* extremal black holes, there is always a fake superpotential. We prove this for large black holes, and conjecture and motivate this for small black holes.

4.1 Fake supersymmetry from Hamilton-Jacobi

When the energy is zero (extremal black holes and domain walls), Hamilton's characteristic function obeys (3.9)

$$\frac{1}{4}(\partial_U \mathcal{W})^2 + \epsilon G^{ij}\partial_i \mathcal{W}\partial_j \mathcal{W} = -\mathrm{e}^{2aU} V_{\mathrm{DW/BH}}(\phi)\,. \tag{4.1}$$

This relation suggests that there could be a simple factorised form for $\mathcal{W}$

$$\mathcal{W} = \mathrm{e}^{aU} W(\phi)\,. \tag{4.2}$$

Exactly this assumption reproduces the (fake) supersymmetry flow equations (2.10) and the defining relation for W (2.11) from the Hamilton-Jacobi equations (3.8), (3.9). The rewriting of the action as a sum of squares (3.10) using Hamilton's principal function then becomes the usual rewriting as a sum of squares using the (fake) superpotential.

A new conserved quantity. Let us make the condition for having a fake superpotential more precise. Consider the effective action (3.3) with $c^2 = 0$ (Hamiltonian is zero on-shell). The Hamilton-Jacobi equations are

$$4\dot{U} = \partial_U \mathcal{W}\,, \qquad \dot{\phi}^i = \epsilon G^{ij}\partial_j \mathcal{W}\,. \tag{4.3}$$

Now observe that

$$\mathcal{Q} \equiv \frac{4}{a}\dot{U} - \mathcal{W} \tag{4.4}$$

is a constant of motion. This follows from:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{Q} &= \frac{4}{a}\ddot{U} - (\partial_i \mathcal{W})\dot{\phi}^i - (\partial_U \mathcal{W})\dot{U} \\ &= -\mathrm{e}^{2aU} V_{\mathrm{BH/DW}}(\phi) - \epsilon G_{ij}\dot{\phi}^i\dot{\phi}^j - 4\dot{U}^2\end{aligned} \tag{4.5}$$

which is zero by virtue of the energy constraint ($\mathcal{H} = 0$). In the second line we used the Hamilton-Jacobi equations (4.3) and the U-equation of motion ($4\ddot{U} = -a\mathrm{e}^{2aU} V_{\mathrm{BH/DW}}(\phi)$). It follows that for non-extremal black hole solutions ($\mathcal{H} \neq 0$) this quantity is not conserved.

Now we consider flows for which $\mathcal{Q} = 0$. Then we can derive that

$$\partial_U \mathcal{W} = a\mathcal{W}\,, \tag{4.6}$$

by the Hamilton-Jacobi equations (4.3). If we integrate this equation we find

$$\mathcal{W} = \mathrm{e}^{aU} W(\phi)\,. \tag{4.7}$$

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We therefore find the elegant result

$$\text{Fake SUSY} \qquad \Leftrightarrow \qquad \mathcal{Q} = 0\,.$$

For non-extremal black holes ($c^2 \neq 0$) it follows straightforwardly that the factorisation property (4.2) cannot hold, which was one of the central observations of [20]. This is not that surprising since non-extremal black holes can never be supersymmetric in any possible supergravity theory.

There is a subtlety to the above statements. The principal function $\mathcal{W}$ is by definition only determined up to a constant. Therefore in principle we could set $\mathcal{Q} = 0$ solution by solution, by adding a moduli-dependent constant to $\mathcal{W}$. However, there is only fake-supersymmetry if the parameters can be chosen so that the constant part of $\mathcal{Q}$ is solution-independent, and it can be chosen to be *identically* zero. We clarify this point in the appendix, with some explicit examples for which there is no solution-independent way of getting $\mathcal{Q} = 0$ and there is no fake supersymmetry.

4.2 Fake supersymmetry and regularity

Regular extremal black holes have an $\mathrm{AdS}_2 \times S^2$ horizon. At this horizon one can show that $\mathcal{Q} = 0$. For that we use that all scalars are fixed at their attractor values

$$\phi(\tau = -\infty) = \phi_H\,, \qquad \partial_i \mathcal{W}(\phi_H) = 0\,. \tag{4.8}$$

From the energy condition we then find that

$$\mathcal{W}(U, \phi_H) = 2\mathrm{e}^U \sqrt{-V_{\mathrm{BH}}(\phi_H)}\,, \tag{4.9}$$

such that $\mathcal{Q} = 0$. Since $\mathcal{Q}$ is conserved it is zero throughout the flow and the factorisation occurs everywhere. *Hence we have established that regular extremal flows must be fake supersymmetric.* This is the usual assumption for the flow equations of regular extremal black holes and here we provided a proof that this is a necessary condition. Note that when $\mathcal{Q} = 0$, the asymptotic value of $\mathcal{W}$ then gives the ADM mass through (4.4):

$$\mathcal{W}(\tau = 0) = 4M_{\mathrm{ADM}}\,. \tag{4.10}$$

A second class of extremal black holes that are still of interest are the so-called extremal *small* black holes. These are zero entropy black holes, meaning they have vanishing horizon size.[3] Equivalently, the black hole singularity is light-like. Since this can be obtained by taking a limit of a regular solution we might postulate similarly that also small extremal black holes are necessarily fake supersymmetric. This limit is not a full proof, and instead we make this into a conjecture for small extremal black holes.

In summary, we make the following statement: extremal flows with $\mathcal{Q} = 0$ comprise all black holes with an $\mathrm{AdS}_2 \times S^2$ horizon (large black holes) and black holes for which the horizon coincides with the singularity (small black holes). When $\mathcal{Q} \neq 0$, the solution is unphysical and has one or more naked singularities (we give such examples in the appendix). In the next section we will discuss some interesting new features of small black holes in detail.

[3] In string theory, these black hole develop a horizon through α' corrections.

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5 Small black hole horizons

For an extremal black hole with a macroscopic horizon area, the attractor mechanism applies: the scalars at the horizon flow to constants that are only functions of the charges. We prove that for small black holes, a similar story holds. When fake supersymmetry is valid, the scalars follow a geodesic flow near the horizon, even though there is a scalar potential.

5.1 Definition of a small black hole horizon

The near-horizon geometry of a small black hole is conformal to $\mathrm{AdS}_2 \times S^2$:

$$ds^2_{\mathrm{NH}} = \rho^\alpha \left(R^2_{\mathrm{AdS}} \left(-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right) + R^2_{S^2}\, d\Omega^2 \right). \tag{5.1}$$

The constant radii of AdS_2 and S^2 do not have to be equal. The conformal factor ρ^α is a simple consequence of expanding the general conformal factor and keeping the leading term. One can show that the constant α must be positive. To illustrate this, consider the D0-D4 STU black hole:

$$ds^2 = -e^{2U} dt^2 + e^{-2U}(dr^2 + r^2 d\Omega^2)\,, \tag{5.2}$$

with warp factor

$$e^{-2U} = \sqrt{H_0 H^1 H^2 H^3}\,. \tag{5.3}$$

The four harmonic functions H are of the form:

$$H^I = 1 + \frac{Q^I}{r}\,. \tag{5.4}$$

When one of the charges is zero, the near-horizon geometry ($\rho \to 0$) is exactly of the form (5.1), with $\rho = \sqrt{r}$ and $\alpha = 1$.

The small black hole near-horizon geometry can be interpreted as a 'scaling solution', similar to DW and cosmological scaling solutions (see e.g. [24, 25]). The defining property of a scaling solution is the existence of a conformal Killing vector. Such a vector defines a local transformation that preserves the metric up to a constant rescaling. For the above metric (5.1) one can easily verify that the transformation

$$\rho \to \mathrm{e}^\lambda \rho\,, \qquad t \to \mathrm{e}^{-\lambda} t\,, \tag{5.5}$$

leaves the metric invariant up to a *constant* factor

$$g_{\mu\nu} \to \mathrm{e}^{\alpha\lambda} g_{\mu\nu}\,. \tag{5.6}$$

It is useful to contrast this with large black holes. Large black holes ($\alpha = 0$) interpolate from Minkowski space at infinity to $\mathrm{AdS}_2 \times S^2$ at the horizon. The general solution can therefore be understood as a flow between two fixed points. The fixed points themselves are characterized by a sudden increase in bosonic (and sometimes fermionic) symmetries. For example, $\mathrm{AdS}_2 \times S^2$ realises the conformal group. Small black holes are similar in that

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respect since, at the horizon, the solution is a scaling solution and it is characterised by the same increase in bosonic symmetries, the only difference is that the symmetries rescale the metric up to a constant. For a recent discussion on the connection between scaling solutions and black brane horizons we refer to [26] and for its applications to holography, see [27].[4]

5.2 From scaling to Killing and geodesic flows

For scaling solutions, the on-shell action scales with on overall factor as well. Hence, an equivalent definition of scaling solutions is that each term in the effective action scales in the same way. What does this imply for the scalar fields? We formalize this, following the strategy of [25].

Consider the continuous transformation with parameter λ such that

$$g_{\mu\nu} \to e^{\lambda} g_{\mu\nu}\,, \tag{5.7}$$

and the action scales in the same way

$$S \to e^{\lambda} S\,. \tag{5.8}$$

The small black hole near-horizon geometries sit in this class. We recall the argument of [24] that the velocity field of the scalars defines a Killing flow. The scaling of the action implies that the velocity squared on the scalar tangent space is independent of the scaling parameter λ:

$$\frac{d}{d\lambda}\left[G_{ij}\dot{\phi}^i\dot{\phi}^j\right] = 0\,. \tag{5.9}$$

If the first-order derivative of the scalars with respect to λ defines a vector field: (there is no explicit λ dependence)

$$\xi^i \equiv \frac{d\phi^i}{d\lambda}(\phi)\,, \tag{5.10}$$

then we get that the vector field ξ is Killing since

$$(\mathcal{L}_\xi G_{ij})\dot{\phi}^i\dot{\phi}^j = \frac{d}{d\lambda}\left[G_{ij}\dot{\phi}^i\dot{\phi}^j\right] = 0\,. \tag{5.11}$$

Hence we find that for scaling solutions, $\xi^i = \dot{\phi}^i$ is a Killing vector field:

$$\nabla_{(i}\xi_{j)} = 0\,. \tag{5.12}$$

The role of fake supersymmetry now becomes clear. When there is a fake superpotential

$$\xi^i = G^{ij}\partial_j W\,, \tag{5.13}$$

we find that also

$$\nabla_{[i}\xi_{j]} = \partial_{[i}\xi_{j]} = 0\,, \tag{5.14}$$

and hence $\xi^i\nabla_i\xi_j = 0$: the scalars satisfy a geodesic equation.

[4] From the extensive literature on scaling solutions it is clear that scaling solutions occur indeed as fixed point solutions to specific autonomous systems.

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This interesting connection between scaling solutions and geodesics on moduli space was first made in [28]. If we apply our conjecture that regular small black holes are always fake supersymmetric, then we find that *the scalar fields of regular small black holes follow a geodesic flow near the horizon.*

Even though the action (2.4) has a scalar potential, near the horizon of a small black hole the flow of the scalars naively seems free since they describe a geodesic. This "paradox" gets resolved when one realises that the geodesic curve is not an arbitrary geodesic. It is a very specific geodesic that is such that $V_{\rm BH}(\phi)$ scales in exactly the same way as the kinetic term. Since a generic potential added to some sigma model cannot have such a property, this simply means that scaling solutions require special potentials, consistent with the fact that small black holes are solutions to specific sets of charges.

6 Discussion

In this paper we analysed the difference between the first-order Hamilton-Jacobi equations (which always exist), and the standard fake supersymmetry equations. As we emphasized, not all extremal black hole flows and domain wall flows are fake supersymmetric. To our knowledge, the first example where this was shown appeared in [11]. This example describes an axion-dilaton domain wall. We have given simpler domain wall and black hole examples in the appendix, which are not fake supersymmetric.[5] A common feature of these examples is the presence of cyclic scalars for which the momentum is a non-zero constant. It is straightforward to demonstrate that this is the reason why there is no fake superpotential. The argument proceeds as follows. Denote the cyclic scalar as ϕ_c, its conjugate constant momentum as p_c and the other scalars as ϕ^i. From the Hamilton-Jacobi equation

$$p_c = \partial_{\phi_c} \mathcal{W}(\phi_c, \phi^i, U)\,, \tag{6.1}$$

we find that

$$\mathcal{W}(\phi_c, \phi^i, U) = \phi_c p_c + \tilde{\mathcal{W}}(U, \phi)\,, \tag{6.2}$$

and is hence never of the factorised form $\mathcal{W} \neq \mathrm{e}^{aU} W(\phi_c, \phi^i)$.

This does not necessarily imply that all solutions that fail to be fake supersymmetry are necessarily having cyclic scalars. That is why we formalised the condition for having fake supersymmetry in an elegant condition: a certain conserved charge $\mathcal{Q}$ has to vanish. This conserved charge is given by

$$\mathcal{Q} = \frac{4}{a}\dot{U} - \mathcal{W}\,, \tag{6.3}$$

with $\mathcal{W}$ Hamilton's characteristic function. Since we wrote down a very general Hamiltonian system (3.1), (3.2), (3.3) it is non-trivial that one can demonstrate the existence of a universal conserved quantity, different from the energy. The fact that we were able to

[5] Note that domain wall examples of sections A.1, A.2 are the same, up to signs, as the black hole examples of sections A.3, A.4. This was chosen on purpose to show the similarity between domain wall flows and extremal black hole flows.

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do this is because our Hamiltonian system is not completely arbitrary, the warp factor U appears in a specific way in the metric on configuration space (3.1) and it factors out in the potential (3.2).

With this precise formulation of fake supersymmetry (versus Hamilton-Jacobi) we were able to derive a physical meaning of fake supersymmetry for extremal black holes: it is a necessary condition for having physically acceptable solutions. For example, the models with cyclic scalars with non-zero momentum have the same kind of unphysical behavior of the black hole warp factor as for over-extremal solutions.[6] We have presented a rigorous proof of this for extremal black holes that have a finite horizon (large black holes), and we have argued, by taking a limit to vanishing horizon size, that the same applies to small black holes. However it would be more satisfying to have a rigorous proof for small black holes as well. Perhaps a useful playground to test this conjecture for small black holes is to investigate solutions in supergravity theories with symmetric scalar manifolds. Then there is full understanding of the space of solutions [29], simple integration algorithms have been developed [30] and the link with the first-order formalism has been investigated in detail, see for example [30, 31].

We furthermore pointed out that the near-horizon geometry of small black holes is of a universal "scaling form", which means it is characterised by an increase in symmetries that do not preserve the metric but rescale it with a constant factor. Using this insight, we have borrowed techniques from scaling solutions in cosmology [24, 25, 28] to uncover a general pattern in the behavior of the scalar fields when they flow towards the horizon of a small black hole. Instead of reaching specific constants, as they do for large black holes, the scalars start to follow a specific geodesic curve on moduli space. This is counter-intuitive since the scalars are subject to a non-zero potential. But just as for scaling attractors in cosmology one has to restrict to specific geodesic curves that are such that the potential scales in the same way as the scalar kinetic term. This is the extension of the well-known attractor mechanism for large black holes: the behavior of the scalars near the horizon is dictated by the increase in symmetry (scaling) and determined by the charges only (the geodesic is of a specific kind). It should be possible to generalise these results to general black p-branes in D dimensions, along the lines of [32].

If we come back to the main motivation for studying fake supersymmetry, which is finding first-order equations of motion to facilitate the search for solutions, then there is clearly no reason to worry about the existence of a fake superpotential, since the first-order Hamilton-Jacobi equations always exist locally. However, generically the problem of finding Hamilton's principal function that governs the first-order HJ flow equations is as difficult as integrating the second-order equations once. So from that point of view there is not much technical gain. What counts, in our opinion, is the physical meaning of having fake supersymmetry. For domain walls, it guarantees stability of the solution [1], and as we showed in this paper, for extremal black holes it is a necessary condition for having physically sound solutions.

[6] A prototypical example of this phenomenon would be extremal black holes in $\mathcal{N} = 2$ supergravity with non-constant hypermultiplet scalars.

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It would be interesting to apply the above insights to the Hamiltonian system defined by the so-called baryonic branch of the Klebanov-Strassler background [33]. For this system there seems an unsettled issue about the existence of first-order gradient flow equations [34, 35].

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A Examples

In this section, we consider some explicit examples to clarify the notions of the previous sections. We focus on domain wall solutions and extremal black holes, for two simple types of effective Lagrangians: the metric scalar U with its potential and a free scalar, and the metric scalar coupled to an axion-dilaton system, with a potential that depends on U and the dilaton.

A.1 Domain wall example 1: Λ plus free scalar

Consider gravity coupled to a negative cosmological constant Λ and a free scalar ϕ. The effective action is simply

$$S = \int \mathrm{d}z \big(4\dot{U}^2 - \dot{\phi}^2 - \mathrm{e}^{2\sqrt{3}U}\Lambda\big)\,. \tag{A.1}$$

The solution for ϕ is straightforward

$$\phi(z) = p\,z + \phi(0)\,, \tag{A.2}$$

with p a constant, describing the scalar's momentum. Without loss of generality we consider it to be non-negative. From the energy constraint we can integrate the equation for U once

$$\dot{U} = \pm\frac{1}{2}\sqrt{p^2 - \mathrm{e}^{2\sqrt{3}\,U}\Lambda}\,. \tag{A.3}$$

By redefining z we can always fix the sign above. We take the plus sign. The explicit solution is then given by

$$\mathrm{e}^{-\sqrt{3}U} = \frac{\sqrt{-\Lambda}}{|p|}\sinh\left(-\frac{\sqrt{3}}{2}|p|\,z\right). \tag{A.4}$$

When $p = 0$ we find

$$\mathrm{e}^{-\sqrt{3}U} = -\frac{\sqrt{-3\Lambda}}{2}\,z\,, \tag{A.5}$$

and after the coordinate transformation $z = -\frac{\sqrt{-3\Lambda}}{2}\,\rho^3$, we find the AdS_4 metric in standard Poincaré coordinates.

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To find $\mathcal{W}(U,\phi)$ for the general solution we use the above two integrations (A.2), (A.3)

$$\partial_\phi \mathcal{W} = -p\,, \qquad \partial_U \mathcal{W} = 2\sqrt{p^2 - \mathrm{e}^{2\sqrt{3}\,U}\Lambda}\,, \tag{A.6}$$

such that

$$\mathcal{W}(U,\phi) = -p\phi - 2pU + \frac{2}{\sqrt{3}}\sqrt{p^2 - \Lambda \mathrm{e}^{2\sqrt{3}U}} + \frac{2p}{\sqrt{3}}\ln\left(\sqrt{p^2 - \Lambda \mathrm{e}^{2\sqrt{3}U}} - p\right). \tag{A.7}$$

When $p = 0$ we indeed find the factorised form

$$\mathcal{W}(U,\phi) = \frac{2}{\sqrt{3}}\sqrt{-\Lambda}\,\mathrm{e}^{\sqrt{3}U}. \tag{A.8}$$

When $p \neq 0$, $\mathcal{W}$ does not factorize and there is no fake supersymmetry in the standard sense.

The constant of motion $\mathcal{Q}$ is

$$\mathcal{Q} = p\,\phi(0) - \frac{p}{\sqrt{3}}\ln(-\Lambda)\,. \tag{A.9}$$

It is clearly moduli dependent for $p \neq 0$ (appearance of $\phi(0)$). Following the arguments of section 4, we cannot set $\mathcal{Q} = 0$ in a solution-independent way and this explains why $\mathcal{W}$ does not factorize.

A.2 Domain wall example 2: the Sonner-Townsend model

The first known example of a domain wall (and cosmological) solution, that can not be derived from a fake superpotential in the usual sense, was found by Sonner and Townsend in [11]. The domain wall solution is a scaling solution and therefore it represents a fixed point of a more general domain wall flow, which is not known analytically. We briefly review this solution and simplify its presentation.

The essential ingredient is again a free scalar, but this time it is an axionic field σ that couples to the dilaton ϕ, as follows

$$\mathcal{L} = 4\dot{U}^2 - \dot{\phi}^2 - \mathrm{e}^{\mu\phi}\dot{\sigma}^2 - \Lambda \mathrm{e}^{2\sqrt{3}U + \lambda\phi}, \tag{A.10}$$

where the constants μ, λ define the couplings of the model. The target space spanned by ϕ and σ is the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The $\mathrm{SL}(2,\mathbb{R})$ symmetry of the kinetic term is broken by the dilaton potential. Since the domain wall solution is a scaling solution, the scalars follow a Killing flow, but the lack of a fake superpotential implies that the Killing flow is not a geodesic flow, as explained in the section 5.

The scaling Ansatz is

$$U = a\ln z + U_0\,, \qquad \phi = b\ln z + \phi_0\,, \tag{A.11}$$

where a, b, U_0, ϕ_0 are constants. The axion equation of motion implies

$$\dot{\sigma} = \frac{d}{z^{\mu b}}\,, \tag{A.12}$$

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with d a constant. If we furthermore demand the scaling condition, which means that the σ-kinetic term scales similar to the other terms in the actions, we can fix b

$$b = \frac{2}{\mu}\,. \tag{A.13}$$

The U and ϕ equations of motion lead to three algebraic relations amongst the four remaining integration constants a, d, U_0, ϕ_0:

$$a = -\frac{1}{\sqrt{3}}\left(1+\frac{\lambda}{\mu}\right), \tag{A.14}$$

$$d^2 = 4\mathrm{e}^{-\mu\phi_0}\left(-\frac{1}{\mu^2}+\frac{\lambda}{3\mu}\left(1+\frac{\lambda}{\mu}\right)\right), \tag{A.15}$$

$$4\left(1+\frac{\lambda}{\mu}\right)+3\Lambda\mathrm{e}^{2\sqrt{3}U_0+\lambda\phi_0}=0\,. \tag{A.16}$$

The Hamiltonian constraint is automatically fulfilled with these relations. These relations imply certain sign restrictions on the possible choices for μ, λ, Λ, which we do not discuss.

As in the previous example we can show that a fake superpotential ($\mathcal{W}$ factorizes as $\mathcal{W} = e^{\sqrt{3}U}W(\phi,\sigma)$) requires one of the integration constants to be zero ($d = 0$) and the system collapses to the single-dilatonic domain wall flow. The expression for $\mathcal{W}$ for the general solution with $d \neq 0$ is quite involved and not that insightful.

A.3 "Black hole" example 1: Maxwell plus free scalar

Consider Einstein-Maxwell theory with a free scalar added to it. Electric solutions have the following expression for the field strength $F = \mathrm{d}(\chi(\tau)\mathrm{d}t)$ where χ is the electric potential. Its equation of motion is $\mathrm{e}^{-2U}\dot{\chi} = q$, where q is the electric charge. This means that the black hole effective potential V_{BH} is simply a constant $V_{\mathrm{BH}} = -q^2$. The effective action is completely analogous to that of section A.1:

$$S = \int \mathrm{d}z\big(4\dot{U}^2+\dot{\phi}^2+q^2\mathrm{e}^{2U}\big)\,. \tag{A.17}$$

The solution for ϕ is

$$\phi(\tau) = p\tau+\phi(0)\,, \tag{A.18}$$

where p is a constant which we take to be non-negative without loss of generality. From the energy condition we have that,

$$2\dot{U} = \pm\sqrt{q^2\mathrm{e}^{2U}-p^2}\,. \tag{A.19}$$

The choice of plus and minus sign is a gauge fixing and will determine how the radial variable τ is related to the usual radius r. The solution (for the plus sign) is

$$\mathrm{e}^{-U} = \frac{|q|}{p}\sin\left(\frac{1}{2}\,p\,\tau\right). \tag{A.20}$$

Let us verify that $p = 0$ gives the standard electric extremal RN black hole. The solution for U becomes

$$\mathrm{e}^{-U} = -\frac{|q|}{2}\tau+1\,, \tag{A.21}$$

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where we fixed an integration constant to be 1 for simplification. The coordinate transformation $r = \pm\left(\frac{1}{\tau} - \frac{|q|}{2}\right)$, gives the following metric

$$\mathrm{d}s^2 = -\left(1 + \frac{|q|}{2r}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{|q|}{2r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{A.22}$$

which we indeed recognize as the standard text book expression for the extremal RN black hole.

From the first-order equations we can find $\mathcal{W}$ to be

$$\mathcal{W}(\phi, U) = p\,\phi + 2\sqrt{q^2 \mathrm{e}^{2U} - p^2} - 2p \arctan\left(\frac{1}{p}\sqrt{q^2 \mathrm{e}^{2U} - p^2}\right). \tag{A.23}$$

In the limit $p \to 0$ this consistently reduces to

$$\mathcal{W}(\phi, U) = 2q\mathrm{e}^{U}. \tag{A.24}$$

For $\mathcal{Q}$ we find

$$\mathcal{Q} = p\,\phi(0) - 2pk\pi\,, \qquad k \in \mathbb{Z}\,. \tag{A.25}$$

This is only manifestly zero when $p = 0$. When $p \neq 0$ it depends on the modulus at infinity $\phi(0)$ and there is no fake supersymmetry.

It is not difficult to verify that the solution has a naked singularity. Moving from $\tau = 0$ at spatial infinity to finite τ we hit a singularity of the metric at $\tau = \frac{2\pi}{|p|}$. Note that this metric singularity is not lightlike (it is not a horizon). One can easily verify that the curvature invariant $R_{abcd}R^{abcd}$ blows up near the metric singularity.

A.4 "Black hole" example 2: dilatonic black hole plus axion

For black hole flows we can have the exact analogy with the Sonner-Townsend domain wall scaling solution of section A.2. This can be engineered by extending the usual dilaton black hole solution with an axion σ. The action is

$$\mathcal{L} = 4\dot{U}^2 + \dot{\phi}^2 + \mathrm{e}^{\mu\phi}\dot{\sigma}^2 - q^2\mathrm{e}^{2U+\lambda\phi}. \tag{A.26}$$

The derivation follows exactly the same rules as with the domain wall. The scaling Ansatz is again

$$U = a\ln\tau + U_0\,, \qquad \phi = b\ln\tau + \phi_0\,, \qquad \dot{\sigma} = \frac{d}{\tau^2}\,. \tag{A.27}$$

Where the equations of motion imply the following relations between the integration constants:

$$a = -\left(1 + \frac{\lambda}{\mu}\right), \tag{A.28}$$

$$b = \frac{2}{\mu}\,, \tag{A.29}$$

$$d^2 = -4\mathrm{e}^{-\mu\phi_0}\left(\frac{1}{\mu^2} + \frac{\lambda}{\mu}\left(1 + \frac{\lambda}{\mu}\right)\right), \tag{A.30}$$

$$4\left(1 + \frac{\lambda}{\mu}\right) + q^2\mathrm{e}^{2U_0+\lambda\phi_0} = 0\,. \tag{A.31}$$

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This is, as far as we know, a new solution, but by itself does not represent a black hole solution since it is not asymptotically flat. Being a scaling solution one might expect that it describes the near horizon region of a small black hole. However, our general theorem about the absence of fake supersymmetry implies that this solution cannot be interpreted that way. If one extends the solution to the general solution that interpolates to this scaling solution, one will encounter naked singularities in the bulk.

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