DOCTORAL DISSERTATION
Doctoral Program in Physics (34th cycle)

On AdS_4 Holography
Towards applications to 2+1 dimensional graphene-like systems

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Politecnico di Torino, 2022

Abstract

This thesis focuses on the study of rigid supersymmetry and supergravity theories with the aim of exploiting them in the analysis of 2+1 dimensional condensed matter systems, like graphene.
In the first part, we perform a holographic analysis, in the context of the AdS/CFT conjecture, of the four dimensional pure $\mathcal{N} = 2$ anti-de Sitter supergravity. It is carried out by including all the contributions coming from fermionic fields and studying the behaviour of the bulk fields and parameters at the boundary of asymptotically locally AdS spacetimes. Furthermore, we construct the corresponding currents of the conformal field theory and show, by following the prescription of the AdS/CFT correspondence, that they are in fact conserved at the quantum level.
In the second part of the thesis, inspired by a duality discovered by Kapustin and Saulina, we construct the superspace Lagrangian for an $\mathcal{N} = 4$ rigid supersymmetric theory of hypermultiplets, whose superspace isometry is encoded in the $\mathrm{D}^2(2, 1; \boldsymbol{\alpha})$ exceptional supergroup. After projecting the Lagrangian to spacetime, in order to enlight some features of the model, we perform two different twists on the spinorial fields of the hypermultiplets. The first one relates our work to that of Kapustin and Saulina, generalising it to the case in presence of a cosmological constant. The second allows to obtain a Lagrangian whose structure admits the "unconventional supersymmetry".

Contents

Chapter 1

Introduction

Science has its motive power in many aspects of human being: wild curiosity for what surrounds him, attempt to control Nature and its manifestations, a strong inclination in looking for recursive elements inducing general rules, and surely many other elements better understood by Humanities researchers. Among the large amount of achievements reached because of these fundamental features, we can certainly include the mathematical language, whose origin is lost in the mists of time. Still today people debate about its essence: for someone it is only a human mental language, casually capable of grasping some aspects of Universe phenomena, for other ones the Book of Nature is really written in a mathematical language. Despite this very interesting controversy, throughout history mathematics has primarily proven its great worth as an instrument to describe and manipulate the world. Indeed, we have evidence of its use since the dawn of human civilisation: from applications to engineering problems to record time and formulate calendars, not forgetting to mention its exploitation for purposes of taxation, commerce and trade.

Whereas geometrical beauty and harmony basically preempted the data of reality in the analysis of Nature until the introduction of scientific method, the relationship has overturned with the emergence of science in its modern fashion, and mathematics has started to come after the observation of phenomena and collection of experimental data, in an effort of modelling the latter by considering its founding properties.

During its journey, physics, considered by the most as the "queen of sciences", has ventured further and further into the comprehension of infinitely small and infinitely large. However, because of the increasing velocity of the progress of knowledge, already in the twentieth century there has been a sort of trend reversal: mathematics, with its elegance and principles of symmetry, has led theoretical physicists towards new models in an attempt to go beyond the acquired understanding, partly due to the absence of suitable technologies to obtain new experimental data at the frontiers of research, partly to guide experiments in testing the results of the new theories. For instance, by following this path, the procedure of spontaneous symmetry breaking and the Higgs model were formulated in the framework of gauge invariant quantum field theories in 1964, with the aim of understanding theoretically the mechanism responsible of the non vanishing masses for vector bosons in the Standard Model. In this specific case, the principle of gauge symmetry brought physicists to suggest the existence of a new particle, the Higgs boson, effectively detected about fifty years later at CERN, whose discovery completed the so-called Standard Model of particle physics,

which describes three of the four fundamental forces of Nature (electromagnetism, weak and strong nuclear forces).
In 1915, Einstein, driven by another symmetric postulate called *equivalence principle*, built the theory of General Relativity and improved our comprehension of the last fundamental interaction, namely the gravitational one, until to that time explained through the Newtonian theory. General Relativity cured some discrepancies already known between theory and experimental data, for instance the "anomalous" precession of the perihelion of Mercury, but it also predicted some new effects never observed before, like the light rays bending and the gravitational waves, the latter being detected *only* one hundred years after the formulation of Einstein's theory in 2016 by the Advanced LIGO (Laser Interferometer Gravitational-Wave Observatory).

These two models represent our best understanding of the physical world today, from its microscopic phenomena to cosmological manifestation. However, they are not complete. In fact, the Standard Model has an amount of problems which goes from phenomena not predicted from the theory (dark energy, dark matter, neutrino masses and matter-antimatter asymmetry) to experimental results not explained, like the significant discrepancy between the theoretical and the measured value of muon's anomalous magnetic dipole moment (called, in short, "muon $g-2$").
Thus, in light of these mismatches, an extension of Standard Model is deemed necessary, but it also brings along some theoretical problems, which physicists have to deal with. For instance, in quantum field theories, coupling constants and masses are not generally constant. Indeed, they are said to run with energy, that means their value depends on the energy of the considered process. The Standard Model is a gauge theory based on the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, which thus has three independent coupling constants. If a unifying and more fundamental theory, built on a gauge group which includes $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as subgroup, exists, it should be characterised by an energy scale where the three coupling constants, following their renormalisation group evolution, coincide. On the contrary, coupling constants in Standard Model never meet at a common energy scale. On the other side, when the running of masses is considered, in particular that of the Higgs boson, one would expect that the large quantum contributions to the latter would inevitably make the mass huge, unless an incredible fine-tuning cancellation between the quadratic radiative corrections and the bare mass happens. The worry concerns future theories of fundamental particles, where an excessive fine-tuning of parameters could be carried out. In the community of theoretical physics, this is called *hierarchy problem*.
A further complication comes from the strong interaction sector. In particular, the Lagrangian of Standard Model would allow for a CP (charge conjugation+parity transformation) violating term. Experimentally, however, no such violation has been found, implying that the coefficient of this term is very close to zero, which can be regarded as a peculiar fine-tuning. This issue is known as "Strong CP problem".

On the other hand, there are no experimental reasons to improve our gravitational description, yet. However, General Relativity can be thought as the classical level of a more fundamental quantum theory, and once the second quantisation procedure is implemented on Einstein's theory, its non renormalisability nature appears, in contrast with Standard Model. This aspect can immediately be evinced by considering the negative mass dimension of the gravity coupling constant $\kappa = \sqrt{8\pi G_N}$, G_N being Newton's constant. The latter feature has a dramatic impact on the superficial degree of divergence of correlation functions. Indeed, the number of divergent Feynman diagrams gradually increases with loops and, in turn, a growing amount of counterterms has to be disposed to manage the ultraviolet poles. Therefore, the quantum theory of General Relativity would ultimately not be predictive.

Possible extensions including both the Standard Model and General Relativity in the framework of quantum field theories must necessarily undergo to the limits imposed by two famous theorems. The former [1], formulated by Coleman and Mandula, puts restrictions on the form of the algebra underlying an interacting relativistic quantum field theory in the presence of massive particles and allowing only bosonic generators. In this case, symmetries are limited to the direct sum of the Poincaré transformations and, possibly, a finite-dimensional compact Lie algebra, representing the so-called internal symmetries. If fermionic generators are allowed, the situation is governed by Haag-Łopuszański-Sohnius theorem [2], and a graded algebra, also called "superalgebra", with spinorial charges Q^i_α can be considered, where α is a spinorial spacetime index and $i = 1, \ldots, \mathcal{N}$ labels the number of supersymmetries in the theory. Therefore, this kind of theories realises the most general model possible within the framework of the few assumptions made in the hypotheses of Coleman-Mandula and Haag-Łopuszański-Sohnius theorems. Supersymmetry (SUSY) is a dynamical theory based on a superalgebra. Due to the structure of the supersymmetric algebra and field transformations, every bosons is required to have a fermionic supersymmetric partner and viceversa: the two classes of elementary particles found in Nature are, in a sense, unified.

Let us go back to the aforementioned issues of Standard Model. The introduction of supersymmetry in the construction of a unified quantum theory brings several benefits, which can solve or, at least, soften these problems. As a first example, the radiative corrections to the Higgs mass produced by the renormalisation procedure are logarithmically divergent with the energy in the Minimal Supersymmetric extension of Standard Model (MSSM), rather than quadratically. This feature is caused by the cancellation occurring between the loops contributions of every particle and its super-partner, and significantly improves the situation concerning the hierarchy problem.

Another important result of the MSSM concerns the running coupling constants of electomagnetism, weak and strong nuclear forces. In fact, when their values are extrapolated to high energies, they join together at a scale of about $2 \cdot 10^{16}$ GeV.

Eventually, since the MSSM contains new particles, it is possible to speculate on suitable candidates for dark matter.

The parameters of global supersymmetry transformations are constant anti-commuting Majorana spinors ϵ_α. When the latter are promoted to be arbitrary spinorial functions of spacetime (i.e. local parameters), gravity has to be included through the introduction of a dynamical metric field $g_{\mu\nu}$, associated to the spin 2 particle (*graviton*). Indeed, the anticommutator of two local supersymmetry transformations closes on infinitesimal diffeomorphism on spacetime, which is the symmetry principle underlying General Relativity. Theories featuring local supersymmetry are called supergravities (SUGRA).

The simplest model in this setup is the $\mathcal{N} = 1$ (also said *minimal*) pure supergravity in

four dimensions, constructed with a gravitational multiplet only. The latter is composed by the previously mentioned graviton and a spin 3/2 vector-spinor field called *gravitino*, usually denoted by Ψ^α_μ.
There exist several generalisations to the case just described, which can in turn be mixed up. Firstly, theories which admit a vacuum different from the Minkowski spacetime, as de Sitter [3] or anti-de Sitter backgrounds. Secondly, theories with more than one supersymmetry or in higher dimensions can be considered, the former being defined *extended* supergravities. A further possibility is represented by the coupling of different kind of multiplets (hyper, vector, ...) to the gravitational one, creating richer structures. Eventually, more realistic phenomenological models can be obtained in the broad world of *gauged* supergravities[1], derived by promoting a suitable global symmetry group of the theory to a local symmetry group, gauged by some vector fields already present in the multiplets. This procedure is often exploited to introduce a scalar potential or a cosmological constant in supersymmetric models.

$\mathcal{N}$-extended supergravities contain $\mathcal{N}$ gravitini, each of them associated to a supercharge. Furthermore, the more the number of supersymmetries increases, the larger the supermultiplets are. This poses a constraint on the amount of allowed supersymmetries for consistent theories coupled to gravity. In particular, in a two-derivative field theory, the maximum spin of a particle has to be 2, which in turn implies $\mathcal{N}_{\text{max}} = 8$ in four dimensions. The maximal ungauged supersymmetric gravity theory $\mathcal{N} = 8$ is unique, namely the field content and interactions are completely fixed by supersymmetry, and there are indications that it could be perturbatively finite, its finiteness being tested until four loops [5][2]. However, while rigid supersymmetry makes radiative corrections less severe, the renormalisibility is not guaranteed for quantum theory of supergravities and, in general, they suffer ultraviolet divergences as General Relativity.
For the reasons just mentioned, as the time went on, theoretical physicists started to think that supergravity theories might be the low-energy limit of a more fundamental UV complete quantum theory, meant to unify the known interactions to date: the superstring theory was recognised as a suitable candidate for that purpose. In this framework, the elementary objects are closed and opened strings of finite length $\ell_s = \sqrt{\alpha'}$ and tension $T \sim 1/\ell_s$, whose oscillation modes give rise to particles and interactions. Indeed, the spectrum of the closed string accommodates the graviton, whereas that of the open string provides gauge vectors. The superstring theory is consistently constructed as a conformal two dimensional sigma-model (*worldsheet*) embedded in a ten dimensional spacetime (*target space*). A perturbation expansion of the theory can be expressed in terms of the string coupling constant g_s, which is not a free parameter of the model, but a dynamical quantity fixed by the vacuum expectation value of the dilaton field on the chosen background. The low-energy limit of the superstring theory is obtained by taking the limit $\ell_s \to 0$ (or, equivalently, by sending the tension to infinity). In this regime, strings appear as point-like objects (i.e. particles) and their dynamics is captured by a ten dimensional supergravity, which provides an effective theory. A solution of the latter theory is the target space where the string model

[1]A beautiful and up-to-date review on this topic can be found in [4].
[2]Further developments on the theory at more than four loops can be found in [6].

is embedded. A classical supergravity description can be exploited if, besides the string length limit, one also restricts at tree-level in the coupling constant g_s.
Five different models were proven to be consistently formulated in the framework of the superstring theory on different backgrounds: type I, type IIA, type IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$. This fact could be thought at odds with the initial hope of finding a unique unifying quantum theory. However, in the 90's, it turned out that these five versions could be related each other through some correspondences, called T, S and more generally U-dualities. The former concerns the compactifications which can be performed on the five models and, for instance, allows to map type IIA into type IIB. The second is a strong/weak coupling duality, linking two theories with coupling constant g_s and $1/g_s$. For example, the S-duality relates the strong and the weak regime of type IIB. Lastly, U-dualities include a mixing of the previous two.
In this scenario, an idea took hold: the different superstring theories could be effective descriptions of the same microscopic degrees of freedom and specific limits of a more general theory, the *M-theory*. To date the fundamental degrees of freedom of the latter are unknown, but, in the low-energy regime, M-theory falls into the unique eleven dimensional supergravity [7], whose fields content has already been discovered. Furthermore, the discovery of the dualities has brought to light the existence, besides the strings, of other fundamental objects in superstring theory, that are solitonic extended objects called D-branes.
Although until today no experimental support exists for phenomenological models in the setup of superstring theory, a renewed interest for this mathematical framework sparked in 1997, when Maldacena proposed the AdS/CFT conjecture [8] for the first time, which would have been revealed one of the main developments of the last decades in theoretical physics. It states that there exists a correspondence between a theory of quantum gravity, formulated in terms of string theory or M-theory, defined on a background given by the product of an anti-de Sitter space and a compact manifold, and a particular supersymmetric quantum field theory called superconformal field theory, living on the boundary of anti-de Sitter.
The interesting feature of this conjecture is the fact that, for certain values of the parameters, one has a duality between a strongly coupled field theory and a weakly coupled gravity theory, which translates in the possibility of analysing new unexplored regime in quantum field theories, as the strongly coupled one. Indeed, it is possible to get information on the latter by doing classical computations in (super)gravity and using the rules of the correspondence. We will further elaborate on the original formulation of the AdS/CFT conjecture in the following of this Introduction.

The thesis is organised as follows.
In the rest of this introductory part we illustrate the so-called rheonomic approach, a useful tool for the construction of supergravity theories (Section 1.1) and we discuss the AdS/CFT conjecture in the case of its best studied example (Section 1.2). Furthermore, we briefly introduce the wide topic of Analogue Gravity, mainly focusing on the 2+1 dimensional condensed matter systems, like graphene, and we describe the main features of unconventional supersymmetry, a model capable to connect condensed matter and high energy physics (Section 1.3). These topics are the fundamental ingredients exploited for

our analysis in the core of the thesis.
In Chapter 2 we develop in detail the holographic framework for an $\mathcal{N} = 2$ pure anti-de Sitter supergravity model in four dimensions, including all the contributions from the fermionic fields, by constructing the corresponding superconformal currents and showing that they satisfy the related Ward identities.
In Chapter 3 we develop a three dimensional $\mathcal{N} = 4$ theory of rigid supersymmetry describing the dynamics of a set of hypermultiplets on a curved AdS_3 worldvolume background, whose supersymmetry is captured by the supergroup $\mathrm{D}^2(2, 1; \boldsymbol{\alpha})$. Furthermore, to unveil some remarkable features of this model, we perform two twists, involving the $\mathrm{SL}(2, \mathbb{R})$ factors of the theory.
Eventually, we conclude both Chapter 2 and 3 with comments on the obtained results and a discussion of the possible future developments for the works carried out during the PhD.

1.1 Rheonomic approach to supergravity theories

The construction of global and local supersymmetric Lagrangians has been one of the main issue to deal with in the community of theoretical physicists after the introduction of the concept of supersymmetry. Some methods have been developed during the years, like Noether coupling and superfields ones. The former is based on a recursive lengthy algorithm, whose cumbersome increases significantly as one considers more supersymmetric or higher dimensional theories. The difficulties of superfields construction reside in the fact that it is scarcely understood for models with a generic number of supersymmetries.
In this Section, we will focus on describing the basic facts about one of these methods, called *rheonomic* or “geometric” approach, introduced in [9,10]. Its major strengths are the clarity of the basic principles on which it is established and the automated algorithm exploited to build Lagrangians. Through the rheonomic approach, we will be able to write the action for the four dimensional anti-de Sitter $\mathcal{N}=2$ theory in Chapter 2 and that for hypermultiplets on $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superspace in Chapter 3, despite the latter will be characterised by rigid supersymmetry. Indeed, the rheonomic approach, which was originally formulated for supergravity, turns out to be, bearing in mind some cautions, a suitable procedure also for global supersymmetric models.

Let us consider Lie supergroups G whose superalgebras $\mathfrak{g}$ can be split as

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{i}\oplus\mathfrak{o}\,, \tag{1.1.1}$$

where $\mathfrak{h},\mathfrak{i},\mathfrak{o}$ are, respectively, the Lorentz bosonic subalgebra (plus, possibly, a further internal algebra), the bosonic subspace of translations and the fermionic subspace of supersymmetric transformations. We can deduce the latter partition from the following relations

$$\begin{aligned}&[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}\,,\quad[\mathfrak{h},\mathfrak{i}]\subset\mathfrak{i}\,,\quad[\mathfrak{h},\mathfrak{o}]\subset\mathfrak{o}\,,\\&[\mathfrak{i},\mathfrak{i}]\subset\mathfrak{h}\,,\quad[\mathfrak{i},\mathfrak{o}]\subset\mathfrak{o}\,,\quad\{\mathfrak{o},\mathfrak{o}\}\subset\mathfrak{h}\oplus\mathfrak{i}\,,\end{aligned} \tag{1.1.2}$$

fulfilled by $\mathfrak{h},\mathfrak{i},\mathfrak{o}$. Furthermore, $\mathfrak{h}$ can be recognised as the gauge subalgebra of the model, $\mathfrak{i}$ as the inner space (indeed its dimension is equal to the number of spacetime directions), whereas $\mathfrak{o}$ corresponds to the outer space. In fact, $\mathfrak{h}\oplus\mathfrak{i}$ is the bosonic subalgebra of $\mathfrak{g}$. Since the structure just depicted is always verified in the known theories, we narrow the field of superalgebras to those which satisfy these requirements.
A Lie superalgebra can also be expressed, in the Maurer-Cartan formalism, in terms of 1-forms σ^A, dual to the generators T_A of $\mathfrak{g}$,

$$\sigma^A\left(T_B\right)=\delta^A_B\,,\qquad \mathrm{d}\sigma^A(T_B,T_C)=-\sigma^A\left([T_B,T_C]\right)\,, \tag{1.1.3}$$

through the Maurer-Cartan equations

$$\mathrm{d}\sigma^A+\frac{1}{2}C^A{}_{BC}\,\sigma^B\wedge\sigma^C=0\qquad A,B,C=1,\dots,\mathrm{dim}\mathfrak{g}\,, \tag{1.1.4}$$

whose left-hand side is recognised as the curvatures (or field strengths) R^A of σ^A. The latter equations are relations dual to the superalgebra (anti)commutators and, with the

additional requirement $\mathrm{d}^2 = 0$, enforces Jacobi identities on the structrure constants.
Lie superalgebras can describe vacuum solutions of gauge theories because of their "rigidity", due to the fact that their (pseudo-)Riemannian geometry is fixed in terms of their structure constants $C^A{}_{BC}$. Indeed, it is clear from the definition of the curvatures that they vanish for Lie superalgebras. The 1-forms associated to the generators of $\mathfrak{i}$ and $\mathfrak{o}$ are usually identified, respectively, with the vielbein V and the gravitini Ψ.

Until now we have considered, referring to [10] classification, group manifolds G. However, whether one wants to construct a model where the structure of superspace is dynamical, "soft" group manifolds $\tilde{G}$ have to be introduced. These are "softened" versions of the previous G, namely they closes with structure functions depending on the spacetime point, rather than structure constants, and $\tilde{G}$ is locally diffeomorphic to G. In turn, soft 1-forms μ^A, dual to the soft generators, have to be considered, but, in this case, the Maurer-Cartan equations are no longer automatically satisfied and become

$$R^A = \mathrm{d}\mu^A + \frac{1}{2} C^A{}_{BC}\, \mu^B \wedge \mu^C \neq 0 \,. \tag{1.1.5}$$

The curvatures R^A obey to the so-called Bianchi identities (also known as integrability conditions), which are a direct follow-up of d^2-nilpotency

$$\nabla R^A \equiv \mathrm{d}R^A + C^A{}_{BC}\, \mu^B \wedge R^C = 0 \,, \tag{1.1.6}$$

∇ being the G-covariant derivative. The 1-forms μ^A are identified with the fields of the theory in the gravitational multiplet.
We are now able to interpret the gauge transformation laws of μ^A generated by an infinitesimal vector $\epsilon = \epsilon^A T_A$ as diffeomorphisms

$$\delta\mu^A \equiv \pounds_\epsilon \mu^A = \left(\iota_\epsilon \mathrm{d} + \mathrm{d}\iota_\epsilon\right) \mu^A = \nabla\epsilon^A + \iota_\epsilon R^A \,, \tag{1.1.7}$$

where $\pounds_\epsilon$, ι_ϵ denote, respectively, the Lie derivative and the contraction along ϵ. If we want to reproduce the usual gauge theories, where infinitesimal gauge transformations are given by covariant derivative of the gauge parameters, the curvatures must satisfy the *horizontality* condition

$$\iota_\epsilon R^A = 0 \,, \tag{1.1.8}$$

which can also be read as the fact that the curvatures have non vanishing components only on spacetime directions of the softened Lie supergroup.
Conversely, when gravitational theories are considered, the horizontality property for some of the curvatures (particularly, those associated to translations and local supersymmetry transformations) no longer holds and the transformation law for the gauge potentials (vielbein and gravitino), which can be still expressed as a Lie derivative, is no longer a gauge transformation. On the contrary, local Lorentz transformations and other internal symmetries keep their gauge status: from this one can understand the reason why $\mathfrak{h}$ is called *gauge* subalgebra.

To summarise the discussion[3] up to here in few words, general relativity and supergravity can not be interpreted as genuine gauge theories.

In light of the previous analysis, the geometric approach to supergravity treats supersymmetry transformations as diffeomorphisms in fermionic directions on a manifold having both bosonic and fermionic dimensions. Despite the benefits it brings in constructing Lagrangians, performing computations and deriving transformation laws for the fields, this formulation can in principle introduce new extra dynamic information with respect to the standard spacetime procedure. Indeed, physical actions are formulated on bosonic manifolds and the same happens in the rheonomic approach, where a bosonic submanifold of the entire superspace is determined by choosing $\theta^\alpha = \mathrm{d}\theta^\alpha = 0$, θ^α being the fermionic coordinates. Thus, a procedure thanks to which spacetime information are extended to the whole superspace without introducing new physical content is needed. This is called *rheonomic extension mapping* and consists in the requirement that all the fields have to be uniquely determined in terms of spacetime quantities. We will further elaborate on this point in a moment.

Besides the latter condition, rheonomic Lagrangians have to satisfy further criteria. First of all, they must possess a number of symmetries: coordinate transformations, rigid scale and $\mathfrak{h}$-gauge invariances. The former is a well-known feature inherited from general relativity.
The second one is due to the fact that the Lie superalgebra structure is left unchanged by the following transformation generated by a scaling factor $w \neq 0$

$$T_{\mathfrak{h}} \to T_{\mathfrak{h}}, \quad T_{\mathfrak{i}} \to w^{-1} T_{\mathfrak{i}}, \quad T_{\mathfrak{o}} \to w^{-1/2} T_{\mathfrak{o}} \,. \tag{1.1.9}$$

As a consequence, the dual 1-forms $\mu^{\mathfrak{h}}$, $\mu^{\mathfrak{i}}$, $\mu^{\mathfrak{o}}$ and the respective curvatures $R^{\mathfrak{h}}$, $R^{\mathfrak{i}}$, $R^{\mathfrak{o}}$ transform as

$$\begin{aligned} &\mu^{\mathfrak{h}} \to \mu^{\mathfrak{h}}, \quad \mu^{\mathfrak{i}} \to w\mu^{\mathfrak{i}}, \quad \mu^{\mathfrak{o}} \to w^{1/2}\mu^{\mathfrak{o}}, \\ &R^{\mathfrak{h}} \to R^{\mathfrak{h}}, \quad R^{\mathfrak{i}} \to wR^{\mathfrak{i}}, \quad R^{\mathfrak{o}} \to w^{1/2} R^{\mathfrak{o}} \,. \end{aligned} \tag{1.1.10}$$

Lagrangians will have to be constructed by assembling the pieces in order to obtain an expression with the right scaling under this map.
The latter symmetry corresponds to the requirement that Lagrangians must be the same after a transformation under the real gauge algebra of the theory, namely $\mathfrak{h}$, which usually is identified as a Lorentz algebra plus an internal one.
Another issue concerns the Hodge dual operator. It is usually exploited in the construction of kinetic terms (for instance for scalar fields), but when the Lagrangian is formulated as

[3]Since we are more interested in showing results and present principles and methodologies to build rheonomic Lagrangians than discussing the accurate mathematical structure in which the subject has been formalised, we refrain from introducing the framework of principal bundles (for an exhaustive discussion, see [11]). In fact, we would have needed the latter since the moment in which we wanted to implement dynamics within the Lie superalgebras setup (i.e. to obtain usual gauge theories). Furthermore, when one tries to correctly reproduce the physical transformation laws of gravitational models, the mechanism of principal bundle breakdown has to be taken into account. A description of this formalism in the context of fibre bundles was developed in details in Section 5 of [12].

a D-form in superspace, as in the case of the standard[4] rheonomic approach, Hodge dual operation has not a mathematical definition. This problem can be overcome in most of the cases through a simple trick, that is by introducing an auxiliary field whose equations of motion reproduce the original expression with the Hodge dual operator. The reader can find an explicit example in Chapter 3.
Eventually, $R^A = 0$ must be included in the set of solutions of the theory as a particular case in which the soft group manifolds fall into the "rigid" setup of Lie superalgebras, in which case all the invariances of the theory are realised as global symmetries.

The aforementioned principles brought the authors of [10] to define five building rules for the construction of rheonomic Lagrangians.

A) Geometricity: Lagrangians should be D-forms, D being the spacetime dimension, constructed out of the soft 1-forms μ^A and, possibly, of some 0-forms, as for the implementation of the Hodge dual trick, by using only the diffeomorphic invariant operators d and $\wedge$, which are respectively the exterior derivative and the wedge product. Furthermore, we say that a Lagrangian is strongly geometrical whether it includes only 1-forms, otherwise it is named geometrical "tout-court". In fact, 0-forms are mainly exploited when one couples matter multiplets (spin 0 and 1/2 fields) with the gravity one or when spin 0 and 1/2 appears directly in the gravitational multiplet. A particular example where we explicitly build a Lagrangian for hypermultiplets, although without coupling them to gravity, will be shown in Chapter 3.
Starting from a geometrical Lagrangian $\mathcal{L}$, the action is obtained as its integral on a D dimensional bosonic hypersurface M immersed in the superspace $\mathcal{M}$

$$I = \int_{M \subset \mathcal{M}} \mathcal{L} \,. \tag{1.1.11}$$

Furthermore, rule **A)** implies that strongly geometrical Lagrangians should be polynomials in the curvature R^A and the degree of the polynomial in R^A is at most $\left\lfloor \frac{D}{2} \right\rfloor$ (i.e. integer part of $\frac{D}{2}$) since $\mathcal{L}$ is a D-form. Regarding ordinary D dimensional supergravity theories, Lagrangians are always polynomials of degree 2 in R^A, otherwise one would get spacetime equations of motion of order greater than 2.

B) $\mathfrak{h}$-gauge Invariance: Lagrangians must be $\mathfrak{h}$-invariant, namely every term should be a scalar under $\mathfrak{h}$ gauge transformations. This can be further paraphrased by stating that $\mathfrak{h}$ indices have to be saturated.
Moreover, coefficients in front of every term must be constructed only with $\mathfrak{h}$-invariant tensors and it can be proved that bare $\mu^{\mathfrak{h}}$ 1-forms can appear only if their global coefficient is a closed (D-1)-form.

C) Homogeneous Scaling Law: Each term in D dimensional Lagrangians must scale under the scaling law (1.1.10) as $\left[w^{D-2}\right]$, which corresponds to the scale-weight of the

[4]In fact, a new formalism to define the Hodge dual operator and extend the action integral to superspace has been developed in [13, 14] during the last years.

Einstein term. This rule can also be viewed as a request of independence of both equations of motion and Bianchi identities from the scaling factor w.

D) Vacuum Existence Field equations of Lagrangians should admit the solution $R^A = 0$ and, therefore, be at least linear in the curvature 2-forms.

E) Rheonomy: The curvature 2-forms can be projected along the soft 1-forms as

$$R^A = R^A{}_{BC}\, \mu^B \wedge \mu^C \,. \tag{1.1.12}$$

The intrinsic components $R^A{}_{BC}$ are divided into inner components, when both B and C span $\mathfrak{i}$ indices, gauge components, if at least one of the two indices belongs to $\mathfrak{h}$, and outer components, which are the leftover ones. Schematically:

$$\begin{aligned} &\text{inner components: } R^A{}_{\mathfrak{i}'\mathfrak{i}} \\ &\text{outer components: } \left\{R^A{}_{\mathfrak{i}\mathfrak{o}}, R^A{}_{\mathfrak{o}'\mathfrak{o}}\right\} \equiv R^A{}_{\mathfrak{t}\mathfrak{o}} \\ &\text{gauge components: } \left\{R^A{}_{\mathfrak{i}\mathfrak{h}}, R^A{}_{\mathfrak{o}\mathfrak{h}}, R^A{}_{\mathfrak{h}'\mathfrak{h}}\right\} \equiv R^A{}_{B\mathfrak{h}} \,, \end{aligned} \tag{1.1.13}$$

with $\mathfrak{t} \equiv \mathfrak{i} \oplus \mathfrak{o}$.
For the theory extended to superspace to have the same physical content as the theory on spacetime, some constraints have to be imposed on the parametrisation of the supercurvatures: this is what is named a set of *rheonomic constraints*. Indeed, in the expansion of the curvature 2-forms in superspace, the rheonomic prescription requires that the outer components must be expressed, on-shell, as linear tensor combinations of the inner components, that is, mathematically speaking,

$$R^A{}_{\mathfrak{t}\mathfrak{o}} = C^{A|\mathfrak{i}'\mathfrak{i}}_{\mathfrak{t}\mathfrak{o}B} R^B{}_{\mathfrak{i}'\mathfrak{i}} \,. \tag{1.1.14}$$

In addition to the latter constraints, one has also the horizontality condition

$$R^A{}_{B\mathfrak{h}} = 0 \,, \tag{1.1.15}$$

i.e. the gauge components vanish. However, there is no need to impose it, since it holds in light of rule **B)**. Hence, the essential requirement in rule **E)** is the validity of (1.1.14), which, in turn, can be read as

$$R^A = R^A{}_{\mathfrak{i}'\mathfrak{i}} V^{\mathfrak{i}'} \wedge V^{\mathfrak{i}} + C^{A|\mathfrak{i}'\mathfrak{i}}_{\mathfrak{i}''\mathfrak{o}B} R^B{}_{\mathfrak{i}'\mathfrak{i}} V^{\mathfrak{i}''} \wedge \Psi^{\mathfrak{o}} + C^{A|\mathfrak{i}'\mathfrak{i}}_{\mathfrak{o}'\mathfrak{o}B} R^B{}_{\mathfrak{i}'\mathfrak{i}} \Psi^{\mathfrak{o}'} \wedge \Psi^{\mathfrak{o}} \,, \tag{1.1.16}$$

where, for concreteness, we assumed that the inner directions are spanned by the vielbein $\mu^{\mathfrak{i}} \equiv V^{\mathfrak{i}}$, whereas the outer ones are spanned by the gravitini $\mu^{\mathfrak{o}} \equiv \Psi^{\mathfrak{o}}$.

In Chapter 2 and 3, the rules of the geometric approach described here will be put into practice to build the Lagrangians for the $\mathcal{N} = 2$ AdS$_4$ supergravity and the D$^2(2, 1; \boldsymbol{\alpha})$ global supersymmetric theory.

1.2 Gauge/gravity duality in a nutshell

Since its inception, the anti-de Sitter (AdS) / Conformal Field Theory (CFT) holographic correspondence [8, 15, 16] has provided an important tool to investigate the strong coupling regime of field theories on a fixed background using classical supergravity on (possibly asymptotically) anti-de Sitter (AAdS) spacetimes in one dimension higher. This is a powerful framework since, being an intrinsically non perturbative strong/weak coupling duality, it opens a window on aspects of the gauge theory which are otherwise not accessible.

In its original formulation, the duality was conjectured as a correspondence between the full type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ four dimensional Super Yang-Mills theory on the boundary of the AdS_5 spacetime. This statement was proposed in [8] for the first time, where the author justified the correspondence by showing a groundbreaking argument. We briefly review it.

Let us consider the type IIB supergravity action in ten dimensions and focus on the sector of the Lagrangian relevant to describe a Dp-brane solution. Such action, in the string frame, reads

$$S = \frac{1}{(2\pi)^7 \ell_s^8} \int \mathrm{d}^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4\partial_\mu \phi\, \partial^\mu \phi\right) - \frac{\sqrt{g}}{2} \left(F_1^2 + \frac{F_3^2}{3!} + \frac{F_5^2}{5!}\right) , \tag{1.2.1}$$

where R and ϕ come from the Neveu-Schwarz sector and are, respectively, the Ricci scalar of the ten dimensional manifold and the dilaton, while $F_1 = \mathrm{d}A_0$, $F_3 = \mathrm{d}A_2$, $F_5 = \mathrm{d}A_4$ are the field strengths of the Ramond-Ramond fields. The vacuum expectation value of the scalar field $\langle e^\phi \rangle = g_s$ shall be deemed to be the coupling constant in the supergravity theory. Interestingly, as can be clearly seen by writing the action in the Einstein frame, there is no coupling between the dilaton and the Ramond-Ramond fields once one considers D3-branes, which will be exactly the case under inspection. The prefactors are inherited from the string theory, $\ell_s = \sqrt{\alpha'}$ being the string length scale.

An ansatz for the solution can be written as

$$\begin{aligned} \mathrm{d}s^2 &= H_p^{-1/2}(r) \left(-\mathrm{d}t^2 + \ldots + \mathrm{d}x_p^2\right) + H_p^{1/2}(r) \left(\mathrm{d}y_1^2 + \ldots + \mathrm{d}y_{9-p}^2\right) \\ A_{tx_1 \ldots x_p} &= H_p(r) \\ e^\phi &= g_s H_p^{\frac{3-p}{4}}(r) \end{aligned} \tag{1.2.2}$$

where $r^2 \equiv y_1^2 + \ldots + y_{9-p}^2$ is the transverse distance from the brane and the expression for $H_p(r)$ is fixed once the ansatz is plugged in the equations of motion:

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}} . \tag{1.2.3}$$

The latter correctly reproduces the Minkowski space in the limit $r \to \infty$. In order to fix the constant L in (1.2.3), one has to consider the quantisation condition on the brane charge, taking into account all the normalisation factors

$$\frac{1}{(2\pi \ell_s)^{p+1} g_s} \int_{\mathbb{S}^{8-p}} * F_{p+2} = N \in \mathbb{N} \implies L^{7-p} = (2\pi)^{p-2} f_p \ell_s^{7-p} g_s N , \tag{1.2.4}$$

with f_p a numerical factor dependent on the value p and $\mathbb{S}^{8-p}$ identifying a $(8-p)$-sphere. Besides its interpretation of brane charge, N can be physically understood as the number of overlapped Dp-branes in $r=0$.
Let us consider now the $p=3$ case and take the limit $r \to 0$, commonly known as near brane or "near horizon" limit. This is precisely the point where the AdS/CFT correspondence emerges. Indeed, taking into account the regime

$$\alpha' \to 0 \quad \& \quad \frac{r}{\alpha'} = u = \text{fixed}\,, \tag{1.2.5}$$

one suppresses the corrections in α' (or equivalently in powers of the curvature) in the supergravity action and the brane theory decouples from the bulk:

$$\kappa_{10}^2 = 8\pi G_{10} = 64\pi^7 g_s^2 (\alpha')^4 \to 0\,. \tag{1.2.6}$$

Furthermore, the previous expressions for $\mathrm{d}s^2$, H_3 and L take the form

$$\begin{aligned}
\mathrm{d}s^2 &= \alpha' \left[(4\pi g_s N)^{1/2} \left(\frac{\mathrm{d}u^2}{u^2} + \mathrm{d}s^2\left(\mathbb{S}^5\right) \right) + \frac{u^2}{(4\pi g_s N)^{1/2}} \mathrm{d}x^\mu \mathrm{d}x_\mu \right] \qquad (1.2.7)\\
H_3 &= 1 + \frac{L^4}{r^4} \quad \overset{r\to 0}{\Longrightarrow} \quad \frac{4\pi g_s N}{u^4 (\alpha')^2}\\
L^4 &= \frac{(2\pi)^4 g_s (\alpha')^2 N}{4\mathrm{Vol}(\mathbb{S}^5)} = 4\pi g_s (\alpha')^2 N
\end{aligned}$$

where $\mathrm{Vol}(\mathbb{S}^5)$ and $\mathrm{d}x^\mu \mathrm{d}x_\mu$ stands, respectively, for the volume of the 5-sphere and the four dimensional Minkowski metric.
The near brane metric represents an $\mathrm{AdS}_5 \times \mathbb{S}^5$ manifold, which preserves the maximum amount of supersymmetries of the theory (32 supercharges) and the constant L plays the role of curvature radius for both AdS_5 and $\mathbb{S}^5$. The limit in which supergravity can be trusted is

$$\frac{L^4}{\ell_s^4} \gg 1 \implies 4\pi g_s N \gg 1\,, \tag{1.2.8}$$

namely when the string length is much smaller than the spacetime curvature radius.
As it is well-known, supergravity models suffer ultraviolet divergences and the quantisation of superstring on $\mathrm{AdS}_5 \times \mathbb{S}^5$ is an issue not completely understood yet. For these reasons, it is convenient getting in a framework where supergravity can be treated as a classical theory. In order to suppress quantum effects from the string, g_s is required to be fixed and small, implying

$$4\pi g_s N \gg 1 \quad \& \quad g_s \ll 1 \implies N \gg 1\,, \tag{1.2.9}$$

from which the notorious large N limit derives.

Until now we have interpreted the Dp-brane as a supergravity solution, which can be regarded as a closed strings point of view. However, these objects may be also seen as extended entities where open strings ends are attached. When one considers N Dp-branes separated by some distances denoted by r and takes the same limits of (1.2.5), they decouple from the bulk and the low energy effective theory defined on them is described by a (p+1) dimensional Super Yang Mills (SYM) action with gauge group U(N). A way to understand the latter statement is looking at the field content on the worldvolume of the brane, which means neglecting those fields defined in the transverse directions. It is composed of a non abelian gauge field, $9-p$ scalars in the adjoint representation of U(N) and the right number of gauginos in the adjoint representation to balance on-shell bosonic and fermionic degrees of freedom.

A relation between the coupling constant g_{YM} of the SYM and the parameters appearing in string theory can be found as follows. Let us remember that one Dp-brane has an effective description as a Dirac-Born-Infeld theory

$$S_{DBI} = -T_p \int \mathrm{d}^{p+1}x\, e^{-\phi}\sqrt{\det\left(g + 2\pi\alpha' F\right)}\,, \tag{1.2.10}$$

where F is the 2-form field strength of a gauge potential 1-form and $T_p = \frac{1}{(2\pi)^p(\alpha')^{p+1}g_s}$ is the tension of the brane determined through string computations. Once one expands the previous expression in powers of α', requiring $\alpha' \to 0$ and g_s small but fixed, the kinetic term of a U(1) gauge vector field appears. Particularly, the coefficient in front of it is, by definition, the coupling constant of the theory. Considering $N > 1$ branes, the exact non-Abelian form of the Dirac-Born-Infeld action is not known, but its form is fixed by symmetries and supersymmetry at two-derivative level. Indeed, specialising to the $p = 3$ case, the only four dimensional two-derivative gauge theory with the correct number of Poincaré supercharges (16) is the $\mathcal{N} = 4$ SYM[5]. Then, by comparing the coefficients in front of the kinetic term, we get

$$g_{YM}^2 = 4\pi g_s \ll 1\,. \tag{1.2.11}$$

One may think that we are selecting the weak coupling regime of this theory. However, this is not true. Indeed, as proved by 't Hooft argument, once one considers the large N limit, the effective coupling constant of Feynman diagrams of the theory is $\lambda = g_{YM}^2 N \gg 1$ and the perturbative series of Feynman diagrams is effectively reorganised in terms of their topology, with λ as effective coupling constant, the planar diagrams giving the dominant contribution. Thus, in the limit in which the classical effective low energy description of the (super)gravity side can be trusted, the corresponding regime of the dual theory is strongly coupled.

A first and basic test to take seriously the correspondence would be the matching between the symmetry groups of the two theories. In fact, on the gravity side we have the isometry group of $\mathrm{AdS}_5 \times \mathbb{S}^5$, namely $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$, and 32 supercharges of the theory preserved entirely by the solution. On the gauge side, the $d = 4$ conformal group $\mathrm{SO}(2,4)$, the R-symmetry group SU(4) and 32 supercharges, related to the $\mathcal{N} = 4$ supersymmetries, appear. Now, we need just to remember the known isomorphism SO(6)$\sim$SU(4) and the check is concluded. All the generators can be assembled in a single supergroup, which turns out to be $\mathrm{SU}(2,2|4)$ [17].

The holographic correspondence has been extended to more general backgrounds of the form $\mathrm{AdS}_D \times \mathcal{M}_{\mathrm{int}}$, possibly with less supersymmetry, which can be embedded in other

[5]Half of the 32 Poincaré supercharges of IIB supergravity are broken by the D3-branes.

string theories or M-theory, such as the maximally supersymmetric $\mathrm{AdS}_4 \times \mathbb{S}^7$ and $\mathrm{AdS}_7 \times \mathbb{S}^4$ solutions of the eleven dimensional supergravity and variants thereof. A valuable approach to the study of holography on a background of the form $\mathrm{AdS}_D \times \mathcal{M}_{\text{int}}$ is to restrict to an effective D dimensional low energy supergravity originating from superstring/M-theory compactified on the internal manifold $\mathcal{M}_{\text{int}}$. Such procedure, called Kaluza-Klein reduction, leads to an infinite tower of fields, which can be split in a finite number of "light" modes and an infinite amount of "heavy" ones. Since we want to focus on the low energy regime of the theory, we need to implement a *consistent* truncation, namely we set all the heavy modes to zero in the equations of motion, leaving us with those for the light ones. This is possible only when the latter don't source the heavy modes in the field equations [18].
The dimensional reduced supergravity admits the AdS_D part of the higher dimensional background as a vacuum and typically is of *gauged* type. The geometry of $\mathcal{M}_{\text{int}}$ determines the amount of supersymmetry preserved by this AdS_D vacuum and the general features of the effective theory. In this setting, the AdS/CFT conjecture can be restated as a holographic relation between the AdS_D supergravity and a $d = D - 1$ dimensional superconformal field theory (SCFT) at the boundary of the AdS geometry[6]. Most interestingly, the duality has been extended, on the gravity side, from global AdS to backgrounds which have an AAdS geometry, reproducing the renormalisation group flow of the dual theory to an infrared (IR) conformal fixed point, the energy scale being fixed by the radial coordinate on the D dimensional spacetime. Indeed, the essential ingredient for this correspondence is the conformal structure of the boundary of AAdS spaces. These are spacetimes with negative curvature and whose metric has a pole of order two in the asymptotic region or, more precisely, conformally compact manifolds [19, 20]. Supergravity solutions that are asymptotically (locally) AdS can be holographically interpreted generically either as explicit deformations of SCFTs or as models in which the superconformal symmetry is spontaneously broken.

Several important results have been obtained in the holographic study of strongly coupled quantum field theories, within the so-called *bottom-up* approach. This latter consists in crafting an appropriate D dimensional AAdS gravity background of a suitably chosen gravity theory, which can reproduce interesting non perturbative phenomena of a boundary field theory, with some given general properties. In this approach emphasis is not given to the higher dimensional ultraviolet (UV) completion of the (super)gravity theory, which typically has a minimal amount of supersymmetry, if any. Moreover, only certain features of the dual field theory are known, which are suitably fixed by the chosen background through the holographic correspondence.

As opposed to the bottom-up one, the so-called *top-down* approach is restricted to gravity theories whose higher dimensional UV completions in superstring or M-theory are known. This has the advantage that the dual CFT is often known. In most cases supergravity models considered in this setting feature an extended amount of supersymmetry (i.e. no less than eight supercharges), which makes them more constrained in field content

[6]The spectrum of the fields in the D dimensional supergravity theory corresponds only to a sector of the operators on the dual field theory side.

and interactions and, therefore, more predictive.[7]

From a formal point of view, the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence states that the CFT_d partition function is equal to the gravitational partition function in AAdS space in one dimension higher [15, 16],

$$Z_G[\hat{\Phi} \to \hat{\Phi}_{(0)}] = Z_{\mathrm{CFT}}[\mathcal{J} \equiv \hat{\Phi}_{(0)}] \,. \tag{1.2.12}$$

In the above formula, $Z_G[\hat{\Phi}_{(0)}]$ is the quantum partition function of the gravity theory in AAdS space, as a function of the boundary value $\hat{\Phi}_{(0)}$ of the bulk field $\hat{\Phi}$, while $Z_{\mathrm{CFT}}[\mathcal{J}]$ is the quantum partition function of the corresponding CFT, in which the source $\mathcal{J}$ of a local operator $\mathcal{O}(x)$, dual to $\hat{\Phi}$, is identified with $\hat{\Phi}_{(0)}$.

Let us recall the definition of the quantum effective action $W[\mathcal{J}]$ for a d dimensional CFT on $\partial\mathcal{M}$ in terms of the partition function $Z_{\mathrm{CFT}}[\mathcal{J}]$

$$Z_{\mathrm{CFT}}[\mathcal{J}] = \mathrm{e}^{\mathrm{i}W[\mathcal{J}]} = \int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}I[\phi] + \mathrm{i}\int_{\partial\mathcal{M}} \mathrm{d}^d x \, \mathcal{O}(\phi)\cdot\mathcal{J}} \,, \tag{1.2.13}$$

where the symbol $\phi(x)$ collectively denotes the fundamental fields of the CFT on which the functional integration is performed. The action $I[\phi]$ should already be renormalised, that is, finite in the UV region. Even though W is a (non local) function of the external source $\mathcal{J}(x)$, the physical information of the theory is contained in the n-point functions of the operators $\mathcal{O}(\phi(x))$,

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{\mathrm{CFT}} = Z^{-1}_{\mathrm{CFT}}[0] \left. \frac{\delta^n Z_{\mathrm{CFT}}[\mathcal{J}]}{\mathrm{i}\delta\mathcal{J}(x_1) \cdots \mathrm{i}\delta\mathcal{J}(x_n)} \right|_{\mathcal{J}=0} \,. \tag{1.2.14}$$

In particular, different correlators are related by Ward identities which express the symmetries in the CFT at the quantum level.

Let us expand on the identification (1.2.12) in the special case of a pure AdS gravity theory in which the only bulk field is the metric $\hat{g}_{\hat{\mu}\hat{\nu}}(x)$ defined on the AAdS spacetime, to be denoted by $\mathcal{M}^{d+1}$. In this case the gravitational partition function has the form

$$Z_{\mathrm{G}}[g_{(0)}] = \int \mathcal{D}\hat{g} \, \mathrm{e}^{\mathrm{i}I_{\mathrm{ren}}[\hat{g}]} \simeq \mathrm{e}^{\mathrm{i}I_{\mathrm{on-shell}}[g_{(0)}]} \,. \tag{1.2.15}$$

Up to a conformal transformation, $g_{(0)\mu\nu}$ is the value at the conformal boundary of the bulk field $\hat{g}_{\hat{\mu}\hat{\nu}}(x)$, on which Dirichlet boundary conditions are imposed: $\left.\delta g_{(0)\mu\nu}\right|_{\partial\mathcal{M}} = 0$. The gravitational action $I_{\mathrm{ren}}[\hat{g}]$ has to be consistent with the boundary conditions and has to be finite in the asymptotic (IR) region. In equation (1.2.15) the classical approximation, for weak gravitational couplings, is performed, in which the partition function can be evaluated on the classical solution, by a saddle point approximation, giving rise to the on-shell action $I_{\mathrm{on-shell}}[g_{(0)}]$. The boundary metric $g_{(0)\mu\nu}$ becomes the source in the boundary CFT.

The AdS/CFT correspondence in the classical approximation of gravity identifies the quantum effective action W as

$$W[g_{(0)}] \simeq I_{\mathrm{on-shell}}[g_{(0)}] \,, \tag{1.2.16}$$

[7] For a discussion on bottom-up versus top-down approaches see, e.g., [20].

where the boundary metric becomes a background field in the CFT, so that the energy-momentum tensor operator $T^{\mu\nu}(\phi, g_{(0)})$ also depends on it. The expectation value of the latter can be calculated as a 1-point function from the effective theory,

$$\langle T^{\mu\nu} \rangle_{\mathrm{CFT}} = \frac{2}{\sqrt{|g_{(0)}|}} \frac{\delta I_{\mathrm{on-shell}}[g_{(0)}]}{\mathrm{i}\, \delta g_{(0)\mu\nu}} = \tau^{\mu\nu}\,, \tag{1.2.17}$$

and $\tau^{\mu\nu}$ is the holographic stress tensor in the gravity side.

The conformal Ward identities in the CFT have the form

$$\nabla_{(0)\mu} \tau^{\mu\nu} = 0\,, \qquad \tau^{\mu}_{\ \mu} = \mathcal{A}\,, \tag{1.2.18}$$

where $\mathcal{A}$ is the Weyl anomaly [21]. This quantum result can, therefore, be obtained in the classical regime of AdS gravity.

Holographic renormalisation

For the above formalism to be well-defined, a field theory has to be finite at short distances. However, a general feature of quantum field theory is that UV (and IR) divergences can appear at quantum level in the correlation functions. In order to guarantee the consistency of the theory, these unphysical effects are usually removed through the procedure of renormalisation. In the framework of the AdS/CFT correspondence, which is in fact a UV/IR duality, i.e. the ultraviolet regime of the field theory is related to the infrared one of the gravity side and vice versa, it is natural to think that the UV poles of CFT n-point functions (1.2.14) could be cancelled holographically, by adding appropriate boundary counterterms in the dual theory. Indeed, a first systematic method in this direction was implemented at the beginning of the century [22,23] and was then applied to various bosonic theories, in particular to gravitational actions coupled to bosonic matter fields.[8] Briefly, the procedure consists in regulating the bulk on-shell (super)gravity action by introducing a cut-off on the radial coordinate, adding appropriate boundary counterterms to eliminate the divergences and then removing the cut-off.[9]

In the subsequent years, the holographic renormalisation scheme was implemented also for actions including fermionic fields. In [26] the authors studied the case of the four dimensional $\mathcal{N} = 1$ supergravity including contributions from the gravitini, while in [27] the boundary counterterms for the minimal $\mathcal{N} = 2$ gauged supergravities in $D = 4$ and $D = 5$ have been analysed, restricting to quadratic order in fermions in the action, by using a Hamiltonian approach. Five dimensional supersymmetric holographic renormalisation has also been considered in [28].

A different approach to the holographic renormalisation was developed in [29], where it was named *topological regularisation*. It was proven to give the same results as the standard procedure in pure gravity in four dimensions, having however the quality of giving a topological meaning to the resummation of the holographic counterterms series expansion.

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[8] A very good review of the subject can be found in [19].

[9] For completeness, let us mention that Gibbons and Hawking had already proposed to add a boundary contribution in 1977, namely the Gibbons-Hawking(-York) term, in order to have a well-defined variational principle for gravity theories [24,25].

A detailed comparison of both counterterm series has been developed in pure AdS gravity in any dimension in [30]. In particular, the topological counterterm needed to regularise four dimensional gravity turns out to be the Gauss-Bonnet term and it is also able to restore the diffeomorphisms invariance, broken by the presence of the boundary [31–33]. Moreover, the addition of this contribution allows to express the renormalised action in the MacDowell-Mansouri form [34].

The above papers treat gravity in the second order formalism. However, an alternative formulation to the latter is the first order formalism, where the spin connection is considered as an independent field from the vielbein [35–40]. In this approach, the powerful tool of exterior calculus and the differential form language can be employed, yielding a geometrical description of gravity. The same approach was used in [41] to extend the results of [31–33] to supergravity and to find the counterterms needed to restore the local supersymmetry[10], broken by the presence of a boundary, for the cases of pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS$_4$ supergravities. The boundary terms found in [41] to restore supersymmetry (that is interpreted as diffeomorphisms in the fermionic directions of superspace in the geometric approach) are in fact the supersymmetric extension of the Gauss-Bonnet term, which was necessary to restore diffeomorphisms invariance in the case of gravity. Correspondingly, those boundary terms were precisely the ones needed to rewrite the total supergravity action in a supersymmetric MacDowell-Mansouri form.
However, while the topological regularisation was shown to be able to renormalise the bulk action for the pure gravity case, the same has not been proven yet for its supersymmetric extension, in particular for the pure $\mathcal{N} = 2$ AdS$_4$ supergravity. In Chapter 2 of this thesis, we proceed from the foregoing works to achieve this goal but, in contrast to [27], we consider the full contribution from the gravitini and start from a rather general setup for what concerns gauge fixings, in view of possible future developments.

Conformal Field Theory at the Boundary of anti-de Sitter

From a different, but complementary, point of view, we explore a relation between the classical local symmetries of an AdS gravity defined on the bulk manifold $\mathcal{M}^D$ and the quantum symmetries in a field theory defined on $\partial\mathcal{M}$. The latter match the *asymptotic symmetries*, at radial infinity, of the gravitational background. In our approach, they appear as *residual symmetries* left over after the gauge fixing of bulk local symmetries and whose parameters take value on $\partial\mathcal{M}$. This matching of symmetries is justified from the group theoretical point of view. Namely the isometries of the AdS vacuum in $D = (d+1)$ dimensional asymptotically AdS spaces are described by the $\mathrm{SO}(2,d)$ group, whose generators are $\mathbf{J}_{ab}$, $\mathbf{J}_a$[11].

The d dimensional boundary breaks the bulk local symmetries in the x^d (radial) direction that naturally leads to the $d+1$ decomposition of the Lorentz indices into

[10]This is due to the fact that, differently from quantum field theories defined on manifold without boundary, we have not the freedom of setting to zero the asymptotic value of the bulk fields in the context of AdS/CFT correspondence, since the latter are interpreted as sources for the operators in the CFT. By requiring their values to vanish, it would mean switching off generic sources in the field theory.

[11]Here, we restrict for simplicity the discussion to the bosonic sector, but its supersymmetric extension will be considered in Chapter 2.

$a = (i, d)$. In that way, the bulk isometry group is isomorphic to the conformal group with generators $\mathbf{J}_{ij}$, $\mathbf{P}_i = \mathbf{J}_i + \mathbf{J}_{id}$, $\mathbf{K}_i = \mathbf{J}_i - \mathbf{J}_{id}$, $\mathbf{D} = \mathbf{J}_d$[12]. Therefore, choosing suitable boundary conditions for the AdS gravity fields in D dimensional bulk, which are $\hat{\omega}^{ab}$ (along $\mathbf{J}_{ab}$) and $\frac{1}{\ell}V^a$ (along $\mathbf{J}_a$), we can identify its d dimensional boundary field content as sourcing the operators of the CFT. Using the isomorphism, the boundary background fields, i.e. sources $\mathcal{J} = \{\omega^{ij}, E^i, B, \mathcal{S}^i\}$ associated with the conformal generators, have the form

$$\begin{aligned}
\mathbf{J}_{ij}: &\quad \omega^{ij} \sim \hat{\omega}^{ij}\,, \\
\mathbf{P}_i: &\quad E^i \sim V^i_+ = \frac{1}{2}\left(\ell\hat{\omega}^{id} + V^i\right)\,, \\
\mathbf{D}: &\quad B \sim V^d\,, \\
\mathbf{K}_i: &\quad \mathcal{S}^i \sim V^i_- = \frac{1}{2}\left(\ell\hat{\omega}^{id} - V^i\right)\,.
\end{aligned}$$

This near-boundary rescaling is the first step in removing the long-distance divergences present in (super)gravity theory in asymptotically AdS spaces, equivalent to renormalisation of the holographic CFT. From this discussion, we draw the following conclusions. First, a

[12] A d dimensional conformal theory is a model invariant under the following transformation of the metric

$$g'_{\mu\nu} = \Omega(x) g_{\mu\nu}\,. \tag{1.2.19}$$

The symmetries obeying this relation are translations, Lorentz transformations, dilation and special conformal transformations, to which the following generators are, respectively, associated:

$$\mathbf{J}_\mu = -\mathrm{i}\partial_\mu\,, \quad \mathbf{J}_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)\,, \quad \mathbf{D} = -\mathrm{i}x^\mu\partial_\mu\,, \quad \mathbf{K}_\mu = -\mathrm{i}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\,. \tag{1.2.20}$$

The latter satisfy the algebraic structure

$$\begin{aligned}
[\mathbf{J}_{\mu\nu}, \mathbf{J}_{\rho\sigma}] &= \mathrm{i}\big(\eta_{\mu\rho}\mathbf{J}_{\nu\sigma} - \eta_{\mu\sigma}\mathbf{J}_{\nu\rho} + \eta_{\nu\sigma}\mathbf{J}_{\mu\rho} - \eta_{\nu\rho}\mathbf{J}_{\mu\sigma}\big) \\
[\mathbf{J}_{\mu\nu}, \mathbf{J}_\rho] &= \mathrm{i}\big(\eta_{\mu\rho}\mathbf{J}_\nu - \eta_{\nu\rho}\mathbf{J}_\mu\big) \\
[\mathbf{J}_\mu, \mathbf{J}_\nu] &= [\mathbf{J}_{\mu\nu}, \mathbf{D}] = 0 \\
[\mathbf{J}_{\mu\nu}, \mathbf{K}_\rho] &= \mathrm{i}\big(\eta_{\mu\rho}\mathbf{K}_\nu - \eta_{\nu\rho}\mathbf{K}_\mu\big) \\
[\mathbf{K}_\mu, \mathbf{J}_\nu] &= -2\mathrm{i}\mathbf{J}_{\mu\nu} - 2\mathrm{i}\eta_{\mu\nu}\mathbf{D} \\
[\mathbf{D}, \mathbf{J}_\mu] &= \mathrm{i}\mathbf{J}_\mu \\
[\mathbf{D}, \mathbf{K}_\mu] &= -\mathrm{i}\mathbf{K}_\mu
\end{aligned} \tag{1.2.21}$$

and can be reorganised in a single matrix as

$$\mathbf{J}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbf{J}_{\mu\nu} & \frac{\mathbf{K}_\mu - \mathbf{J}_\mu}{2} & -\frac{\mathbf{K}_\mu + \mathbf{J}_\mu}{2} \\ -\frac{\mathbf{K}_\mu - \mathbf{J}_\mu}{2} & 0 & \mathbf{D} \\ \frac{\mathbf{K}_\mu + \mathbf{J}_\mu}{2} & -\mathbf{D} & 0 \end{pmatrix} \qquad \hat{\mu}, \hat{\nu} = 0, \ldots, d+1\,, \tag{1.2.22}$$

which, in turn, fulfill a simple commutation rule

$$[\mathbf{J}_{\hat{\mu}\hat{\nu}}, \mathbf{J}_{\hat{\rho}\hat{\sigma}}] = \mathrm{i}\big(\eta_{\hat{\mu}\hat{\rho}}\mathbf{J}_{\hat{\nu}\hat{\sigma}} - \eta_{\hat{\mu}\hat{\sigma}}\mathbf{J}_{\hat{\nu}\hat{\rho}} + \eta_{\hat{\nu}\hat{\sigma}}\mathbf{J}_{\hat{\mu}\hat{\rho}} - \eta_{\hat{\nu}\hat{\rho}}\mathbf{J}_{\hat{\mu}\hat{\sigma}}\big)\,, \qquad \eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-1, 1, \ldots, 1, -1)\,. \tag{1.2.23}$$

The latter relation identifies $\mathrm{SO}(2, d)$, which indeed is the isometry group of AdS_D ($D = d+1$). This rearrengement proves the isomorphism existing between the symmetry groups of AdS_D and a conformal theory in d dimension.

full linearly realised conformal group on the boundary can be made manifest only in first order formalism, where the spin connection is an independent field. Second, the conformal structure on the boundary naturally introduces two geometric quantities in d dimensions, a dilation gauge field B and the Schouten tensor $\mathcal{S}^i$. They will play an important role for the analysis of symmetries of this holographic correspondence in Chapter 2.

1.3 Analogue Gravity, graphene and unconventional SUSY

In 1915 general relativity was conceived to unify special relativity and Newton's universal law of gravitation, as well as giving an answer for the discrepancy between some predictions of the latter theory and experimental observations. In order to get a satisfactory description, general relativity, thanks to the application of the branch of mathematics called differential geometry, represents gravity as a distortion of spacetime geometry where objects exist and move. This feature makes gravity unique among the other fundamental forces, which are interpreted by means of (quantum) fields defined on a fixed background.
This special feature of gravity has brought many problems in physicists intent of unifying the four fundamental interactions known nowadays, but it has also given the possibility to exploit this model in situations different from those for which it was thought in the first instance. Indeed, since the earliest years of general relativity life, formal analogies between Einstein's theory and completely different contexts have surprisingly been found, like the study of sound waves in a moving fluid, supersonic fluid flow, phononic Hawking radiation and many others phenomena typically (but not only) based on condensed matter systems [42]. The sector of theoretical physics which focuses on the analysis of such analogies goes under the name of "Analogue Gravity", as the reader can intuitively understand. When we talk about analogies, we should interpret them as mathematical parallelisms, rather than identities, which instead could be considered as completely physical equivalences. In particular, an analogue model can capture and reflect a sufficient number of features of general relativity, but not necessarily all of them.
Some analogue models can be interesting for experimental reasons, since they allow to experience effects which are only theoretically hypothesised in their high energy counterpart. Other ones provide new light on puzzling theoretical questions. On the contrary, new insights carried out within the context of general relativity can be exploited to better comprehend aspects of analogue models. Therefore, in principle, there is not an *a priori* preferential direction in the information flow.

The first paper which allows us to discuss of Analogue Gravity is [43], where Gordon tried to analyse dielectric media by using an "effective metric". More precisely, the latter has the feature of mimicking the former and reads

$$g_{\mu\nu}^{\text{effective}} = \eta_{\mu\nu} + \Big(1 + n^{-2}(x)\Big)\, V_\mu V_\nu \,, \tag{1.3.1}$$

where $\eta_{\mu\nu}$ is the usual flat metric, V_μ is the constant 4-velocity of the medium and $n(x)$ is the value of the refractive index in the point of coordinate x.
The idea of effective metric techniques was momentarily forgotten for about thirty years, but it came back in [44], when Pham showed that, under certain conditions, Maxwell's equations could be expressed in terms of a particular effective metric

$$g_{\mu\nu}^{\text{effective}} = g_{\mu\nu} + \left(1 - \frac{1}{\epsilon\mu}\right) V_\mu V_\nu \,, \tag{1.3.2}$$

with $g_{\mu\nu}$, ϵ and μ as, respectively, the spacetime metric, the permeability and the permittivity. The analogy was further developed during the following years, coming to the achievement

that an arbitrary gravitational field can be represented as an optical medium if the latter is subjected to the rather unphysical restriction that the electric permittivity is proportional to the magnetic permeability.
Another result was obtained with the extension of the previous analogue models to the case of flowing fluid media, namely assuming that V_μ is no longer constant [45].

In the 1980s analogue models found another application in the study of propagation of shockwaves in astrophysical contexts. Indeed, in [46] it was shown that the wave equation for linear perturbations of a relativistic perfect fluid on a Schwarzschild background exhibits a relativistic wave equation on an effective metric, which can also admit horizons. The latter paper can be considered a step forward for the Analogue Gravity framework precisely because of the introduction of such a general relativistic Schwarzschild background.

The seminal paper of 1981, which divided the "historical" phase of Analogue Gravity from the modern one, is commonly identified with "Experimental black hole evaporation" [47], where Unruh developed a model based on fluid flow and translated it in the general-relativistic black holes setting, inquiring crucial issues concerning Hawking radiation. In fact, [47] seems to be one of the first papers where the Hawking radiation nature was disclosed: it appears that it is not an effect related to general relativity framework or formalism, but it occurs when we are in presence of a quantum field theory defined on a curved background with a horizon.
During the 1990s and the first decade of 2000, many works tried to make more precise the correspondence between black holes and Analogue Gravity theories, by exploring the mapping of concepts like horizon, ergosphere and surface gravity, discussing the implications of Bekenstein–Hawking entropy for analogue models [48–50] and looking for experimental evidences [51].

In the aforementioned scenario, an analogy between condensed matter systems on a lattice and general relativity were carried out. In fact, it turned out that, in the continuum limit, the spatial distortions set up by defects in crystals (i.e. change in the arrangement of atoms) can be modeled through geometric properties of an affine space with torsion and curvature. Indeed, no perfect lattices can be created in the laboratory, since chemical, electrical and structural imperfections always appear. The latter show up as foreign atoms, a surplus or deficiency of electrons, as well as a lack of local symmetry.
To be more specific, we will focus on defects for the two spatial dimensions case, as graphene-like materials, because we will be interested in them in the continuation of this thesis. In this instance, codimesion-2 defects, that in three spatial dimensions would be line-like, as dislocations and disclinations, become point-like. Dislocations can be found when a number of vacancies (missing atoms) or interstitials (an excess of atoms) occur within the crystal and destroy the traslational invariance by multiples of the lattice vectors. Furthermore, they can be mathematically characterised through the Burgers vector. To better understand the meaning of the latter, let us consider a geometrical circuit in an ideal lattice and map it into a disturbed crystal because of the presence of a dislocation line: the Burgers vector measures the failure to close of the image of the circuit. Interestingly, it can be proved that the dislocation density is directly related to the torsion tensor of an affine space[13].

[13] For further details see [52].

On the other side, disclinations are a different kind of defects, capable of destroying the global rotational order of a crystal, saving it locally nevertheless. They occur when the coordination number (namely the nearest neighbours) of some atoms change because of local strain and twist. For instance, in a hexagonal lattice like graphene, one can produce configurations with 5-folded and 7-folded disclinations. It is known that the disclination density can be related to the Einstein tensor, which, in turn, is formed from the curvature tensor. As a result, the disclination density is linked to the curvature of an affine space[14].

Up to this point, only geometric properties of spacetime have been played a role in the analogy. The latter can be further extended by introducing fermionic matter on the gravity side, which would allow to include the description of the electronic properties of these materials. In particular, we will take into account a regime where the charge carriers, modelled as pseudo-particle wavefunctions, have long wavelengths compared to the characteristic lattice length, feeling the background as a continuum. Otherwise, the effective geometrical description, provided by general relativity, breaks down and one has to be aware that the structure is actually composed of discrete pieces, i.e. atoms and molecules. For sake of concreteness, we will analyse the specific case of pure, isolated graphene, where the real space substrate consists of a two dimensional honeycomb lattice of carbon atoms. The latter is, in turn, composed by two inequivalent sublattices, whose atoms locations are designated as sites A and B. Indeed, if one chooses two basis vectors $\mathbf{a_1}$, $\mathbf{a_2}$ on the plane and applies a linear combination (with integer coefficients n_1, n_2) of them on the position of an atom, at most two vertices of the hexagon can be reached (Figure 1.1): these are considered sites A, whereas the remaining ones are called sites B. We call $\mathbf{R} = n_1\mathbf{a_1} + n_2\mathbf{a_2}$ the linear combination of the basis vectors.

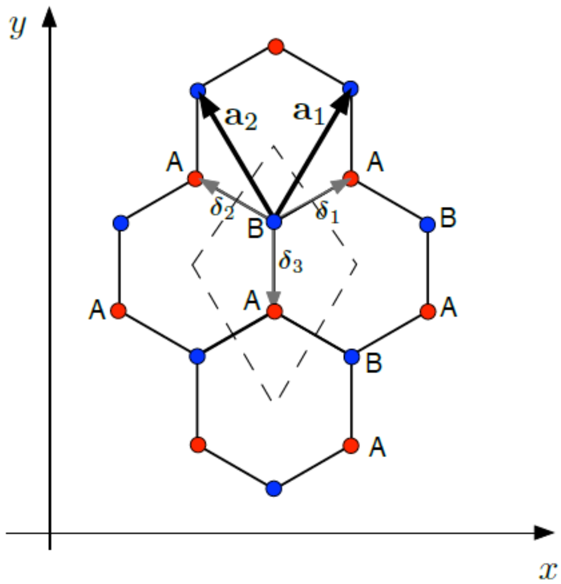

Figure 1.1: The real space substrate of graphene represented as a honeycomb structure, picture taken from [53].

[14]Also in this case, we refer to [52] for further information.

A reciprocal lattice in momentum space can also be defined through the equation $e^{\mathbf{iK}\cdot\mathbf{R}} = 1$, where $\mathbf{K}$ is a generic vector which connects two points of this grid. Just like for the real space, two basis vectors $\mathbf{b_1}$, $\mathbf{b_2}$ can be constructed, requiring an orthogonality condition with $\mathbf{a_1}$, $\mathbf{a_2}$, and two sets of inequivalent sites are recognised, usually called **K** and **K'**[15]. The reciprocal lattice has a honeycomb structure, but rotated by $\frac{\pi}{2}$ with respect to the physical space one.
Let us now consider the spectrum of the Hamiltonian in the so-called nearest neighbors tight-binding approximation. It can be proved that two bands, π and π^*, exists and they look like two Mexican hats upside down with respect to each other. Furthermore, π and π^* touch each other (namely, they have the same energy level) in six points, corresponding exactly to **K** and **K'**, forming the so-called Dirac cones. **K** and **K'** are, in turn, denominated Dirac points.
The fermions behaviour can be approximated by the Taylor expansion of the Hamiltonian around the Dirac ponts, since the band-width of π and π^* is about 15-18 eV, which is far greater than the typical energy values that come into play with applied voltages and temperatures. In this setup, the pseudo-particles behave in a "relativistic" way: this can be deduced from the linear dispersion relation between the energy and the quasi-momentum. On account of their relativistic behaviour, the pseudo-particles follow a "Dirac equation"[16], which is massless if the graphene layer is pure and isolated, whereas is massive in presence of defects or external potentials. Interestingly, as a consequence of this mathematical description, the charge carriers possess an additional twofold degree of freedom called "pseudo-spin" number.

On the gravity side, the inclusion of fermions is implemented through the exploitation of local supersymmetric theories. However, the gravitini appearing in this framework are spin 3/2 massless fields, whereas, in the case of graphene, pseudo-particles propagating on the lattices are described through spin 1/2 particles. Therefore, it would seem that the gravitini have no counterpart on the condensed matter side. Luckily, a substantial improvement in this direction has been achieved in [54] and goes by the name of "unconventional supersymmetry" (we will refer to theories featuring this symmetry also as AVZ models). In this paper, the authors assumed that all bosonic and fermionic fields enter as part of the connection, transforming in an adjoint representation under local "supersymmetry", rather than a vector one. Indeed, it is often possible to combine an adjoint and a vector representation of a certain group into an adjoint representation of a larger group. Once this procedure is carried out, the resulting theories could have completely different supermultiplets and dynamical features with respect to the "usual" supersymmetry. Indeed, it results that spin 1/2 and 1 fields are fundamental in such a model, whereas the others can be composite. Furthermore, the number of bosonic and fermionic degrees of freedom are no longer constrained to be the same, since their amount is now determined by that of bosonic and fermionic generators to which the fields are coupled in the connection.
For instance, in [54] the simplest case of a Chern-Simons 3-form is analysed, providing

[15] In the present discussion, with an abuse of notation, we are denoting points and vectors with the same symbols. It will be clear from the context which object we are referring to.

[16] Here, we have used the quotation marks since the charge carriers propagate in the honeycomb lattice with Fermi velocity $v_F \simeq 10^6$ m/s, which plays the role of the speed of light in the original Dirac equation.

a Lagrangian for the OSp(2|2) superconnection in 2+1 dimensions. As a result, they found no gravitini, despite supersymmetry appears as an off-shell (gauge) symmetry of the Lagrangian. The latter, apart from a torsional term, describes a standard theory for a charged Dirac particle minimally coupled to a U(1) potential and interacting with a fixed geometry. Indeed, since the model is three dimensional, the metric (that is to say the vielbein) has no degrees of freedom.
The (possibly massive) spin 1/2 fields χ_I^{AVZ} are related to the gauge connection $\Psi_{I\mu}$ of the odd symmetries through the so-called *matter ansatz*:

$$\Psi_{I\mu} = \mathrm{i}\gamma_i e^i{}_\mu \chi_I^{AVZ} \,, \tag{1.3.3}$$

$e^i{}_\mu$ being the rigid dreibein of the spacetime where the Chern-Simons Lagrangian is integrated on. By implementing (1.3.3), we are, in fact, projecting out the spin 3/2 component of the Rarita-Schwinger field and keeping the spin 1/2 one, as can be understood from the expression fulfilled by Ψ_μ,

$$\left(\delta^\mu_\nu - \frac{1}{3}\gamma_\nu\gamma^\mu\right)\Psi_\mu = 0 \,, \tag{1.3.4}$$

where $\left(\delta^\mu_\nu - \frac{1}{3}\gamma_\nu\gamma^\mu\right)$ is identified with the spin 3/2 projector. Thus, it can be concluded that the unconventional supersymmetry uses the discarded spin 1/2 sector of supergravity. In the context of condensed matter systems, the dreibein can be thought as a non dynamical background, even though not necessarily trivial, provided by the material underlayer on which the fermionic excitations propagate.
The emergence of such dreibein in (1.3.3) resembles the introduction of a metric through a covariant gauge fixing, required to implement the BRST quantisation procedure. In fact, as we will further explore in Chapter 3, this intuition inspired the paper [55], where the matter ansatz was tried to be read as a gauge fixing of a Chern-Simons theory.

Chapter 2

$\mathcal{N} = 2$ AdS$_4$ SUGRA, holography and Ward identities

In Section 1.2 the essential notions about the gauge/gravity duality were introduced. In particular, we mentioned the issues of holographic renormalisation and recovery of the supersymmetry in theories defined on a manifold with a boundary. We reviewed two different approaches to the topics: the standard renormalisation scheme and the topological regularisation. The latter allows to obtain the same results as the former in pure gravity, whereas its formalism was partially extended to supergravity in [41]. Indeed, the counterterms needed to restore the local supersymmetry were found for the cases of pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS$_4$ supergravities. However, while the topological regularisation was shown to be able to renormalise the bulk action for the pure gravity case, the same was not been proven yet for its supersymmetric extension.
The purpose of the investigation presented in this Chapter, based on [56], is to generalise the holographic analysis of [26] to an extended supergravity, namely to a pure $\mathcal{N} = 2$ model[1], by analysing the consistency of the boundary theory. In order to do so, we show that the Ward identities of the dual field theory are satisfied, as expected for a SCFT in three dimensions. This result would prove the actual renormalisation of the theory.

As far as the asymptotic symmetries and the gauge fixing conditions defining them are concerned, we shall keep our analysis as general as possible. More precisely, we shall be taking a "cautious approach", only imposing those gauge fixing conditions which appear to be strictly necessary for the consistent definition of the asymptotic symmetries. The reason for this relies on one of the motivations which have inspired the present analysis, namely the application of the AdS$_4$/CFT$_3$ holographic approach to the study of the AVZ model, described in Section 1.3. The latter has been embedded, as a boundary theory, in pure $\mathcal{N} = 2$ AdS$_4$ supergravity in [57], although a fully fledged holographic correspondence has not been developed yet. The present investigation represents a preliminary result in this direction. Having this in mind, we will avoid imposing the constraint $\gamma^\mu \psi_\mu = 0$ on the gravitino field at the boundary since, in the AVZ model, this condition has to be relaxed,

[1]Some aspects of the minimal $\mathcal{N} = 2$ gauged supergravity in the context of holography were already discussed in [27].

as the dynamical fermion of the theory is identified with the contraction $\gamma^\mu\psi_\mu$ itself. This fermion satisfies a Dirac equation and, as previously mentioned, was shown to be well-suited for the description of the electronic properties of graphene-like materials near the Dirac points [54, 58]. Holographically embedding the AVZ model in $\mathcal{N}=2$ AdS$_4$ supergravity and eventually in $\mathcal{N}>2$ theories paves the way for a top-down approach to the study of this condensed-matter system in a gauge/gravity framework.

This Chapter is organised as follows. In Section 2.1 we review the asymptotic symmetries in Einstein AdS$_4$ gravity for purpose of introducing the first order formalism and in Section 2.2 we summarise the geometric approach to pure $\mathcal{N}=2$ AdS$_4$ supergravity in the presence of a boundary. Sections 2.3 and 2.4 are devoted to the near-boundary analysis of supergravity fields and local parameters. Then, in Section 2.5, we write out the superconformal currents and Ward identities, proving that the latter are indeed satisfied off-shell on the curved background when the bulk equations of motion are imposed. We conclude with a discussion of the obtained results and some final remarks. Useful formulas and conventions are gathered in Appendix A.1, while details on computations are collected in Appendix A.2 and Appendix A.3.

2.1 Asymptotic symmetries in Einstein AdS$_4$ gravity

We start our discussion with a review of the results in pure AdS gravity and reformulating them in first order framework.

Asymptotically AdS spaces $\mathcal{M}^D$ in $D=d+1$ dimensions are conformally compact Einstein spaces [19] which can be described in terms of local coordinates $x^{\hat\mu}=(x^\mu,x^d)$, x^μ ($\mu=0,\dots d-1$) and $z=x^d$ being, respectively, local coordinates on the boundary $\partial\mathcal{M}$ and the radial coordinate. In this frame, the asymptotic AdS boundary is located at $z=0$. In a neighborhood of $z=0$, asymptotically AdS backgrounds admit a metric $\hat g_{\hat\mu\hat\nu}$ (with a mostly negative signature) in the Fefferman-Graham (FG) form,[2]

$$\mathrm{d}s^2=\hat g_{\hat\mu\hat\nu}\,\mathrm{d}x^{\hat\mu}\mathrm{d}x^{\hat\nu}=\frac{\ell^2}{z^2}\left(-\,\mathrm{d}z^2+g_{\mu\nu}(x,z)\,\mathrm{d}x^\mu\mathrm{d}x^\nu\right)\,, \tag{2.1.1}$$

where ℓ is the AdS radius, $g_{\mu\nu}$ is regular on the boundary and admits a power expansion in the radial coordinate z,

$$g_{\mu\nu}=g_{(0)\mu\nu}(x)+\frac{z^2}{\ell^2}\,g_{(2)\mu\nu}(x)+\cdots\,. \tag{2.1.2}$$

In pure AdS gravity, only even powers in z appear in the series, until the order z^{d-1}. In general, addition of matter fields, as is the case in supergravity, requires more general powers in the z-expansion of the metric, depending on the value of the AdS mass of the field. By solving order by order the Einstein equations, the coefficients in the expansion are

[2]The most general asymptotically AdS metric contains also the subleading $\hat g_{z\mu}$ terms, in particular $\hat g_{z\mu}=\mathcal{O}(z)$ in three dimensions [59] and $\hat g_{z\mu}=\mathcal{O}(z^2)$ in four dimensions [60]. They can always be set to zero by choosing FG coordinate frame on a patch near the boundary.

determined as local functions or derivatives of $g_{(0)\mu\nu}$. For instance, $g_{(2)\mu\nu}$ depends linearly on the curvature, in a combination that produces the boundary Schouten tensor $\mathcal{S}_{\mu\nu}(g_{(0)})$,

$$g_{(2)\mu\nu} = \ell^2 \mathcal{S}_{\mu\nu} = \ell^2 \left(\mathring{\mathcal{R}}_{\mu\nu} - \frac{1}{2(d-1)}\, g_{(0)\mu\nu}\, \mathring{\mathcal{R}} \right) , \tag{2.1.3}$$

where $\mathring{\mathcal{R}}^{\mu}{}_{\nu\lambda\sigma}(g_{(0)})$ is the boundary Riemann curvature and $\mathring{\mathcal{R}}_{\mu\nu}$ and $\mathring{\mathcal{R}}$ are the corresponding Ricci tensor and Ricci scalar, respectively.[3] On top of this, there is a term $z^d \log z$ in odd spacetime dimensions D. Furthermore, only the local part of the mode $g_{(d)\mu\nu}$ can be resolved from the equations of motion, as it is proportional to the holographic stress tensor of the theory [19, 22].

The invariance of (2.1.1) under radial diffeomorphisms leads to the Penrose-Brown-Henneaux (PBH) transformations [61]. The full set of residual symmetries includes, besides the PBH transformations, the boundary transversal diffeomorphisms. The latter have the form of asymptotic symmetries, namely their parameters take value on $\partial\mathcal{M}$ and the AdS gravity is invariant under the action of these transformations at asymptotic infinity.

In an explicit form, by using the definition of the Lie derivative $\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \pounds_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}}$ for diffeomorphisms generated by the parameters $\hat{\xi}^{\hat{\mu}}$, the FG gauge fixing implies

$$\begin{aligned} \delta\hat{g}_{zz} &= 0 \quad \Rightarrow \quad \hat{\xi}^z = z\sigma(x)\,, \\ \delta\hat{g}_{\mu z} &= 0 \quad \Rightarrow \quad \hat{\xi}^\mu = \xi^\mu(x) + \frac{z^2}{2\ell}\, g_{(0)}^{\mu\nu}\partial_\nu\sigma + \mathcal{O}(z^4)\,, \end{aligned} \tag{2.1.4}$$

where $\xi^\mu(x)$ and $\sigma(x)$ are arbitrary local parameters on the boundary.
From $\delta\hat{g}_{\mu\nu} = \frac{\ell^2}{z^2}\,\delta g_{\mu\nu}$, we obtain the transformation law of the first terms in the asymptotic expansion of (2.1.2)

$$\begin{aligned} \delta g_{(0)\mu\nu} &= \pounds_\xi g_{(0)\mu\nu} - 2\sigma\, g_{(0)\mu\nu}\,, \\ \delta g_{(2)\mu\nu} &= \pounds_\xi g_{(2)\mu\nu} - \ell\nabla^{(0)}_{(\mu}\nabla^{(0)}_{\nu)}\sigma\,. \end{aligned} \tag{2.1.5}$$

In the first equation of (2.1.5), it is clear that radial diffeomorphisms induce Weyl transformations on the boundary described by the parameter $\sigma(x)$. This purely kinematic treatment allows to determine the local part of the coefficients in the series (2.1.2), without resorting to the asymptotic resolution of the field equations. In fact, it is carried out by integrating the Weyl parameter from the transformation law.
Our interest in the asymptotic symmetries is due to the fact that they produce conservation laws which are mapped into holographic Ward identities for the boundary CFT.

Holographic gauge fixing in first order formalism

Concerning our specific case, we work in first order formalism in $D = 4$, where the independent fields are 1-forms on $\mathcal{M}^4$. Indeed, one has the vielbein $V^a = V^a_{\hat{\mu}}(x)\,\mathrm{d}x^{\hat{\mu}}$, stemmed from the metric $\hat{g}_{\hat{\mu}\hat{\nu}} = \eta_{ab}\, V^a_{\hat{\mu}} V^b_{\hat{\nu}}$ (with the Minkowski metric η_{ab}), and the spin

[3] The conventions adopted on curvatures can be found in Appendix A.1.

connection $\hat{\omega}^{ab}=\hat{\omega}^{ab}_{\hat{\mu}}(x)\,\mathrm{d}x^{\hat{\mu}}$. World indices on four dimensional spacetime are denoted by hatted Greek letters $\hat{\mu},\hat{\nu},\ldots=0,1,2,3$, whereas the corresponding anholonomic tangent space indices are labeled by Latin letters $a,b,\ldots=0,1,2,3$.

Besides general coordinate transformations $\delta x^{\hat{\mu}}=-\hat{\xi}^{\hat{\mu}}$, which define local translations with parameters $p^a=\hat{\xi}^{\hat{\mu}}V^a_{\hat{\mu}}$, the theory is endowed with local Lorentz invariance, whose parameters are $j^{ab}=-j^{ba}$. The AdS gravity in first order formalism is invariant under the general transformations[4]

$$\begin{aligned}\delta V^a &= \hat{\mathcal{D}}p^a-j^{ab}V_b+i_p\hat{T}^a\,,\\ \delta\hat{\omega}^{ab} &= \hat{\mathcal{D}}j^{ab}+\frac{2}{\ell^2}\,p^{[a}V^{b]}+i_p\hat{R}^{ab},\end{aligned}\tag{2.1.6}$$

where $\hat{\mathcal{D}}(\hat{\omega})$ is the Lorentz-covariant derivative, $\hat{\mathcal{R}}^{ab}(\hat{\omega})$ is the Lorentz curvature and $\hat{T}^a=\hat{\mathcal{D}}V^a$ is the torsion 2-form. We have also introduced the SO$(2,3)$ curvature $\hat{R}^{ab}=\hat{\mathcal{R}}^{ab}-\frac{1}{\ell^2}\,V^aV^b=\frac{1}{2}\hat{R}^{ab}_{\hat{\mu}\hat{\nu}}\mathrm{d}x^{\hat{\mu}}\mathrm{d}x^{\hat{\nu}}$ and the contraction operators $i_p\hat{R}^{ab}=p^cV^{\hat{\nu}}_{\ c}\hat{R}^{ab}_{\hat{\nu}\hat{\mu}}\mathrm{d}x^{\hat{\mu}}$, $i_p\hat{T}^a$. For the non supersymmetric case, discussed in this Section, we will assume that the gravitational field is torsionless, namely $i_p\hat{T}^a=0$.

In order to extend the discussion to AAdS spacetimes in first order formalism, we have to specify the form of V^a and $\hat{\omega}^{ab}$. To this end, we have ten local parameters (p^a,j^{ab}) at our disposal to gauge fix. This *holographic gauge fixing* will provide the radial expansion of gauge fields and parameters. Furthermore, the residual transformations (which leave invariant the gauge fixing) have to induce boundary Weyl dilations and transformations of boundary fields, which, in turn, lead to the conservation laws.
In this framework, the radial components of the gravitational fields are considered as Lagrange multipliers, similarly to the lapse and shift functions in the Arnowitt–Deser–Misner (ADM) formulation of gravity [62][5]. The simplest choice $V^a_z=0$, $\hat{\omega}^{ab}_z=0$ leads to a trivial theory on the boundary with an non invertible vielbein.
On the contrary, a suitable gauge fixing for spacetime diffeomorphisms p^a and Lorentz transformation j^{ab} is

$$V^a_z=\frac{\ell}{z}\,\delta^a_3,\qquad \hat{\omega}^{ab}_z=0\,.\tag{2.1.7}$$

The latter conditions, in principle, are sufficient to determine local symmetries. However, in AdS space, the vielbein should be chosen so that it reproduces the FG metric (2.1.1). For this reason, we assume an adapted frame where the boundary is orthogonal to the radial coordinate,

$$V^3_{\ \mu}=0\,.\tag{2.1.8}$$

The latter condition can be relaxed as long as the fall-off of the field $V^3_{\ \mu}(x)$ is consistent with the behaviour of AAdS spaces. As shown in [36, 63], this field plays a role in the

[4] This transformation law comes from the combination of local Lorentz transformations and the Lie derivative $\pounds_pA=\mathrm{d}\,(i_pA)+i_pF$, where A and F denote, respectively, a generic gauge field and its associated field strength.

[5] Radial expansions and holography in gravitational theories on Riemann-Cartan spaces were developed in [37] and applied, for instance, in three [37–39], four [40], and five [37] bulk dimensions for different setups. They were discussed for arbitrary dimensions in [36].

explicit construction of the conformal algebra for the dual CFT. By setting V^3_μ to zero, the Weyl rescalings of the boundary are still there, but their realisation become non linear, as the associated field turns into a composite field.

As mentioned before, the choice (2.1.7) is holographic whether it produces a radial expansion of the boundary fields. Let us denote the $3+1$ decomposition of Lorentz indices as $a=(i,3)$ $(i=0,1,2)$ and use the following convention for the Levi-Civita tensor on $\mathcal{M}^4$ projected to the boundary $\partial\mathcal{M}$,

$$\epsilon^{ijk3} = -\epsilon^{ijk}\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = -1\,. \tag{2.1.9}$$

In AAdS spacetimes, the vielbein behaves as

$$V^i{}_\mu = \frac{\ell}{z}\,\hat{E}^i{}_\mu(x,z)\,, \tag{2.1.10}$$

where $\hat{E}^i{}_\mu$ is finite at the boundary $z=0$ and it can be, in turn, expanded in a power series in its vicinity,

$$\hat{E}^i{}_\mu \;=\; E^i_{(0)\ \mu} + \frac{z^2}{\ell^2}\,E^i_{(2)\ \mu} + \frac{z^3}{\ell^3}\,E^i_{(3)\ \mu} + \mathcal{O}(z^4)\,. \tag{2.1.11}$$

We rename the coefficients $E^i_{(0)\ \mu} \equiv E^i{}_\mu$, $E^i_{(2)\ \mu} \equiv S^i{}_\mu$ and $E^i_{(3)\ \mu} \equiv \tau^i{}_\mu$, due to physical implications they will have later. Hence, the expansion (2.1.11) becomes

$$\begin{aligned} \hat{E}^i{}_\mu &= E^i{}_\mu + \frac{z^2}{\ell^2}\,S^i{}_\mu + \frac{z^3}{\ell^3}\,\tau^i{}_\mu + \mathcal{O}(z^4)\,, \\ \hat{E}^\mu_i &= E^\mu_i - \frac{z^2}{\ell^2}\,S^\mu_i - \frac{z^3}{\ell^3}\,\tau_i{}^\mu + \mathcal{O}(z^4)\,, \end{aligned} \tag{2.1.12}$$

where E^μ_i is the inverse of the vielbein $E^i{}_\mu$[6]. The latter two tensors project the indices between the boundary spacetime and its tangent space and satisfy the relations

$$e = \det[V^a_{\hat{\mu}}] = \frac{\ell^4}{z^4}\,\hat{e}_3\,, \quad \hat{e}_3 = \det[\hat{E}^i{}_\mu]\,, \quad e_3 \equiv \det[E^i{}_\mu]\,. \tag{2.1.13}$$

Let us notice that the linear terms in z are absent in the induced vielbein $\hat{E}^i{}_\mu$, in order to reproduce $g_{(1)\mu\nu}=0$ in pure gravity. Furthermore, it is convenient to make use of the residual Lorentz transformations to make $S^{ij}=S^i{}_\mu E^{\mu j}$ and $\tau^{ij}=\tau^i{}_\mu E^{\mu j}$ symmetric, namely to set $S^{[ij]}=0$ and $\tau^{[ij]}=0$ [26]. If the Lorentz parameters at the boundary are expanded as

$$j^{ij} = \theta^{ij} + \frac{z}{\ell}\,j^{ij}_{(1)} + \frac{z^2}{\ell^2}\,j^{ij}_{(2)} + \frac{z^3}{\ell^3}\,j^{ij}_{(3)} + \mathcal{O}(z^4)\,, \tag{2.1.14}$$

[6]Strictly speaking, the inverse vielbein $(E^{-1})^\mu_i \equiv E^\mu_i$ has the property $E^\mu{}_i = g^{\mu\nu}_{(0)}\eta_{ij}E^j{}_\nu = E_i{}^\mu$, following from the invertibility and symmetric properties of the metric. It implies that one can overlook the order of the indices in the vielbein and its inverse. The same argument holds for the bulk vielbein $V^i{}_\mu$ and its inverse V^μ_i, but *not* for the higher order terms in the expansion, that are not necessarily invertible.

from the Lorentz transformations (2.1.6) we find $j^{ij}_{(1)}=0$ and

$$\begin{aligned} \delta_j E^i_\mu &= -\theta^{ij}E_{j\mu}\,, & \delta_j S^i{}_\mu &= -\theta^{ij}S_{j\mu} - j^{ij}_{(2)}E_{j\mu}\,, \\ \delta_j E^{\mu i} &= -\theta^{ij}E^\mu{}_j\,, & \delta_j \tau^i{}_\mu &= -\theta^{ij}\tau_{j\mu} - j^{ij}_{(3)}E_{j\mu}\,. \end{aligned} \tag{2.1.15}$$

Here, $\theta^{ij}(x)$ are an asymptotic parameters which will become a holographic symmetry. On the contrary, the antisymmetric parts of S^{ij} and τ^{ij} are independent of θ^{ij},

$$\delta_j S^{[ij]} = -j^{ij}_{(2)}\,, \qquad \delta_j \tau^{[ij]} = -j^{ij}_{(3)}\,. \tag{2.1.16}$$

Therefore, they are related only to the subleading Lorentz transformations and they can always be set to zero,

$$S^{[ij]}=0\,, \qquad \tau^{[ij]}=0\,. \tag{2.1.17}$$

However, we will not assume yet that $j^{ij}_{(2)}$ and $j^{ij}_{(3)}$ vanish, since they might not be independent parameters. We will come back to this issue later, after all independent asymptotic symmetries have been identified (see (2.1.51)).

In fact, the above procedure can be extended to make all coefficients in the expansion of $V^i{}_\mu$ symmetric. Without going into details, θ^{ij} can be shown to decouples from the transformation of $E^{[ij]}_{(n)} \equiv E^{\mu[j}E^{i]}_{(n)\mu}$, which implies that we are always allowed to set $E^{[ij]}_{(n)}=0$ for $n\geq 0$. As a net result, all modes $E^i_{(n)\mu}$ can be symmetric tensors,

$$E^{[ij]}_{(n)}=0\,, \qquad n\geq 0\,. \tag{2.1.18}$$

Hence, the expansion defined by the above considerations is consistent with the FG frame (2.1.1) and

$$\begin{aligned} g_{(0)\mu\nu} &= E_{i\nu}E^i{}_\mu\,, \\ g_{(2)\mu\nu} &= 2S_{\mu\nu} = \ell^2\mathcal{S}_{\mu\nu}\,, \\ g_{(3)\mu\nu} &= 2\tau_{\mu\nu}\,. \end{aligned} \tag{2.1.19}$$

Recalling that in Einstein AdS gravity we know the expression of the coefficients $g_{(n)\mu\nu}$ $(n>0)$ in terms of the source $g_{(0)\mu\nu}$ [22, 61], we are able to identify $E^i{}_\mu$ as the vielbein at the conformal boundary, $S^i{}_\mu = \frac{\ell^2}{2}\mathcal{S}^i{}_\mu$ as proportional to the Schouten tensor and $\tau^i{}_\mu$ as the holographic stress tensor.

On the other hand, in absence of supersymmetry, the torsion constraint $\hat{\mathcal{D}}V^a=0$ determines the spin connection to be (see (A.1.1))

$$\hat{\omega}^{ab}_{\hat{\mu}} = V^{\hat{\nu}b}\left(-\partial_{\hat{\mu}}V^a_{\hat{\nu}} + \hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}}V^a_{\hat{\lambda}}\right)\,. \tag{2.1.20}$$

In our notation, $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}}$ is the affine Levi-Civita connection in the bulk, which is symmetric in $(\hat{\mu}\hat{\nu})$ and torsionless. The radial components of the spin connection are consistent with the gauge fixing (2.1.7), once one assumes (2.1.18) is satisfied. The boundary components of the spin connection become

$$\hat{\omega}^{ij}_\mu \;=\; \hat{E}^{\nu j}\left(-\partial_\mu\hat{E}^i{}_\nu + \mathring{\Gamma}^\lambda_{\nu\mu}(g)\hat{E}^i{}_\lambda\right) = \mathring{\omega}^{ij}_\mu(x,z)\,,$$

$$\hat{\omega}^{i3}_\mu \;=\; \frac{1}{z}\,\hat{E}^i{}_\mu - \frac{1}{2}\,k_{\mu\nu}\hat{E}^{\nu i}\,, \tag{2.1.21}$$

where $\mathring{\omega}^{ij}_\mu(x,0)=\mathring{\omega}^{ij}_\mu(E)$ is the torsionless spin connection on the boundary, $\mathring{\Gamma}^\lambda_{\nu\mu}(g)$ is the affine Levi-Civita connection at the boundary and we defined the auxiliary tensor

$$k_{\mu\nu}\equiv\partial_z g_{\mu\nu}=\mathcal{O}(z)\,,\qquad \partial_z g^{\mu\nu}=-k^{\mu\nu}\,. \tag{2.1.22}$$

Both $\mathring{\Gamma}^\lambda_{\nu\mu}(g)$ and $k_{\mu\nu}$ are regular quantities at $z=0$.
In a more explicit form, we can write

$$\begin{aligned}
\hat{\omega}^{ij}_\mu &= \mathring{\omega}^{ij}_\mu(x,z)=\mathring{\omega}^{ij}_\mu(x)+\frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu}(S,E)+\frac{z^3}{\ell^3}\,\omega^{ij}_{(3)\mu}(\tau,E)+\mathcal{O}(z^4)\,,\\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,E^i{}_\mu-\frac{z}{\ell^2}\,\tilde{S}^i{}_\mu-\frac{2z^2}{\ell^3}\,\tilde{\tau}^i{}_\mu+\mathcal{O}(z^3)\,,
\end{aligned} \tag{2.1.23}$$

where

$$\tilde{S}^i{}_\mu\equiv S_\mu{}^i=S^i{}_\mu\,,\qquad \tilde{\tau}^i{}_\mu\equiv\frac{1}{4}\left(\tau^i{}_\mu+3\tau_\mu{}^i\right)=\tau^i{}_\mu\,, \tag{2.1.24}$$

the last step being valid only upon imposing the Lorentz gauge fixing (2.1.18). Therefore, the tensors $\tilde{S}^i{}_\mu$ and $\tilde{\tau}^i{}_\mu$ can be chosen symmetric and equal to $S^i{}_\mu$ and $\tau^i{}_\mu$ in pure AdS gravity. We will see later, in (2.3.15), that the group theory definition of the boundary Schouten tensor is $\mathcal{S}^i{}_\mu=\frac{1}{\ell^2}\,(S^i{}_\mu+\tilde{S}^i{}_\mu)$ and it reduces to $\frac{2}{\ell^2}\,S^i{}_\mu$ after the identification in (2.1.24).

When the bulk torsion vanishes, the 1-forms $\omega^{ij}_{(2)}=\omega^{ij}_{(2)\mu}\,\mathrm{d}x^\mu$ and $\omega^{ij}_{(3)}=\omega^{ij}_{(3)\mu}\,\mathrm{d}x^\mu$ are not arbitrary, but they can be expressed in terms of $S^i=S^i{}_\mu\,\mathrm{d}x^\mu$ and $\tau^i=\tau^i{}_\mu\,\mathrm{d}x^\mu$ as

$$E_j\wedge\omega^{ij}_{(2)}=\mathring{\mathcal{D}}S^i\,,\qquad E_j\wedge\omega^{ij}_{(3)}=\mathring{\mathcal{D}}\tau^i\,, \tag{2.1.25}$$

where $\mathring{\mathcal{D}}$ denotes the covariant derivative with respect to the connection $\mathring{\omega}^{ij}_\mu(E)$.

Eventually, let us analyse the fall-off of the curvature. Asymptotically AdS spaces require the curvature to be asymptotically constant. A direct checkup confirms that the near-boundary form of the AdS curvature is

$$\begin{aligned}
\hat{R}^{i3}_{\mu\nu} &= -z\,\mathcal{C}^i{}_{\mu\nu}+\mathcal{O}(z^2)\,, & \hat{R}^{i3}_{\mu z} &= \frac{3z}{\ell^3}\,\tau^i{}_\mu+\mathcal{O}(z^2)\,,\\
\hat{R}^{ij}_{\mu\nu} &= W^{ij}_{\mu\nu}-\frac{12z}{\ell^3}\,E^{[i}_{\ [\mu}\tau^{j]}_{\ \nu]}+\mathcal{O}(z^2)\,, & \hat{R}^{ij}_{\mu z} &= -\frac{2z}{\ell^2}\,\omega^{ij}_{(2)\mu}-\frac{3z^2}{\ell^3}\,\omega^{ij}_{(3)\mu}+\mathcal{O}(z^3)\,,
\end{aligned} \tag{2.1.26}$$

where $\mathcal{C}^i=\frac{1}{2}\,\mathcal{C}^i{}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=\mathring{\mathcal{D}}\mathcal{S}^i$ is the three dimensional Cotton tensor. The latter appears once one exploits (2.1.25) to substitute $\omega^{ij}_{(2)}$ in the expression $\hat{R}^{i3}=-\frac{z}{\ell^2}\left(\mathring{\mathcal{D}}\tilde{S}^i+E_j\wedge\omega^{ij}_{(2)}\right)+\mathcal{O}(z^2)$. Similarly, $\hat{R}^{ij}$ depends on the tensor $\tau^i+2\tilde{\tau}^i$ and reduces to the above result upon setting $\tau^i=\tilde{\tau}^i$.
Furthermore, the Weyl tensor vanishes in three dimensions,

$$W^{ij}=\mathring{\mathcal{R}}^{ij}-2E^{[i}\wedge\mathcal{S}^{j]}=0\,, \tag{2.1.27}$$

so that the three dimensional Bianchi identity can equivalently be written as

$$E^{[i}\wedge\mathcal{C}^{j]}=0\,, \tag{2.1.28}$$

yielding a tracelessness condition for the Cotton tensor, $\mathcal{C}^{i}{}_{ij}=0$[7].

An important consequence of $W^{ij}_{\mu\nu}=0$ in three dimensions is that, from (2.1.26), we get $\hat{R}^{ab}\Big|_{z=0}=0$. In the next Sections we will explore how the latter condition is modified in the supergravity context.

Residual symmetries

The gauge fixing adopted above leads to the asymptotic form of the boundary fields (2.1.10), (2.1.12) and (2.1.23). Hence, we now look for transformations which do not change the frame choice (2.1.7). From (2.1.6) it follows

$$\begin{aligned}
0 &= \delta V^{3}_{\;z}=\partial_z p^3\,, && (2.1.29)\\
0 &= \delta V^{i}_{\;z}=\partial_z p^i+\frac{\ell}{z}\,j^{i3}\,, && (2.1.30)\\
0 &= \delta V^{3}_{\;\mu}=\partial_\mu p^3-\hat{\omega}^{i3}_{\mu}p_i+j^{i3}V_{i\mu}\,, && (2.1.31)\\
0 &= \delta\hat{\omega}^{i3}_{z}=\frac{1}{\ell z}\,p^i+\partial_z j^{i3}+i_p\hat{R}^{i3}_{z}\,, && (2.1.32)\\
0 &= \delta\hat{\omega}^{ij}_{z}=\partial_z j^{ij}+i_p\hat{R}^{ij}_{z}\,. && (2.1.33)
\end{aligned}$$

In order to solve the above equations, we need to compute the asymptotic expansion of the contraction of the AdS curvature (2.1.26)

$$\begin{aligned}
i_p\hat{R}^{i3}_{z} &= p^j\left(\frac{3z^2}{\ell^4}\,\tau^{i}{}_{j}+\mathcal{O}(z^3)\right)\,,\\
i_p\hat{R}^{i3}_{\mu} &= -p^3\left(\frac{3z^2}{\ell^4}\,\tau^{i}{}_{\mu}+\mathcal{O}(z^3)\right)+p^j\left(\frac{z^2}{\ell}\,E^{\nu}{}_{j}\mathcal{C}^{i}{}_{\mu\nu}+\mathcal{O}(z^3)\right)\,,\\
i_p\hat{R}^{ij}_{z} &= p^k\left(-\frac{2z^2}{\ell^3}\,E^{\mu}{}_{k}\omega^{ij}_{(2)\mu}-\frac{3z^3}{\ell^4}\,E^{\mu}{}_{k}\omega^{ij}_{(3)\mu}+\mathcal{O}(z^4)\right)\,. \qquad (2.1.34)
\end{aligned}$$

In light of the latter expressions, (2.1.29)–(2.1.33) acquire the form

$$\begin{aligned}
0 &= \partial_z p^3\,, && (2.1.35)\\
0 &= \partial_z j^{i3}+\frac{1}{\ell z}\,p^i+\frac{3z^2}{\ell^4}\,p^j\left(\tau^{i}{}_{j}+\mathcal{O}(z)\right)\,, && (2.1.36)\\
0 &= \partial_z p^i+\frac{\ell}{z}\,j^{i3}\,, && (2.1.37)\\
0 &= \partial_\mu p^3-\hat{\omega}^{i3}_{\mu}p_i+j^{i3}V_{i\mu}\,, && (2.1.38)\\
0 &= \partial_z j^{ij}+p^k\left(-\frac{2z^2}{\ell^3}\,E^{\mu}{}_{k}\omega^{ij}_{(2)\mu}-\frac{3z^3}{\ell^4}\,E^{\mu}{}_{k}\omega^{ij}_{(3)\mu}+\mathcal{O}(z^4)\right)\,. && (2.1.39)
\end{aligned}$$

[7]For the sake of completeness, we mention that the Cotton tensor is also covariantly constant in three dimensions. For more properties about it in Riemannian geometries, see [64].

Equation (2.1.35) can be readily solved as

$$p^3 = -\ell\sigma(x)\,, \tag{2.1.40}$$

with the boundary parameter $\sigma(x)$ introduced as an integration constant. The equations (2.1.36) and (2.1.37) can be decoupled by eliminating j^{i3} and we find the differential equation for p^i:

$$0 = \partial_z^2 p^i + \frac{1}{z}\,\partial_z p^i - \frac{1}{z^2}\,p^i - \frac{3z}{\ell^3}\,p^j\left(\tau^i{}_j + \mathcal{O}(z)\right)\,. \tag{2.1.41}$$

The solution for both parameters reads

$$\begin{aligned} p^i &= \frac{\ell}{z}\,\xi^i + \frac{z}{\ell}\,b^i + \frac{z^2}{\ell^2}\,\xi^j\tau^i{}_j + \mathcal{O}(z^3)\,, \\ j^{i3} &= \frac{1}{z}\,\xi^i - \frac{z}{\ell^2}\,b^i - \frac{2z^2}{\ell^3}\,\xi^j\tau^i{}_j + \mathcal{O}(z^3)\,, \end{aligned} \tag{2.1.42}$$

where $\xi^i(x)$ and $b^i(x)$ are new integration constants. Then (2.1.39) leads to the solution for the Lorentz parameter

$$j^{ij} = \theta^{ij} + \frac{z^2}{\ell^2}\,\xi^\mu\omega^{ij}_{(2)\mu} + \frac{z^3}{\ell^3}\,\xi^\mu\omega^{ij}_{(3)\mu} + \mathcal{O}(z^4)\,, \tag{2.1.43}$$

with $\theta^{ij}(x)$ another arbitrary function defined on the boundary and identified with the Lorentz parameter.

The last equation to be solved is the asymptotic condition (2.1.38) which –once all the previous solutions are plugged in– becomes

$$0 = \delta V^3_\mu = -\ell\partial_\mu\sigma + \frac{2}{\ell}\,\xi_i S^i{}_\mu - \frac{2}{\ell}\,b^i E_{i\mu} + \mathcal{O}(z^2)\,. \tag{2.1.44}$$

At leading order, (2.1.44) implies the parameter b^i not to be independent, i.e.

$$b_i = -\frac{\ell^2}{2}\,E^\mu{}_i\partial_\mu\sigma + S^j{}_i\xi_j\,. \tag{2.1.45}$$

Overall, the radial expansion of the gauge parameters in absence of fermions takes the form

$$\begin{aligned} p^3 &= -\ell\sigma(x)\,, \\ p^i &= \frac{\ell}{z}\,\xi^i(x) + \frac{z}{\ell}\,b^i + \frac{z^2}{\ell^2}\,\xi^j\tau^i{}_j + \mathcal{O}(z^3)\,, \\ j^{i3} &= \frac{1}{z}\,\xi^i(x) - \frac{z}{\ell^2}\,b^i - \frac{2z^2}{\ell^3}\,\xi^j\tau^i{}_j + \mathcal{O}(z^3)\,, \\ j^{ij} &= \theta^{ij}(x) + \frac{z^2}{\ell^2}\,\xi^\mu\omega^{ij}_{(2)\mu} + \frac{z^3}{\ell^3}\,\xi^\mu\omega^{ij}_{(3)\mu} + \mathcal{O}(z^4)\,, \end{aligned} \tag{2.1.46}$$

with $b^i(\sigma,\xi_j)$ given by (2.1.45).
It is worth emphasising that $\omega^{ij}_{(2)}$ and $\omega^{ij}_{(3)}$ satisfy (2.1.25). Particularly, we find that the higher order spin connection components fulfill

$$\mathcal{C}^i{}_{\mu\nu} = \frac{4}{\ell^2}\,\omega^i_{(2)[\mu\nu]}\,, \qquad \mathring{\mathcal{D}}_{[\mu}\tau^i{}_{\nu]} = E_{j[\mu}\,\omega^{ij}_{(3)\nu]}\,. \tag{2.1.47}$$

Therefore, we recognise $\sigma(x)$, $\xi^i(x)$, $\theta^{ij}(x)$ as the independent boundary parameters, which are associated, respectively, to dilations, diffeomorphisms and Lorentz transformations. This can be inferred from the variation of the boundary fields

$$\begin{aligned}
\delta E^i{}_\mu &= \mathring{\mathcal{D}}_\mu \xi^i + \sigma E^i{}_\mu - \theta^{ij} E_{j\mu}\,, \\
\delta S^i{}_\mu &= \mathring{\mathcal{D}}_\mu b^i - \sigma S^i{}_\mu - \theta^{ij} S_{j\mu} + \frac{\ell^2}{2}\,\xi^\nu\,\mathcal{C}^i{}_{\nu\mu}\,, \\
\delta \tau^i{}_\mu &= \mathring{\mathcal{D}}_\mu \left(\xi^j \tau^i{}_j\right) - 2\sigma\tau^i{}_\mu - \theta^{ij}\tau_{j\mu} + 2\xi^\nu \mathring{\mathcal{D}}_{[\nu}\tau^i{}_{\mu]}\,,
\end{aligned} \tag{2.1.48}$$

and the spin connection

$$\delta \mathring{\omega}^{ij}_\mu = \mathring{\mathcal{D}}_\mu \theta^{ij} - 2E^{\nu[i}E^{j]}{}_\mu \partial_\nu \sigma + \frac{4}{\ell^2}\left(-\xi_k E^{[i}{}_\mu S^{j]k} + \xi^{[i}S^{j]}{}_\mu\right)\,. \tag{2.1.49}$$

It is straightforward to check that the obtained residual symmetries match the usual PBH transformations (2.1.5) in the metric formalism, where the coefficient $g_{(d)\mu\nu} \equiv g_{(3)\mu\nu}$ transforms homogeneously as

$$\delta g_{(3)\mu\nu} = \pounds_\xi g_{(3)\mu\nu} - \sigma g_{(3)\mu\nu}\,. \tag{2.1.50}$$

After the computation of the full set of asymptotic parameters (2.1.45)–(2.1.46) and the transformation law of the boundary fields (2.1.48), we are now ready to come back to the conditions (2.1.17) and discuss their consistency with respect to the residual transformations. As we can see from the expansion (2.1.43), if we restrict to Lorentz transformations, namely we set $p^i = 0$, (2.1.16) implies $j^{ij}_{(2)} = j^{ij}_{(3)} = 0$. According to (2.1.43), the choices $S^{[ij]} = 0$, $\tau^{[ij]} = 0$ are naturally preserved by the Lorentz part of the residual symmetry group. We need, however, to check the consistency of these conditions against a generic residual symmetry transformation, including the diffeomorphisms on the boundary, parametrised by $\xi^i = E^i{}_\mu\,\xi^\mu$. For instance, the condition $S^{[ij]} = 0$ yields

$$\begin{aligned}
\delta S^{[ij]}\Big|_{S^{[ij]}=0} &= \Big(-\theta^i{}_k\,S^{[kj]} + \theta^j{}_k\,S^{[ki]} - 2\sigma S^{[ij]} + E^{i\mu}\,S^{[jk]}\mathring{\mathcal{D}}_\mu \xi_k \\
&\qquad - E^{j\mu}\,S^{[ik]}\mathring{\mathcal{D}}_\mu \xi_k + \omega_{(2)}{}^{[i}{}_k{}^{j]}\,\xi^k - j^{ij}_{(2)} + \frac{\ell^2}{4}\,\mathcal{C}^{kji}\,\xi_k\Big)\Big|_{S^{[ij]}=0} \\
&= -3\,\omega^{[ij|k]}_{(2)}\,\xi_k = -\frac{3\ell^2}{4}\,\mathcal{C}^{[i|jk]}\,\xi_k = 0\,.
\end{aligned} \tag{2.1.51}$$

As a result, we see that the condition $S^{[ij]} = 0$ is consistent, since its variation is proportional to $\mathcal{C}_{[i|jk]}$, which in turn vanishes. A similar analysis, which we avoid to report, can be made for the condition $\tau^{[ij]} = 0$.

Conservation laws for conformal symmetry

In Riemann-Cartan AdS gravity, the leading orders of the bulk fields $E^i{}_\mu$, ω^{ij}_μ are arbitrary functions on the three dimensional boundary: they act as sources in the dual field theory. From (1.2.16), we can generalise the quantum effective action to first order formalism,

$$W[E,\omega] = -\mathrm{i}\ln Z[E,\omega]\,, \tag{2.1.52}$$

in such a way that the (external) gravitational sources $E^i{}_\mu$ and ω^{ij}_μ are coupled to currents on $\partial\mathcal{M}$, namely the energy-momentum tensor $J^\mu{}_i$ and the spin current $J^\mu{}_{ij}$, expressed as

$$\delta W = \int \left(\delta E^i \wedge J_i + \frac{1}{2}\,\delta\omega^{ij} \wedge J_{ij}\right) , \tag{2.1.53}$$

in the formalism of differential forms. Here, we have introduced the 2-form currents $J = \frac{1}{2}\,J_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, while the usual 1-form Noether currents ${}^*J = J_\mu\,\mathrm{d}x^\mu$ are defined as their Hodge star duals

$$J^\mu = \frac{1}{2e_3}\,\epsilon^{\mu\nu\lambda}J_{\nu\lambda}\,. \tag{2.1.54}$$

Regardless of the presence of supersymmetry, the spin connection is not an independent source. For this reason the associated current vanishes, i.e. $J_{ij} = 0$. Once we plug this information in (2.1.53), together with the transformations (2.1.48) expressed in the differential form language

$$\delta E^i = \mathring{\mathcal{D}}\xi^i + \sigma E^i - \theta^{ij}E_j\,, \tag{2.1.55}$$

we get

$$0 = \delta W = \int \left[-\xi^i\mathring{\mathcal{D}}J_i + \left(\sigma E^i - \theta^{ij}E_j\right)\wedge J_i\right]\,. \tag{2.1.56}$$

This implies the following *classical conservation laws* for the conformal symmetry in $d = 3$

$$\begin{aligned}
\xi^i :\quad & 0 = \mathring{\mathcal{D}}J_i\,, && \text{(conserved } J_{\mu\nu}\text{)}\\
\sigma :\quad & 0 = E^i \wedge J_i\,, && \text{(traceless } J_{\mu\nu}\text{)}\\
\theta^{ij} :\quad & 0 = E_i \wedge J_j - E_j \wedge J_i\,. && \text{(symmetric } J_{\mu\nu}\text{)}
\end{aligned} \tag{2.1.57}$$

Let us notice that the full Weyl symmetry on the boundary is expressed in terms of the traceless Belinfante-Rosenfeld tensor $J^\mu{}_i$ and the second relation is not modified at the quantum level, since no conformal anomaly exists in three dimensions.

It is interesting to see that, as observed in [36], since ω^{ij} and $S^i = \frac{\ell^2}{2}\,\mathcal{S}^i$ are composite fields and the dilation gauge field $B = \frac{1}{\ell}\,V^3{}_\mu\mathrm{d}x^\mu$ is vanishing, the full conformal group is encoded in the conservation laws on the boundary, but it is non linearly realised. We can retrieve its linear realisation by treating ω^{ij}, S^i and B as independent fields, considering b^i as a free parameter and adding the special conformal and the dilation currents, respectively $J_{(K)i}$ and $J_{(D)}$, in the variation of the action (2.1.53) via the couplings $\delta\mathcal{S}^i \wedge J_{(K)i}$ and $\delta B \wedge J_{(D)}$.

As a result, we obtain the generalised form of the transformation laws (2.1.57)

$$\begin{aligned}
\xi^i \quad &:\quad \mathcal{D}J_i = B \wedge J_i + \mathcal{S}^j \wedge J_{ij} + \ell\mathcal{S}_i \wedge J_{(D)}\,,\\
\sigma \quad &:\quad \ell\mathrm{d}J_{(D)} = -E^i \wedge J_i + \mathcal{S}^i \wedge J_{(K)i}\,,\\
\theta^{ij} \quad &:\quad \mathcal{D}J_{ij} = 2E_{[i} \wedge J_{j]} + 2\mathcal{S}_{[i} \wedge J_{(K)j]}\,,\\
b^i \quad &:\quad \mathcal{D}J_{(K)i} = E^j \wedge J_{ij} - \ell\,E_i \wedge J_{(D)} - B \wedge J_{(K)i}\,,
\end{aligned} \tag{2.1.58}$$

where $\mathcal{D}$ is the covariant derivative with respect to the Lorentz connection $\omega^{ij} = \mathring{\omega}^{ij} - 2B^{[i} \wedge E^{j]}$. An extension of the FG formalism and enhancement of the boundary theory to include the Weyl current has been analysed in [63]. The superconformal group approach to the holographic currents issue in $d=3$ is discussed in Section 2.5.

We will extend the above analysis to the supersymmetric case in the following Sections.

2.2 Pure $\mathcal{N}=2$ AdS$_4$ supergravity

The spacetime field content of the four dimensional pure $\mathcal{N}=2$ supergravity is given by the vielbein $V^a_{\hat{\mu}}$, the gravitino $\Psi^\alpha_{\hat{\mu}A}$ (we will generally omit the spinor index $\alpha=1,\dots,4$), the SO(1,3) spin connection $\hat{\omega}^{ab}_{\hat{\mu}}$ and the graviphoton $\hat{A}_{\hat{\mu}}$. We follow the same conventions of the previous section for the latin $(a,b,\dots)$ and greek $(\hat{\mu},\hat{\nu},\dots)$ indices, whereas $A,\dots=1,2$ refer to indices in the fundamental representation of the R-symmetry group. Let us recall that the R-symmetry group for the ungauged theory is U(2), but the Fayet-Iliopoulos term in the SU(2) sector, which depends on the AdS radius ℓ as $P\propto 1/\ell$, explicitly breaks the R-symmetry to SO(2) for AdS$_4$ supergravity.
The graviphoton and gravitini are, respectively, represented by an abelian gauge field and Majorana spinors. The conventions on fermions can be found in Appendix A.1.

A geometric formulation of the theory in $\mathcal{N}=2$ *superspace*, in the presence of a negative cosmological constant and allowing for non trivial boundary conditions, was given in [41].[8] In that setting, the field content is expressed in terms of 1-forms in *superspace* $\mathcal{M}^{4|8}$ and is composed by the supervielbein (V^a,Ψ_A), defining an orthonormal basis of the $\mathcal{N}=2$ superspace, the Lorentz spin connection $\hat{\omega}^{ab}$ and the graviphoton gauge connection $\hat{A}$.
Preliminarily, let us remark that in this Section, to make contact with the outcomes of [41], to which we generally refer for the description of the bulk setting, we will first present the results in the geometric superspace approach and then we translate them in the spacetime point of view. Indeed, the whole holographic analysis will be realised within a spacetime approach to supergravity.
In the rheonomic approach, about which we discussed in Section 1.1, the superfields are functions of all the coordinates of superspace $\mathcal{M}^{4|8}(x^{\hat{\mu}},\theta^{\alpha A})$, where $x^{\hat{\mu}}$ are commuting bosonic coordinates and $\theta^{\alpha A}$ are fermionic Grassmann coordinates

$$\begin{aligned}
V^a(x,\theta) &= V^a_{\hat{\mu}}(x,\theta)\mathrm{d}x^{\hat{\mu}} + V^a_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,,\\
\hat{\omega}^{ab}(x,\theta) &= \hat{\omega}^{ab}_{\hat{\mu}}(x,\theta)\mathrm{d}x^{\hat{\mu}} + \hat{\omega}^{ab}_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,,\\
\Psi^A_\alpha(x,\theta) &= \Psi^A_{\alpha\hat{\mu}}(x,\theta)\mathrm{d}x^{\hat{\mu}} + \Psi^A_{\alpha|\beta B}(x,\theta)\mathrm{d}\theta^{\beta B}\,,\\
\hat{A}(x,\theta) &= \hat{A}_{\hat{\mu}}(x,\theta)\mathrm{d}x^{\hat{\mu}} + \hat{A}_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,.
\end{aligned}\tag{2.2.1}$$

[8] In this Chapter we generally adopt the notation of [41], where, in particular, the metric is mostly minus. However, with respect to that paper, we made some changes which make the formulas more transparent and better adapted to match the notation in three dimensions. More precisely, the graviphoton gauge connection is defined with a prefactor, $A\to -\frac{1}{\sqrt{2}}\hat{A}$, while the four dimensional Lorentz spin connection and curvature are expressed with different symbols and extra minus signs: $\omega^{ab}\to-\hat{\omega}^{ab}$, $R^{ab}\to-\hat{\mathcal{R}}^{ab}$. We will use Majorana spinors both in four as well as in three dimensions and redefine the constants appearing in the quoted paper as $L=\frac{1}{\sqrt{2}}$ and $\frac{1}{\ell}=2e=\frac{P}{\sqrt{2}}=\sqrt{-\frac{\Lambda}{3}}$, where Λ is the cosmological constant and ℓ is the AdS$_4$ radius.

Moreover, the action is written as an integral of the Lagrangian 4-form over a bosonic subspace $\mathcal{M}^4$ of the entire superspace $\mathcal{M}^{4|8}$, namely

$$I=\int_{\mathcal{M}^4\subset\mathcal{M}^{4|8}}\mathcal{L}\,. \tag{2.2.2}$$

Indeed, in the geometric framework, the Lagrangian 4-form is invariant under general coordinate transformations in superspace and supersymmetry transformations on spacetime, which are associated with diffeomorphisms in the fermionic directions of superspace; one can thus exploit "general super-coordinate transformations" to freely choose, as the bosonic submanifold of integration in superspace, any $\mathcal{M}^4\subset\mathcal{M}^{4|8}$ (see also [65] for details on this point).
The bulk Lagrangian 4-form for the pure $\mathcal{N}=2$ theory is given by[9] [41, 66]

$$\begin{aligned}\mathcal{L}^{\text{bulk}}=&\frac{1}{4}\hat{\mathcal{R}}^{ab}V^cV^d\epsilon_{abcd}+\overline{\Psi}^A\Gamma_a\Gamma_5\hat{\rho}_AV^a+\frac{\mathrm{i}}{2}\left(\hat{F}+\frac{1}{2}\,\overline{\Psi}^A\Psi^B\epsilon_{AB}\right)\overline{\Psi}^C\Gamma_5\Psi^D\epsilon_{CD}\\&-\frac{\mathrm{i}}{2\ell}\,\overline{\Psi}^A\Gamma_{ab}\Gamma_5\Psi_AV^aV^b-\frac{1}{8\ell^2}\,V^aV^bV^cV^d\epsilon_{abcd}\\&+\frac{1}{4}\left(\tilde{F}^{cd}V^aV^b\hat{F}-\frac{1}{12}\,\tilde{F}_{lm}\tilde{F}^{lm}V^aV^bV^cV^d\right)\epsilon_{abcd}\,,\end{aligned} \tag{2.2.3}$$

where we will generally omit writing of the wedge product in long expressions to lighten the notation. This Lagrangian is written in a first order approach for the gauge field $\hat{A}$.

A consistent definition of the action in the presence of non trivial boundary conditions requires the full Lagrangian to include a boundary contribution [24, 67], namely

$$\mathcal{L}=\mathcal{L}^{\text{bulk}}+\mathcal{L}^{\text{boundary}}\,. \tag{2.2.4}$$

The boundary term has to ensure both a well-defined action principle (for suitable AAdS boundary conditions) and the regularity of the full action in the asymptotic region. Holographic techniques renormalise a gravity theory in a covariant way by setting a cut-off on the spacetime at the finite radius z. The variation of the action is expressible in terms of the variation of the sources at the conformal boundary. Due to the asymptotic behaviour of the fields, the variational problem on the boundary sources induces infinities which have to be cancelled by the introduction of counterterms. Asymptotic regularity, then, is dictated by a well-posed variational principle [68]. Holographic renormalisation was first introduced in [21] and further developed in [19, 22, 23], while the counterterms for Einstein-Hilbert AdS gravity were obtained in [69–72][10]. The prescription has been applied to supergravity theories, as well, in particular for computation of the superconformal anomaly [27] (for computations in the field theory side see, e.g., [73, 74]).

In our context, it is more convenient to adopt a geometric approach to the renormalisation problem, originally formulated in [31–33], which considers the addition of the topological Euler-Gauss-Bonnet term to the bulk gravity action. The corresponding coupling

[9] The definition of the quantities in (2.2.3) can be found in (2.2.8) and (2.2.13).

[10] However, the counterterm prescription given in these references does not deal with the logarithmic divergence coming from the bulk action.

is fixed by requiring the vanishing of the AdS curvature on the boundary. In [29, 30] it was shown that adding this topological term in four dimensions is equivalent to the holographic renormalisation program.[11] Since the method is deeply rooted in first order formulation, clearly it is particularly suitable for embedding holographic renormalisation in supergravity and, specially, within the geometrical approach in superspace.

A generalisation of the approach to the supersymmetric case was given in [41] and analogous results for the $\mathcal{N}=1$ case were previously obtained in [26]. The supersymmetric extension of the Euler-Gauss-Bonnet term is unique for a given theory with $\mathcal{N}$ supersymmetries, and it is a total derivative, corresponding to a boundary term taking values in the fermionic directions of superspace[12].

For the case at hand, the boundary Lagrangian is given by the supersymmetric generalisation of the Euler-Gauss-Bonnet term,

$$\mathcal{L}^{\text{boundary}} = -\frac{\ell^2}{8}\left(\hat{\mathcal{R}}^{ab}\hat{\mathcal{R}}^{cd}\epsilon_{abcd} + \frac{8\mathrm{i}}{\ell}\,\hat{\bar{\rho}}^A\Gamma_5\hat{\rho}_A - \frac{2\mathrm{i}}{\ell}\,\hat{\mathcal{R}}^{ab}\overline{\Psi}^A\Gamma_{ab}\Gamma_5\Psi_A + \frac{4\mathrm{i}}{\ell^2}\,\mathrm{d}\hat{A}\,\overline{\Psi}^A\Gamma_5\Psi^B\epsilon_{AB}\right). \tag{2.2.5}$$

The supercurvatures appearing in (2.2.3) and (2.2.5) are defined by

$$\begin{aligned}
\hat{\mathcal{R}}^{ab} &= \mathrm{d}\hat{\omega}^{ab} + \hat{\omega}^{ac}\wedge\hat{\omega}_c{}^b\,, && (2.2.6)\\
\hat{\rho}_A &= \hat{\mathcal{D}}\Psi_A - \frac{1}{2\ell}\,\hat{A}\epsilon_{AB}\wedge\Psi^B = \mathrm{d}\Psi_A + \frac{1}{4}\,\Gamma_{ab}\,\hat{\omega}^{ab}\wedge\Psi_A - \frac{1}{2\ell}\,\hat{A}\epsilon_{AB}\wedge\Psi^B\,, && (2.2.7)\\
F &= \mathrm{d}\hat{A} - \overline{\Psi}^A\wedge\Psi^B\epsilon_{AB}\,. && (2.2.8)
\end{aligned}$$

Most notably, the same full Lagrangian can be equivalently rewritten in terms of the OSp(2|4) curvatures, which are defined as

$$\begin{aligned}
\hat{\mathbf{R}}^{ab} &= \hat{\mathcal{R}}^{ab} - \frac{1}{\ell^2}\,V^aV^b - \frac{1}{2\ell}\,\delta^{AB}\overline{\Psi}_A\Gamma^{ab}\Psi_B\,,\\
\hat{\mathbf{R}}^{a} &= \hat{\mathcal{D}}V^a - \frac{\mathrm{i}}{2}\,\overline{\Psi}^A\Gamma^a\Psi_A\,,\\
\hat{\boldsymbol{\rho}}_A &= \hat{\rho}_A - \frac{\mathrm{i}}{2\ell}\,\delta_{AB}\Gamma_a\Psi^BV^a\,,\\
\hat{\mathbf{F}} &= F\,.
\end{aligned} \tag{2.2.9}$$

When expressed in terms of the supercurvatures (2.2.9), apart from subtleties related to the extension of the action integral to superspace (see [13, 14]), the full Lagrangian acquires the following form à la MacDowell-Mansouri [34], that is quadratic in the super AdS curvatures $F^\Lambda = \left(\hat{\mathbf{R}}^a, \hat{\mathbf{R}}^{ab}, \hat{\boldsymbol{\rho}}_A, \hat{\boldsymbol{F}}\right)$,

$$\mathcal{L} = -\frac{\ell^2}{8}\hat{\boldsymbol{R}}^{ab}\wedge\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \mathrm{i}\ell\hat{\bar{\boldsymbol{\rho}}}^A\Gamma_5\wedge\hat{\boldsymbol{\rho}}_A + \frac{1}{4}\hat{\boldsymbol{F}}\wedge{}^*\hat{\boldsymbol{F}}\,. \tag{2.2.10}$$

[11] This renormalisation procedure also allows to make contact with the concept of Renormalised Volume for asymptotically hyperbolic spaces in a more mathematical framework [75].

[12] It is still an open question whether there is a topological index in the superspace associated to this invariant. A useful tool to face this problem could be the integral form approach in superspace developed in [13, 14].

The quantity ${}^*\hat{\boldsymbol{F}}$ denotes the Hodge-dual on spacetime of the field strength $\hat{\boldsymbol{F}}$, namely

$$
{}^*\hat{\mathbf{F}} = \frac{1}{2}\,{}^*\hat{\mathbf{F}}_{\hat{\mu}\hat{\nu}}\,\mathrm{d}x^{\hat{\mu}}\wedge\mathrm{d}x^{\hat{\nu}} = \frac{e}{4}\,\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\,\hat{\mathbf{F}}^{\hat{\rho}\hat{\sigma}}\,\mathrm{d}x^{\hat{\mu}}\wedge\mathrm{d}x^{\hat{\nu}}\,, \tag{2.2.11}
$$

and the 4-form Lagrangian (2.2.10) depends on the fields $\hat{\Phi}^\Lambda = (V^a, \hat{\omega}^{ab}, \Psi_A, \hat{A})$ only through their field strengths F^Λ.

The super AdS curvatures (2.2.9) satisfy on-shell the Bianchi identities

$$
\begin{aligned}
\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{ab} &= \frac{2}{\ell^2}V^{[a}\hat{\boldsymbol{R}}^{b]} + \frac{1}{\ell}\overline{\Psi}^A\Gamma^{ab}\hat{\boldsymbol{\rho}}_A\,,\\
\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{a} &= \hat{\boldsymbol{R}}^a{}_bV^b + \mathrm{i}\overline{\Psi}^A\Gamma^a\hat{\boldsymbol{\rho}}_A\,,\\
\hat{\mathcal{D}}\hat{\boldsymbol{\rho}}^{A} &= \frac{1}{2\ell}\hat{A}\epsilon^{AB}\hat{\boldsymbol{\rho}}_B - \frac{\mathrm{i}}{2\ell}\Gamma_aV^a\hat{\boldsymbol{\rho}}^A + \frac{1}{4}\hat{\boldsymbol{R}}_{ab}\Gamma^{ab}\Psi_A - \frac{1}{2\ell}\hat{\boldsymbol{F}}\epsilon^{AB}\Psi_B + \frac{\mathrm{i}}{2\ell}\Gamma_a\Psi^A\hat{\boldsymbol{R}}^a\,,\\
\mathrm{d}F &= 2\epsilon^{AB}\overline{\Psi}_A\hat{\boldsymbol{\rho}}_B\,.
\end{aligned} \tag{2.2.12}
$$

In the case at hand, the on-shell rheonomic parametrisation of the supercurvatures (2.2.9) results to be given by the following expressions,

$$
\begin{aligned}
\hat{\mathbf{R}}^a &= 0\,,\\
\hat{\mathbf{F}} &= \tilde{F}_{ab}V^aV^b\,,\\
\hat{\boldsymbol{\rho}}^A &= \tilde{\rho}^A_{ab}V^aV^b - \frac{\mathrm{i}}{2}\,\Gamma^a\Psi^BV^b\tilde{F}_{ab}\epsilon^{AB} - \frac{1}{2}\Gamma_5\Gamma^a\Psi^BV^b\,{}^*\tilde{F}_{ab}\epsilon^{AB}\,,\\
\hat{\mathbf{R}}^{ab} &= \tilde{R}^{ab}_{cd}V^cV^d - \overline{\Theta}^{ab}_{A|c}\Psi_AV^c - \frac{1}{2}\,\overline{\Psi}_A\Psi_B\epsilon_{AB}\tilde{F}^{ab} - \frac{\mathrm{i}}{2}\overline{\Psi}_A\Gamma_5\Psi_B\epsilon_{AB}\,{}^*\tilde{F}^{ab}\,,
\end{aligned} \tag{2.2.13}
$$

where the quantities $\tilde{F}_{ab}$, $\tilde{\rho}^A_{ab}$, $\tilde{R}^{ab}_{cd}$ and the spinor-tensor $\Theta^{ab|c}_A$ are computed in Appendix A.3.

Let us notice that the quantities $\tilde{R}^{ab}_{cd}$, $\tilde{\rho}^A_{ab}$ and $\tilde{F}_{ab}$, appearing in the parametrisations (2.2.13), are the so-called *supercovariant field strengths* and they differ in general from the spacetime projections of the supercurvatures, namely $\hat{\mathbf{R}}^{ab}_{\hat{\mu}\hat{\nu}} \neq 2\tilde{R}^{ab}_{cd}\,V^c_{\hat{\mu}}V^c_{\hat{\nu}}$, $\hat{\boldsymbol{\rho}}^A_{\hat{\mu}\hat{\nu}} \neq 2\tilde{\rho}^A_{ab}\,V^a_{\hat{\mu}}V^b_{\hat{\nu}}$. However, since in the present case the parametrisation of $\hat{\mathbf{F}}$ takes contribution only from the 2-vielbein sector, we have $\hat{\mathbf{F}}_{\hat{\mu}\hat{\nu}} = 2\tilde{F}_{ab}\,V^a_{\hat{\mu}}V^b_{\hat{\nu}}$.

The symmetries of the action are diffeomorphisms, local Lorentz transformations, supersymmetry and U(1) gauge transformations, with the corresponding parameters p^a, j^{ab}, ϵ^A and λ, respectively. Taking the above discussion into account, the transformation laws of the bulk fields read

$$
\begin{aligned}
\delta V^a &= \hat{\mathcal{D}}p^a - j^{ab}V_b + \mathrm{i}\,\overline{\epsilon}_A\Gamma^a\Psi^A\,,\\
\delta\hat{\omega}^{ab} &= \hat{\mathcal{D}}j^{ab} + \frac{2}{\ell^2}\,p^{[a}V^{b]} + 2\,\tilde{R}^{ab}{}_{cd}p^cV^d + \overline{\Theta}^{ab}_{A|c}\Psi^Ap^c + \frac{1}{\ell}\,\overline{\epsilon}^A\Gamma^{ab}\Psi_A\\
&\quad -\overline{\Theta}^{ab}_{A|c}\epsilon^AV^c + \epsilon^{AB}\tilde{F}^{ab}\overline{\Psi}_A\epsilon_B + \mathrm{i}\,\epsilon^{AB}\,{}^*\tilde{F}^{ab}\overline{\Psi}_A\Gamma_5\epsilon_B\,,\\
\delta\Psi^A &= -\frac{1}{4}\,j^{ab}\Gamma_{ab}\Psi^A - \frac{\mathrm{i}}{2\ell}\,\Gamma_a\Psi^Ap^a + 2\,\tilde{\rho}^A_{ab}p^aV^b + \frac{\mathrm{i}}{2}\,\Gamma^a\Psi_Bp^b\tilde{F}_{ab}\epsilon^{AB}\\
&\quad +\frac{1}{2}\,\Gamma_5\Gamma^a\Psi_B\,{}^*\tilde{F}_{ab}\,p^b\epsilon^{AB} + \frac{\hat{\lambda}}{2\ell}\,\epsilon^{AB}\Psi_B + \hat{\mathcal{D}}\epsilon^A - \frac{1}{2\ell}\,\hat{A}\epsilon^{AB}\epsilon_B
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathrm{i}}{2\ell}\,\Gamma_a\epsilon^A V^a - \frac{\mathrm{i}}{2}\,\epsilon^{AB}\tilde{F}_{ab}V^b\Gamma^a\epsilon_B - \frac{1}{2}\,\epsilon^{AB}\,{}^*\tilde{F}_{ab}\,\Gamma_5\Gamma^a\epsilon_B V^b\,,\\
\delta\hat{A} \;&=\; \mathrm{d}\hat{\lambda} + 2\,\bar{\epsilon}^A\Psi^B\epsilon_{AB} + 2\,\tilde{F}_{ab}p^a V^b\,. \qquad (2.2.14)
\end{aligned}
$$

The latter generalises to the supersymmetric case the transformation laws (2.1.6).

In this framework, the supersymmetry invariance of the Lagrangian is expressed by the vanishing of the Lie derivative of the Lagrangian for infinitesimal diffeomorphisms in the fermionic directions, that is, $\delta_\epsilon\mathcal{L} = \pounds_\epsilon\mathcal{L} = \imath_\epsilon\mathrm{d}\mathcal{L} + \mathrm{d}(\imath_\epsilon\mathcal{L}) = 0$. When the spacetime geometry has a boundary $\partial\mathcal{M}$, where the superfields do not vanish, then the condition $\imath_\epsilon\mathcal{L}|_{\partial\mathcal{M}} = 0$ is not automatically satisfied and determines the precise expression of the boundary contributions to the Lagrangian and the conditions[13]

$$
\hat{\mathbf{R}}^{ab}|_{\partial\mathcal{M}} = 0\,, \quad \hat{\boldsymbol{\rho}}_A|_{\partial\mathcal{M}} = 0\,, \quad \hat{\mathbf{F}}|_{\partial\mathcal{M}} = 0\,, \quad \hat{\mathbf{R}}^a|_{\partial\mathcal{M}} = 0\,. \qquad (2.2.15)
$$

Thus, to preserve supersymmetry, the OSp(2|4) supercurvatures (2.2.9) are constrained on $\partial\mathcal{M}$ to their vacuum values (2.2.15), which are indeed the Maurer-Cartan equations of a rigid OSp(2|4) background. Note that OSp(2|4) is also the supergroup of global superconformal transformations on $\mathcal{N}=2$ three dimensional superspace, so that the above relations can be understood from the boundary point of view, in light of the AdS/CFT duality, as the conditions for superconformal invariance of the theory at the asymptotic boundary.

Let us write out the equations of motion of the theory. They can be equivalently derived from the bulk Lagrangian (2.2.3) or the full one (2.2.10), the two expressions differing by the Bianchi identities (2.2.12), which are satisfied on-shell.
By using the bulk Lagrangian (2.2.3), one finds

$$
\begin{aligned}
\delta\hat{\omega}^{ab}:&\quad V^c\hat{\boldsymbol{R}}^d\epsilon_{abcd} = 0 \quad\Rightarrow\quad \hat{\boldsymbol{R}}^a = 0\,,\\
\delta V^a:&\quad \frac{1}{2}\,V^b\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \overline{\Psi}^A\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_A + {}^*\tilde{F}_{ab}\,V^b\hat{\boldsymbol{F}} - \frac{1}{12}\,\tilde{F}^{ef}\tilde{F}_{ef}V^bV^cV^d\epsilon_{abcd} = 0\,,\\
\delta\overline{\Psi}^A:&\quad 2\Gamma_aV^a\Gamma_5\hat{\boldsymbol{\rho}}_A - \epsilon_{AB}\Psi^{B*}\hat{\boldsymbol{F}} + \mathrm{i}\epsilon_{AB}\hat{\boldsymbol{F}}\Gamma_5\Psi^B = 0\,,\\
\delta\hat{A}:&\quad \mathrm{d}^*\hat{\boldsymbol{F}} - 2\mathrm{i}\epsilon^{AB}\overline{\Psi}_A\Gamma_5\hat{\boldsymbol{\rho}}_B = 0\,,
\end{aligned}
\qquad (2.2.16)
$$

whereas considering the variation of the full Lagrangian (2.2.10), which includes the boundary contributions, the Euler-Lagrange equations for the vielbein and the gauge field have the same expressions and those for the spin connection and the gravitino get replaced by the (equivalent) expressions

$$
\delta\hat{\omega}^{ab}:\quad -\frac{1}{2}\,\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} + \mathrm{i}\,\overline{\Psi}^A\Gamma_{ab}\Gamma_5\hat{\boldsymbol{\rho}}_A = 0\,, \qquad (2.2.17)
$$

$$
\delta\overline{\Psi}^A:\quad \frac{\ell}{4}\,\Gamma^{ab}\Psi_A\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - 2\mathrm{i}\ell\Gamma_5\hat{\mathcal{D}}\hat{\boldsymbol{\rho}}_A + \mathrm{i}\Gamma_5\hat{A}\epsilon_{AB}\hat{\boldsymbol{\rho}}^B + \Gamma_aV^a\Gamma_5\hat{\boldsymbol{\rho}}_A - \epsilon_{AB}\Psi^{B*}\hat{\boldsymbol{F}} = 0\,. \qquad (2.2.18)
$$

[13]To be more precise, a boundary condition is required to be imposed on the fundamental fields of a theory in order to have a well-defined variational principle. In fact, part of this Chapter will be devoted to obtain the fall-off of the fundamental fields of our supergravity theory (see (2.4.27)), which is part of the just mentioned boundary condition.

For the rest of this Chapter, we will restrict our analysis to spacetime, which translates into the projection of the 4-form Lagrangian from superspace to spacetime (defined by the $\theta^\alpha = 0$, $\mathrm{d}\theta^\alpha = 0$ hypersurface $\mathcal{M}^4$) and the restriction of all the superfields, including the bosonic vielbein V^a and the gravitino $\Psi_{\alpha A}$, to their lowest components

$$V^a(x) = V^a_{\hat\mu}(x)\mathrm{d}x^{\hat\mu}\,, \quad \hat\omega^{ab}(x) = \hat\omega^{ab}_{\hat\mu}(x)\mathrm{d}x^{\hat\mu}\,, \quad \Psi^A(x) = \Psi^A_{\hat\mu}(x)\mathrm{d}x^{\hat\mu}\,, \quad \hat A(x) = \hat A_{\hat\mu}(x)\mathrm{d}x^{\hat\mu}\,,$$

by the restrictions

$$\begin{aligned} V^a(x) &= V^a(x,\theta)|_{\theta=\mathrm{d}\theta=0} = V^a_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \hat\omega^{ab}(x) &= \hat\omega^{ab}(x,\theta)|_{\theta=\mathrm{d}\theta=0} = \hat\omega^{ab}_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \Psi^A(x) &= \Psi^A(x,\theta)|_{\theta=\mathrm{d}\theta=0} = \Psi^A_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \hat A(x) &= \hat A(x,\theta)|_{\theta=\mathrm{d}\theta=0} = \hat A_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,. \end{aligned} \tag{2.2.19}$$

2.3 Near-boundary analysis of the supergravity fields

In the present Section, we are going to apply the holographic techniques, outlined in Section 2.1, to the four dimensional supergravity theory described in Section 2.2.
Given the pure, $\mathcal{N}=2$ supergravity theory, we can deduce the symmetries of its holographically dual QFT in a similar fashion as described in Section 2.1 for AdS_4 gravity. The laws (2.2.14) now depend on the local parameters p^a, j^{ab}, $\hat\lambda$ and ϵ_A and we will use this freedom to fix the Lagrange multipliers and the non dynamic variables, , associated with the radial components of the fields.

We have to choose a suitable gauge that generalises (2.1.7). The asymptotic behaviour of the vielbein in the supergravity extension is the same as for the pure gravity case, because it is determined solely by the metric (2.1.1). Since the gravitini source the torsion field, we can evaluate the asymptotic behaviour of the spin connection in supergravity from the vanishing supertorsion condition in (2.2.16), as explicitly worked out in Appendix A.2. Similarly, the gravitini also act as a source for the electromagnetic field, which determines the fall-off of the graviphoton connection, discussed in Appendix A.2.

Thus, we are left with the analysis of the asymptotic behaviour of the gravitini. To this end, it is convenient to express them in terms of chiral components with respect to the matrix Γ^3,

$$\Psi = \Psi_+ + \Psi_-\,,$$

where the eigenstates $\Psi_\pm$ of the matrix Γ^3 are defined by (A.1.17). The conventions of gamma matrices are given in Appendix A.1.

The asymptotic behaviour of the gravitini is determined by the supertorsion constraints, associated with supersymmetry both in four and three dimensional spacetimes. As a consequence, we are interested in gravitini whose fall-off is $\Psi_{\mu\pm} = \mathcal{O}(z^{\mp 1/2})$ and $\Psi_{z\pm} = \mathcal{O}(z^{\pm 1/2})$, as introduced in [57]. From a group theoretical point of view, the same result is obtained from the request of covariance with respect to the $\mathrm{OSp}(2|4)$ supergroup, which in particular implies, as we will discuss in general terms in Section 2.5, a definite scaling $(\pm 1/2)$ under the subgroup $\mathrm{SO}(1,1) \subset \mathrm{OSp}(2|4)$. The latter parametrises radial rescalings in the bulk and dilations on the boundary. The gravitini expansions are mathematically written as

$$\Psi_{A\mu\pm} = \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}} \varphi_{A\mu\pm}(x,z)\,, \qquad \Psi_{Az\pm} = \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}} \varphi_{A\pm z}(x,z)\,, \tag{2.3.1}$$

where the Majorana fermions $\varphi_{A\mu\pm}$ and $\varphi_{Az\pm}$ are *regular* functions at the boundary and can be expanded, in turn, in power series of z. The second relation in (2.3.1) is consistent with the condition that singles out the spin 3/2 components in the gravitini,

$$\Gamma^a \Psi_{A\hat{\mu}}\, V^{\hat{\mu}}{}_a = 0\,, \tag{2.3.2}$$

which, in the FG frame (2.1.1), reads

$$\left(\Gamma^i \Psi_{A\mu}\right)_\pm V^\mu{}_i + \left(\Gamma^3 \Psi_{Az}\right)_\pm V^z{}_3 = 0\,. \tag{2.3.3}$$

We will not use the above equation in our computations, but let us notice that, if we relax it, more general asymptotics for the gravitini components $\Psi_{Az\pm}$ can in principle be considered. An exploration in this direction could be relevant in view of our interest for the implementation of the unconventional supersymmetry in a holographic SCFT.

Since $\Psi_{A\mu\pm}$ and the transformed field $\Psi_{A\mu\pm} + \delta_\epsilon \Psi_{A\mu\pm}$, given by (2.2.14), have to be of the same order in z, we obtain that $\delta_\epsilon \Psi_{A\mu\pm} \sim \hat{\mathcal{D}}_\mu \epsilon_{A\pm} \sim \epsilon_{A\pm}$ are of the same order, namely

$$\epsilon_{A\pm} = \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}} H_{A\pm}(x,z)\,, \tag{2.3.4}$$

where the Majorana spinor $H_{A\pm}(x,z)$ is regular on the boundary[14].

Regarding the bosonic fields, $\hat{\omega}^{ij}$ and $\hat{A}$ have no scaling with respect to $\mathrm{SO}(1,1) \subset \mathrm{OSp}(2|4)$, while V^i, $\hat{\omega}^{i3}$ have not a definite one. To make this scaling dependence manifest in the supersymmetric theory, it is convenient to define bosonic quantities with definite $\mathrm{SO}(1,1)$ scaling near the boundary:

$$V^i_{\pm\hat{\mu}} = \frac{1}{2}\left(\ell \hat{\omega}^{i3}_{\hat{\mu}} \pm V^i{}_{\hat{\mu}}\right)\,, \tag{2.3.5}$$

where V^i_+ and V^i_- have, respectively, scaling $+1$ and -1. Their asymptotic behaviour is

$$V^i_{\pm\mu} = \left(\frac{z}{\ell}\right)^{\mp 1} E^i_{\pm\mu}(x,z)\,, \tag{2.3.6}$$

and the regular functions $E^i_\pm$ have the following power expansion in z,

$$E^i_{+\mu} \;=\; E^i{}_\mu + \frac{z^2}{\ell^2}\frac{S^i{}_\mu - \tilde{S}^i{}_\mu}{2} + \frac{z^3}{\ell^3}\frac{\tau^i{}_\mu - 2\tilde{\tau}^i{}_\mu}{2} + \mathcal{O}(z^4)\,,$$

[14] In order for the boundary theory to be supersymmetry invariant, we notice that, from (2.2.14), the supersymmetry parameters must be asymptotically proportional to Killing spinors of the asymptotic background. H_{+A} will be Killing spinors of the chosen background, while H_{-A} will be superconformal Killing spinors.

$$E^i_{-\mu} \;=\; -\frac{\ell^2}{2}\,\mathcal{S}^i{}_\mu - \frac{z}{\ell}\,\frac{\tau^i{}_\mu + 2\tilde{\tau}^i{}_\mu}{2} + \mathcal{O}(z^2)\,. \tag{2.3.7}$$

Unless stated differently, all the regular functions on the boundary appearing in this Chapter, $f = \{w^i,\ w^{ij},\ \varphi_{A\mu\pm},\ \varphi_{Az\pm},\ H_{A\pm}, \ldots\}$, are generically expanded in a power series as

$$f(x,z) = \sum_{n=0}^{\infty}\left(\frac{z}{\ell}\right)^n f_{(n)}(x) = f_{(0)}(x) + \frac{z}{\ell}\,f_{(1)}(x) + \frac{z^2}{\ell^2}\,f_{(2)}(x) + \cdots\,. \tag{2.3.8}$$

By using these conventions, the asymptotic expansion of the spin connection is computed in Appendix A.2. It is found, for instance see (A.2.7), that a suitable gauge fixing which includes gravitini has $\hat{\omega}^{ab}_z \neq 0$, but it is still subleading on the boundary. We choose arbitrary functions $\hat{\omega}^{i3}_z = w^i(x,z)$ and $\hat{\omega}^{ij}_z = \frac{z}{\ell}\,w^{ij}(x,z)$ in such a way that they are consistent with the vanishing supertorsion condition, but we will treat them off-shell as independent variables in first order formulation of supergravity.

In order to ensure that the gauge fixing of $\hat{A}_z$ is consistent with the supergravity dynamics imposed later, it has to satisfy the radial component of the graviphoton equation in (2.2.16), which is analysed in Appendix A.2. It turns out that having two independent components $\Psi_{Az\pm}$ is not restrictive enough in the context of holography, since it would not allow the components of the gravitini on $\partial\mathcal{M}$, $\varphi^A_{\pm\mu}$, to be the only source of the bulk electromagnetic field, $\mathcal{F} = \mathrm{d}A$ on $\partial\mathcal{M}$, as happens in Einstein-Maxwell gravity,

$$\hat{\mathbf{F}}_{\mu\nu} = 0 \quad\Rightarrow\quad \mathcal{F}_{\mu\nu} = 4\epsilon_{AB}\,\overline{\varphi}^A_{+[\mu}\varphi^B_{-\nu]}\,, \tag{2.3.9}$$

which has a U(1) gauge parameter not diverging on $\partial\mathcal{M}$, namely $\hat{\lambda} = \mathcal{O}(z^0)$. Then, as explained in Appendix A.2, the leading order of the component $\hat{A}_z$, denoted by $\frac{\ell}{z}\,A_{(-1)z}$, is related to the leading order of the component Ψ_{-Az}, namely $\varphi_{-Az(0)}$. The general solution, given by (A.2.49), requires that either both functions vanish or $A_{(-1)z}$ to be constant and $\varphi_{(0)-Az}$ determined in terms of it.

If we are interested in a theory consistent with supersymmetry on the boundary, we have two options. The first one is to impose the stronger condition

$$\Psi_{Az-} = 0\,, \tag{2.3.10}$$

whereas the second chance is to change the asymptotic structure of the U(1) sector, allowing for a divergent leading contribution in $\hat{A}_z$.

To sum up, the results of Appendix A.2 show that the holographic gauge fixing conditions on the local parameters $p^a,\ j^{ab},\ \lambda,\ \epsilon_A$ in AdS space have the form

$$\begin{aligned}
V^3_z &= \frac{\ell}{z}\,, \qquad \hat{\omega}^{i3}_z = w^i(x,z)\,, \qquad \Psi_{\pm Az} = \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}}\varphi_{\pm Az}(x,z)\,,\\
V^i_z &= 0\,, \qquad \hat{\omega}^{ij}_z = \frac{z}{\ell}\,w^{ij}(x,z)\,,\quad \hat{A}_z = \frac{\ell}{z}\,A_{(-1)z}(x) + \frac{z}{\ell}\,A_{(1)z}(x) + \mathcal{O}(z^3)\,,
\end{aligned} \tag{2.3.11}$$

where we can distinguish three particular cases

$$\begin{aligned}
\Psi_{z\pm} \neq 0 \quad&\Rightarrow\quad \hat{A}_z = \mathcal{O}(1/z)\,,\quad w^i = \mathcal{O}(1)\,,\qquad w^{ij} = \mathcal{O}(1)\,,\\
\Psi_{z-} = 0 \quad&\Rightarrow\quad \hat{A}_z = \mathcal{O}(z)\,,\qquad w^i = \mathcal{O}(z^2)\,,\qquad w^{ij} = \mathcal{O}(1)\,,\\
\Psi_{z\pm} = 0 \quad&\Rightarrow\quad \hat{A}_z = \mathcal{O}(z)\,,\qquad w^i = 0\,,\qquad\quad w^{ij} = \mathcal{O}(1)\,.
\end{aligned} \tag{2.3.12}$$

We can expand the gauge fixing functions in powers of z, since they depend on the radial and boundary coordinates. For the spinors we use the notation

$$\begin{aligned}
\Psi^A_{+z} &= \sqrt{\frac{z}{\ell}}\,\varphi^A_{+z}(x,z)=\sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}\psi^A_{+z}\\0\end{pmatrix}+\frac{z}{\ell}\begin{pmatrix}\zeta^A_{+z}\\0\end{pmatrix}+\mathcal{O}(z^2)\right],\\
\Psi^A_{-z} &= \sqrt{\frac{\ell}{z}}\,\varphi^A_{-z}(x,z)=\sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}0\\\psi^A_{-z}\end{pmatrix}+\frac{z}{\ell}\begin{pmatrix}0\\\zeta^A_{-z}\end{pmatrix}+\mathcal{O}(z^2)\right].
\end{aligned}\tag{2.3.13}$$

It is important to emphasise that we assume the gauge fixing functions $\Psi^A_z(x)$ and $\hat{A}_z(x)$ not to transform under local transformations. This is equivalent to say that their transformation laws can always be reabsorbed in higher order terms of the asymptotic transformations. In contrast, the quantities $w^i(x)$ and $w^{ij}(x)$, introduced in (2.3.11), do transform, because on-shell they have to allow for the vanishing supertorsion condition. However, in first order formalism, they enter off-shell at the same footing as other gauge fixing functions, with the only difference that we do not require them to be invariant under residual transformations. Indeed, by using the explicit expressions in (A.2.6) and (A.2.7), it is straightforward, by varying the supertorsion, to check that δw^i, $\delta w^{ij}\neq 0$, but it is always possible to set $w^i=w^{ij}=0$ consistently (namely with $\delta w^i=\delta w^{ij}=0$). Moreover, let us notice that when $w^i\neq 0$, $\delta w^i\neq 0$ as well, and the same is true for w^{ij}. Nonetheless, δw^i and δw^{ij} always appear at higher order and they do not influence the near-boundary expressions.

The conditions (2.3.11) produce the following generic asymptotic behaviour of the boundary fields,

$$\begin{aligned}
V^i{}_\mu &= \frac{\ell}{z}\,E^i{}_\mu+\frac{z}{\ell}\,S^i{}_\mu+\frac{z^2}{\ell^2}\,\tau^i{}_\mu+\mathcal{O}(z^3)\,,\\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,E^i{}_\mu-\frac{z}{\ell^2}\,\tilde{S}^i{}_\mu-\frac{2z^2}{\ell^3}\,\tilde{\tau}^i{}_\mu+\mathcal{O}(z^3)\,,\\
\hat{\omega}^{ij}_\mu &= \omega^{ij}_\mu(x,z)=\omega^{ij}_\mu+\frac{z}{\ell}\,\omega^{ij}_{(1)\mu}+\frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu}+\mathcal{O}(z^3)\,,\\
\hat{A}_\mu &= A_\mu(x,z)=A_\mu+\frac{z}{\ell}\,A_{(1)\mu}+\frac{z^2}{\ell^2}\,A_{(2)\mu}+\mathcal{O}(z^3)\,,\\
\Psi^A_{\mu+} &= \sqrt{\frac{\ell}{z}}\,\varphi^A_{\mu+}(x,z)=\sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}\psi^A_{\mu+}\\0\end{pmatrix}+\frac{z}{\ell}\begin{pmatrix}\zeta^A_{\mu+}\\0\end{pmatrix}+\frac{z^2}{\ell^2}\begin{pmatrix}\Pi^A_{\mu+}\\0\end{pmatrix}+\mathcal{O}(z^3)\right],\\
\Psi^A_{\mu-} &= \sqrt{\frac{z}{\ell}}\,\varphi^A_{\mu-}(x,z)=\sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}0\\\psi^A_{\mu-}\end{pmatrix}+\frac{z}{\ell}\begin{pmatrix}0\\\zeta^A_{\mu-}\end{pmatrix}+\mathcal{O}(z^2)\right],
\end{aligned}\tag{2.3.14}$$

where all functions defined on $\partial\mathcal{M}$ are finite at $z=0$. The spinors acquire a half-integer power expansion in z because their bilinears, which arise from the supersymmetry transformation of the bosons, have integer power expansion in z. We also allow for linear terms in z, absent in pure AdS gravity, since, in principle, they could be switched on by supersymmetry.

Even though the supertorsion is zero, the torsion $\hat{T}^a$ does not vanish, so that $\hat{\omega}^{ab}_\mu$ cannot be entirely determined by the bosonic vielbein. In particular, the leading order relation

$\hat{\omega}^{i3}_\mu \sim \frac{1}{\ell} V^i{}_\mu$ (see Appendix A.2) is inherited from the Riemannian geometry ($k_{\mu\nu} \sim \frac{1}{\ell}\,\hat{g}_{\mu\nu}$). For the supersymmetric case, the subleading terms in the expansion, $\tilde{S}^i{}_\mu$ and $\tilde{\tau}^i{}_\mu$, are different from the Riemannian counterparts $S^i{}_\mu$ and $\tau^i{}_\mu$, and the boundary Schouten tensor is now defined as

$$\mathcal{S}^i{}_\mu = \frac{1}{\ell^2}\,(S^i{}_\mu + \tilde{S}^i{}_\mu)\,, \tag{2.3.15}$$

which is the gauge field associated with special conformal transformations, as we will identify below. Similarly, we will later see that $-(\tau^i{}_\mu + 2\tilde{\tau}^i{}_\mu)/\ell$ becomes the holographic stress tensor, up to fermionic terms. Thus, let us notice that there is an obstruction in the symmetrisation of $\mathcal{S}^i{}_\mu$ and the holographic stress tensor, because the terms $\tilde{S}^i{}_\mu$ and $\tilde{\tau}^i{}_\mu$ are not a priori symmetric in the presence of the gravitini.

The Schouten tensor in $d = 3$ and its superconformal extension

We have already seen in the previous Sections that the Schouten tensor plays an important role in pure AdS gravity, as it describes the first near-boundary correction of the metric, given by (2.1.3). From the CFT side, it arises as a component of the superconformal connection, as shown at the beginning of Section 2.5. In this paragraph, we will focus on its geometric properties, derived in the context of conformal gravity (for a review, see [76]).

Let us consider a d dimensional manifold characterised by a metric $g_{\mu\nu}$ and a torsionful affine connection $\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda_{\mu\nu} - K^\lambda{}_{\mu\nu}$, where $\mathring{\Gamma}^\lambda_{\mu\nu}$ is the Levi-Civita connection and $K^\lambda{}_{\mu\nu} = g^{\rho\lambda}\,(T_{\rho\mu\nu} + T_{\rho\nu\mu} - T_{\mu\nu\rho})$ is the contorsion tensor, $T_{\mu\nu}{}^\lambda \equiv \Gamma^\lambda{}_{[\mu\nu]}$ being the torsion tensor. The Schouten tensor obtained from the conformal constraint equation on the conformal curvature components is defined by [76]

$$\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} \; - \frac{1}{2(d-1)}\,g_{\mu\nu}\,\mathcal{R}\,, \tag{2.3.16}$$

where $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are, respectively, the Ricci curvature tensor and the Ricci scalar constructed from the torsionful affine connection $\Gamma^\lambda{}_{\mu\nu}$. This formula coincides with (2.1.3) in pure AdS gravity: in the latter case the Ricci tensor is symmetric, which implies that so is $\mathcal{S}_{\mu\nu}$. In presence of torsion, however, the Schouten tensor has both symmetric and antisymmetric parts,

$$\begin{aligned}
\mathcal{S}_{(\mu\nu)} &= \mathcal{R}_{(\mu\nu)} - \frac{1}{2(d-1)}\,g_{\mu\nu}\mathcal{R}\,, \\
\mathcal{S}_{[\mu\nu]} &= \mathcal{R}_{[\mu\nu]}\,.
\end{aligned} \tag{2.3.17}$$

In particular, in $d = 3$, we can explicitly evaluate its symmetric and antisymmetric parts as

$$\begin{aligned}
\mathcal{S}_{(\mu\nu)} &= \mathring{\mathcal{R}}_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\mathring{\mathcal{R}} - \frac{1}{2}\,g_{\mu\nu}T_\lambda T^\lambda + T_\mu T_\nu + \tilde{T}_{\lambda\rho\nu}\left(\tilde{T}^{\lambda\rho}{}_\mu - \tilde{T}_\mu{}^{\lambda\rho}\right) - \tilde{T}_{\lambda\rho\mu}\tilde{T}_\nu{}^{\lambda\rho} \\
&\quad -\frac{1}{2}\,g_{\mu\nu}\tilde{T}_{\lambda\rho\sigma}\left(\frac{1}{2}\,\tilde{T}^{\lambda\rho\sigma} + \tilde{T}^{\lambda\sigma\rho}\right) - \nabla_{(\mu}T_{\nu)} + 2\,\nabla_\lambda\tilde{T}_{(\mu}{}^\lambda{}_{\nu)}\,, \\
\mathcal{S}_{[\mu\nu]} &= T^\lambda\left(\tilde{T}_{\mu\lambda\nu} + \tilde{T}_{\mu\nu\lambda} - \tilde{T}_{\nu\lambda\mu}\right) + 2\tilde{T}_{\lambda\rho[\nu}\tilde{T}_{\mu]}{}^{\lambda\rho} + \nabla_\lambda\tilde{T}_{\mu\nu}{}^\lambda + \nabla_{[\mu}T_{\nu]}\,,
\end{aligned} \tag{2.3.18}$$

where we have also exploited the trace decomposition of the torsion tensor $T_{\lambda\mu}{}^{\nu} = \delta_{[\mu}{}^{\nu}T_{\lambda]} + \tilde{T}_{\lambda\mu}{}^{\nu}$, with T_λ and $\tilde{T}_{\lambda\mu}{}^{\nu}$ its trace and traceless parts, respectively. Here, $\nabla = \nabla(\mathring{\Gamma})$ denotes the derivative with respect to the Levi-Civita affine connection and $\mathring{\mathcal{R}}_{\mu\nu}$ and $\mathring{\mathcal{R}}$ are the Ricci tensor and curvature scalar of the Levi-Civita connection, respectively.

When the torsion is non vanishing, such as in presence of fermions, in general we have $\mathcal{S}_{[\mu\nu]} \neq 0$ and the symmetric part $\mathcal{S}_{(\mu\nu)}$ acquires the torsionful term.[15] Thus, in the context of supergravity, we expect the "super-Schouten tensor" (2.3.15) to be not symmetric and a superconformal extension of the expression (2.3.17).

The equations written above are general and valid for any Riemann-Cartan manifold. In our specific case, we have the following quantities that arise from the asymptotic expansion,

$$\begin{aligned} S_{\mu\nu} &= E_{i\mu}S^{i}{}_{\nu}\,, \qquad \tau_{\mu\nu} = E_{i\mu}\tau^{i}{}_{\nu}\,, \\ \tilde{S}_{\mu\nu} &= E_{i\mu}\tilde{S}^{i}{}_{\nu}\,, \qquad \tilde{\tau}_{\mu\nu} = E_{i\mu}\tilde{\tau}^{i}{}_{\nu}\,, \\ \mathcal{S}_{\mu\nu} &= E_{i\mu}\mathcal{S}^{i}{}_{\nu}\,. \end{aligned} \tag{2.3.19}$$

It can be shown from (A.2.6) that, when $\varphi^{A}_{-z}=0$, the tensors $\tilde{S}_{\mu\nu}$ and $\tilde{\tau}_{\mu\nu}$ acquire the form

$$\begin{aligned} \tilde{S}_{\mu\nu} &= S_{\nu\mu} - \ell\,\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(0)-\nu]} + \mathrm{i}\ell\,\overline{\varphi}_{(0)A+(\nu}\Gamma_{\mu)}\varphi^{A}_{(0)+z}\,, \\ \tilde{\tau}_{\mu\nu} &= \frac{\tau_{\mu\nu}+3\tau_{\nu\mu}}{4} + \frac{\ell}{2}\left(-\overline{\varphi}_{A+[\mu}\varphi^{A}_{-\nu]} + \mathrm{i}\overline{\varphi}^{A}_{+(\mu}\Gamma_{\nu)}\varphi_{A+z}\right)_{(1)}\,, \end{aligned} \tag{2.3.20}$$

where the last line is relevant for the holographic stress tensor, whose direct relation to $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ will be shown in Section 2.5.

It means that, even if we symmetrise $S_{\mu\nu}$ and $\tau_{\mu\nu}$ by suitable gauge fixing of the residual Lorentz transformations, the fermions $\psi_{A\pm\mu}$ become obstructions to make the tensors $\tilde{S}_{\mu\nu}$ and $\tilde{\tau}_{\mu\nu}$ symmetric for arbitrary ψ_{A+z}, because of their antisymmetric parts

$$\begin{aligned} \tilde{S}_{[\mu\nu]} &= S_{[\nu\mu]} - \ell\,\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(0)-\nu]}\,, \\ \tilde{\tau}_{[\mu\nu]} &= \frac{1}{2}\,\tau_{[\nu\mu]} - \frac{\ell}{2}\left(\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(1)-\nu]} + \overline{\varphi}_{(1)A+[\mu}\varphi^{A}_{(0)-\nu]}\right)\,. \end{aligned} \tag{2.3.21}$$

Focusing on the Schouten tensor (2.3.15), we find, for its generalisation to the superconformal case, the expression we will refer to as "super-Schouten" in the following,

$$\mathcal{S}_{\mu\nu} = \frac{2}{\ell^2}\,S_{(\mu\nu)} - \frac{1}{\ell}\,\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(0)-\nu]} + \frac{\mathrm{i}}{\ell}\,\overline{\varphi}_{(0)A+(\nu}\Gamma_{\mu)}\varphi^{A}_{(0)+z}\,, \tag{2.3.22}$$

which in turn implies

$$\begin{aligned} \mathcal{S}_{(\mu\nu)} &= \frac{2}{\ell^2}\,S_{(\mu\nu)} + \frac{\mathrm{i}}{\ell}\,\overline{\varphi}_{(0)A+(\mu}\Gamma_{\nu)}\varphi^{A}_{(0)+z}\,, \\ \mathcal{S}_{[\mu\nu]} &= -\frac{1}{\ell}\,\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(0)-\nu]}\,. \end{aligned} \tag{2.3.23}$$

This result matches (2.3.18) and shows that the symmetric part of the super-Schouten tensor includes not only the metric term, $S_{(\mu\nu)}$, but also the fermionic bilinears. Moreover,

[15] The antisymmetric contribution is still vanishing in the special case when the torsion contains only one component, the trace T_λ, which should be also covariantly constant.

the antisymmetric part does not vanish for arbitrary fermions $\psi_{\pm\mu}$. Therefore, we are not able to symmetrise the super-Schouten tensor, as this procedure would lead to conditions on the leading terms of the boundary gravitini, which have to stay unconstrained.

Similarly, the term relevant for the holographic stress tensor,

$$\begin{aligned}\tau_{\mu\nu}+2\tilde{\tau}_{\mu\nu} &= 3\tau_{(\mu\nu)}+\ell\,\Big(-\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(1)-\nu]}-\overline{\varphi}_{(1)A+[\mu}\varphi^{A}_{(0)-\nu]}\\ &\quad +\mathrm{i}\overline{\varphi}^{A}_{(0)+(\mu}E^{i}_{\nu)}\Gamma_i\varphi_{(1)A+z}+\mathrm{i}\overline{\varphi}^{A}_{(0)+(\mu}E^{i}_{\nu)}\Gamma_i\varphi_{(1)A+z}\Big)\,, \end{aligned} \tag{2.3.24}$$

is not symmetric in general, the antisymmetric part being

$$\tau_{[\mu\nu]}+2\tilde{\tau}_{[\mu\nu]}=-\ell\,\Big(\overline{\varphi}_{(0)A+[\mu}\varphi^{A}_{(1)-\nu]}+\overline{\varphi}_{(1)A+[\mu}\varphi^{A}_{(0)-\nu]}\Big)\,. \tag{2.3.25}$$

We will further discuss about symmetry of the holographic stress tensor in Section 2.5.

2.4 Field transformations and asymptotic symmetries

So far, we have chosen the Lagrange multipliers and other non dynamic variables (2.3.11), which generate the asymptotic expansion of the fields (2.3.14). Hereafter, we will focus on the case $\Psi_{Az-}=0$. A stronger condition $\Psi_{Az\pm}=0$, referred to as "FG gauge", was considered in [26] in the context of $\mathcal{N}=1$ AdS_4 supergravity. However, an advantage of the choice $\Psi_{Az+}\neq 0$ is to provide more freedom that could be used, in principle, to simplify complicated fermionic expressions. We will see, though, that the presence of this particular field will not modify the asymptotic behaviours in the theory.

Boundary conditions on the curvatures

The OSp(2|4) supercurvatures vanish at the boundary in asymptotically AdS spacetimes, as expressed by the conditions (2.2.15). In particular, the supertorsion vanishes exactly and its consequences are discussed in Appendix A.2. The other supercurvature conditions at the boundary, whose expressions have been rewritten in (2.5.3), boil down to the following constraints on $\partial\mathcal{M}$,

$$\begin{aligned}\mathcal{D}E^i-\frac{\mathrm{i}}{2}\,\overline{\psi}^{A}_{+}\wedge\gamma^i\psi_{A+}&=0\,,\\ \mathcal{R}^{ij}-2\,E^{[i}\wedge\mathcal{S}^{j]}-\frac{1}{\ell}\,\overline{\psi}^{A}_{+}\wedge\gamma^{ij}\psi_{A-}&=0\,,\\ \nabla\psi^{A}_{+}+\frac{\mathrm{i}}{\ell}\,E^i\wedge\gamma_i\psi^{A}_{-}&=0\,,\end{aligned} \tag{2.4.1}$$

where $\mathcal{R}^{ij}$ is the Riemann curvature tensor 2-form at the boundary and $\mathcal{S}^i$ is the boundary super-Schouten 1-form defined in (2.3.15).

The first equation ensures the vanishing boundary supertorsion, by fixing the boundary torsion $T^i=\mathcal{D}E^i$ in terms of the gravitini. The second equation involves the boundary Weyl tensor $W^{ij}=\mathcal{R}^{ij}-2\,E^{[i}\wedge\mathcal{S}^{j]}$ and it can be interpreted as the requirement for the super Weyl tensor to vanish on the boundary.

The three equations can be explicitly solved in terms of the boundary fields ω^{ij}, $\mathcal{S}^i$ and ψ_-^A. Since the spin connection has been solved in Appendix A.2, we focus on the other two composite fields here. From the third relation of (2.4.1), we get the conformino

$$\psi_{-A\,\mu} = -\frac{\ell}{2e_3}\,\epsilon^{\lambda\nu\rho}\gamma_\lambda\gamma_\mu\nabla_\nu\psi_{+A\,\rho}\,, \tag{2.4.2}$$

whereas we solve for the super-Schouten tensor the second equation

$$\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\mathcal{R} - \frac{1}{\ell}\left(\overline{\psi}_{+A\,\rho}\gamma^\rho{}_\mu\psi_{-A\,\nu} - \overline{\psi}_{+A\,\nu}\gamma^\rho{}_\mu\psi_{-A\,\rho} - \frac{1}{2}\,g_{\mu\nu}\overline{\psi}_{+A\,\rho}\gamma^{\rho\lambda}\psi_{-A\,\lambda}\right). \tag{2.4.3}$$

This result implies that the super-Schouten tensor $\mathcal{S}^i{}_\mu$ and its superpartner, the conformino $\psi_{-A\,\mu}$, are not independent sources on $\partial\mathcal{M}$, since they can be expressed in terms of the supervielbein $(E^i{}_\mu, \psi_{+A\,\mu})$ and their curvatures.

In the end, let us comment on the Schouten tensor. At first sight, it could look like we are dealing with several different expressions for the Schouten tensor. Its definition (2.3.15) has a geometric origin, as explained in Section 2.5, and it is a component of the $d=3$ superconformal field associated with the conformal boosts. From the point of view of the $D=4$ bulk fields, the Schouten tensor comes from the combination of the vielbein and the spin connection in the negative grading quantity with respect to $\mathrm{O}(1,1)\subset\mathrm{SO}(2,4)$. The vanishing supertorsion condition leads to the $\mathcal{R}$-independent kinematic relation between the super-Schouten tensor (2.3.22) and $S_{(\mu\nu)}$ in the superconformal case. On the contrary, the asymptotically AdS condition and the vanishing supercurvatures on the boundary (2.4.1) lead to the $\mathcal{R}$-dependent Schouten tensor (2.4.3). Matching the two formulas allows to express $S_{(\mu\nu)}$ in terms of the boundary curvature $\mathcal{R}_{\mu\nu}$ plus fermion bilinears, that has to be fulfilled on-shell. In pure AdS gravity, for instance, it comes down to the known relation $\mathcal{S}_{\mu\nu} = \frac{2}{\ell^2}\,S_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\mathcal{R}$, obtained by solving the Einstein equations near the boundary. Thus, these two equations have different origin, but they have to be consistent on-shell.
On the other hand, the definition of the Schouten tensor (2.3.16) is that usually found in the literature [76], obtained from the conformal constraint equation. The superconformal version of this constraint leads to the super-Schouten tensor (2.4.3) together with its superpartner (2.4.2).

Rheonomic parametrisations

The transformation laws (2.2.14) depend explicitly on the contractions of the supercurvatures. A proper way to account for all contributions requires to know the near-boundary behaviour of the rheonomic parametrisations appearing in (2.2.14).
The simplest method to proceed is to project the expressions (2.2.13) for the rheonomic parametrisation of the supercurvatures on the spacetime manifold and identify their asymptotic behaviour with the one of the spacetime projections of the supercurvatures (2.2.8). One can start from the U(1) field strength, whose parametrisation in (2.2.13) takes value on the 2-vielbein component only in the case at hand. One then proceeds to find $\tilde{\rho}^A_{ab}$ from the curvature of the gravitino, which can be further used to compute $\Theta^{ab}_{A|c}$ and $\tilde{R}^{ab}{}_{cd}$ in the last

expression of (2.2.13).
By following this procedure, we determine the asymptotic behaviour of all the supercovariant field strengths, whose derivation is fully carried out in Appendix A.3. The asymptotic expansion of $\tilde{F}_{ab}$ and $\tilde{\rho}^A_{ab}$ leads to

$$
\begin{aligned}
\tilde{F}_{ij} &= \mathcal{O}(z^3)\,, & \tilde{F}_{i3} &= -\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^2 A_{(1)\mu}E^\mu_i + \mathcal{O}(z^3)\,,\\
\tilde{\rho}^A_{ij+} &= \mathcal{O}(z^{5/2})\,, & \tilde{\rho}^A_{i3+} &= -\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}} E^\mu_i \zeta^A_{\mu+} + \mathcal{O}(z^{5/2})\,,\\
\tilde{\rho}^A_{ij-} &= \mathcal{O}(z^{5/2})\,, & \tilde{\rho}^A_{i3-} &= \mathcal{O}(z^{5/2})\,.
\end{aligned} \tag{2.4.4}
$$

In order to find a radial power expansion of $\tilde{R}^{ab}{}_{cd}$, one needs the $\Theta^{ab}_{A|c}$ coefficients, which are found by inserting (2.4.4) into the definition (A.3.5), as shown in Appendix A.3. After lengthy but straightforward calculation, one obtains

$$
\begin{aligned}
\tilde{R}^{i3}{}_{jk} &= \frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2 E^\mu_{[j}E^\nu_{k]}\overline{\psi}^A_{\mu+}\left(\gamma^i\zeta_{A\nu+} + \gamma^l\zeta_{A\rho+}E_{l\nu}E^{i\rho}\right) + \mathcal{O}(z^3)\,,\\
\tilde{R}^{ij}{}_{k3} &= -\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^2 E^\mu_k\left(\omega^{ij}_{(1)\mu} - \mathrm{i}\,\overline{\psi}^A_{\mu+}\gamma^{[i}E^{j]\nu}\zeta_{A\nu+}\right) + \mathcal{O}(z^3)\,,\\
\tilde{R}^{i3}{}_{j3} &= \mathcal{O}(z^3)\,, \qquad \tilde{R}^{ij}{}_{kl} = \mathcal{O}(z^3)\,.
\end{aligned} \tag{2.4.5}
$$

It is worthwhile noticing that all expansions (2.4.4) and (2.4.5) are subleading in z and, when they are slower than $\mathcal{O}(z^{5/2})$, this is due to the presence of $\omega^{ij}_{(1)\mu}$ and $\zeta^A_{\mu+}$. We will show below that the higher order residual symmetries can be used to cancel out such linear terms, similarly as in pure AdS gravity.

Residual symmetries

We look for the residual symmetries of the form (2.2.14) that leave the gauge fixing unaltered on the boundary,

$$
\delta V^a_{\ z} = 0\,, \quad \delta\hat{\omega}^{ij}_z = \mathcal{O}(z)\,, \quad \delta\hat{\omega}^{i3}_z = \mathcal{O}(z^2)\,, \quad \delta\hat{A}_z = 0\,, \quad \delta\Psi_{\pm Az} = 0\,. \tag{2.4.6}
$$

The non dynamic fields in (2.3.11) are functions of the boundary coordinates through w^i, w^{ij}, φ_{+Az} and $\hat{A}_z$. In (2.4.6), we assume that $\hat{A}_z(x)$ and $\Psi_{\pm Az}(x)$ do not change under general coordinate transformations, even though they depend on x^μ. We will show that this assumption will not break the boundary symmetries, but only modify subleading parameters. On the other hand, the functions $w^i(x)$ and $w^{ij}(x)$ change under the coordinate transformations because, on-shell, they have to satisfy the supertorsion constraint. In fact, it would have been more natural to allow all x^μ-dependent quantities to transform non trivially under boundary coordinate transformations, but we do not account it for simplicity. On the contrary, allowing Ψ_{+Az} to transform as a dynamical field might be related to the implementation of unconventional supersymmetry (discussed in Section 1.3) on the boundary, where a spinor $\chi(x^\mu)$ arises from the gauge fixing of the gravitini [55]. We do not investigate further on this point and we leave it for possible future developments.

The symmetry parameters can be expanded as in (2.3.8), where we keep the same notation for the leading orders of the bosonic parameters as in (2.1.46),

$$\begin{aligned}
p^i &= \frac{\ell}{z}\,\xi^i + \frac{z}{\ell}\,p^i_{(1)} + \frac{z^2}{\ell^2}\,p^i_{(2)} + \mathcal{O}(z^3),\\
p^3 &= -\ell\sigma + \frac{z}{\ell}\,p^3_{(1)} + \frac{z^2}{\ell^2}\,p^3_{(2)} + \frac{z^3}{\ell^3}\,p^3_{(3)} + \mathcal{O}(z^4),\\
j^{ij} &= \theta^{ij} + \frac{z}{\ell}\,j^{ij}_{(1)} + \frac{z^2}{\ell^2}\,j^{ij}_{(2)} + \frac{z^3}{\ell^3}\,j^{ij}_{(3)} + \mathcal{O}(z^4)\,,\\
j^{i3} &= \frac{1}{z}\,\xi^i + \frac{z}{\ell}\,j^{i3}_{(1)} + \frac{z^2}{\ell^2}\,j^{i3}_{(2)} + \mathcal{O}(z^3)\,,\\
\hat{\lambda} &= \lambda + \frac{z}{\ell}\,\lambda_{(1)} + \mathcal{O}(z^2)\,,\\
\epsilon^A_+ &= \sqrt{\frac{\ell}{z}}\,H_+(x,z) = \sqrt{\frac{\ell}{z}}\begin{pmatrix}\eta^A_+\\0\end{pmatrix} + \sqrt{\frac{z}{\ell}}\begin{pmatrix}\eta^A_{(1)+}\\0\end{pmatrix} + \mathcal{O}(z^{3/2})\,,\\
\epsilon^A_- &= \sqrt{\frac{z}{\ell}}H_-(x,z) = \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\\eta^A_-\end{pmatrix} + \left(\frac{z}{\ell}\right)^{\frac{3}{2}}\begin{pmatrix}0\\\eta^A_{(1)-}\end{pmatrix} + \mathcal{O}(z^{5/2})\,.
\end{aligned} \tag{2.4.7}$$

Before we get started, the first subleading Lorentz parameter can be consistently set to zero in the above expansion,

$$j^{ij}_{(1)} = 0\,. \tag{2.4.8}$$

As a first step to find the asymptotic symmetries, we will analyse the linear terms in the transformation laws. The equation $\delta\hat{\omega}^{ij}_z = 0$ from (2.4.6) leads to the simple differential expression

$$\partial_z j^{ij} - \frac{1}{\ell}\,\xi^\mu\omega^{ij}_{(1)\mu} - \frac{\mathrm{i}}{\ell}\,\xi^\mu\overline{\psi}^A_{\mu+}E^{\nu[i}\gamma^{j]}\zeta_{A\nu+} + \frac{\mathrm{i}}{\ell}\,\overline{\eta}^A_+E^{\nu[i}\gamma^{j]}\zeta_{A\nu+} + \mathcal{O}(z) = 0\,, \tag{2.4.9}$$

which, taken at the leading order, amounts to solving the algebraic equation

$$\xi^\mu\omega^{ij}_{(1)\mu} = \mathrm{i}\left(\overline{\eta}^A_+ - \xi^\mu\overline{\psi}^A_{\mu+}\right)E^{\nu[i}\gamma^{j]}\zeta_{A\nu+}\,. \tag{2.4.10}$$

Since ξ^i and η^A_+ should stay arbitrary and we know that ω^{ij}_μ is a composite field (explicitly computed in Appendix A.2) which has not the linear term, $\omega^{ij}_{(1)\mu} = 0$, we can choose a vanishing configuration for $\zeta_{A\mu+}$:

$$\omega^{ij}_{(1)\mu} = 0\,,\qquad \zeta_{A\mu+} = 0\,. \tag{2.4.11}$$

This choice has also been made for the $\mathcal{N}=1$ supergravity in [26]. In our case, when $\mathcal{N}=2$, it becomes the unique solution both when $\Psi_{-z}=0$ and $\Psi_{-z}\neq 0$ (for more detailed discussion, see (A.2.44) in Appendix A.2). It is crucial that the value of these fields does not change after a generic local transformation, namely $\delta\omega^{ij}_{(1)\mu} = \delta\zeta_{A\mu+} = 0$, and we discuss about it in the next paragraph.

Another constraint on the parameters arises from the fact that the FG coordinate frame (2.1.1) does not admit the finite terms in the expansions of $V^i_{\ \mu}$ and $\hat{\omega}^{i3}_\mu$. Local invariance preserves this frame only if

$$0 = \delta V^i_{(0)\mu} = -\frac{1}{\ell}\,E^i_{\ \mu}\,p^3_{(1)} \quad\Rightarrow\quad p^3_{(1)} = 0\,. \tag{2.4.12}$$

Then, using the expansion of the rheonomic parametrisations given in Appendix A.3, we find that $\delta\hat{\omega}^{i3}_{(0)\mu} = -\frac{1}{\ell^2} E^i{}_\mu \, p^3_{(1)} = 0$ is satisfied as well.

On the other hand, the invariance of $\Psi^A_{\pm z}$ under (2.4.6) yields at the leading order

$$
\begin{aligned}
0 &= \delta\Psi^A_{+z} \quad \overset{\text{order}\sqrt{\frac{\ell}{z}}}{\Longrightarrow} \quad 0 = \frac{1}{\ell}\left(\eta^A_{(1)+} - \xi^\mu \zeta^A_{\mu+}\right) , \qquad (2.4.13)\\
0 &= \delta\Psi^A_{-z} \quad \overset{\text{order}\sqrt{\frac{z}{\ell}}}{\Longrightarrow} \quad 0 = \frac{1}{\ell}\left(\eta^A_{(1)-} - \xi^\mu \zeta^A_{\mu-}\right) + \frac{\mathrm{i}}{4\ell}\epsilon^{AB} A_{(1)\mu}\gamma^\mu \left(\eta_{B+} - \xi^\nu \psi_{B\nu+}\right),
\end{aligned}
$$

which can be solved, by exploiting (2.4.11), as

$$
\eta^A_{(1)+} = 0 \, , \qquad \eta^A_{(1)-} = \xi^\mu \zeta^A_{\mu-} - \frac{\mathrm{i}}{4}\,\epsilon^{AB} A_{(1)\mu}\gamma^\mu \left(\eta_{B+} - \xi^\nu \psi_{B\nu+}\right) . \qquad (2.4.14)
$$

Furthermore, the transformation law of the radial component of the graviphoton implies

$$
0 = \delta\hat{A}_z = \frac{1}{\ell}\,\lambda_{(1)} - \frac{1}{\ell}\,A_{(1)\mu}E^\mu{}_i \xi^i + \mathcal{O}(z) \quad \Rightarrow \quad \lambda_{(1)} = A_{(1)\mu}\xi^\mu \, . \qquad (2.4.15)
$$

Eventually, let us require $\delta\hat{\omega}^{i3}_z = 0$ and $\delta V^i{}_z = 0$ in (2.4.6). At finite order, they have the form

$$
0 = \delta V^i_{(0)z} = \ell \, \delta\hat{\omega}^{i3}_{(0)z} = j^{i3}_{(1)} + \frac{1}{\ell}\,p^i_{(1)} + w^{ij}_{(0)}\,\xi_j + \mathrm{i}\,\bar{\eta}_{+A}\gamma^i\,\psi^A_{+z} \, . \qquad (2.4.16)
$$

There are two unknown parameters, $p^i_{(1)}$ and $j^{i3}_{(1)}$, and only one equation, that leads to an arbitrary vector $K^i(x)$ in the solution, associated with the special conformal transformations on $\partial\mathcal{M}$, as we will prove later. The solution for the first order parameters is

$$
\begin{aligned}
p^i_{(1)} &= \ell m^i + \frac{\ell^2}{2}\,K^i \equiv b^i, \qquad (2.4.17)\\
\ell j^{i3}_{(1)} &= \ell m^i - \frac{\ell^2}{2}\,K^i \equiv -\tilde{b}^i \, ,
\end{aligned}
$$

where $m^i(x)$ is a function that depends on the gauge fixing,

$$
m^i(x) = -\frac{1}{2}\,\left(w^{ij}_{(0)}\xi_j + \mathrm{i}\,\bar{\eta}^A_+\gamma^i\psi_{Az+}\right) . \qquad (2.4.18)
$$

At the linear order in z, we get

$$
\begin{aligned}
0 &= \delta V^i_{(1)z} = j^{i3}_{(2)} + \frac{2}{\ell}\,p^i_{(2)} + n^i \, ,\\
0 &= \ell \, \delta\hat{\omega}^{i3}_{(1)z} = 2j^{i3}_{(2)} + \frac{1}{\ell}\,p^i_{(2)} + s^i \, , \qquad (2.4.19)
\end{aligned}
$$

where we denoted

$$
\begin{aligned}
n^i(x) &= w^{ij}_{(1)}\xi_j + \mathrm{i}\,\bar{\eta}_{+A}\gamma^i\,\zeta^A_{+z} \, ,\\
s^i(x) &= -\frac{1}{\ell}\,\xi^\mu(\tau - 4\tilde{\tau})^i{}_\mu + \mathrm{i}\;\bar{\eta}_{+A}\gamma^i\,\zeta^A_{+z} - \xi^\mu E^{\nu i}\bar{\psi}_{+A\mu}\zeta^A_{-\nu} - \mathrm{i}\,\xi^\mu\bar{\psi}_{+A\mu}\gamma^i\zeta^A_{+z}
\end{aligned}
$$

$$-\frac{\mathrm{i}}{4}\xi^\mu E^{\nu i}\epsilon_{AB}\overline{\psi}^A_{\mu+}\gamma^\rho\psi^B_{\nu+}A_{(1)\rho}+E^{\mu i}\overline{\eta}^A_+\left(\frac{\mathrm{i}}{4}\epsilon_{AB}\gamma^\rho\psi^B_{+\mu}A_{(1)\rho}+\zeta_{A-\mu}\right). \quad (2.4.20)$$

The function $w^{ij}_{(1)}$ can be determined from the vanishing supertorsion equation (see (A.2.14) in Appendix A.2),

$$w^{ij}_{(1)}=-\frac{2}{\ell}\,(\tau-\tilde{\tau})^{ij}-\mathrm{i}\,E^{\mu j}\overline{\psi}_{+A\mu}\gamma^i\,\zeta^A_{+z}\,. \quad (2.4.21)$$

The solution for the second order parameters $p^i_{(2)}$ and $j^{i3}_{(2)}$ is unique,

$$\begin{aligned} p^i_{(2)} &= \frac{\ell}{3}\left(s^i-2n^i\right)\,,\\ \ell j^{i3}_{(2)} &= \frac{\ell}{3}\left(n^i-2s^i\right)\,. \end{aligned} \quad (2.4.22)$$

In the following computations, we will need only the combination of the parameters

$$\begin{aligned} \ell j^{i3}_{(2)}-p^i_{(2)} &= \ell\left(n^i-s^i\right)=-\xi^\mu(\tau+2\tilde{\tau})^i{}_\mu+\ell\xi^\mu E^{\nu i}\bar{\psi}_{+A\mu}\zeta^A_{-\nu} \quad (2.4.23)\\ &\quad+\frac{\mathrm{i}\ell}{4}\,\xi^\mu E^{\nu i}\epsilon_{AB}\overline{\psi}^A_{\mu+}\gamma^\rho\psi^B_{\nu+}A_{(1)\rho}-\ell E^{\mu i}\overline{\eta}^A_+\left(\frac{\mathrm{i}}{4}\,\epsilon_{AB}\gamma^\nu\psi^B_{+\mu}A_{(1)\nu}+\zeta_{A-\mu}\right). \end{aligned}$$

After all the above considerations and writing only the relevant terms, the residual local parameters can be written as

$$\begin{aligned} p^3 &= -\ell\sigma+\mathcal{O}(z^2)\,,\\ p^i &= \frac{\ell}{z}\,\xi^i+\frac{z}{\ell}\,b^i+\frac{z^2}{\ell^2}\,p^i_{(2)}+\mathcal{O}(z^3)\,,\\ j^{i3} &= \frac{1}{z}\,\xi^i-\frac{z}{\ell^2}\,\tilde{b}^i+\frac{z^2}{\ell^2}\,j^{i3}_{(2)}+\mathcal{O}(z^3)\,,\\ j^{ij} &= \theta^{ij}+\mathcal{O}(z^2)\,,\\ \hat{\lambda} &= \lambda+\frac{z}{\ell}\,A_{(1)\mu}\xi^\mu+\mathcal{O}(z^2)\,,\\ \epsilon^A_+ &= \sqrt{\frac{\ell}{z}}\begin{pmatrix}\eta^A_+\\ 0\end{pmatrix}+\mathcal{O}(z^{1/2})\,,\\ \epsilon^A_- &= \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\ \eta^A_-\end{pmatrix}+\mathcal{O}(z^{3/2})\,. \end{aligned} \quad (2.4.24)$$

The parameters $p^i_{(2)}$ and $j^{i3}_{(2)}$ will play a role in cancellation of terms in the next step, but will not influence the transformation law of the holographic fields. We also expect that the conservation laws do not depend on m^i, since it is a gauge fixing function.
Let us notice that in absence of the gravitini one has $b^i=\tilde{b}^i=\frac{\ell^2}{2}\,K^i$, $w^{ij}=0$, namely the result coincides with the pure AdS case (2.1.46).

Therefore, the independent residual parameters in the $\mathcal{N}=2$ AdS$_4$ supergravity are

$$\sigma(x),\ \xi^i(x),\ \theta^{ij}(x)\,,\ \lambda(x)\,,\ \eta^A_\pm(x)\,,$$

associated, respectively, with the dilations, diffeomorphisms, Lorentz, Abelian, and supersymmetry transformations in the holographically dual theory.

The parameters b^i and $\tilde{b}^i$ have not been taken into account because $b^i - \tilde{b}^i = 2\ell m^i$ is non physical, while $b^i + \tilde{b}^i = \ell^2 K^i$ is not an independent quantity, due to the condition (2.1.8). The invariance of the latter implies

$$0 = \delta V^3_{\ \mu} = -\ell\partial_\mu\sigma - \ell E^i_{\ \mu}K_i + \ell\xi_i\mathcal{S}^i_{\ \mu} + \overline{\eta}_{A+}\psi_{-A\mu} - \overline{\eta}_{A-}\psi_{+A\mu} + \mathcal{O}(z)\,. \tag{2.4.25}$$

The finite part of the above equation can be solved in terms of $K^i = (b^i + \tilde{b}^i)/\ell^2$ as

$$K^i = \frac{1}{\ell}\,E^{\mu i}\left(-\ell\partial_\mu\sigma + \ell\xi_j\,\mathcal{S}^j_{\ \mu} + \overline{\eta}_{A+}\psi^A_{-\mu} - \overline{\eta}_{A-}\psi^A_{+\mu}\right)\,, \tag{2.4.26}$$

confirming that it is not an independent local parameter. This analysis completes the establishment of the radial expansion for the asymptotic parameters up to the relevant order.

Transformation law of the holographic fields

It remains to determine the transformation law of the boundary fields. This is fundamental for their identification with the sources in the boundary CFT.

The bulk fields (2.3.14) can be cast in the form

$$\begin{aligned}
V^i_{\ \mu} &= \frac{\ell}{z}\,E^i_{\ \mu} + \frac{z}{\ell}\,S^i_{\ \mu} + \frac{z^2}{\ell^2}\,\tau^i_{\ \mu} + \mathcal{O}(z^3)\,,\\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,E^i_{\ \mu} - \frac{z}{\ell^2}\,\tilde{S}^i_{\ \mu} - \frac{2z^2}{\ell^3}\,\tilde{\tau}^i_{\ \mu} + \mathcal{O}(z^3)\,,\\
\hat{\omega}^{ij}_\mu &= \omega^{ij}_\mu + \frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu} + \mathcal{O}(z^3)\,,\\
\hat{A}_\mu &= A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,,\\
\Psi^A_{\mu+} &= \sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}\psi^A_{\mu+}\\ 0\end{pmatrix} + \frac{z^2}{\ell^2}\begin{pmatrix}\Pi^A_{\mu+}\\ 0\end{pmatrix} + \mathcal{O}(z^3)\right],\\
\Psi^A_{\mu-} &= \sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}0\\ \psi^A_{\mu-}\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}0\\ \zeta^A_{\mu-}\end{pmatrix} + \mathcal{O}(z^2)\right].
\end{aligned} \tag{2.4.27}$$

Directly from (2.2.14), by writing the boundary 1-forms in the basis (2.4.27) on $\partial\mathcal{M}$, we find for the transformation law of the bosonic fields

$$\begin{aligned}
\delta E^i &= \mathcal{D}\xi^i + \sigma E^i - \theta^{ij}E_j + \mathrm{i}\,\overline{\eta}^A_+\gamma^i\psi_{+A}\,,\\
\delta\omega^{ij} &= \mathcal{D}\theta^{ij} + 2\xi^{[i}\mathcal{S}^{j]} + 2\,K^{[i}E^{j]} + \frac{1}{\ell}\,\overline{\eta}^A_+\gamma^{ij}\psi_{-A} + \frac{1}{\ell}\,\overline{\eta}^A_-\gamma^{ij}\psi_{+A}\,,\\
\delta A &= \mathrm{d}\lambda + 2\epsilon_{AB}\,\overline{\eta}^A_+\psi^B_- + 2\epsilon_{AB}\,\overline{\eta}^A_-\psi^B_+\,,
\end{aligned} \tag{2.4.28}$$

and for the gravitino

$$\delta\psi_{+A} = \mathcal{D}\eta_{A+} + \frac{\mathrm{i}}{\ell}\,E^i\gamma_i\eta_{A-} - \frac{\mathrm{i}}{\ell}\,\xi^i\gamma_i\psi_{-A} + \frac{1}{2}\,\sigma\psi_{+A}$$

$$-\frac{1}{4}\,\theta^{ij}\gamma_{ij}\varphi_{A+}+\frac{1}{2\ell}\,\lambda\epsilon_{AB}\,\psi_+^B-\frac{1}{2\ell}\,A\,\epsilon_{AB}\eta_+^B\,. \tag{2.4.29}$$

The super-Schouten tensor and its superpartner, the conformino, are composite fields that appear at the subleading order in (2.2.14) and transform as

$$\begin{aligned}
\delta\mathcal{S}^i &= \mathcal{D}K^i-\sigma\mathcal{S}^i-\theta^{ij}\mathcal{S}_j+\frac{2\mathrm{i}}{\ell^2}\,\overline{\eta}_-^A\Gamma^i\psi_{-A}+\mathcal{E}^i\,,\\
\delta\psi_{-A} &= \mathcal{D}\eta_{A-}+\frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i\gamma_i\eta_{A+}-\frac{\mathrm{i}\ell}{2}\,K^i\gamma_i\psi_{+A}-\frac{1}{2}\,\sigma\psi_{-A}\\
&\quad -\frac{1}{4}\,\theta^{ij}\gamma_{ij}\varphi_{-A}+\frac{1}{2\ell}\,\lambda\epsilon_{AB}\psi_-^B-\frac{1}{2\ell}\,A\epsilon_{AB}\eta_-^B+\Sigma_A\,.
\end{aligned} \tag{2.4.30}$$

The equations (2.4.28)–(2.4.30), together with the transformation law of $B\equiv V^3_{\mu}\mathrm{d}x^\mu$, given by (2.4.25), define the full set of $\mathcal{N}=2$ superconformal transformations of the boundary 1-forms E^i, B, $\mathcal{S}^i$, ω^{ij}, A, $\psi_{\pm A}$.

Here, similarly to the Cotton tensor, which arises in the transformation law of the pure AdS gravity from the Lie derivative, as discussed at the end of Section 2.1, the tensor $\mathcal{E}^i={\mathcal{E}^i}_\mu\,\mathrm{d}x^\mu$ [16] and the spinor $\Sigma^A=\Sigma^A_\mu\,\mathrm{d}x^\mu$ appear at order z and $z^{1/2}$ respectively,

$$\begin{aligned}
{\mathcal{E}^i}_\mu &= \frac{2}{\ell}\,\tilde{R}^{i3}_{(3)jk}\xi^k{E^j}_\mu+\frac{1}{\ell}\,\overline{\Theta}{}^{i3}_{(5/2)-A|j}\left(\eta_+^A{E^j}_\mu-\psi^A_{+\mu}\xi^j\right),\\
{\Sigma^A}_\mu &= 2E^\nu{}_{[i}E^\lambda{}_{j]}\left(\nabla_\nu\psi^A_{\lambda-}+\frac{\mathrm{i}\ell}{2}\,{\mathcal{S}^k}_\nu\gamma_k\psi^A_{\lambda+}\right)\xi^i{E^j}_\mu\,.
\end{aligned} \tag{2.4.31}$$

In order to explicitly relate them to the Cotton tensor, we recall that in pure gravity, from a geometrical point of view, the linear term of $\hat{R}^{i3}_{\mu\nu}$ is related to the Cotton tensor through (2.1.26). Thus, the $\mathcal{N}=2$ supersymmetric extension of the Cotton tensor (${\mathcal{C}^i}_{\mu\nu}$) and its superpartner, the Cottino ($\Omega^A_{\mu\nu}$), are the first subleading terms in the corresponding supercurvature expansions, defined by

$$\begin{aligned}
\hat{\mathbf{R}}^{i3}_{\mu\nu} &= -z\,{\mathcal{C}^i}_{\mu\nu}+\mathcal{O}(z^2)\,,\\
\hat{\boldsymbol{\rho}}^A_{-\mu\nu} &= \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\ \Omega^A_{\mu\nu}\end{pmatrix}+\mathcal{O}(z^{3/2})\,,
\end{aligned} \tag{2.4.32}$$

giving rise, by means of (2.2.9), to the expressions

$${\mathcal{C}^i}_{\mu\nu} = 2\mathcal{D}_{[\mu}{\mathcal{S}^i}_{\nu]}-\frac{2\mathrm{i}}{\ell^2}\,\overline{\psi}{}^A_{-[\mu}\gamma^i\psi_{-A|\nu]}\,, \tag{2.4.33}$$

$$\Omega^A_{\mu\nu} = 2\nabla_{[\mu}\psi^A_{-\nu]}-\mathrm{i}\ell\,\gamma_i\psi^A_{+[\mu}{\mathcal{S}^i}_{\nu]}\,. \tag{2.4.34}$$

An easy way to connect the above quantities to the transformation law of the super-Schouten tensor and the conformino is by using the rheonomic parametrisations of the supercurvatures $\hat{\mathbf{R}}^{i3}_{\mu\nu}$ and $\hat{\boldsymbol{\rho}}^A_{-\mu\,\nu}$, given by the last two equations in (2.2.13). Taking all the terms into account, the super-Cotton tensor and the Cottino are expressed as

$$-\ell\,{\mathcal{C}^i}_{\mu\nu} = \hat{\mathbf{R}}^{i3}_{(1)\mu\nu}=2\tilde{R}^{i3}_{(3)jk}{E^j}_\mu{E^k}_\nu-2\overline{\psi}_{+A[\mu}{E^j}_{\nu]}\Theta^{i3}_{(5/2)-A|j}\,, \tag{2.4.35}$$

[16] In our conventions, the z-expansion coefficients of the 4-spinor-tensor $\Theta^{ab|c}_A$ are expressed in terms of bispinor-tensors $\Theta^{ab|c}_{(n)\pm A}$. Similarly, the 4-spinors $\tilde{\rho}^{ab}_A$ have bispinor coefficients $\tilde{\rho}^{ab}_{(n)\pm A}$.

$$\Omega^A_{\mu\nu} = \hat{\boldsymbol{\rho}}^A_{(1/2)-\mu\,\nu} = 2\tilde{\rho}^A_{(5/2)-ij}E^i{}_\mu E^j{}_\nu = 4\Theta^{i3}_{(5/2)-A|j}E_{i[\mu}E^j{}_{\nu]}\,.$$

The last step makes use of the explicit expressions in Appendix A.3 to decompose the spinor-tensor coefficient $\Theta^{i3|j}_{(5/2)-A}$ into its symmetric part, $-2\mathrm{i}\gamma^{(i}\tilde{\rho}^{j)3}_{(5/2)A+}$ and the antisymmetric part $\frac{1}{2}\,\Omega^{Aij}$. As a result, the additional terms in the transformation law (2.4.30) are recognised to be the contractions of the super-Cotton tensor and Cottino with respect to the boundary superdiffeomorphism parameters ξ^i and η^A_+,

$$\begin{aligned}\Sigma^A &= i_\xi\Omega^A\,,\\ \mathcal{E}^i &= i_\xi\mathcal{C}^i + \frac{1}{\ell}\left(\overline{\eta}^A_+ - \overline{\psi}_{+A\nu}\xi^\nu\right)\Theta^{i3}_{(5/2)-A|j}\,E^j\,.\end{aligned} \tag{2.4.36}$$

Finally, we obtain an expected result for $\delta\mathcal{S}^i$ and $\delta\psi_{-A}$. The contribution of the symmetric part of the spinor-tensor $\Theta^{i3|j}_{(5/2)-A}$ is non physical, as it depends on the gauge fixing functions ψ_{+zA} and $A_{(1)z}$. We can, in principle, further gauge fix the higher order parameters such that $\tilde{\rho}^{i3}_{(5/2)A+}$ vanishes as a consequence of $\hat{\boldsymbol{\rho}}^A_{(1/2)\mu z+} = 0$. However, the result does not have observable consequences near the boundary, thus we will not proceed in this direction.

Let us notice that not all the contractions of the OSp(2|4) supercurvatures appear in the transformation laws (2.4.28)–(2.4.30), but only the ones that have origin in the negative grading supercurvatures. This is due to the fact that, after imposing (2.2.15), all the OSp(2|4) supercurvatures vanish on $\partial\mathcal{M}$, except two, $\hat{\mathbf{R}}^{i3}_{\mu\nu}$ and $\hat{\boldsymbol{\rho}}^A_{-\mu\,\nu}$. Indeed, (2.2.15) leads to the weaker condition on these supercurvatures,

$$\epsilon_{ijk}\,\mathcal{C}^i{}_{[\mu\nu}E^k{}_{\rho]} + 2\overline{\psi}_{+A[\mu}\gamma_j\Omega_{A\nu\rho]} = 0\,, \tag{2.4.37}$$

which implies in particular $\gamma_{[\mu}\Omega^A_{\nu\rho]} = 0$ and, consequently, $\gamma^\nu\Omega^A_{\nu\rho} = 0$.
As a matter of fact, non trivial $\mathcal{C}^i$ and Ω_A on $\partial\mathcal{M}$ mean that the holographic SCFT is not invariant under local OSp(2|4) transformations, for the same reason as SO(2,3) is not a local symmetry of the bulk gravity –namely, they are only general coordinate transformations rewritten in a gauge-covariant form, as we saw in Section 1.1. This explains the origin of the contractions of supercurvatures in the transformation laws.

If we consider the set of fields $E^i{}_\mu$, $\psi_{+A\mu}$, ω^{ij}_μ, A_μ, we see that the first three transform, respectively, as a boundary vielbein, a spin connection and a gravitino, charged with respect to the SO(2) R-symmetry connection A_μ. Correspondingly, the parameters ξ^i, η_{+A}, θ^{ij} and λ are associated with boundary diffeomorphisms, supersymmetry, Lorentz, and SO(2) gauge transformations.
On the other hand, the boundary function σ, with respect to which all the above fields have definite weights (1 for $E^i{}_\mu$, 1/2 for $\psi_{+A\mu}$ and 0 for ω^{ij}_μ and A_μ), is identified with the local parameter associated with Weyl dilations, since it produces a rescaling of the vielbein and, therefore, of the metric. In the same fashion, superconformal transformations are characterised by the local parameter η_{-A}, with the corresponding gauge field ψ_{A-}. The parameter K^i, although not independent within the gauge choice $V^3{}_\mu = 0$, corresponds to special conformal transformations, whose associated gauge connection is the super-Schouten tensor.

Consistency of the subleading gauge fixings

On top of the previous analysis of the asymptotic parameters, it remains to look for potential inconsistencies in the vanishing of linear terms, in particular $V^3_{(1)\mu} = \omega^{ij}_{(1)\mu} = \zeta^A_{\mu+} = 0$. By using the transformation law of the gauge fields, it is straightforward to find

$$
\begin{aligned}
\delta V^3_{(1)\mu} &= \frac{2}{\ell}\,\xi^\nu(\tau + 2\tilde{\tau})_{[\nu\mu]} + 2\xi^\nu \overline{\psi}^A_{+[\nu}\zeta_{\mu]-A} = 0\,,\\
\delta\zeta^A_{\mu+} &= -\frac{\mathrm{i}}{\ell}\,\gamma_i\zeta^A_{-\mu}\xi^i - \frac{1}{2\ell}\,\psi^A_{\mu+}p^3_{(1)} + 2\tilde{\rho}^A_{(5/2)+ij}\xi^i E^j{}_\mu - \frac{1}{4\ell}\,\xi^\rho A_{(1)\rho}\epsilon^{AB}\psi_{B\mu+}\,, \\
&\quad + \frac{\mathrm{i}}{4\ell}\,\gamma^i\psi_{B+\mu}\epsilon_{ijk}A_{(1)\rho}E^{\rho k}\xi^j + \frac{\lambda_{(1)}}{2\ell}\,\epsilon^{AB}\psi_{B+\mu} + \frac{\mathrm{i}}{\ell}\,\gamma_i\eta^A_{(1)-}E^i{}_\mu - \frac{1}{2\ell}\,A_{(1)\mu}\epsilon^{AB}\eta_{B+}\\
&\quad + \frac{1}{4\ell}\,A_{(1)\mu}\epsilon^A B\eta_{B+} - \frac{\mathrm{i}}{4\ell}\,\epsilon^{AB}\epsilon_{ijk}\gamma^i\eta_{B+}E^j{}_\mu A_{(1)\rho}E^{\rho k} = 0\,,
\end{aligned}
\tag{2.4.38}
$$

where the first condition holds by virtue of (2.3.25) and the second one follows from plugging in the expressions of $\tilde{\rho}^A_{ij}$, $\lambda_{(1)}$, $\eta^A_{(1)-}$ and $p^3_{(1)} = 0$. Eventually, the variation of (A.2.12) enables to solve

$$
\delta\omega^{ij}_{(1)\mu} = \mathrm{i}\,E^{\nu i}E^{\lambda j}E_{k\mu}\,\delta\overline{\zeta}^A_{+[\nu}\gamma^k\psi^A_{\lambda]+} - 2\mathrm{i}\,E^{\nu[i}\delta\overline{\zeta}_{+A[\mu}\gamma^{j]}\psi^A_{\nu]+}\,,
\tag{2.4.39}
$$

from which we recognise that $\delta\zeta_{A\mu+} = 0$ implies $\delta\omega^{ij}_{(1)\mu} = 0$.

2.5 Superconformal currents in the holographic quantum theory

In the previous Section, we showed that the residual symmetries of the pure $\mathcal{N}=2$ AdS$_4$ supergravity are given by the three dimensional superconformal transformations. According to the AdS/CFT correspondence, these are also asymptotic symmetries underlying the dual superconformal field theory.

The superconformal group on a three dimensional manifold includes Lorentz transformations (with the local parameter θ^{ij}), coordinate transformations (ξ^i), dilations (σ), special conformal transformations (K^i), supersymmetry trasformations (η_{A+}), special superconformal transformations (η_{A-}) and the R-symmetry (λ). Within a gauge theory, the corresponding gauge fields are the spin connection ω^{ij}_μ, the vielbein $E^i{}_\mu$, the dilation gauge field B_μ, the super-Schouten tensor $\mathcal{S}^i{}_\mu$, the gravitino $\psi^A_{+\mu}$, the conformino $\psi^A_{-\mu}$ and the graviphoton A_μ.

It is useful to present this *superconformal structure* of the boundary by listing all the transformations, associated local parameters and gauge fields (sources in SCFT), and the conserved currents (quantum operators in SCFT) in the following table:

Transformation	**Local parameter**	**Source**	**Current**
Lorentz	θ^{ij}	ω_μ^{ij}	$J^\mu{}_{ij} = 0$
Translation	ξ^i	$E^i{}_\mu$	$J^\mu{}_i$
Dilation	σ	$B_\mu = 0$	$J^\mu{}_{(D)} = 0$
Special conformal	K^i	$\mathcal{S}^i{}_\mu$	$J^\mu_{(K)i} = 0$
Abelian R-symmetry	λ	A_μ	J^μ
Supersymmetry	η_{A+}	$\psi_{A+\mu}$	$J^\mu{}_{A+}$
Superconformal	η_{A-}	$\psi_{A-\mu}$	$J^\mu{}_{A-} = 0$

When all sources are independent, the currents are also independent. When one imposes the constraints over supercurvatures with a purpose to eliminate non physical degrees of freedom, some parameters result to be realised non linearly and the corresponding sources become composite fields, with the associated currents vanishing.

In supergravity, the spin connection is a composite field determined by a constraint on the translation curvature (supertorsion). The gauge field of special conformal transformations (super-Schouten tensor) and its supersymmetric partner (conformino) are also composite, obtained from the constraint on the conformal supercurvatures, equations (2.4.2) and (2.4.3). Our particular gauge fixing $B_\mu = V^3_\mu = 0$ eliminates the dilation gauge field and the corresponding dilation current. The inclusion of B_μ has been discussed in pure AdS gravity in [36].

Before moving on to the explicit analysis of quantum symmetries in a three dimensional field theory holographically dual to $\mathcal{N} = 2$ AdS$_4$ supergravity, let us first understand more precisely its superalgebra structure.

$d = 3$ superconformal algebra

The superisometry group OSp(2|4) of the vacuum of the bulk theory is encoded in the definition of its curvatures $\hat{\mathbf{R}}^\Lambda = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^a, \hat{\boldsymbol{\rho}}^A, \hat{\mathbf{F}}\}$,

$$\hat{\mathbf{R}}^\Lambda \equiv \mathrm{d}\boldsymbol{\mu}^\Lambda + \frac{1}{2} C_{\Sigma\Gamma}{}^\Lambda\, \boldsymbol{\mu}^\Sigma \wedge \boldsymbol{\mu}^\Gamma \,, \tag{2.5.1}$$

where $C_{\Sigma\Gamma}{}^\Lambda$ are the $\mathfrak{osp}(2|4)$ structure constants and $\boldsymbol{\mu}^\Lambda = \{\hat{\omega}^{ab},\, V^a,\, \Psi_A,\, \hat{A}\}$ the Cartan 1-forms. Asymptotic expansions of the supercurvatures $\hat{\mathbf{R}}^\Lambda$ are given in Appendix A.2. Moreover, $\mathfrak{osp}(2|4)$ also describes the superconformal structure of the boundary. This is made manifest by decomposing the Cartan 1-forms in irreducible representations with respect to the SO(1,1)×SO(2,1) subgroup of OSp(2|4), where SO(2,1) is (the connected component of) the Lorentz group at the boundary and SO(1,1) is the isometry group, which acts as a rescaling on the coordinate z in the FG parametrisation: $z \to \mathrm{e}^\sigma z$. This decomposition requires splitting the index a into $(i, 3)$, where $i = 0, 1, 2$. Furthermore, V^i and $\hat{\omega}^{i3}$ naturally combine into $V^i_\pm$ introduced in (2.3.5), which have definite scalings with respect to the $\mathrm{SO}(1,1)$ group. Eventually, since the spinorial representation of the generator T_0 of the $\mathrm{SO}(1,1)$ group is

$$(T_0)^\alpha{}_\beta = -\frac{\mathrm{i}}{2}\,(\Gamma^3)^\alpha{}_\beta \,, \tag{2.5.2}$$

the four dimensional gravitini naturally split into $\Psi_{\pm A}$ with definite radial chirality. In terms of the $\mathrm{SO}(1,1)\times\mathrm{SO}(2,1)$ irreducible forms $\hat\omega^{ij}$, V^i_+, V^i_-, V^3, A, $\Psi^A_\pm$, where we recall the expressions (2.3.5), the bulk supercurvatures [58] given by (2.2.9) become

$$\begin{aligned}
\hat{\mathbf{R}}^{ij} &= \hat{\mathcal{R}}^{ij} + \frac{4}{\ell^2}\, V^{[i}_+ \wedge V^{j]}_- - \frac{1}{\ell}\,\overline{\Psi}{}^A_+ \wedge \Gamma^{ij}\Psi_{A-}\,,\\
\hat{\mathbf{R}}^i_\pm &= \hat{\mathcal{D}}V^i_\pm \mp \frac{1}{\ell}\, V^i_\pm \wedge V^3 \mp \frac{\mathrm{i}}{2}\,\overline{\Psi}{}^A_\pm \wedge \Gamma^i\Psi_{A\,\pm}\,,\\
\hat{\mathbf{R}}^3 &= \mathrm{d}V^3 + \frac{2}{\ell}\, V^i_+ \wedge V_{-i} + \overline{\Psi}{}^A_- \wedge \Psi_{A+}\,,\\
\hat{\mathbf{F}} &= \mathrm{d}\hat{A} - 2\epsilon_{AB}\,\overline{\Psi}{}^A_+ \wedge \Psi^B_-\,,\\
\hat{\boldsymbol{\rho}}^A &= \hat{\mathcal{D}}\Psi^A_\pm \pm \frac{\mathrm{i}}{\ell}\, V^i_\pm \wedge \Gamma_i\Psi^A_\mp \pm \frac{1}{2\ell}\, V^3 \wedge \Psi^A_\pm - \frac{1}{2\ell}\,\epsilon_{AB}\hat{A}\wedge\Psi^B_\pm\,.
\end{aligned} \tag{2.5.3}$$

The right hand sides of the above equations encode the algebraic structure of the superconformal algebra in $d=3$, where V^3 is the 1-form associated with the Weyl transformations, V^i_+ the ones associated with the spacetime translations, V^i_- with the conformal boosts, Ψ^A_+ with the supersymmetries, Ψ^A_- with the superconformal transformations [77,78]. The connection components $\hat\omega^{ij}$ correspond to the Lorentz algebra at the boundary. The precise connection to the Cartan 1-forms of the superconformal algebra in $d=3$ is that the leading order 1-form in the z-expansion of the above bulk quantities are identified with the Cartan 1-forms dual to the corresponding superconformal generators. Let us summarise below the correspondence between the $D=4$ gauge field and $d=3$ superconformal field:

$$\begin{array}{lcll}
\hat\omega^{ij} & \to & \omega^{ij} & \text{Lorentz symmetry}\,,\\
V^3 & \to & B & \text{Weyl symmetry}\,,\\
V^i_+ & \to & E^i & \text{spacetime translations}\,,\\
V^i_- & \to & \mathcal{S}^i & \text{conformal boosts}\,,\\
\Psi^A_+ & \to & \psi^A_+ & \text{supersymmetry}\,,\\
\Psi^A_- & \to & \psi^A_- & \text{superconformal symmetry}\,,\\
\hat{A} & \to & A & \mathrm{SO}(2)\ \text{R-symmetry}\,.
\end{array}$$

This can also be understood as the boundary conditions set imposed on the bulk fields in an asymptotically AdS space.

Let us make this connection more precise. To this end, we perform the redefinitions (2.3.1) and (2.3.6) and define the gauge vector associated with the Weyl rescalings as follows,

$$B = \frac{1}{\ell}\left(V^3 - \ell\frac{\mathrm{d}z}{z}\right) = B_\mu(x)\,\mathrm{d}x^\mu\,. \tag{2.5.4}$$

Note that, in order for B to be non vanishing, we have to generalise the FG parametrisation (2.1.1) to allow for a non trivial component V^3_μ for the vielbein. After rescaling the various fields by z/ℓ factors according to their $\mathrm{O}(1,1)$ grading, the $\mathrm{d}z/z$ term in V^3, within the definitions of the field strengths, cancel out. Next, we recall the relation between the $d=3$ super-Schouten tensor and E^i_- given by the second equation of (2.3.7),

$$\mathcal{S}^i = -\frac{2}{\ell^2}\; E^i_-\Big|_{z=0}\,. \tag{2.5.5}$$

By rescaling the field strengths associated with $\Psi_\pm$ and $V^i_\pm$, in (2.5.3), correspondingly, we can evaluate the right hand side at $z=0$, $\mathrm{d}z=0$ and find the following supercurvatures in the dual field theory (see Appendix A.2),

$$
\begin{aligned}
\mathbf{R}^{ij} &= \mathcal{R}^{ij} - 2\,E^{[i}\wedge\mathcal{S}^{j]} - \frac{1}{\ell}\,\overline{\psi}{}^A_+\wedge\gamma^{ij}\psi_{A-}\,,\\
\mathbf{R}^i_+ &= \mathcal{D}E^i + B\wedge E^i - \frac{\mathrm{i}}{2}\,\overline{\psi}{}^A_+\wedge\gamma^i\psi_{A+}\,,\\
\mathcal{C}^i \equiv -\frac{2}{\ell^2}\,\mathbf{R}^i_- &= \mathcal{D}\mathcal{S}^i - B\wedge\mathcal{S}^i - \frac{\mathrm{i}}{\ell^2}\,\overline{\psi}{}^A_-\wedge\gamma^i\psi_{A-}\,,\\
\mathbf{R} &= \mathrm{d}B - E^i\wedge\mathcal{S}_i + \frac{1}{\ell}\,\overline{\psi}{}^A_-\wedge\psi_{A+}\,,\\
\mathbf{F} &= \mathrm{d}A - 2\epsilon_{AB}\,\overline{\psi}{}^A_+\wedge\psi^B_-\,,\\
\boldsymbol{\rho}^A_+ &= \mathcal{D}\psi^A_+ + \frac{1}{2}\,B\wedge\psi^A_+ + \frac{\mathrm{i}}{\ell}\,E^i\wedge\gamma_i\psi^A_- - \frac{1}{2\ell}\,\epsilon_{AB}A\wedge\psi^B_+\,,\\
\Omega^A \equiv \boldsymbol{\rho}^A_- &= \mathcal{D}\psi^A_- - \frac{1}{2}\,B\wedge\psi^A_- + \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i\wedge\gamma_i\psi^A_+ - \frac{1}{2\ell}\,\epsilon_{AB}A\wedge\psi^B_-\,,
\end{aligned}
\tag{2.5.6}
$$

where $\mathcal{D}$ is the Lorentz-covariant derivative. Each $\mathcal{D}$ always appears in the combination $\mathcal{D}+\Delta B$ of the Weyl-covariant derivative, as naturally expected from a theory with local Weyl symmetry. The Weyl weight Δ of the corresponding field is equal to its scaling dimension, namely $\Delta(E^i_\pm)=\pm1$, $\Delta(\psi^A_\pm)=\pm\frac{1}{2}$, $\Delta(\mathcal{S}^i)=-1$ and $\Delta(\omega^{ij})=\Delta(A)=\Delta(B)=0$. This feature can be used to reconstruct the B-terms in the transformations laws (2.4.28)–(2.4.30), similarly as it was done in the pure AdS gravity case given by (2.1.58).
Eventually, let us note that, for $B=0$, the third and the last equations of (2.5.6) yield the definitions of $\mathcal{C}^i$ and Ω^A in (2.4.33) and (2.4.34), respectively.

Superconformal currents

To explore the quantum symmetries in a SCFT dual to supergravity with $\Psi^A_{z-}=0$, we apply the AdS/CFT correspondence summarised in Section 1.2 to the case when the boundary fields are $\mathcal{J}^\Lambda(x)=\{E^i{}_\mu(x),\,\omega^{ij}_\mu(x),\,\psi_{+A\,\mu}(x),\,A_\mu(x)\}$. They become sources for the corresponding operators in the dual SCFT. Generalising (2.1.52) to the supergravity case, the bulk action in the classical supergravity approximation is identified with the effective action of the dual boundary theory as

$$
I_{\text{on-shell}}[E^i,\,\omega^{ij},\,\psi^A_+,\,A] = W[E^i,\,\omega^{ij},\,\psi^A_+,\,A] = -\mathrm{i}\,\ln(Z[E^i,\,\omega^{ij},\,\psi^A_+,\,A])\,. \tag{2.5.7}
$$

The sources $\mathcal{J}^\Lambda$ couple to the operators in quantum field theory $J^\mu_\Lambda=\{J^\mu{}_i,\,J^\mu{}_{ij},\,J^\mu_{A+},\,J^\mu\}$, which are the energy-momentum tensor, spin current, supercurrent, and U(1)-current, respectively. The latter are identified with the 1-point functions of the Noether currents in the presence of arbitrary sources, associated with the residual symmetries of the boundary action. However, we shall refrain from writing explicitly the symbol $\langle\cdots\rangle_{\text{CFT}}$. We will also express the currents in terms of their Hodge-dual 2-forms in the boundary theory, to be denoted by the same symbol, as defined by (2.1.54).

The explicit expression of these currents is inferred from the variation of the effective action with respect to the sources,

$$\delta W = \int_{\partial\mathcal{M}} \delta\mathcal{J}^\Lambda \wedge J_\Lambda = \int_{\partial\mathcal{M}} \left(\delta E^i \wedge J_i + \frac{1}{2}\,\delta\omega^{ij} \wedge J_{ij} + \overline{J}^A_+ \wedge \delta\psi_{A+} + J \wedge \delta A\right)\,, \tag{2.5.8}$$

which is the generalisation of (2.1.53) to supergravity.
Invariance of the boundary effective action with respect to the residual symmetries of the boundary theory implies conservation laws to be satisfied by the currents. As we shall prove, they are satisfied by virtue of the "constraint" equations of motion in the bulk. Namely, in the radial foliation of spacetime, the bulk equations of motion are divided into the ones describing the radial "evolution" (that were used to determine radial expansions of the bulk fields) and the "constraints", which do not contain radial derivatives ∂_z and should give rise to conservation laws in the holographic QFT.

Therefore, our program is to firstly obtain the expressions of the currents and the corresponding conservation laws. Eventually, using the bulk equations of motion, we shall show that these conditions are indeed satisfied at the quantum level and they represent the Ward identities in the SCFT.

In this derivation it is somewhat convenient to retain, in the computation of δW, a four dimensional notation, writing it in terms of the bulk fields and their curvatures, keeping in mind that, in the boundary integral, they are meant to be functions of the corresponding boundary values through the supergravity solution. So, when we write $\delta\hat{\omega}^{ab}$, $\delta\Psi^A$, $\delta\hat{A}$,, we mean the variations of the bulk fields in a supergravity solution, originating from a variation of the corresponding boundary conditions. By using the compact form (2.2.10) of the full supergravity action and the field equations, we find

$$\delta W = \delta I_{\text{on-shell}} = \int_{\partial\mathcal{M}} \left(-\frac{\ell^2}{4}\,\delta\hat{\omega}^{ab}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - 2\mathrm{i}\ell\delta\overline{\Psi}^A\Gamma_5\hat{\boldsymbol{\rho}}_A + \frac{1}{2}\,\delta\hat{A}\;{}^*\hat{\boldsymbol{F}}\right)\Bigg|^{\text{on-shell}}_{z=\mathrm{d}z=0}\,, \tag{2.5.9}$$

where we have explicitly indicated that the quantities in the integral are to be computed on the boundary $\partial\mathcal{M}$, namely at $z=\mathrm{d}z=0$. Using the boundary expansion of the four dimensional fields in (2.4.27), we can write the above variation in the form (2.5.8) (recall that we have set $\omega_{(1)}$ and ζ^A_+ to zero) and read off the explicit form of the external current 2-forms on $\partial\mathcal{M}$,

$$\begin{aligned}
J_i &= \frac{1}{2}\,\epsilon_{ijk}\left[\frac{2}{\ell}\,E^j\wedge(\tau^k+2\tilde{\tau}^k) + \overline{\psi}^A_+\wedge\gamma^{jk}\zeta_{A-}\right],\\
J_{ij} &= 0\,,\\
J &= \frac{1}{2}\,\epsilon_{ijk}\,\tilde{F}^{i3}\,V^j\wedge V^k\Big|_{z=0}\,,\\
J^A_+ &= -2\,\mathrm{i}\,E^i\wedge\gamma_i\zeta^A_- + A_{(1)}\wedge\epsilon_{AB}\,\psi^B_+\,,
\end{aligned} \tag{2.5.10}$$

where $\tilde{F}_{ab}$ are the components of the supercovariant field strength associated with the graviphoton, see (2.2.13). The current associated with the Lorentz transformation (J_{ij})

vanishes because it corresponds to a composite field (ω_μ^{ij}), but it has been treated as independent in first order formulation of gravity. The other composite fields ($\mathcal{S}^i{}_\mu$ and $\psi_{A-\mu}$) have not been taken into account as sources.

From the above expressions for the conserved current 2-forms, J_Λ, we can obtain the Noether currents J^μ_Λ as the Hodge-dual 3-vectors ${}^*J_\Lambda = J_{\Lambda\mu}\,dx^\mu$ defined by (2.1.54). The non vanishing currents are

$$\begin{aligned}
J^\mu_i &= -\frac{1}{\ell}\,\Big((\tau^\mu_i + 2\tilde{\tau}^\mu_i) - E^\mu_i(\tau^k{}_k + 2\tilde{\tau}^k{}_k)\Big) + \frac{\mathrm{i}}{e_3}\,\epsilon^{\mu\nu\rho}\,\bar{\psi}^A_{+\nu}\gamma_i\zeta_{A-\rho}\,,\\
J^\mu_{A+} &= -\frac{2\mathrm{i}}{e_3}\,\epsilon^{\mu\nu\rho}\gamma_\nu\zeta_{A-\rho} + \frac{1}{e_3}\,\epsilon^{\mu\nu\rho}\,A_{(1)\nu}\epsilon_{AB}\,\psi^B_{+\rho}\,,\\
J^\mu &= -g^{\mu\nu}_{(0)}\,\tilde{F}_{\nu z} = \frac{1}{2\ell}\,g^{\mu\nu}_{(0)}\,A_{(1)\nu}\,,
\end{aligned}\tag{2.5.11}$$

where in the first equation the traces $\tau^k{}_k$, $\tilde{\tau}^k{}_k$ are defined using the vielbein tensor (e.g. $\tau^k{}_k \equiv \tau^k{}_\mu\,E^\mu_k$). In the last equation we have used the fact that the contribution of A_z to $\tilde{F}_{\mu z}$ is subleading in z, while the fermion bilinears do not contribute at $z = 0$, having set $\varphi_{-Az} = 0$.

In particular, the holographic stress tensor is $J_{\mu\nu} = J_{\mu i}E^i{}_\nu$. Recall that, in the CFT$_d$ dual to pure AdS$_{d+1}$ gravity, this tensor is proportional to the (symmetric) metric coefficient $g_{(d)\mu\nu} \propto \tau_{\mu\nu}$ whose trace is zero. Indeed, the above result in pure gravity with the traceless $\tau^\mu_i = \tilde{\tau}^\mu_i$ reduces to $J^{\text{pure GR}}_{\mu\nu} = -\frac{3}{\ell}\,\tau_{\mu\nu}$. In the SCFT$_3$, the relevant bosonic coefficient is $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ and it is generally no longer symmetric because of $\tilde{\tau}_{\mu\nu}$. Furthermore, the trace of $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ is not necessarily zero –it has to be computed from the conservation law of the local Weyl symmetry.

In supergravity, the holographic stress tensor contains the fermionic contribution. Which particular fermionic coefficient becomes holographic can be determined by simple power counting in the variation of the action. Since the on-shell action is always a boundary term, the Jacobian e, given by (2.1.13), expressed in terms of the boundary Jacobian e_3 has a factor $1/z^4$, but on the boundary ($z =$ constant) it becomes $1/z^3$. Thus, the holographic order –the one that contributes to the holographic current– is always the *third* order in z of the variation of the on-shell Lagrangian density on the three dimensional boundary. For the metric, it means the third coefficient in the expansion ($\tau_{\mu\nu}$). For fermions, it translates into $\Psi_{(3/2)-\mu} = \zeta_{-\mu}$. Similarly, the third coefficient on the boundary of the Maxwell Lagrangian comes from $(\partial_z\hat{A}_\mu)^2$, implying that the finite part of $\partial_z\hat{A}_\mu$, that is $\hat{A}_{(1)\mu}$, enters the holographic current. In d dimensions, the respective holographic orders are $\tau^i{}_\nu = \hat{E}^i_{(d)\nu}$, $\Psi_{(d/2)-\mu}$, and $\hat{A}_{((d-1)/2)\mu}$. They are the last terms in the near-boundary power expansion of the variation of the action which do not vanish when $z = 0$.

Conservation laws in SCFT

We observe that, in the boundary expansion of the integrand form in (2.5.9), the divergent terms vanish by virtue of the conditions (2.2.15) that, in components, are given by (2.4.1). These conditions therefore guarantee consistency of the holographic construction. Namely, both the currents and the conservation laws become finite, confirming that the bulk supergravity has been properly regularised in the asymptotic region.

Since the leading terms in the boundary expansion of the bulk curvatures vanish for (2.4.1), from (2.5.9) it follows that the currents in (2.5.10) are expressed in terms of the subleading terms in the same expansions. The reader can check, for instance, that

$$J_{ij}=-\ell^2\,\epsilon_{ijk}\,\hat{\mathbf{R}}^{k3}_{(0)}\ ,\quad J_i=-\frac{\ell}{2}\,\epsilon_{ijk}\,\hat{\mathbf{R}}^{jk}_{(1)}\ ,\quad J_{A+}=-2\ell\,\hat{\boldsymbol{\rho}}_{(1/2)\,A+}\,. \tag{2.5.12}$$

Next, we seek for the form of conservation laws associated with the residual symmetry discussed in Section 2.4, in case when the quantum effective action is invariant. After that, we will have to check whether the obtained supercurrents indeed satisfy these conservation laws and, since they are quantum, in fact they will give the Ward identities. The corresponding transformations are parametrised by ξ^i, θ^{ij}, λ, $\eta^A_\pm$. This means that δW evaluated on the corresponding symmetry transformations of the fields must vanish and amounts to the following conservation laws for the Noether currents which are the generalisation of the pure gravity laws (2.1.57) (we omit the wedge symbol),

$$\begin{aligned}
\mathcal{D}J_i &= \mathcal{S}^j\,J_{ij}-\frac{\mathrm{i}}{\ell}\overline{J}{}^A_+\gamma_i\psi_{A-}+\mathcal{S}^k{}_i\,J_{kj}\,E^j-\frac{\mathrm{i}\ell}{2}\,\mathcal{S}^j{}_i\,\overline{J}{}^A_-\gamma_j\psi_{A+}\,,\\
\mathcal{D}J_{ij} &= 2\,E_{[i}\,J_{j]}-\frac{\mathrm{i}}{2}\,\overline{J}{}^A_+\gamma_{ij}\psi_{A+}-\frac{\mathrm{i}}{2}\,\overline{J}{}^A_-\gamma_{ij}\psi_{A-}\,,\\
0 &= \partial_\mu\left[E^{\mu i}\left(J_{ij}\,E^j-\frac{\mathrm{i}\ell}{2}\,\overline{J}{}^A_-\gamma_i\psi_{A+}\right)\right]+E^i\,J_i+\frac{1}{2}\,\overline{J}{}^A_+\,\psi_{A+}-\frac{1}{2}\,\overline{J}{}^A_-\,\psi_{A-}\,,\\
\mathrm{d}J &= \frac{1}{2\ell}\,\epsilon_{AB}\left(\overline{J}{}^A_+\,\psi_{B+}+\overline{J}{}^A_-\,\psi_{B-}\right)\,,\\
\nabla J_{A+} &= \frac{1}{2\ell}\,\gamma^{ij}\psi_{A-}\,J_{ij}+\mathrm{i}\,\gamma^i\,\psi_{A+}\,J_i-\frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i\,\gamma_i\,J_{A-}+2\,\epsilon_{AB}\,\psi_{B-}\,J+\frac{1}{\ell}\,\psi^i_{A-}\,J_{ij}\,E^j\\
&\quad-\frac{\mathrm{i}}{2}\,\psi^i_{A-}\,\overline{J}_{B-}\gamma_i\psi_{B+}\,,\\
\nabla J_{A-} &= \frac{1}{2\ell}\,\gamma^{ij}\psi_{A+}\,J_{ij}+2\,\epsilon_{AB}\,\psi_{B+}\,J-\frac{\mathrm{i}}{\ell}\,E^i\,\gamma_i\,J_{A+}-\frac{1}{\ell}\,\psi^i_{A+}\,J_{ij}\,E^j+\frac{\mathrm{i}}{2}\,\psi^i_{A+}\,\overline{J}_{B-}\gamma_i\psi_{B+}\,.
\end{aligned} \tag{2.5.13}$$

Note that the above conservation laws reduce to those in (2.1.57) in the pure gravity case, namely in the absence of the fermionic superpartners and of the U(1) gauge field. This is better seen from the pure gravity laws (2.1.58), when the dilation gauge field is $B=0$ and the conformal current is $J_{(K)i}=0$. Then, the dilation current $J_{(D)}$ is not independent and can be solved from the last (algebraic) equation in (2.1.58), leading to the identities $\ell\mathcal{S}_iJ_{(D)}=\mathcal{S}_i{}^kJ_{kj}E^j$ and $\ell\mathrm{d}J_{(D)}=\partial_\mu\left(E^{\mu i}J_{ij}E^j\right)$. The obtained set of equations matches (2.5.13) when all spinors are set to zero and $\mathcal{S}_{ij}$ is symmetric. In addition, it is explicit from (2.5.13) that the fermions are sources of the electromagnetic current J.

As a final comment we observe that, in supergravity, invariance of the boundary action under Weyl transformations is guaranteed by the third equation of (2.5.13) which, taking into account (2.5.10), amounts to the condition

$$E^i\wedge J_i=-\frac{1}{2}\,\overline{J}{}^A_+\wedge\psi_{A+}+\frac{1}{2}\,\overline{J}{}^A_-\wedge\psi_{A-}=-\frac{1}{2}\,\overline{J}{}^A_+\wedge\psi_{A+}\,. \tag{2.5.14}$$

Let us now use the explicit form of the currents, given in (2.5.10), to write (2.5.14) in components. By exploiting (2.5.11), we find the trace of the bosonic part of the holographic stress tensor, namely

$$(2\tilde{\tau}+\tau)^{l}{}_{l} = -\mathrm{i}\,\ell\,\epsilon^{ijk}\bar{\psi}^{A}_{+j}\gamma_i\zeta_{A-\,k}\,. \tag{2.5.15}$$

By exploiting the properties of the gamma matrices, the reader can verify that the above relation is consistent with (3.34) of [26].

Notice that neither the holographic stress tensor $J_{\mu\nu}$ nor its bosonic part $\tau_{\mu\nu}+2\tilde{\tau}_{\mu\nu}$ have vanishing trace as in pure gravity. This does not mean that we have the trace anomaly because the value of the trace $J^i\wedge E_i$, given in (2.5.14), is fixed by the structure of the superalgebra. This is consistent with the result in $\mathcal{N}=1$ supergravity [26].
Similarly, $J_{\mu\nu}$ and $\tau_{\mu\nu}+2\tilde{\tau}_{\mu\nu}$ are not symmetric: the second conservation law in (2.5.13) with $J_{ij}=0$ and $J_-=0$ gives the antisymmetric part as $E_{[i}\wedge J_{j]}=\frac{\mathrm{i}}{4}\overline{J}_+\gamma_{ij}\wedge\psi_+$. A reason is that, with our gauge fixing choice, $J_{\mu\nu}$ is not, as in pure gravity, the traceless Belinfante-Rosenfeld stress tensor. However, we know that, in principle, it is possible to use an ambiguity in definitions of Noether currents to construct a so-called 'improved' stress tensor which would be symmetric and traceless.

The Ward identities

We now prove that the Ward identities are indeed satisfied by using the explicit form of the currents and showing that $\delta W=0$. We remind the reader that, although all expressions are evaluated on-shell in the bulk supergravity, they represent off-shell identities in CFT computed on the curved background. We start by integrating (2.5.9) by parts,

$$\begin{aligned}\delta W = \int\limits_{\partial\mathcal{M}}\Bigg[&\frac{\ell^2}{4}j^{ab}\mathcal{D}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}-\frac{\ell^2}{4}\left(\frac{2}{\ell^2}p^aV^b+\frac{1}{\ell}\bar{\epsilon}^A\Gamma^{ab}\Psi_A\right)\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}+2\mathrm{i}\ell\bar{\epsilon}^A\Gamma_5\mathcal{D}\hat{\boldsymbol{\rho}}_A\\ &-2\mathrm{i}\ell\left(\frac{1}{4}j^{ab}\overline{\Psi}^A\Gamma_{ab}+\frac{\mathrm{i}}{2\ell}p^a\overline{\Psi}^A\Gamma_a+\frac{1}{2\ell}\lambda\epsilon^{AB}\overline{\Psi}_B-\frac{1}{2\ell}\hat{A}\epsilon^{AB}\bar{\epsilon}_B-\frac{\mathrm{i}}{2\ell}\bar{\epsilon}^A\Gamma_aV^a\right)\Gamma_5\hat{\boldsymbol{\rho}}_A\\ &-\frac{1}{2}\lambda\,\mathrm{d}^*\hat{\boldsymbol{F}}+\bar{\epsilon}^A\Psi^B\epsilon_{AB}{}^*\hat{\boldsymbol{F}}\Bigg]\Bigg|^{\text{on-shell}}_{z=\mathrm{d}z=0}.\end{aligned} \tag{2.5.16}$$

We now make use of the Bianchi identities (2.2.12), to obtain

$$\begin{aligned}\delta W = \int\limits_{\partial\mathcal{M}}\Bigg[&\frac{\ell}{4}j^{ab}\overline{\Psi}^A\Gamma^{cd}\hat{\boldsymbol{\rho}}_A\epsilon_{abcd}-\frac{\ell^2}{4}\left(\frac{2}{\ell^2}p^aV^b+\frac{1}{\ell}\bar{\epsilon}^A\Gamma^{ab}\Psi_A\right)\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}\\ &+2\mathrm{i}\ell\left(\frac{1}{2\ell}\hat{A}\epsilon^{AB}\bar{\epsilon}_A\Gamma_5\hat{\boldsymbol{\rho}}_B-\frac{\mathrm{i}}{2\ell}\bar{\epsilon}^A\Gamma_5\Gamma_a\hat{\boldsymbol{\rho}}_AV^a+\frac{1}{4}\hat{\boldsymbol{R}}^{ab}\bar{\epsilon}^A\Gamma_5\Gamma_{ab}\Psi_A-\frac{1}{2\ell}\epsilon^{AB}\hat{\boldsymbol{F}}\bar{\epsilon}_A\Gamma_5\Psi_B\right)\\ &-2\mathrm{i}\ell\left(\frac{1}{4}j^{ab}\overline{\Psi}^A\Gamma_{ab}+\frac{\mathrm{i}}{2\ell}p^a\overline{\Psi}^A\Gamma_a+\frac{1}{2\ell}\lambda\epsilon^{AB}\overline{\Psi}_B-\frac{1}{2\ell}\hat{A}\epsilon^{AB}\bar{\epsilon}_B-\frac{\mathrm{i}}{2\ell}\bar{\epsilon}^A\Gamma_aV^a\right)\Gamma_5\hat{\boldsymbol{\rho}}_A\\ &-\frac{1}{2}\lambda\,\mathrm{d}^*\hat{\boldsymbol{F}}+\bar{\epsilon}^A\Psi^B\epsilon_{AB}{}^*\hat{\boldsymbol{F}}\Bigg]\Bigg|^{\text{on-shell}}_{z=\mathrm{d}z=0}.\end{aligned} \tag{2.5.17}$$

We are now able to write the Ward identities, in the four dimensional notation, which have to hold on-shell. They originate from requiring the vanishing of the coefficient of the independent symmetry parameters in δW. Let us denote the independent asymptotic parameters by $\Lambda(x)=\left\{\theta^{ij},\xi^i,\sigma,\eta^A_\pm,\lambda\right\}$, computed in Subection 2.4 as the radial expansion of the bulk parameters $\hat{\Lambda}(x,z)=\{j^{ab},p^a,\epsilon^A_\pm,\hat{\lambda}\}$. Since in the quantum effective action all divergences cancel out and the subleading terms vanish on the boundary, we can identify the bulk gauge transformations with the boundary ones,

$$\delta W \equiv \delta_\Lambda W = \delta_{\hat{\Lambda}} W\big|^{\text{on-shell}}_{z=\mathrm{d}z=0}\,. \tag{2.5.18}$$

This method makes use of the fact that the quantum effective action has already been renormalised and enables to prove the invariance of the action (and therefore the validity of the Ward identities) by looking directly at the bulk parameters $\hat{\Lambda}$.

Concerning the Lorentz transformations, we can easily verify that the coefficient of the four dimensional parameters j^{ab} vanishes identically due to the identity (A.1.6) for four dimensional gamma matrices whose properties are given in Appendix A.1,

$$\frac{\ell}{4}\,j^{ab}\overline{\Psi}^A\Gamma^{cd}\hat{\boldsymbol{\rho}}_A\,\epsilon_{abcd}-\frac{\mathrm{i}\ell}{2}\,j^{ab}\overline{\Psi}^A\Gamma_{ab}\Gamma_5\hat{\boldsymbol{\rho}}_A=0\,. \tag{2.5.19}$$

Focusing on traslations, namely the terms containing p^a, one finds, up to terms which vanish in the $z\to 0$ limit,

$$-\frac{1}{2}\,p^aV^b\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}+p^a\overline{\Psi}^A\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_A\,. \tag{2.5.20}$$

The above expression vanishes at the boundary by effect of the Einstein equations in the bulk (see the second equation of (2.2.16)),

$$-\frac{1}{2}\,p^aV^b\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}+p^a\overline{\Psi}^A\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_A=\frac{1}{2}\,p^a\epsilon_{abcd}V^b\left(\hat{F}^{cd}\hat{\boldsymbol{F}}-\frac{1}{6}\hat{F}_{ef}\hat{F}^{ef}V^cV^d\right)\,, \tag{2.5.21}$$

since the two terms on the right hand side are zero at $z=0$.

The terms involving the supersymmetry parameter ϵ_A are given by

$$\begin{aligned}
&\mathrm{i}\hat{A}\epsilon^{AB}\overline{\epsilon}_A\Gamma_5\hat{\boldsymbol{\rho}}_B+\overline{\epsilon}^A\Gamma_5\Gamma_a\hat{\boldsymbol{\rho}}_AV^a+\frac{\mathrm{i}\ell}{2}\hat{\boldsymbol{R}}^{ab}\overline{\epsilon}^A\Gamma_5\Gamma_{ab}\hat{\boldsymbol{\rho}}_A-\mathrm{i}\epsilon^{AB}\hat{\boldsymbol{F}}\overline{\epsilon}_A\Gamma_5\Psi_B\\
&-\frac{\ell}{4}\overline{\epsilon}^A\Gamma^{ab}\Psi_A\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd}+\mathrm{i}\hat{A}\epsilon^{AB}\overline{\epsilon}_B\Gamma_5\hat{\boldsymbol{\rho}}_A-\overline{\epsilon}^A\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_AV^a+\overline{\epsilon}^A\Psi^B\epsilon_{AB}\,{}^*\hat{\boldsymbol{F}}\\
&=\overline{\epsilon}^A(-2\Gamma_aV^a\Gamma_5\hat{\boldsymbol{\rho}}_A+\epsilon_{AB}\Psi^B\,{}^*\hat{\boldsymbol{F}}-\mathrm{i}\epsilon_{AB}\hat{\boldsymbol{F}}\Gamma_5\Psi^B)\,.
\end{aligned} \tag{2.5.22}$$

They vanish as a consequence of the equations of motion of the gravitini (2.2.16).

Finally, we evaluate the terms depending on the Abelian transformations parameter $\hat{\lambda}$ and find

$$\hat{\lambda}\left(-\frac{1}{2}\,\mathrm{d}^*\hat{\boldsymbol{F}}-\mathrm{i}\epsilon^{AB}\overline{\Psi}_B\Gamma_5\hat{\boldsymbol{\rho}}_A\right)\,, \tag{2.5.23}$$

which vanishes by virtue of the gauge field equation of motion in (2.2.16).

This proves that, on-shell, $\delta W=0$, namely that the equations (2.5.13), which were derived from $\delta W=0$ in the three dimensional notation, are indeed satisfied. This can be seen as a consequence of the absence of any anomaly (in particular the conformal one) in

$d = 3$. Note that the term in (2.5.21), which is proportional to p^3 and, as shown above, vanishes once the $a = 3$ component of the Einstein equations in the bulk is implemented (see the second equation of (2.2.16)), coincides, once integrated over the boundary, with the variation of the generating functional under a dilation, being $p^3 = -\ell\sigma$ at $z = 0$. Its vanishing provides the trace Ward identity (2.5.14).
Eventually, notice that, in the above derivation, we have neglected the curvature-contraction terms occurring in the general expression of the symmetry variations of the fields (2.2.14), which one can check to give vanishing contributions at the boundary.

2.6 Discussion

In this Chapter, we have developed in detail the holographic framework for an $\mathcal{N} = 2$ pure AdS$_4$ supergravity in the first order formalism, including all the contributions in the fermionic fields. This analysis, which generalises the one of [26, 27], includes an extended discussion of the gauge fixing conditions on the bulk fields which yield the asymptotic symmetries at the boundary. The corresponding currents of the boundary theory are constructed and shown to satisfy the associated Ward identities, once the field equations of the bulk theory are imposed.

Consistency of the holographic setup, in particular the finiteness of the quantum generating functional of the boundary theory, is shown to require the vanishing of the super-AdS curvatures computed at the boundary, which was proven in [41] to be a necessary condition for a consistent definition of the bulk supergravity. In particular, the vanishing of $\hat{\mathbf{R}}^{ij}|_{\partial\mathcal{M}}$ determines the general expression of the super-Schouten tensor $\mathcal{S}^i$ of the boundary theory, which extends the more familiar bosonic expression of standard gravity by the inclusion of gravitini bilinears, see (2.4.3). The same applies to the superpartner of $\mathcal{S}^i$, namely the conformino. Working in the first order formalism, we are able to keep the full superconformal structure of the theory manifest in principle, even if only a part of it is realised as a symmetry of the theory on $\partial\mathcal{M}$, as the rest appears as a non linear realisation on $\partial\mathcal{M}$. Furthermore, an important role in our analysis is played by the supertorsion constraint $\hat{\mathbf{R}}^a = 0$, where $\hat{\mathbf{R}}^a$ was defined in (2.2.9), which determines the bulk spin connection. In particular, the radial component, $\hat{\mathbf{R}}^3 = 0$, of this condition poses general constraints on the sources of the boundary CFT. In the FG parametrisation of the bulk background, that condition implies a non vanishing antisymmetric component of the super-Schouten tensor, proportional to the gravitini bilinear $\overline{\psi}_{A+[\mu}\psi_{A-\nu]}$, see (2.3.23). This shows that, in general, the superconformal structure and the conformino field $\psi_{A-\mu}$ pose an obstruction to the symmetrisation of $\mathcal{S}_{\mu\nu}$. For a special choice of background, namely $\psi_{A-\mu} \propto \psi_{A+\mu}$, $\overline{\psi}_{A+[\mu}\psi_{A-\nu]} = 0$ and the super-Schouten tensor becomes symmetric, i.e. $\mathcal{S}^i \wedge E_i = 0$. This latter property restricts $\mathcal{S}^i$ to be proportional to E^i. The manifest SCFT symmetry is then broken to the symmetry of the chosen background which, in our case, is a maximally symmetric spacetime: AdS$_3$ ($\mathcal{S}^i \neq 0$, $\psi_{\pm\mu} \neq 0$), dS$_3$ ($\mathcal{S}^i \neq 0$, $\psi_{\pm\mu} = 0$) or Mink$_3$ ($\psi_{-\mu} = \mathcal{S}^i = 0$), and provides the vacuum of the boundary theory.[17] The three (super)algebras associated with the symmetries of these backgrounds are defined by suitable

[17] The AdS$_3$ and dS$_3$ cases are distinguished by the sign of the proportionality factor between $\mathcal{S}^i$ and E^i.

projections on the OSp(2|4) asymptotic symmetry group.

As far as the gauge fixing conditions are concerned, we refrain from imposing $\gamma^\mu \psi_{\pm\mu} = 0$ in SCFT, having in mind generalisations of standard holography where this condition is relaxed in the boundary theory. This has a bearing on the radial gauge fixing condition on the gauge field. This generalisation is needed in particular to apply the holographic analysis to the AVZ model as boundary field theory, where the only propagating degrees of freedom are associated with a spin 1/2 field χ, which is identified with the contraction $\gamma^\mu \psi_\mu$ itself. This theory is naturally defined on an AdS$_3$ background. In [55] it was shown that the spinor χ is actually the Nakanishi-Lautrup field associated with the covariant gauge fixing of the odd local symmetries in a three dimensional Chern-Simons theory with gauge supergroup $\mathrm{OSp}(2|2) \times \mathrm{SO}(2,1)$[18]. This opens a window on the definition of the dual field theory of which the AVZ model provides an effective description. We shall pursue this objective in a future investigation.

Other future directions of research would be an extension of the present analysis to $\mathcal{N} > 2$ bulk supergravity, along the lines of [58], or the $D > 4$ bulk dimensions where, for odd D, quantum anomalies would arise in a boundary SCFT. Furthermore, a generalisation of the present results to the case where the FG choice of parametrisation is relaxed, which would allow the full superconformal symmetry of the boundary theory to be linearly realised, will also be object of our investigation.

Eventually, it still remains as an open problem the question of rendering the AdS supergravity action finite in the presence of matter multiplets by adding topological bulk terms.

[18] We will further elaborate on this point in the next Chapter of the thesis.

Chapter 3

Twisting D(2,1; α) Superspace

The discovery of dualities between theories apparently unrelated has been one of the major themes of the last decades in theoretical high energy physics research. Starting from the string dualities, which connect the five distinct versions of superstring theory (type I, type IIA, type IIB, heterotic SO(32) and heterotic $E_8 \times E_8$), passing by the AdS/CFT correspondence, which allows to describe a model defined on the boundary of an anti-de Sitter spacetime through the features of the bulk theory, a great effort has been devoted to advance in this direction.

As far as we are concerned, our attention was captured by an interesting relation unveiled in [79], which connects two three dimensional Chern-Simons theories. The first side of the duality is occupied by a Rozansky-Witten theory [80][1] with flat hyper-Kähler manifold and coupled to a Chern-Simons field based on a gauge group $\mathcal{G}$. Note that the latter, as shown by Kapustin and Saulina, can also be recast as a *topologically twisted* three dimensional $\mathcal{N}=4$ superconformal Chern-Simons theory based on $\mathcal{G}$ and coupled to a set of hypermultiplets[2] with flat hyper-Kähler target space. A *topological twist* consists of an identification between two different compact groups under which a supersymmetric theory is invariant, one of them being a Lorentz subgroup. Thus, the twist identifies

[1] A Rozansky-Witten theory is a $\mathcal{N}=4$ supersymmetric topological sigma-model with a hyper-Kähler manifold X as target space. The bosonic scalar fields ϕ^i are local coordinate on the latter, where a metric g_{ij} is defined, while the fermions are a Grassmanian scalar η^I and a Grassmanian gauge vector Ψ^I_μ. The action is integrated on an oriented manifold M endowed with a metric $h_{\mu\nu}$ and reads

$$S = \int_M (L_1 + L_2)\sqrt{h}\mathrm{d}^3x$$

with

$$\begin{gathered}
L_1 = \frac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \epsilon_{IJ}\Psi^I_\mu\nabla^\mu\eta^J\,, \\
L_2 = \frac{1}{2}\frac{1}{\sqrt{h}}\epsilon^{\mu\nu\rho}\left(\epsilon_{IJ}\Psi^I_\mu\nabla_\nu\Psi^J_\rho + \frac{1}{3}\Omega_{IJKL}\Psi^I_\mu\Psi^J_\nu\Psi^K_\rho\eta^L\right)\,, \\
i,j = 1,\ldots,4n\,, \quad \mu,\nu,\ldots = 1,2,3\,, \quad I,J,\ldots = 1,\ldots,2n\,,
\end{gathered}$$

where $I, J, \ldots$ label a $2n$ dimensional representation of $Sp(n)$ and Ω_{IJKL} is a completely symmetric matrix related to the Riemann curvature tensor of X, which vanishes when we consider a flat hyper-Kähler manifold as target space.

[2] This new class of theories was discovered by Gaiotto and Witten in [81] for the first time.

a spacetime symmetry with an internal one, which entails the selection of a (typically topological) specific subsector of the model. In the case at hand, the *topological twist* is performed between the rotational group $\mathrm{SU}(2)_L$ and the R-symmetry one $\mathrm{SU}(2)_R$. The restatement of the Rozansky-Witten model as a *topologically twisted* Gaiotto-Witten theory is particularly meaningful for the analysis developed in this part of the thesis, since the first step to reproduce Kapustin and Saulina's results in our framework will be to construct the hypermultiplet Lagrangian.
On the other hand of the duality, a 3D Chern-Simons theory based on a supergroup $\mathcal{SG}$ appears, henceforth called super-Chern-Simons theory. The maximal bosonic subgroup of $\mathcal{SG}$ is $\mathcal{G}$ and the fermionic part of the symmetry is gauge fixed. Throughout this Chapter, the reader will recognise our counterpart of this side of the duality as the model we obtain after the first of the two twist.
In this equivalence, the BRST transformations exploited to gauge fix the odd symmetries in $\mathcal{SG}$ are shown to derive from the topologically twisted supersymmetry transformations in the three dimensional $\mathcal{N}=4$ superconformal theory. Furthermore, the ghosts and anti-ghosts are interpreted as scalars in the hypermultiplets, whereas the hyperini, once twisted, take the role of Nakanishi-Lautrup fields and odd part of connection in the super-Chern-Simons theory.

Along the lines of this work, an Achucarro-Townsend AdS_3 supergravity [82][3], described as a Chern-Simons theory on an AdS_3 supergroup $\mathcal{SG}' = \mathrm{OSp}(2|2)\times\mathrm{SL}(2,\mathbb{R})$, was considered in [55]. The authors performed a BRST procedure to gauge fix the fermionic symmetries of $\mathcal{SG}'$ and related it to the ansatz (1.3.3), carried out in a covariant setting. We shall denote the gauge connection associated with these odd gauge symmetries by $\Psi^{\alpha}_{I\mu}\,\mathrm{d}x^{\mu}$, where $I=1,2$ is an SO(2) index and $\alpha=1,2$ is an $\mathrm{SL}(2,\mathbb{R})$ index. The gauge fixing constraint is implemented by fermionic Nakanishi-Lautrup fields η^{α}_{I} and introduces a dependence of the theory on the three dimensional worldvolume metric, where by worldvolume we mean the base space which the Chern-Simons action is integrated on. By the same token, we shall refer to the $\mathcal{SG}$-algebra-valued Chern-Simons connections as target space fields.
Substantial differences with respect to the models considered in antecent literature are the presence of a cosmological constant in the three dimensional theories and the Lorentzian signature of the manifold where the Chern-Simons form is integrated on, while the latter is a Riemannian one in [79, 80]. As a consequence, while Kapustin and Saulina's supergroup comprises compact SU(2) factors in the bosonic subalgebra, in [55] $\mathrm{SL}(2,\mathbb{R})$ factors appear in the maximal bosonic subgroup of $\mathcal{SG}'$, due to the fact that AdS_3 isometry group is $\mathrm{SO}(2,2)\sim\mathrm{SL}(2,\mathbb{R})'_1\times\mathrm{SL}(2,\mathbb{R})'_2$, which is non compact. This implies that the two theoretical constructions differ in the chosen topological twist, which involves different real forms of the Lorentz and R-symmetry groups.
Another important issue related to the cosmological constant in the construction of [55], following the prescription in [54], is that the worldvolume spin connection is identified with an $\mathrm{SL}(2,\mathbb{R})$-gauge connection in $\mathcal{SG}$. Different choices for this identification amount to the

[3]In this letter, Achucarro and Townsend proved that $\mathcal{N}=p+q$ supergravity theories based on AdS_3 supergroups $\mathrm{OSp}(p|2)\times\mathrm{OSp}(q|2)$ can be written as integral on the Chern-Simons 3-form associated to the supergroup.

presence or not of a non trivial spacetime torsion on the worldvolume, which naturally has an AdS_3 geometry.

The BRST procedure implies the introduction of scalar ghost and anti-ghost fields, to be denoted, respectively, by ϕ_I^α and $\bar{\phi}_I^\alpha$ with the same index structure as the odd-symmetry parameters of $\mathcal{SG}$, but with opposite spin-statistics. In this case, the gauge field $\Psi_{I\mu}^\alpha$ is a Grassmanian vector, so that the corresponding BRST parameters are ordinary commuting scalars carrying labels αI. This was observed in [80], where it was also shown that these fields naturally parametrise a hyper-Kähler manifold. To get this identification, one has to interpret ϕ_I^α and $\bar{\phi}_I^\alpha$ as independent complex fields. Then, one can associate the ghost quantum number with an SU(2) fundamental representation, labeled by a new index $A = 1, 2$, so that the ghost/anti-ghost fields can be grouped in a doublet $\phi_I^{\alpha A}$, where $\phi_I^{\alpha 1} = \mathrm{Re}(\phi_I^\alpha)$ and $\phi_I^{\alpha 2} = \mathrm{Im}(\phi_I^\alpha)$.[4]
A peculiarity of three dimensional Chern-Simons theories is the presence, besides the BRST symmetries generated by $\mathcal{S}, \bar{\mathcal{S}}$, of additional "vector-BRST" global symmetries [55, 83–86], whose generators are denoted here by $\mathcal{S}_i$, $\bar{\mathcal{S}}_i$, $i = 0, 1, 2$. Similarly to the scalar ghosts, the BRST generators can be grouped in SU(2)-doublets $\mathcal{S}^A$, $\mathcal{S}_i^A$.
Notice that Ψ_{Ii}^α and η_I^α transform with respect to the worldvolume Lorentz group $\mathrm{SL}(2,\mathbb{R})_L$ as triplets and singlets, respectively. Therefore, following [55], if we identify $\mathrm{SL}(2,\mathbb{R})_L$ with the diagonal of the two $\mathrm{SL}(2,\mathbb{R})$ factors in the worldvolume AdS_3 isometry group, denoted just above as $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$, we can arrange these two fields into a single set of Grassmann fields $\Lambda_I^{\alpha\alpha'\dot{\alpha}'}$,

$$\Lambda_I^{\alpha\alpha'\dot{\alpha}'} = \mathrm{i}\,\gamma^{i\,\alpha'\dot{\alpha}'}\,\Psi_{i\,I}^\alpha + a\,\epsilon^{\alpha'\dot{\alpha}'}\,\eta_I^\alpha\,, \tag{3.0.1}$$

where $\gamma^{i\,\alpha'\dot{\alpha}'}$ and $\epsilon^{\alpha'\dot{\alpha}'}$ are $\mathrm{SL}(2,\mathbb{R})_L$-invariant tensors on the worldvolume, intertwining between the two fundamental representations of $\mathrm{SL}(2,\mathbb{R})'_1$ and $\mathrm{SL}(2,\mathbb{R})'_2$, respectively labeled by $\alpha' = 1, 2$ and $\dot{\alpha}' = 1, 2$. The above choice of the worldvolume Lorentz symmetry inside the bosonic symmetry group corresponds, in our setting, to the topological twist performed in [79, 80, 87].
In light of the above twist, the Chern-Simons BRST operators $\mathcal{S}^A$ and $\mathcal{S}_i^A$ can be viewed as components of a single operator with index structure $\mathcal{Q}_{\alpha'\dot{\alpha}'A}$, behaving as supercharges. The latter can then be treated as a global supersymmetry of the worldvolume theory of hypermultiplets. The parameters of this supersymmetry have index structure $\epsilon^{\alpha'\dot{\alpha}'A}$ and transform in the $(\mathbf{2},\,\mathbf{2},\,\mathbf{2})$ of the symmetry group $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2 \times \mathrm{SU}(2)$, where the latter factor has the role of the manifest R-symmetry of the theory and it is reinterpreted, once a twist is performed, as the group acting on the ghost number of the fields. The number of supercharges is eight, corresponding to an $\mathcal{N} = 4$ supersymmetry on $D = 3$.
These considerations and the AdS_3 geometry of our background suggest a superspace description based on an AdS_3 supergroup whose maximal bosonic subgroup is a suitable real form of $\mathrm{SL}(2,\mathbb{C})^3$ and the odd generators transform in the product of the bi-spinor representation of each $\mathrm{SL}(2,\mathbb{C})$ factor. This naturally hints towards the exceptional supergroup $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$, whose maximal bosonic subgroup is indeed $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2 \times \mathrm{SU}(2)$ and whose odd generators $\mathcal{Q}_{\alpha'\dot{\alpha}'A}$ transform in the product $(\mathbf{2},\,\mathbf{2},\,\mathbf{2})$ of the fundamental representations of the three factors [88].

[4]With respect to [55], we use here a different SU(2) basis to be labeled by the index A.

This construction bears important differences with respect to the one in [79], due to the fact that we work on a worldvolume with Lorentzian signature and an AdS geometry, with isometry group $\mathrm{SO}(2,2) \sim \mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$.
Our model has $\mathcal{N}=4$ supersymmetry: on a Minkowski worldvolume, this would normally be associated with an SO(4) R-symmetry group commuting with the spacetime symmetry, as stated in [81]. In our case, instead, one of the SU(2) factors of the SO(4) R-symmetry group corresponds to one of the two $\mathrm{SL}(2,\mathbb{R})$ factors in the spacetime isometry group.
As explained above, the worldvolume Lorentz group $\mathrm{SL}(2,\mathbb{R})_L$ is the diagonal subgroup of the $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$ isometry group. As a consequence of this, in our model the spinor index is a composite one, $\alpha'\dot{\alpha}'$, yielding a redundant 4-component description of the spinorial degrees of freedom and reducing the manifest part of R-symmetry to SU(2). We shall refer to these Grassmann-valued fields as *spinorial* fields, independently of their actual $\mathrm{SL}(2,\mathbb{R})_L$ representation. With respect to this supersymmetry, the spinors $\Lambda_I^{\alpha\alpha'\dot{\alpha}'}$ and the scalars $\phi_I^{\alpha A}$ belong to a set of hypermultiplets.

In this Chapter, based on [89], we make a preliminary step towards the explicit construction of the gauge fixed theory defined in [55], choosing a worldvolume superspace based on the supergroup $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$. More specifically, we construct a $D=3$, $\mathcal{N}=4$ model describing a set of hypermultiplets $(\Lambda_I^{\alpha\alpha'\dot{\alpha}'}, \phi_I^{\alpha A})$ on a rigid AdS_3 superspace with symmetry group $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$. Supersymmetric models featuring rigid supersymmetry on a curved background were previously investigated in [90–93].[5]
The hypermultiplets $(\Lambda_I^{\alpha\alpha'\dot{\alpha}'}, \phi_I^{\alpha A})$ transform under the flavour symmetry $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2)$ through the indices α, I, which we introduced here in view of a future generalisation where a gauging of the flavour symmetries will be performed. The model, when rewritten in terms of the twisted quantities $(\Psi^\alpha_{iI}, \eta^\alpha_I)$, will bear resemblance with the construction of [80], restricted to a flat hyper-Kähler geometry, where only odd symmetry generators show up in the Chern-Simons theory.
In [55,79] the presence of a non gauge fixed bosonic subgroup $\mathcal{G}$ within the gauge supergroup $\mathcal{SG}$ induces, in the model with gauge fixed odd symmetry, mass terms for the fermion fields. The latter are a necessary ingredient if we ultimately wish to derive, within our theoretical setting, a model featuring unconventional supersymmetry and describing a massive Dirac field. In the framework we are considering here, the gauge group $\mathcal{G}$ is absent and the fermion masses are related to the non trivial gauging of the R-symmetry group SU(2) within the supergroup $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$. This, in turn, depends on the parameter $\boldsymbol{\alpha}$, since the SU(2) generators enter the anticommutator of two supersymmetry ones with a factor $\boldsymbol{\alpha}+1$. As we shall see, the case $\boldsymbol{\alpha}+1=0$ is a singular limit, where the structure of the superalgebra changes. In the general case, where $\boldsymbol{\alpha}+1\neq 0$, the group SU(2) is non trivially gauged, the gauge coupling coinciding with the same parameter $\boldsymbol{\alpha}+1$. The corresponding gauge fields A^x are part of the worldvolume supergravity sector which is frozen in the rigid limit we are

[5] Here, it is worth emphasising that the supergroup $\mathrm{D}^2(2,1;\alpha)$, its singular values included, describes, in the present construction, the super-isometry group of the worldvolume theory and should not be mistaken with the Achucarro-Townsend AdS_3 supergroup. The latter is, in general, the product of two supergroups, each containing one of the two $\mathrm{SL}(2,\mathbb{R})$ factors of the AdS_3 isometry group. Let us recall that, in our case, the choice of $\mathrm{D}^2(2,1;\alpha)$ as the worldvolume supergroup was forced by the transformation property of the supersymmetry generators under the $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2 \times \mathrm{SU}(2)$ symmetry of the worldvolume theory.

considering. They are, in other words, solutions, together with the supervielbein and the spin connection, to the Maurer-Cartan equations of the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ superalgebra.[6] As a consequence, their field strengths are proportional, through the gauge coupling constant, to a non exact cocycle in the fermionic directions of superspace (see (3.1.16) below). By supersymmetry, the non trivial gauging of SU(2) induces, even in the absence of dynamical gauge fields, spin 1/2 fermion shift matrices $\mathbb{N}^{\alpha}_{IA}$ and a mass term for the fermion fields $\Lambda_I^{\alpha\alpha'\dot{\alpha}'}$, which are all proportional, through the coupling constant, to $\boldsymbol{\alpha}+1$. This gauging is also responsible for a scalar potential, which is in fact a mass term for the scalar fields. In other words the SU(2) group plays, in our construction, a role to some extent analogous to the one played by the gauge group $\mathcal{G} \subset \mathcal{SG}$ in [55] in determining the masses of the dynamical fields.

We find a supersymmetric spacetime Lagrangian, whose superspace extension features a *quasi-invariance* under supersymmetry, meaning an invariance up to a total derivative term, which, in our case only affects the fermionic directions. We shall elaborate on this point in Section 3.2. Moreover, related to this issue, the interpretation of the supersymmetries in terms of the BRST symmetry generators $\mathcal{S}^A$ and their vector counterpart $\mathcal{S}^A_i$ is not straightforward for generic values of $\boldsymbol{\alpha}$, since $\mathcal{S}^A$ do not anticommute on fields with non trivial ghost charge[7], thus failing to define a cohomology. We retrieve a direct BRST interpretation of the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ supersymmetries, and thus an apparent connection to the construction of [55] and [79,80], only for the special singular value $\boldsymbol{\alpha}=-1$, for which SU(2) is effectively ungauged and becomes an external automorphism of the superalgebra. In fact, for such value of the parameter, the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ algebra reduces to

$$\mathrm{D}^2(2,1;\,\boldsymbol{\alpha}=-1) \simeq \mathfrak{sl}(2|2) \oplus \mathfrak{su}(2)$$

and the $\mathfrak{su}(2)$ factor becomes an outer automorphism of the $\mathfrak{sl}(2|2)$ algebra.

For the construction of the theory, we adopt the geometric approach to supersymmetry and supergravity (see Section 1.1 and [10,65]), which allows to obtain the superspace Lagrangian and the supersymmetry transformations of the fields. The equations of motion derived from the Lagrangian impose the standard Klein-Gordon equation for the scalar fields, with mass term proportional to the AdS radius and a massive Dirac-like equation for the spinor fields $\Lambda_I^{\alpha\alpha'\dot{\alpha}'}$.

By performing a first twist, analogous to the one in [55,79], the spacetime Lagrangian obtained from the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ supergroup can precisely be rewritten in terms of the quantities Ψ^{α}_{Ii} and η^{α}_I, besides the scalars $\phi^{\alpha A}_I$. As a result, we find that it describes a Chern-Simons term in the connection 1-form Ψ^{α}_I together with a gauge fixing term defined by the Nakanishi-Lautrup field η^{α}_I plus a a kinetic term for the scalar fields $\phi^{\alpha A}_I$, analogously to the results in [79,80]. Inspection of the Dirac equation, in its twisted form, shows that the only massive degrees of freedom in the fermionic sector are encoded in the field η^{α}_I.

We eventually perform a second twist to make contact with the model of [54]. In this case, we identify one of the two $\mathrm{SL}(2,\mathbb{R})$ factors in the isometry group of the AdS_3 worldvolume with a part of the aforementioned flavour symmetry group. This peculiar choice mixes target

[6]Hereafter, with an abuse of notation, we use the same symbols to denote the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ supergroup and the associated superalgebra.

[7]For instance we have $\mathcal{S}^2\bar{\phi} \propto (\boldsymbol{\alpha}+1)\,\phi$.

space and worldvolume indices and allows to decompose the hyperini in terms of new fields $\hat{\chi}^{\alpha}_{iI}$ and χ^{α}_{I}. The spacetime Lagrangian obtained in this way contains the Chern-Simons term for a spin 3/2 field, $\mathring{\chi}_{Ii}$, and describes the coupling of this field to two propagating spin 1/2 particles χ_{1I}, χ_{2I}.
The theory is still consistent if we implement one of the constraints needed for the unconventional supersymmetry, that is if we set the spin 3/2 component $\mathring{\chi}_{Ii}$ to zero: the resulting theory then describes a spin 1/2 fermion χ_{2I} satisfying a Dirac equation, whose mass term is proportional again to $(\boldsymbol{\alpha}+1)$ and which sources the spinor χ_{1I}, that can in general be written as a linear combination of χ_{2I} and a massless spin 1/2 fermion. This leads to a generalisation of the ansatz (1.3.3), in which the spin 1/2 field on the right hand side is χ_{1I}, while χ_{2I} is proportional to η^{α}_{I}.

This Chapter is organised as follows. In Section 3.1 we consider the algebraic relations defining the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ superalgebra, its description as an AdS_3 superspace and the matter content of our theory, together with the supersymmetry transformations laws of the fields. In Section 3.2 we derive the supersymmetric Lagrangian, we compute the corresponding field equations and we prove its supersymmetry invariance both in spacetime and superspace. Furthermore, we comment on the hyper-Kähler structure underlying the Lagrangian, in view of a possible generalisation of the theory to a curved scalar manifold.
In Section 3.3 we perform two twists of the spinor fields in the hypermultiplets. The first one is useful to make contact with the results obtained in [80] and [79], whereas the second one allows to implement the AVZ ansatz (1.3.3).
We conclude with some final remarks and possible future developments for this research line.

3.1 Setup for the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ model

As mentioned in the introductory part of this Chapter, our aim is to construct a theory defined on a supersymmetric background, whose symmetry is captured by the superalgebra $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$. We start by defining the structure of this algebra and the superspace based on it.

The $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ superalgebra

$\mathrm{D}(2,1;\,\boldsymbol{\alpha})$ is an exceptional superalgebra whose bosonic subalgebra is $[\mathfrak{sl}(2)]^3$. It is included in the list of the superalgebras defining possible super-AdS backgrounds in three spacetime dimensions, as discussed in [88].
In particular, we are interested in the supergroup referred to as $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ in [88]. As explained in the above cited paper, in this case the bosonic subgroup can be chosen in the following real form:

$$\mathrm{D}^2(2,1;\,\boldsymbol{\alpha}) \supset \underbrace{\mathrm{SL}(2,\mathbb{R})'}_{a=1} \times \underbrace{\mathrm{SL}(2,\mathbb{R})'}_{a=2} \times \underbrace{\mathrm{SU}(2)}_{a=3}\,, \tag{3.1.1}$$

which allows the interpretation of the first two factors as the isometry group of AdS_3, whereas the third represents the manifest part of R-symmetry. For the sake of notational simplicity we shall denote each of the three factors on the right hand side of (3.1.1), generically by $\mathrm{SL}(2)_{(a)}$.
Finally, the generators of the odd part of the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ superalgebra, as previously anticipated, transform in the $(\mathbf{2},\mathbf{2},\mathbf{2})$ representation. A detailed description of the superalgebra can be found for example in [94].

Let us denote by $\mathcal{T}_{(a)}^{i_a}$ $(i_a=1,2,3)$ the generators of the $\mathfrak{sl}(2)_{(a)}\subset \mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ and by $\mathcal{Q}_{\alpha_1\alpha_2\alpha_3}$ $(\alpha_a=1,2)$ the odd ones.
The superalgebra is then expressed by the following (anti)commutation relations:

$$
\begin{aligned}
&\left[\mathcal{T}_{(a)}^{i_a}\,,\,\mathcal{Q}_{\dots\alpha_a\dots}\right]=\left(\mathbb{t}_{(a)}^{i_a}\right)_{\alpha_a}{}^{\beta_a}\,\mathcal{Q}_{\dots\beta_a\dots}\,;\\
&\{\mathcal{Q}_{\alpha_1\alpha_2\alpha_3}\,,\,\mathcal{Q}_{\beta_1\beta_2\beta_3}\}=\sum_{\substack{a,b,c=1\\ a\neq b\neq c}}^{3}\mathrm{i}\,s_a\left(\mathbb{t}_{(a)}^{i_a}\right)_{\alpha_a\beta_a}\epsilon_{\alpha_b\beta_b}\epsilon_{\alpha_c\beta_c}\mathcal{T}_{(a)\,i_a}=\\
&=\mathrm{i}\left[s_1\left(\mathbb{t}_{(1)}^{i_1}\right)_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha_3\beta_3}\mathcal{T}_{(1)i_1}+s_2\left(\mathbb{t}_{(2)}^{i_2}\right)_{\alpha_2\beta_2}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_3\beta_3}\mathcal{T}_{(2)i_2}+s_3\left(\mathbb{t}_{(3)}^{i_3}\right)_{\alpha_3\beta_3}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\mathcal{T}_{(3)i_3}\right],
\end{aligned}
\tag{3.1.2}
$$

where

$$
\left(\mathbb{t}_{(a)}^{i_a}\right)_{\alpha_a\beta_a}=\frac{\mathrm{i}}{2}\left(\gamma_{(a)}^{i_a}\right)_{\alpha_a\beta_a}
\tag{3.1.3}
$$

are representation matrices that, taking into account the different real forms of the three bosonic factors, are defined as

$$
\gamma_{(a=1)}^{i_1}=\gamma_{(a=2)}^{i_2}=(\sigma_2\,,\mathrm{i}\sigma_1\,,\mathrm{i}\sigma_3)\ ,\quad \eta=\mathrm{diag}(+,-,-)\,,
\tag{3.1.4}
$$

$$
\gamma_{(a=3)}^{i_3}=(\sigma_1\,,\sigma_2\,,\sigma_3)\ ,\quad \eta=\mathrm{diag}(+,+,+)\,.
\tag{3.1.5}
$$

We refer to the Appendix B for useful relations involving the gamma matrices.
In particular, the closure of the algebra imposes the following relation between the three real non vanishing parameters s_a:

$$
s_1+s_2+s_3=0\,.
$$

The cases where one of the s_a vanishes are singular limits.
Up to the normalisation of the odd generators, the superalgebra is then characterised by a single parameter $\boldsymbol{\alpha}=s_2/s_1$ [8] and the Jacobi identities can be expressed as the condition

$$
s_3/s_1=-(\boldsymbol{\alpha}+1)\,.
$$

[8] When expressed in terms of $\boldsymbol{\alpha}$, superalgebras determined by the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}^{-1}$, $-1-\boldsymbol{\alpha}$ and $\frac{-\boldsymbol{\alpha}}{1+\boldsymbol{\alpha}}$ are isomorphic, see [94].

When expressed in terms of $\boldsymbol{\alpha}$, the singular limits correspond to $\boldsymbol{\alpha} = -1, 0, \infty$.
As explained in Section 1.1, an equivalent representation of the above superalgebra is given in terms of the superconnection

$$\Omega = \omega_{(a)i_a} \mathcal{T}^{i_a}_{(a)} + \psi^{\alpha_1\alpha_2\alpha_3} \mathcal{Q}_{\alpha_1\alpha_2\alpha_3} \,, \tag{3.1.6}$$

where the bosonic and fermionic Maurer-Cartan 1-forms $\omega_{(a)i_a}$, $\psi^{\alpha_1\alpha_2\alpha_3}$ define the superalgebra in its dual form through the Maurer-Cartan equations

$$\hat{R}^{i_a}_{(a)} \equiv \mathrm{d}\omega^{i_a}_{(a)} - \frac{1}{2}\epsilon^{i_a j_a k_a}\omega_{(a)j_a}\omega_{(a)k_a} + \frac{\mathrm{i}}{2}\, s_a \psi^{\alpha_1\alpha_2\alpha_3} \left(\mathbb{t}^{i_a}_{(a)}\right)^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\beta_2\beta_3} \psi^{\beta_1\beta_2\beta_3} \;=\; 0 \,, \tag{3.1.7}$$

$$\hat{\nabla}\psi^{\alpha_1\alpha_2\alpha_3} \equiv \mathrm{d}\psi^{\alpha_1\alpha_2\alpha_3} + \sum_{a=1}^{3} \omega_{i_a(a)} \wedge \left(\mathbb{t}^{i_a}_{(a)}\right)^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\beta_2\beta_3} \psi^{\beta_1\beta_2\beta_3} \;=\; 0 \,. \tag{3.1.8}$$

The above equations (3.1.7), (3.1.8) can be obtained as Euler-Lagrange equations from the following Lagrangian:

$$\mathcal{L}_\kappa = \frac{\kappa}{2}\left[\sum_{a=1}^{3} \frac{1}{s_a}\left(\omega_{(a)i_a}\mathrm{d}\omega^{i_a}_{(a)} - \frac{1}{3}\epsilon^{i_a j_a k_a}\omega_{(a)i_a}\omega_{(a)j_a}\omega_{(a)k_a}\right) - \mathrm{i}\psi_{\alpha_1\alpha_2\alpha_3}\hat{\nabla}\psi^{\alpha_1\alpha_2\alpha_3}\right] , \tag{3.1.9}$$

where κ is the level of the Chern-Simons action.

$\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superspace description

In the following, we are going to give a superspace interpretation of the Maurer-Cartan equations (3.1.7), (3.1.8): to this end, we interpret the diagonal subgroup $\mathrm{SL}(2,\mathbb{R})'_D \subset \mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$ as the Lorentz group of our background super-geometry. Correspondingly, we choose

$$\omega^i \equiv \frac{1}{2}(\omega^i_{(1)} + \omega^i_{(2)}) \,,$$

as the spin connection and we interpret the 1-form

$$e^i \equiv \frac{L}{2}(\omega^i_{(1)} - \omega^i_{(2)}) \,,$$

as the dreibein, where we have introduced the scale parameter $L \in \mathbb{R}_+$ with dimension of a length. The dreibein e^i, together with $\Psi = \sqrt{L}\,\psi$, which is regarded as a gravitino 1-form field, define the supervielbein of our rigid, but curved superspace background.

Note that, with the above choice, only $\mathrm{SL}(2,\mathbb{R})'_D \subset \mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$ is a manifest spacetime symmetry and the indices i_1, i_2 are both identified with the Lorentz spacetime index i. However, we refrain to identify the spinor indices associated to $\mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$ for the moment. It will be explicitly done through the first twist.
We choose to name the indices referring to the first two factors as follows:

$$i_1 = i_2 = i = 0,1,2\,; \quad \alpha_1 = \alpha' = 1,2\,; \quad \alpha_2 = \dot{\alpha}' = 1,2\,; \quad \alpha'\dot{\alpha}' \equiv (\alpha) = 1,\cdots,4\,.$$

Furthermore, it is useful to introduce the 4×4 matrices

$$\left(\mathbb{T}^i_{(1)}\right)_{(\alpha)(\beta)} \equiv \left(\mathbb{t}^i_{(1)}\right)_{\alpha'\beta'} \otimes \epsilon_{\dot\alpha'\dot\beta'}\,, \qquad \left(\mathbb{T}^i_{(2)}\right)_{(\alpha)(\beta)} \equiv \epsilon_{\alpha'\beta'} \otimes \left(\mathbb{t}^i_{(2)}\right)_{\dot\alpha'\dot\beta'}\,, \tag{3.1.10}$$

whose properties are given in Appendix B, and their linear combinations

$$\begin{aligned} \mathbb{J}^i &= \mathbb{T}^i_{(1)} + \mathbb{T}^i_{(2)}\,, \\ \mathbb{K}^i &= \mathbb{T}^i_{(1)} - \mathbb{T}^i_{(2)}\,, \\ \mathbb{M}^i_{\pm} &= -\frac{\mathrm{i}}{2}\left(\mathbb{T}^i_{(1)} \pm \boldsymbol{\alpha}\mathbb{T}^i_{(2)}\right)\,, \end{aligned} \tag{3.1.11}$$

playing the role of the gamma matrices in the ordinary superspace.

On the other hand, only the part of the $\mathcal{N}=4$ R-symmetry associated with the group SU(2) is manifest and interpreted as internal symmetry group. To distinguish it from the other two bosonic factors in the superalgebra, we relabel the corresponding indices

$$i_3, j_3, \ldots = x, y, \ldots = 1,2,3\,; \quad \alpha_3, \beta_3, \ldots = A, B, \ldots = 1,2\,,$$

we redefine the connection $\omega^{i_3}_{(3)}$ as

$$\omega^{i_3}_{(3)} \Rightarrow A^x$$

and the representation matrix as

$$\left(\mathbb{t}_{(3)x}\right)^A{}_B \equiv \frac{\mathrm{i}}{2}\left(\sigma_1\,,\sigma_2\,,\sigma_3\right)^A{}_B\,. \tag{3.1.12}$$

In light of the above definitions, the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ Maurer-Cartan equations can be written as

$$R^i \equiv \mathrm{d}\omega^i - \frac{1}{2}\epsilon^{ijk}\omega_j\wedge\omega_k = \frac{1}{2L^2}\epsilon^{ijk}e_j\wedge e_k + \frac{1}{2L}\left(\mathbb{M}^i_+\right)_{(\alpha)(\beta)}\Psi^{(\alpha)A}\wedge\Psi^{(\beta)B}\epsilon_{AB}\,, \tag{3.1.13}$$

$$\nabla\Psi^{(\alpha)A} \equiv \mathcal{D}\Psi^{(\alpha)A} + \left(\mathbb{t}^x_{(3)}\right)^A{}_B A^x\wedge\Psi^{(\alpha)B} = -\frac{1}{L}\left(\mathbb{K}^i\right)^{(\alpha)}{}_{(\beta)}e_i\wedge\Psi^{(\beta)A}\,, \tag{3.1.14}$$

$$\mathcal{D}e^i \equiv \mathrm{d}e^i - \epsilon^{ijk}\omega_j\wedge e_k = \frac{1}{2}\left(\mathbb{M}^i_-\right)_{(\alpha)(\beta)}\Psi^{(\alpha)A}\wedge\Psi^{(\beta)B}\epsilon_{AB}\,, \tag{3.1.15}$$

$$\mathcal{F}^x \equiv \mathrm{d}A^x - \frac{1}{2}\epsilon^{xyz}A_y\wedge A_z = \frac{\mathrm{i}}{2L}(\boldsymbol{\alpha}+1)\left(\mathbb{t}^x_{(3)}\right)_{AB}\Psi^{(\alpha)A}\wedge\Psi^{(\beta)B}\delta_{(\alpha)(\beta)}\,, \tag{3.1.16}$$

where the Lorentz covariant derivative $\mathcal{D}\Psi^{(\alpha)A}$ is defined as

$$\mathcal{D}\Psi^{(\alpha)A} = \mathrm{d}\Psi^{(\alpha)A} + \omega^i\mathbb{J}_i^{(\alpha)}{}_{(\beta)}\Psi^{(\beta)A}\,.$$

In (3.1.13)-(3.1.16), the left hand sides can be read as definitions of the superspace curvature, gravitino covariant derivative, supertorsion and gauge field strength respectively, while the right hand sides define their parametrisations as 2-forms in the superspace spanned by the supervielbein e^i, $\Psi^{(\alpha)A}$. In other words, the above relations define our background superspace. In particular, we see that they define a curved AdS_3 background, where L is the AdS radius.

Note that the quantity $g\equiv(\boldsymbol{\alpha}+1)$ plays the role of the coupling constant associated with the SU(2) gauge group. This is better understood by redefining $A^x = g\,A'^x$, in which case the field strength of the rescaled gauge fields read

$$\mathcal{F}'^x \equiv \mathrm{d}A'^x - \frac{g}{2}\epsilon^{xyz}A'_y \wedge A'_z = \frac{\mathrm{i}}{2L}\left(\mathbb{t}^x_{(3)}\right)_{AB}\Psi^{(\alpha)A}\wedge\Psi^{(\beta)B}\,\delta_{(\alpha)(\beta)}\,. \tag{3.1.17}$$

As it is typical of supersymmetric theories, the gauging of an internal symmetry induces, by supersymmetry, additional terms in the supersymmetry transformation laws (*fermion shifts*) of the fermion fields and fermion mass terms, all proportional to the gauge coupling constant g, and a scalar potential proportional to g^2, independently of the fact that in our model the gauge fields do not propagate, being part of the background. We are going to compute these terms in the following Section. Let us notice that, in the limit $g = (\boldsymbol{\alpha}+1) \to 0$, the fermion shifts and the fermion mass terms vanish, together with the scalar potential.
Here, let us also mention that, differently from other AdS superalgebras, the $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ one allows for various different contractions, due to its dependence on the two unrelated parameters L and $\boldsymbol{\alpha}$, and it admits, in particular, the presence of central charges in the contracted structure both on Minkowski and on AdS. A first limit is $L\to\infty$, where we recover a super-Poincaré structure in the presence of non abelian gauge fields; two more contractions involve the limit $g = (\boldsymbol{\alpha}+1)\to 0$, performed *before* or *after* having introduced the redefinition A'^x. In the former case, we end up with an AdS_3 background superspace coupled to non abelian pure gauge SU(2) connections trivially embedded in superspace, while with the latter the gauge fields become abelian and are associated with central charges of the contracted superalgebra on an AdS_3 background. Eventually, one more contraction can be considered: we first redefine $A^x = \frac{1}{L}\tilde{A}^x$ and consequently take the limit $L\to\infty$, obtaining a central extension of a super-Poincaré structure where the central charges are associated with abelian gauge fields.

By exploiting standard coset geometry techniques applied to the supercoset

$$\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})/[\mathrm{SL}(2,\mathbb{R})_L\times\mathrm{SU}(2)]\,,$$

one can express the supervielbein e^i, $\Psi^{(\alpha)A}$, as well as the connection 1-forms, in terms of the differentials $\mathrm{d}x^\mu$, $\mathrm{d}\theta^{(\alpha)A}$. To this end, we can define a supercoset representative

$$\mathbb{L}(x,\,\theta) \equiv \mathbb{L}_F(\theta)\cdot\mathbb{L}_B(x)\,, \tag{3.1.18}$$

where $\mathbb{L}_F(\theta) = \exp(\theta^{(\alpha)A}\,\mathcal{Q}_{(\alpha)A})$ and $\mathbb{L}_B(x) = \exp(t^i(x)\,\mathcal{K}_i)$, t^i being non linearly related to the spacetime coordinates and $\mathcal{K}_i \equiv \mathcal{T}_{(1)i} - \mathcal{T}_{(2)i}$. For the sake of simplicity, let us collectively denote the generators of the bosonic subalgebra $\mathfrak{sl}(2)_1\oplus\mathfrak{sl}(2)_2\oplus\mathfrak{su}(2)$ by $\mathrm{B}_{\mathcal{A}}$. The left invariant 1-form reads

$$\Omega(x,\,\theta,\,\mathrm{d}\theta,\,\mathrm{d}x) = \mathbb{L}^{-1}\,\mathrm{d}\mathbb{L} = \mathbb{L}_B^{-1}(x)\,(\mathbb{L}_F^{-1}\,\mathrm{d}\mathbb{L}_F(\theta,\,\mathrm{d}\theta))\,\mathbb{L}_B(x) + \mathbb{L}_B^{-1}\,\mathrm{d}\mathbb{L}_B(x,\,\mathrm{d}x)\,. \tag{3.1.19}$$

Defining the Lie algebra-valued 1-forms $\Omega_F(\theta,\mathrm{d}\theta)$ and $\Omega_B(x,\mathrm{d}x)$ as follows

$$\begin{aligned}
&\Omega_F(\theta,\mathrm{d}\theta) \equiv \mathbb{L}_F^{-1}\,\mathrm{d}\mathbb{L}_F(\theta,\,\mathrm{d}\theta) = \Omega_F(\theta,\mathrm{d}\theta)^{(\alpha)A}\,\mathcal{Q}_{(\alpha)A} + \Omega_F(\theta,\mathrm{d}\theta)^{\mathcal{A}}\,\mathrm{B}_{\mathcal{A}}\,,\\
&\Omega_B(x,\mathrm{d}x) = \mathbb{L}_B^{-1}\,\mathrm{d}\mathbb{L}_B(x,\,\mathrm{d}x) = \Omega_B(x,\mathrm{d}x)^{\mathcal{A}}\,\mathrm{B}_{\mathcal{A}}\,,
\end{aligned} \tag{3.1.20}$$

we can rewrite the 1-forms in (3.1.19) as

$$\Omega(x,\,\theta,\,\mathrm{d}\theta,\,\mathrm{d}x) = \Omega_F(\theta,\mathrm{d}\theta)^{(\alpha)A}\,(\mathbb{L}_B(x))_{(\alpha)A}{}^{(\beta)B}\,\mathcal{Q}_{(\beta)B} + \Omega_F(\theta,\mathrm{d}\theta)^{\mathcal{A}}\,(\mathbb{L}_B(x))_{\mathcal{A}}{}^{\mathcal{B}}\,\mathrm{B}_{\mathcal{B}}+ \\ + \Omega_B(x,\mathrm{d}x)^{\mathcal{A}}\,\mathrm{B}_{\mathcal{A}} = \frac{e^i}{L}\,\mathcal{K}_i + \frac{1}{\sqrt{L}}\,\Psi^{(\alpha)A}\,\mathcal{Q}_{(\alpha)A} + \omega^i\,\mathcal{J}_i + A^x\,\mathcal{T}_{(3)\,x}\,, \quad (3.1.21)$$

where we have denoted by $(\mathbb{L}_B(x))_{(\alpha)A}{}^{(\beta)B}$ and $(\mathbb{L}_B(x))_{\mathcal{A}}{}^{\mathcal{B}}$ the matrices representing the adjoint action of $\mathbb{L}_B(x)$ on the supersymmetry generators $\mathcal{Q}_{(\alpha)A}$ and $\mathrm{B}_{\mathcal{A}}$, which can be deduced from the structure constants of the superalgebra. Moreover, we defined $\mathcal{J}^i = \mathcal{T}^i_{(1)} + \mathcal{T}^i_{(2)}$.
From the above equation we can read off the supervielbein and connection. In particular we find for e^i and $\Psi^{(\alpha)A}$ the following general formulae:

$$\begin{aligned} e^i &= L\,\left(\Omega_F(\theta,\mathrm{d}\theta)^{\mathcal{A}}\,(\mathbb{L}_B(x))_{\mathcal{A}}{}^{i} + \Omega_B(x,\mathrm{d}x)^i\right)\,, \\ \Psi^{(\beta)B} &= \sqrt{L}\,\Omega_F(\theta,\mathrm{d}\theta)^{(\alpha)A}\,(\mathbb{L}_B(x))_{(\alpha)A}{}^{(\beta)B}\,, \end{aligned} \quad (3.1.22)$$

where the i index in the first equation labels the components along the $\mathcal{K}_i$ generators. We notice that restriction to spacetime is effected by setting $\theta = 0$, $\mathrm{d}\theta = 0$, which in turn implies $\Psi^{(\beta)B} = 0$.[9]

In the following, we will study the dynamics of a set of hypermultiplets in this curved background. To be consistent with the rigid superspace interpretation, the supergravity Lagrangian (3.1.9) must decouple from the matter sector in the rigid limit. To this aim, we set the parameter κ to

$$\kappa = \frac{L}{\ell_P}\,,$$

where we denote by ℓ_P the Planck length. In the rigid limit $\ell_P \to 0$, it is possible to choose $L \gg \ell_P$, so that the supergravity dynamics is fully decoupled from the matter sector.

The matter content of the theory

The model describes the coupling between the rigid supersymmetric background defined above and a set of hypermultiplets, labeled by a couple of flavour indices $a \equiv \alpha I = 1, \cdots, 2n$, composed by a set of scalars $\phi_I^{\alpha A}$ and their spin 1/2 superpartners $\Lambda_I^{\alpha(\alpha)}$.

[9]Here, we refrain from using an explicit matrix representation of the supercoset representative, since it is known not to be needed in order to compute e^i, $\Psi^{(\alpha)A}$ in terms of the coordinates and their differentials. These quantities, indeed, only depend on the structure constants of the superalgebra, as it can be explicitly shown by using the general formula (see Theorem 5 of [95]):

$$e^{-X}\cdot\mathrm{d}\left(e^X\right) = \left(\frac{\mathbf{1} - e^{-\mathrm{Ad}_X}}{\mathrm{Ad}_X}\right)\mathrm{d}X\,,$$

where X is a superalgebra generator, linear function of the superalgebra parameters, $\mathrm{Ad}_X(Y) \equiv [X,Y]$ and $\frac{\mathbf{1}-e^{-\mathrm{Ad}_X}}{\mathrm{Ad}_X} \equiv \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\,(\mathrm{Ad}_X)^k$. In order to evaluate the components of Ω_F along the generators, as polynomials in θ, $\mathrm{d}\theta$, for instance, we just need to choose $X = \theta\cdot\mathcal{Q}$ in the above formula.
Their explicit expression is not needed for the scope of the present investigation.

In the geometric approach to supersymmetry and supergravity in superspace (see Section 1.1), the first step for identifying the model is to extend the notion of the matter fields to superfields in superspace and to define their covariant derivatives in superspace,

$$\nabla\phi_I^{\alpha A} \equiv \mathrm{d}\phi_I^{\alpha A} + A^x \left(\mathbb{t}^x_{(3)}\right)^A{}_B\,\phi_I^{\alpha B}\,, \tag{3.1.23}$$

$$\nabla\Lambda_I^{\alpha(\alpha)} \equiv \mathrm{d}\Lambda_I^{\alpha(\alpha)} + \omega_i \left(\mathbb{J}^i\right)^{(\alpha)}{}_{(\beta)}\,\Lambda_I^{\alpha(\beta)}\,. \tag{3.1.24}$$

The corresponding Bianchi identities, which stem from the d^2-closure, must then hold on-shell in superspace,

$$\nabla^2\phi_I^{\alpha A} = \left(\mathbb{t}^x_{(3)}\right)^A{}_B\,\mathcal{F}_{x(3)}\phi_I^{\alpha B}\,, \tag{3.1.25}$$

$$\nabla^2\Lambda_I^{\alpha(\alpha)} = \left(\mathbb{J}_i\right)^{(\alpha)}{}_{(\beta)}\,R^i\Lambda_I^{\alpha(\beta)}\,. \tag{3.1.26}$$

Note that the above relations are not identically satisfied in superspace, but amount to on-shell constraints, when the covariant derivatives above are parametrised as general 1-forms in superspace. More precisely, their generic parametrisation can be expressed as

$$\nabla\phi_I^{\alpha A} = \nabla_i\phi_I^{\alpha A}e^i + \Psi^{(\alpha)A}\Lambda^\alpha_{(\alpha)I}\,, \tag{3.1.27}$$

$$\nabla\Lambda_I^{\alpha(\alpha)} = \nabla_i\Lambda_I^{\alpha(\alpha)}e^i + \nabla_i\phi_I^{\alpha A}\left(\mathbb{m}^i\right)^{(\alpha)(\beta)}\,\Psi^B_{(\beta)}\epsilon_{AB} + \mathbb{N}^\alpha_{IA}\,\Psi^{(\alpha)A}\,. \tag{3.1.28}$$

The consistency constraints (3.1.25), (3.1.26), once imposed on the derivative of parametrisations (3.1.27), (3.1.28), imply the field equations for the fermion fields

$$\left(\mathbb{T}^i_{(1)} + \boldsymbol{\alpha}\mathbb{T}^i_{(2)}\right)_{(\sigma)(\alpha)}\nabla_i\Lambda_I^{\alpha(\alpha)} + \frac{1}{4L}\left(\boldsymbol{\alpha}-1\right)\left[\Lambda^\alpha_{(\sigma)I} + 4\left(\mathbb{T}^i_{(1)}\mathbb{T}_{(2)i}\right)_{(\sigma)(\alpha)}\Lambda_I^{\alpha(\alpha)}\right] = 0\,, \tag{3.1.29}$$

which can be written in an alternative form as

$$\left(\mathbb{T}^i_{(1)} + \boldsymbol{\alpha}\mathbb{T}^i_{(2)}\right)_{(\sigma)(\alpha)}\left\{\nabla_i\Lambda_I^{\alpha(\alpha)} + \frac{1}{3L(1+\boldsymbol{\alpha})}\left[(1+3\boldsymbol{\alpha})\mathbb{T}_{(1)i} - (3+\boldsymbol{\alpha})\mathbb{T}_{(2)i}\right]^{(\alpha)}{}_{(\beta)}\,\Lambda_I^{\alpha(\beta)}\right\} = 0\,, \tag{3.1.30}$$

and also determine the auxiliary matrices $\mathbb{m}^i$, $\mathbb{N}^\alpha_{IA}$:

$$\left(\mathbb{m}^i\right)^{(\alpha)}{}_{(\beta)} = -\frac{\mathrm{i}}{2}\left(\mathbb{T}^i_{(1)} - \boldsymbol{\alpha}\mathbb{T}^i_{(2)}\right)^{(\alpha)}{}_{(\beta)} = \left(\mathbb{M}^i_-\right)^{(\alpha)}{}_{(\beta)}\,,\quad \mathbb{N}^\alpha_{IA} = \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{4L}\,\epsilon_{AB}\,\phi_I^{\alpha B}\,. \tag{3.1.31}$$

Since in this approach the supersymmetry transformation laws are described geometrically as Lie derivatives along the fermionic directions of superspace, the above procedure allows to easily determine the supersymmetry transformations of the fields, which read

$$\begin{aligned}
\delta_\varepsilon\phi_I^{\alpha A} &= \varepsilon^{(\alpha)A}\Lambda^\alpha_{(\alpha)I}\,,\\
\delta_\varepsilon\Lambda_I^{\alpha(\alpha)} &= \Phi^{\alpha A}_{I;i}\left(\mathbb{M}^i_-\right)^{(\alpha)}{}_{(\beta)}\varepsilon^{(\beta)}_A + \mathbb{N}^\alpha_{IA}\varepsilon^{(\alpha)A}\,,\\
\delta_\varepsilon e^i &= \left(\mathbb{M}^i_-\right)_{(\alpha)(\beta)}\varepsilon^{(\alpha)A}\Psi^{(\beta)}_A\,,\\
\delta_\varepsilon\omega^i &= \frac{1}{L}(\mathbb{M}^i_+)_{(\alpha)(\beta)}\varepsilon^{(\alpha)A}\Psi^{(\beta)B}\epsilon_{AB}\,,
\end{aligned}$$

$$\begin{aligned}\delta_\varepsilon \Psi^{(\alpha)A} &= \nabla\varepsilon^{(\alpha)A} + \frac{1}{L}\,\mathbb{K}_i^{(\alpha)(\beta)}\,e^i\varepsilon^A_{(\beta)}\,,\\ \delta_\varepsilon A^x &= \frac{\mathrm{i}}{L}(1+\alpha)(\mathbb{t}^x_{(3)})_{AB}\varepsilon^{(\alpha)A}\Psi^B_{(\alpha)}\,.\end{aligned} \tag{3.1.32}$$

Let us notice that the condition of background invariance under supersymmetry requires ε to be a Killing spinor, namely that $\delta_\varepsilon\Psi^{(\alpha)A}=0$. The latter, in turn, implies that the supersymmetry parameter should satisfy the following equation:

$$\hat{\nabla}\varepsilon \equiv \nabla\varepsilon + \frac{1}{L}\,\mathbb{K}_i\,e^i\varepsilon = 0\,. \tag{3.1.33}$$

Moreover, it follows that all the background fields have vanishing supersymmetry transformations on spacetime,

$$\delta e^i_\mu = \delta A^x_\mu = \delta\omega^i_\mu = \delta\Psi^{(\alpha)A}_\mu = 0\,. \tag{3.1.34}$$

3.2 Superspace and spacetime Lagrangians

The geometric approach allows to determine the Lagrangian for our dynamical hypermultiplets as a bosonic 3-form in the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superspace. The procedure to be followed is straightforward from a logical point of view, but it can turn out to be quite difficult at a computational level. Firstly, one considers an ansatz for the action with all possible bosonic 3-form terms allowed by the rules of rheonomy and invariant under the symmetries of the theory (see Section 1.1). Then, superspace equations of motion for $\phi^{\alpha A}_I$ and $\Lambda^\alpha_{(\alpha)I}$ are studied, imposing that their inner components reproduce the spacetime equations of motion, whereas the outer ones are required to vanish identically. In particular, it means that when one considers a specific sector of the latter, the coefficients associated to the irreducible representations of the basis have to be zero separately. Once these conditions are put all together, the coefficients in the ansatz are fixed and the superspace Lagrangian reads

$$\begin{aligned}\mathcal{L} =& a_1\left(\nabla\phi^{\alpha A}_I - \Psi^{(\alpha)A}\Lambda^\alpha_{(\alpha)I}\right)\,\Phi^{I;i}_{\alpha A}\,e^j\,e^k\epsilon_{ijk} - \frac{1}{6}a_1\Phi^{\alpha A}_{I;\ell}\Phi^{I;\ell}_{\alpha A}e^i\,e^j\,e^k\epsilon_{ijk}\\ &- 16\frac{a_1}{\boldsymbol{\alpha}^2-1}\Lambda^{\alpha(\alpha)I}\left(\mathbb{M}^i_+\right)_{(\alpha)(\beta)}\Bigg[\frac{1}{2}\nabla\Lambda^{\beta(\beta)}_I e^j + \left(\mathbb{M}^j_-\right)^{(\beta)}_{\ (\gamma)}\left(\nabla\phi^{\beta A}_I - \frac{1}{2}\Psi^{(\delta)A}\Lambda^\beta_{(\delta)I}\right)\Psi^{(\gamma)B}\epsilon_{AB}\\ &- \mathbb{N}^\beta_{IA}(\phi)\Psi^{(\beta)A}e^j\Bigg]\epsilon_{\alpha\beta}\,e^k\epsilon_{ijk}\\ &+ \frac{\mathrm{i}a_1}{3(\boldsymbol{\alpha}+1)}\mathcal{M}_{(\alpha)(\beta)}\Lambda^{\alpha(\alpha)I}\Lambda^{\beta(\beta)}_I\epsilon_{\alpha\beta}e^i\ e^j\ e^k\epsilon_{ijk} - \frac{a_1}{3}\mathcal{V}(\phi)\,e^i\ e^j\,e^k\epsilon_{ijk}\\ &+ \frac{\mathrm{i}a_1}{(\boldsymbol{\alpha}-1)}\phi^{\beta(A}_I\nabla\phi^{\alpha I|B)}\epsilon_{\alpha\beta}\,\Psi^{(\alpha)}_A\Psi^{(\beta)}_B\left[\frac{(1-\boldsymbol{\alpha}+\boldsymbol{\alpha}^2)}{4}\delta_{(\alpha)(\beta)} + \boldsymbol{\alpha}\left(\mathbb{T}^k_{(1)}\mathbb{T}_{(2)k}\right)_{(\alpha)(\beta)}\right]\\ &- \frac{\mathrm{i}(1+\boldsymbol{\alpha})a_1}{8L}\phi^{IA\alpha}\phi^{\beta B}_I\left(\mathbb{J}^i\right)_{(\alpha)(\beta)}\Psi^{(\alpha)C}\Psi^{(\beta)D}e_i\epsilon_{CD}\epsilon_{\alpha\beta}\epsilon_{AB}\,,\end{aligned} \tag{3.2.1}$$

where the hyperini mass matrix and the scalar potential have the following expressions:

$$\mathcal{M}_{(\alpha)(\beta)} = \frac{1}{L}\left[\delta_{(\alpha)(\beta)} + 4\left(\mathbb{T}^i_{(1)}\cdot\mathbb{T}_{(2)i}\right)_{(\alpha)(\beta)}\right]\,, \tag{3.2.2}$$

$$\mathcal{V}(\phi) = -\frac{1}{2L^2}\phi_I^{\alpha A}\phi^{\beta B I}\epsilon_{\alpha\beta}\epsilon_{AB} + \text{constant}\,. \tag{3.2.3}$$

As we are going to show, this Lagrangian is invariant under supersymmetry modulo boundary terms.
We choose the overall normalisation to be $a_1 = \frac{1}{2}$. Thus, the spacetime projection of the superspace Lagrangian (3.2.1) takes the simple expression

$$\begin{aligned}\mathcal{L}_{\text{spacetime}} =& \frac{1}{2}\nabla_\mu\phi_I^{\alpha A}\,\nabla^\mu\phi^{\beta B I}\epsilon_{\alpha\beta}\epsilon_{AB} - \frac{8}{\boldsymbol{\alpha}^2-1}\Lambda^{\alpha(\alpha)I}\left(\mathbb{M}_+^\mu\right)_{(\alpha)(\beta)}\nabla_\mu\Lambda_I^{\beta(\beta)}\,\epsilon_{\alpha\beta} \\ &+ \frac{\mathrm{i}}{(\boldsymbol{\alpha}+1)}\mathcal{M}_{(\alpha)(\beta)}\Lambda^{\alpha(\alpha)I}\Lambda_I^{\beta(\beta)}\epsilon_{\alpha\beta} - \mathcal{V}(\phi)\,.\end{aligned} \tag{3.2.4}$$

Notice that the spacetime Lagrangian describes non mutually interacting scalar and fermion sectors, the interaction terms only appearing in the components of the superspace Lagrangian along the odd directions.
The expression of the scalar potential, which is in fact a mass term for the scalar fields, is fixed by the requirement of supersymmetry of the action, to be discussed in the course of this Section.

Furthermore, we are going to explicitly write down the Euler-Lagrange equations of the spacetime Lagrangian, which provides the field equations of the hypermultiplets, and discuss some of the peculiarities of the superspace Lagrangian (3.2.1), which are not apparent in its spacetime projection (3.2.4).

The spacetime field equations

The scalar field $\phi_I^{\alpha A}$ satisfies the following Klein-Gordon equation of motion:

$$\nabla_\mu\nabla^\mu\,\phi_I^{\alpha A} = +\frac{1}{L^2}\phi_I^{\alpha A}\,, \tag{3.2.5}$$

where the mass is given by the inverse of the AdS radius L. Let us observe that the squared mass of the scalar fields, $m_\phi^2 = -\frac{1}{L^2}$, saturates the Breitenlohner-Freedman (BF) bound [96, 97] in $D = 3$. Being the BF bound satisfied, the vacuum is perturbatively stable against scalar fluctuations.

The equation of motion of $\Lambda_I^{(\alpha)\alpha}$, which can be easily obtained from the Lagrangian (3.2.4), as well as from the Bianchi identities in superspace (see (3.1.29)), reads

$$\left(\mathbb{M}_+^i\right)_{(\alpha)(\beta)}\nabla_i\Lambda_I^{\alpha(\beta)} - \frac{\mathrm{i}}{8}\left(\boldsymbol{\alpha}-1\right)\mathcal{M}_{(\alpha)(\beta)}\Lambda_I^{\alpha(\beta)} = 0\,. \tag{3.2.6}$$

It is a massive Dirac equation with a constant mass proportional to the inverse of the AdS radius L. The mass matrix $\mathcal{M}$ can be diagonalized through conjugation with the orthogonal matrix

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\,, \tag{3.2.7}$$

showing that it has only one eigenvalue different from zero,

$$\mathcal{M}_{\mathcal{D}(\sigma)(\alpha)} = \left(\mathcal{P}^t \mathcal{M} \mathcal{P}\right)_{(\sigma)(\alpha)} = \frac{1}{L} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.2.8}$$

The interpretation of the above result in terms of mass eigenstates will be more transparent in the twisted descriptions of the model that we will give in Section 3.3.

Scalar potential and supersymmetry invariance

In this paragraph, we discuss the supersymmetry of the action of our model, starting from the properties of the Lagrangian both in superspace (3.2.1) and in spacetime (3.2.4). As we are going to see, we find that supersymmetry invariance of the superspace Lagrangian requires a non trivial contribution from the boundary. This means that the bulk Lagrangian is invariant modulo total derivative terms. The latter are relevant to the complete invariance of the action, being our model formulated on a spacetime with AdS geometry, which is not globally hyperbolic. Here, we will be dealing with the invariance of the model in the bulk only, leaving a detailed analysis of the invariance of the action, which includes the boundary contributions, along the lines of [41], to future investigation. For this reason, we expect all contributions $\mathcal{Y}$ in $\delta\mathcal{L}$ to sum up to a total derivative term $\mathrm{d}(\delta\mathcal{Z})$ in such a way that

$$\delta\mathcal{L} = \mathcal{Y} = \underbrace{\mathcal{Y} + \mathrm{d}(\delta\mathcal{Z})}_{0} - \mathrm{d}(\delta\mathcal{Z}) = -\mathrm{d}(\delta\mathcal{Z}) . \tag{3.2.9}$$

By using the transformation laws (3.1.32), restricted to spacetime, and the Killing spinor equation (3.1.33), it can be verified that the spacetime Lagrangian (3.2.4) features off-shell invariance under supersymmetry.
In particular, this invariance is crucial to determine the explicit expression of the scalar potential appearing in (3.2.1) and (3.2.4), as expected. Indeed, invariance of the spacetime Lagrangian (3.2.4) to order $1/L^2$ requires the scalar potential $\mathcal{V}(\phi)$ to satisfy the following condition:

$$\frac{\partial \mathcal{V}}{\partial \phi^{\alpha A}} = -\frac{1}{L^2}\, \epsilon_{\alpha\beta}\epsilon_{AB}\, \phi^{\beta B} , \tag{3.2.10}$$

which yields the expression given in (3.2.3).

The analysis of supersymmetry for the superspace Lagrangian can instead be performed, in a geometric setting, by computing its Lie derivative along odd diffeomorphisms,

$$\delta_\epsilon \mathcal{L} = \pounds_\epsilon \mathcal{L} = \iota_\epsilon(\mathrm{d}\mathcal{L}) + \mathrm{d}\iota_\epsilon(\mathcal{L}) , \tag{3.2.11}$$

and ignoring the total derivative part for the bulk analysis, as explained above. Eventually, we can analyse independently the invariance in different sectors, defined by the inverse powers of the AdS radius L and on different basis elements for 3-forms in superspace. Of particular interest is the sector $\frac{1}{L^2}\epsilon\Psi ee$, which yields the supersymmetric potential Ward

identity [98–100].
The explicit computation of this sector in (3.2.1) yields

$$\mathcal{V}(\phi)\left((\boldsymbol{\alpha}+1)\mathbb{K}^i-(\boldsymbol{\alpha}-1)\mathbb{J}^i\right)+\frac{1}{2L^2}\frac{\boldsymbol{\alpha}+1}{\boldsymbol{\alpha}-1}\phi^2\left((1-\boldsymbol{\alpha})\mathbb{K}^i+(\boldsymbol{\alpha}+1)\mathbb{J}^i\right)$$
$$-\frac{\boldsymbol{\alpha}+1}{L^2}\phi^2\epsilon^{ijk}\mathbb{K}_j\mathbb{J}_k=\mathcal{Y}|_{\frac{1}{L^2}\epsilon\Psi ee}\neq 0\,. \tag{3.2.12}$$

We notice that, while the components on the left hand side along $\mathbb{K}^i$ vanish for the choice of the potential in (3.2.3), the components along $\mathbb{J}^i$ fail to do so. These contributions can be disposed of by adding a suitable total derivative term to the superspace Lagrangian in (3.2.1) of the form

$$\mathrm{d}\mathcal{Z}=\mathrm{d}\left(\Lambda^{\alpha(\alpha)I}\left[r_1\left(\mathbb{J}^i\right)+r_2\epsilon^{ijk}\mathbb{T}_{(1)j}\mathbb{T}_{(2)k}\right]_{(\alpha)(\beta)}\phi_I^{\beta A}\Psi^{(\beta)B}\epsilon_{AB}\epsilon_{\alpha\beta}e_i\right)\,, \tag{3.2.13}$$

where the values of r_1, r_2 are restricted by the requirement that the $\frac{1}{L^2}\epsilon\Psi ee$ component of $\delta_\epsilon\mathcal{L}$ vanishes:

$$\mathcal{V}(\phi)\left((\boldsymbol{\alpha}+1)\mathbb{K}^i-(\boldsymbol{\alpha}-1)\mathbb{J}^i\right)+\frac{1}{2L^2}\frac{\boldsymbol{\alpha}+1}{\boldsymbol{\alpha}-1}\phi^2\left((1-\boldsymbol{\alpha})\mathbb{K}^i+(\boldsymbol{\alpha}+1)\mathbb{J}^i\right)$$
$$-\frac{\boldsymbol{\alpha}+1}{L^2}\phi^2\epsilon^{ijk}\mathbb{K}_j\mathbb{J}_k+\frac{2(\boldsymbol{\alpha}+1)}{L^2}\left(r_1+\frac{1}{2}r_2\right)\phi^2\mathbb{J}^i=0\,. \tag{3.2.14}$$

This leads to the following relation

$$r_1+\frac{r_2}{2}=-\frac{1}{2}\left(\frac{\boldsymbol{\alpha}^2+1}{\boldsymbol{\alpha}^2-1}\right)\,. \tag{3.2.15}$$

In light of the remark in (3.2.9), this signals that the Lagrangian in (3.2.1) is invariant in the bulk modulo a total derivative term $-\mathrm{d}(\delta\mathcal{Z})$.

Comments on the dependence of the Lagrangian on the hyper-Kähler geometry

Although we are restricting to a flat hyper-Kähler manifold, in view of a possible generalisation to a curved one, it would be useful to provide an intrinsic characterisation of the scalar dependence of the Lagragian (in particular of the scalar potential) in terms of quantities characterising the hyper-Kähler geometry. To this end we start recalling the main facts about hyper-Kähler geometry.

A hyper-Kähler manifold [101, 102] of real dimension $4n_H$ is a manifold on which three complex structures are defined J^x, $x=1,2,3$, $(J^x)^2=-\mathbf{1}$, closing an $\mathfrak{su}(2)$ algebra: $[J^x,\, J^y]=\epsilon^{xyz}\,J^z$. The metric h_{uw} is required to be *Hermitian* with respect to any of the three structures. In a local patch with coordinates q^u, $u=1,\ldots,4n_H$, this amounts to the conditions

$$h_{uw}\,J^{x\,w}{}_v+h_{vw}\,J^{x\,w}{}_u=0\,, \tag{3.2.16}$$

where $(J^x)^u{}_v$ represent the action of the complex structures on a coordinate basis of the tangent space and satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz}\, J^z\,. \tag{3.2.17}$$

The manifold is further required to be Kähler with respect to each of the three complex structures. This, in turn, is equivalent to the covariantly constant condition of the matrices $(J^x)^u{}_v$ with respect to the Levi-Civita connection $\tilde{\Gamma}^w_{uv}$ on the manifold,

$$\mathcal{D}_w\,(J^x)^u{}_v = 0\,. \tag{3.2.18}$$

We define the hyper-Kähler 2-forms as follows:

$$\Omega^x = \Omega^x_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\ ,\quad \Omega^x_{uv} = h_{uw}\,J^{x\,w}{}_v = h_{w[u}\,J^{x\,w}{}_{v]}\,. \tag{3.2.19}$$

The hyper-Kähler condition implies that these three 2-forms are closed: $\mathrm{d}\Omega^x = 0$.

In our case the coordinates q^u are identified with the scalar fields of our model $\phi_I^{\alpha A}$. The matrices $(J^x)^u{}_v$ are constant and define the (linear) action of the $\mathfrak{su}(2)$ generators on the index A of the scalars $\phi_I^{\alpha\,A}$,

$$(J^x)^u{}_v = (J^x)^{A\alpha I}{}_{B\beta J} = 2\,(\mathbb{t}^x)^A{}_B\,\delta^\alpha_\beta\,\delta^I_J\,. \tag{3.2.20}$$

The real dimension of the space is 8, corresponding to $n_H = 2$ hypermultiplets.

We can treat the space as a complex manifold with respect to the complex structure $J \equiv J^{x=2}$, which acts on the indices A, B, ... as the matrix $2\,(\mathbb{t}^2)^A{}_B = \mathrm{i}\,(\sigma^2)^A{}_B$

$$J \cdot \phi_I^{\alpha A} = \mathrm{i}\,(\sigma^2)^A{}_B\,\phi_I^{\alpha B}\,. \tag{3.2.21}$$

This choice of the complex structure yields the definition of four complex coordinates ϕ_I^α and their complex conjugates $\bar{\phi}_I^\alpha$,

$$\phi_I^\alpha \equiv \phi_I^{\alpha A=1} + \mathrm{i}\,\phi_I^{\alpha A=2}\,,\quad \bar{\phi}_I^\alpha \equiv \phi_I^{\alpha A=1} - \mathrm{i}\,\phi_I^{\alpha A=2}\,, \tag{3.2.22}$$

such that

$$J \cdot \mathrm{d}\phi_I^\alpha = -\mathrm{i}\,\mathrm{d}\phi_I^\alpha\ ,\quad J \cdot \mathrm{d}\bar{\phi}_I^\alpha = \mathrm{i}\,\mathrm{d}\bar{\phi}_I^\alpha\,. \tag{3.2.23}$$

When interpreting ϕ_I^α, $\bar{\phi}_I^\alpha$ as ghost and anti-ghost fields, the operator $\mathrm{i}\,J$ measures their *ghost charges*, which are $+1$ and -1, respectively. In our model the hyper-Kähler manifold is flat and the metric reads

$$\mathrm{d}s^2 = h_{uv}\,\mathrm{d}q^u\,\mathrm{d}q^v = \epsilon_{\alpha\beta}\,\epsilon_{AB}\,\mathrm{d}\phi_I^{\alpha A}\,\mathrm{d}\phi_I^{\beta B} = \mathrm{i}\epsilon_{\alpha\beta}\,\mathrm{d}\phi_I^\alpha\,\mathrm{d}\bar{\phi}_I^\beta\,. \tag{3.2.24}$$

The Kähler 2-form associated with J reads

$$K = h_{uv}\,J^v{}_w\,\mathrm{d}q^u \wedge \mathrm{d}q^w = -\mathrm{d}\phi^\alpha \wedge \mathrm{d}\bar{\phi}^\beta\,\epsilon_{\alpha\beta}\,, \tag{3.2.25}$$

and the corresponding Kähler potential has the following expression:

$$\mathcal{K}(\phi,\bar{\phi}) \equiv \phi_I^{\alpha A}\phi_I^{\beta B}\epsilon_{\alpha\beta}\epsilon_{AB} = \mathrm{i}\,\epsilon_{\alpha\beta}\,\phi_I^\alpha\,\bar{\phi}_I^\beta\,. \tag{3.2.26}$$

In terms of this potential the metric in the complex basis is given by the known relation for Kähler manifolds,

$$\mathrm{d}s^2 = \left(\frac{\partial^2\mathcal{K}}{\partial\phi^\alpha_I \partial\bar{\phi}^\beta_J}\right) \mathrm{d}\phi^\alpha_I \, \mathrm{d}\bar{\phi}^\beta_J \,. \tag{3.2.27}$$

As for the other complex structures J^x, whose action on the A, B indices can be described in terms of the matrices $2\,(\mathbb{t}^x)^A{}_B$, it is useful to describe the index $x = 1, 2, 3$ in terms of a symmetric couple (AB) and write $(J^x)^C{}_D = \mathrm{i}\,(\mathbb{t}^x)_{AB}\,(J^{(AB)})^C{}_D$, where $(J^{(AB)})^C{}_D \equiv \delta^{(A}_D \epsilon^{B)C}$. The three closed hyper-Kähler 2-forms $\Omega^{(AB)}$ have then the following expression [55]:

$$\Omega^{(AB)} = \mathrm{d}\phi^{\alpha\,A}_I \wedge \mathrm{d}\phi^{\beta\,B}_I \epsilon_{\alpha\beta} \,. \tag{3.2.28}$$

Being closed, locally these forms can be written as the exterior derivative of 1-forms $\mathcal{A}^{(AB)}$: $\Omega^{(AB)} = \mathrm{d}\mathcal{A}^{(AB)}$, where

$$\mathcal{A}^{(AB)} = \phi^{\alpha\,(A}_I \, \mathrm{d}\phi^{\beta\,|B)}_I \epsilon_{\alpha\beta} \,. \tag{3.2.29}$$

Let us now show that the dependence of the Lagrangian on the scalar fields can be described in terms of geometrical quantities which are intrinsic to the hyper-Kähler manifold and this suggests a natural generalisation of its expression to more general non flat hyper-Kähler geometries [80]. We note indeed that the scalar potential $\mathcal{V}(\phi, \bar{\phi})$ can be expressed in terms of $\mathcal{K}(\phi, \bar{\phi})$ as follows:

$$\mathcal{V}(\phi, \bar{\phi}) = -\frac{1}{2L^2}\,\mathcal{K}(\phi, \bar{\phi}) + \text{constant} \,. \tag{3.2.30}$$

Moreover the expression $\phi^{\alpha\,(A}_I \, \nabla\phi^{\beta\,|B)}_I \epsilon_{\alpha\beta}$ in a $\Psi\Psi$-component of the Lagrangian, as well as $\nabla\phi^{\alpha A}_I$ in the spacetime Lagrangian, are respectively interpreted in terms of the connection $\mathcal{A}^{(AB)}$ and the vielbein $\mathcal{U}^{\alpha A}_I = \mathcal{U}^{\alpha A}_{I\,u}\,\mathrm{d}q^u$ 1-forms, in which the exterior derivative d is replaced by the covariant one ∇ due to the gauging of the SU(2) isometry algebra by A^x_μ.
Let us eventually add that, when a curved hyper-Kähler manifold is considered, the supersymmetry transformation laws contain extra contributions depending on the affine connection on the σ-model and the Lagrangian includes an additional term of the form

$$\epsilon^{AB}\, R_{\alpha AI,\beta BJ;\gamma K,\sigma L}\, \Lambda^{(\alpha)\alpha I}\, \Lambda^{(\beta)\beta J}\, \Lambda^{(\gamma)\gamma K}\, \Lambda^{(\sigma)\sigma L} \epsilon_{(\alpha)(\beta)(\gamma)(\sigma)} \,, \tag{3.2.31}$$

where

$$\epsilon_{(\alpha)(\beta)(\gamma)(\delta)} = 4\,(\mathbb{T}^i_{(1)})_{[(\alpha)(\beta)}(\mathbb{T}_{(1)\,i})_{(\gamma)(\delta)]} = -4(\mathbb{T}^i_{(2)})_{[(\alpha)(\beta)}(\mathbb{T}_{(2)\,i})_{(\gamma)(\delta)]} \tag{3.2.32}$$

is the totally antisymmetric SO(2, 2)-invariant tensor and

$$R_{\gamma K,\sigma L} = \frac{1}{2}\, R_{uv;\gamma K,\sigma L} \mathrm{d}q^u \wedge \mathrm{d}q^v = \frac{1}{2}\, R_{\alpha AI,\beta BJ;\gamma K,\sigma L}\, \mathrm{d}\phi^{\alpha AI} \wedge \mathrm{d}\phi^{\beta BJ}$$

is the curvature 2-form with value in the $\mathfrak{usp}(2n) = \mathfrak{usp}(4)$ algebra.

As mentioned above, this observation is useful in view of a generalisation of the Lagrangian to a sigma-model on more general hyper-Kähler manifolds [80]. This task will be undertaken in a future investigation.

3.3 The Twists

In this Section we perform two different twists of the theory. As we shall see, the first one will relate this model to the one of [80], whereas the second one will allow to make contact with the unconventional supersymmetry, explored in Section 1.3 and [54–58].

First twist

In all of the analysis up to now, the manifest invariance of the action is only restricted to the Lorentz group $\mathrm{SL}(2,\mathbb{R})_L$ embedded as the diagonal subgroup

$$\mathrm{SL}(2,\mathbb{R})_L = \mathrm{SL}(2,\mathbb{R})'_D \subset \mathrm{SL}(2,\mathbb{R})'_1 \times \mathrm{SL}(2,\mathbb{R})'_2$$

and to the R-symmetry group SU(2) inside $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$. However, we have used so far a somewhat hybrid notation in the description of the fermionic fields and supersymmetry, by keeping the spinor indices $\alpha', \dot\alpha'$ of $\mathrm{SL}(2,\mathbb{R})'_1$ and $\mathrm{SL}(2,\mathbb{R})'_2$ distinct and thus working with a redundant 4-component description of spinor fields.
Here, we rewrite the spinor fields in irreducible $\mathrm{SL}(2,\mathbb{R})'_D$ components,

$$\Lambda_I^{(\alpha)\alpha} \;\to\; \Psi^\alpha_{Ii}\,,\;\; \eta^\alpha_I\,, \tag{3.3.1}$$

according to the branching

$$(\mathbf{2},\,\mathbf{2}) \;\to\; \mathbf{3}+\mathbf{1}\,. \tag{3.3.2}$$

We will call this decomposition a *twist*, in analogy with the known *topological twist*. In our framework, it amounts to making explicit the choice of the spin connection of $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superspace among the $\mathrm{SL}(2,\mathbb{R})$ connections of the superalgebra.
To express $\Lambda_I^{(\alpha)\,\alpha}$ in terms of its component fields, we introduce the following intertwining matrices:

$$\gamma^i_{(\alpha)} \equiv (\gamma^i)_{\alpha'\dot\alpha'}\,,\;\; \epsilon_{(\alpha)} \equiv \epsilon_{\alpha'\dot\alpha'}\,, \tag{3.3.3}$$

which are clearly invariant under the Lorentz group $\mathrm{SL}(2,\mathbb{R})_L$[10]. By using these quantities, we can decompose $\Lambda_I^{\alpha(\alpha)}$ into the irreducible components Ψ^α_{iI}, η^α_I as follows:

$$\Lambda_I^{\alpha(\alpha)} = \mathrm{i}\,\gamma^{i(\alpha)}\,\Psi^\alpha_{iI} + \epsilon^{(\alpha)}\,\eta^\alpha_I\,. \tag{3.3.4}$$

In the context of the analysis carried out in [55], the two components of $\Lambda_I^{\alpha(\alpha)}$, resulting from the twist, describe, respectively, the gauge field Ψ^α_{iI} associated to the odd gauge symmetries of a Chern-Simons model defined on the supergroup OSp(2|2) and the corresponding Nakanishi-Lautrup field η^α_I. In our case, similarly as in [80], the bosonic subgroup of the gauge supergroup being replaced by a global flavour symmetry, the odd gauge fields Ψ^α_{iI}

[10]To see this for $\gamma^i_{(\alpha)}$, one can verify that

$$(\mathbb{T}^i_{(1)} + \mathbb{T}^i_{(2)})^{(\alpha)}{}_{(\beta)}\,\gamma^{k\,(\beta)} = -\epsilon^{ik\ell}\,\gamma^{(\alpha)}_\ell\,.$$

are the only relics of the Chern-Simons gauge supergroup. Correspondingly, the analogous components of the supersymmetry generators $\mathcal{Q}^{(\alpha)A}$ define what in [79] were identified as the BRST symmetry generator $\mathcal{S}$, the "vector" BRST symmetry generator $\mathcal{S}_i$ and their secondary counterparts $\bar{\mathcal{S}}$, $\bar{\mathcal{S}}_i$:

$$\mathcal{Q}^{(\alpha)A} = \mathrm{i}\,\gamma^{i(\alpha)}\,\mathcal{S}_i^A + \epsilon^{(\alpha)}\,\mathcal{S}^A\,, \tag{3.3.5}$$

where $\mathcal{S}^A \equiv (\mathcal{S}, \bar{\mathcal{S}})$, $\mathcal{S}_i^A \equiv (\mathcal{S}_i, \bar{\mathcal{S}}_i)$.
Let us now compute the anticommutator of the two supersymmetry generators for the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ algebra in (3.1.2), in terms of the twisted operators $\mathcal{S}^A$, $\mathcal{S}_i^A$. We find

$$\left\{\mathcal{S}^A, \mathcal{S}^B\right\} = \frac{\mathrm{i}}{2}\,s_3\,\left(\mathbb{t}_{(3)x}\right)^{AB}\,\mathcal{T}^x_{(3)}\,, \tag{3.3.6}$$

$$\left\{\mathcal{S}_i^A, \mathcal{S}_j^B\right\} = \frac{\mathrm{i}}{4}\epsilon_{ijk}\epsilon^{AB}\left(s_1\mathcal{T}^k_{(1)} + s_2\mathcal{T}^k_{(2)}\right) + \frac{\mathrm{i}}{2}\,s_3\,\eta_{ij}\,\left(\mathbb{t}_{(3)x}\right)^{AB}\,\mathcal{T}^x_{(3)}\,, \tag{3.3.7}$$

$$\left\{\mathcal{S}_i^A, \mathcal{S}^B\right\} = \frac{\mathrm{i}}{4}\epsilon^{AB}\left(s_1\mathcal{T}_{(1)i} - s_2\mathcal{T}_{(2)i}\right)\,. \tag{3.3.8}$$

The above expressions show that, with the exception of the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ singular value $s_3 = 0$ (corresponding to $\boldsymbol{\alpha} = -1$), the scalar generators $\mathcal{S}^A$ do not behave as cohomology operators.

By proving the following relation:

$$(\mathbb{T}^i_{(1)}\mathbb{T}_{(2)\,i})^{(\alpha)}{}_{(\beta)}\,\gamma^{k\,(\beta)} = -\frac{1}{4}\,\gamma^{k\,(\alpha)}\,, \tag{3.3.9}$$

one can verify that $\gamma^{i\,(\alpha)}$ provide three eigenvectors related to a vanishing eigenvalue for the mass matrix $\mathcal{M}_{(\alpha)(\beta)}$:

$$\mathcal{M}^{(\alpha)}{}_{(\beta)}\gamma^{i(\beta)} = \left(\delta^{(\alpha)}_{(\beta)} + 4\,(\mathbb{T}^j_{(1)}\mathbb{T}_{(2)\,j})^{(\alpha)}{}_{(\beta)}\right)\gamma^{i(\beta)} = 0\,. \tag{3.3.10}$$

This implies that the massive degrees of freedom are encoded in η_I^α.

We should now write the equation for $\Lambda_I^{\alpha(\alpha)}$ in terms of Ψ^α_{iI} and η^α_I. To this end it is useful to write the following relations (we suppress the indices α and I):

$$\begin{aligned}(\mathbb{T}^i_{(1)}\Lambda)^{(\alpha)} &= \frac{\mathrm{i}}{2}\,\epsilon^{i\ell k}\,\gamma^{(\alpha)}_\ell\,\Psi_k - \frac{1}{2}\epsilon^{(\alpha)}\Psi^i + \frac{\mathrm{i}}{2}\,\gamma^{i(\alpha)}\,\eta\,,\\ (\mathbb{T}^i_{(2)}\Lambda)^{(\alpha)} &= \frac{\mathrm{i}}{2}\,\epsilon^{i\ell k}\,\gamma^{(\alpha)}_\ell\,\Psi_k + \frac{1}{2}\epsilon^{(\alpha)}\Psi^i - \frac{\mathrm{i}}{2}\,\gamma^{i(\alpha)}\,\eta\,.\end{aligned} \tag{3.3.11}$$

By substituting (3.3.4) in the field equation (3.2.6) and projecting along $(\gamma_p)^{(\sigma)}$ and $\epsilon^{(\sigma)}$, we find[11]

$$(\gamma_p)^{(\sigma)}:\quad (\boldsymbol{\alpha} - 1)\,\nabla_p\eta_I + (1+\boldsymbol{\alpha})\,\epsilon_{pik}\nabla^i\Psi^k_I = 0\,,$$

[11] We use the following identity:

$$(\gamma_p)^{(\sigma)}\,\epsilon_{(\sigma)} = 0, \quad (\gamma_p)^{(\sigma)}\,(\gamma_k)_{(\sigma)} = -2\eta_{pk}, \quad \epsilon^{(\sigma)}\epsilon_{(\sigma)} = 2\,.$$

$$\epsilon^{(\sigma)}: \quad (\boldsymbol{\alpha}-1)\,\nabla_i \Psi^i_I + \frac{2}{L}\,(\boldsymbol{\alpha}-1)\,\eta_I = 0\,. \tag{3.3.12}$$

We observe that for the value $\boldsymbol{\alpha}=1$, which is not singular for $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$[12], η decouples and we end up with only one equation for Ψ^α_i. Let us notice that, if the index α were a spinor index with respect to the Lorentz group, the equation $\epsilon_{ipk}\nabla^p\Psi^{k\alpha}_I=0$ would be the Rarita-Schwinger equation for a massless spin 3/2 field. Recall, however, that in our construction α is an internal gauge index.

The above mentioned equations of motion can be reproduced by the following Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{spacetime}} =&\frac{1}{2}\nabla_i\phi^{\alpha A}_I\,\nabla^i\phi^{\beta BI}\epsilon_{\alpha\beta}\epsilon_{AB} - \mathcal{V}(\phi) + \frac{4\mathrm{i}}{1-\boldsymbol{\alpha}}\epsilon^{ijk}\Psi^{\alpha I}_i\nabla_j\Psi^\beta_{kI}\epsilon_{\alpha\beta}\\ &+\frac{8\mathrm{i}}{(1+\boldsymbol{\alpha})}\left(-\Psi^{i\alpha I}\nabla_i\eta^\beta_I + \frac{1}{L}\eta^{\alpha I}\eta^\beta_I\right)\epsilon_{\alpha\beta}\,,\end{aligned} \tag{3.3.13}$$

which can be obtained from the spacetime Lagrangian (3.2.4) by performing the twist (3.3.4). Note that the η_I-dependent terms in the above Lagrangian are consistent with the interpretation of η_I as the Nakanishi-Lautrup field [55].
The supersymmetry variations of $\phi^{\alpha A}_I$ and these two new fields are

$$\begin{aligned}\delta_\varepsilon\phi^{\alpha A}_I &= \varepsilon^{(\alpha)A}\left(\mathrm{i}\gamma^i_{(\alpha)}\Psi^\alpha_{iI} + \epsilon_{(\alpha)}\eta^\alpha_I\right)\,,\\ \delta_\varepsilon\Psi^\alpha_{iI} &= -\frac{1}{8}\left((1-\boldsymbol{\alpha})\epsilon_{ijk}\nabla^j\phi^{\alpha A}_I + \frac{1+\boldsymbol{\alpha}}{L}\phi^{\alpha A}_I\eta_{ik}\right)\gamma^k_{(\alpha)}\varepsilon^{(\alpha)}_A - \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}\nabla_i\phi^{\alpha A}_I\epsilon_{(\alpha)}\varepsilon^{(\alpha)}_A\,,\\ \delta_\varepsilon\eta^\alpha_I &= -\frac{1+\boldsymbol{\alpha}}{8}\nabla_i\phi^{\alpha A}_I\gamma^i_{(\alpha)}\varepsilon^{(\alpha)}_A + \frac{1}{2}\mathbb{N}^\alpha_{IA}\epsilon_{(\alpha)}\varepsilon^{(\alpha)A}\,.\end{aligned} \tag{3.3.14}$$

These expressions can now be rewritten in terms of the new symmetry parameters arising from the twist we are considering, $\varepsilon^{(\alpha)A}=\mathrm{i}\gamma^{i(\alpha)}\varepsilon^A_i + \epsilon^{(\alpha)}\varepsilon^A$, that is

$$\begin{aligned}\delta_{\varepsilon^{iB}\mathcal{S}_{iB}}\phi^{\alpha A}_I &= \varepsilon^{iA}\Psi^\alpha_{iI}\,,\\ \delta_{\varepsilon^B\mathcal{S}_B}\phi^{\alpha A}_I &= \epsilon^{AB}\eta^\alpha_I\varepsilon_B\,,\\ \delta_{\varepsilon^{lB}\mathcal{S}_{lB}}\Psi^\alpha_{iI} &= \frac{\mathrm{i}}{8}\left((1-\boldsymbol{\alpha})\epsilon_{ijk}\nabla^j\phi^{\alpha A}_I + \frac{1+\boldsymbol{\alpha}}{L}\phi^{\alpha A}_I\eta_{ik}\right)\varepsilon^k_A\,,\\ \delta_{\varepsilon^B\mathcal{S}_B}\Psi^\alpha_{iI} &= -\frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}\nabla_i\phi^{\alpha A}_I\varepsilon_A\,,\\ \delta_{\varepsilon^{iB}\mathcal{S}_{iB}}\eta^\alpha_I &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}\nabla_i\phi^{\alpha A}_I\varepsilon^i_A\,,\\ \delta_{\varepsilon^B\mathcal{S}_B}\eta^\alpha_I &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8L}\phi^\alpha_{AI}\varepsilon^A\,.\end{aligned} \tag{3.3.15}$$

In particular, since on a generic field Φ we have, by linearity, $\delta_{\varepsilon^B\mathcal{S}_B}\Phi = \varepsilon^B\delta_{\mathcal{S}_B}\Phi \equiv \varepsilon^B(\mathcal{S}_B\cdot\Phi)$, from the above equations we obtain

$$(\mathcal{S}^B\cdot\phi^{\alpha A}_I) = \epsilon^{AB}\eta^\alpha_I\,,$$

[12] In fact, for $\boldsymbol{\alpha}=1$ the algebra $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ becomes isomorphic to a real form of $\mathrm{D}(2,1)\sim\mathfrak{osp}(4|2)$ with bosonic subgroup $\mathfrak{so}(2,2)\times\mathfrak{su}(2)$.

$$
\begin{aligned}
(\mathcal{S}^A \cdot \Psi^\alpha_{iI}) &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}\nabla_i \phi^{\alpha A}_I\,,\\
(\mathcal{S}^A \cdot \eta^\alpha_I) &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8L}\phi^{\alpha A}_I\,,
\end{aligned}
\tag{3.3.16}
$$

so that

$$
\begin{aligned}
(\mathcal{S}^{(B}\mathcal{S}^{A)} \cdot \phi^{\alpha C}_I) &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8L}\epsilon^{C(A}\phi^{\alpha B)}_I\,,\\
(\mathcal{S}^{(B}\mathcal{S}^{A)} \cdot \Psi^\alpha_{iI}) &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}\epsilon^{(AB)}\nabla_i\eta^\alpha_I = 0\,,\\
(\mathcal{S}^{(B}\mathcal{S}^{A)} \cdot \eta^\alpha_I) &= \frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8L}\epsilon^{(AB)}\eta^\alpha_I = 0\,.
\end{aligned}
\tag{3.3.17}
$$

The above equations show that the operators $\mathcal{S}^A$ do not anticommute on fields with non vanishing ghost-number. The cohomological structure is retrieved in the singular case $\boldsymbol{\alpha}+1=0$.
Furthermore, we notice that the obtained spacetime Lagrangian can be expressed as in [80], namely

$$
\mathcal{L}_{\text{spacetime}} = \frac{4\mathrm{i}}{1+\boldsymbol{\alpha}}\mathcal{S}^A \cdot \left(\nabla^i \phi^{\alpha B I}\Psi^\beta_{iI}\epsilon_{AB} + \frac{1}{L}\eta^{\alpha I}\phi^{\beta B}_I\epsilon_{AB}\right)\epsilon_{\alpha\beta} + \frac{4\mathrm{i}}{1-\boldsymbol{\alpha}}\epsilon^{ijk}\Psi^{\alpha I}_i\nabla_j\Psi_{k\alpha I}\,.
\tag{3.3.18}
$$

Let us remark that the charges $\mathcal{S}^A$ act on the spacetime Lagrangian similarly as BRST cohomology operators, separating it in a "physical" Lagrangian and a term in their image, despite the fact that the $\mathcal{S}^A$ do not behave as proper cohomological charges, as shown in (3.3.6). In fact, extra contributions due to the peculiar structure of the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superspace show up in the superspace Lagrangian, some of them being associated with the commutator $[\mathcal{S}^A,\nabla]\phi^{B\beta}_I$. However, the latter vanishes on spacetime, as a consequence of (3.1.34). Indeed, its twisted expression implies a trivial spacetime action of the $\mathcal{S}^A$ on the background fields appearing in the covariant derivatives.

Second twist

As discussed in [55], in order to make contact with the model of [54], where an unconventional supersymmetric theory featuring spin 1/2 fields $\chi^{(\mathrm{AVZ})}_I$ was constructed, we perform a second twist which amounts to writing the fields in a covariant way with respect to the diagonal subgroup $\mathrm{SL}(2,\mathbb{R})_D$ of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})'_1$, where the former factor is the flavour symmetry group acting on the index α. This translates into the introduction of the $\mathrm{SL}(2,\mathbb{R})_D$-invariant tensors $\gamma^i_{\alpha\alpha'}$, $\epsilon_{\alpha\alpha'}$ and decomposing $\Lambda^{\alpha\alpha'\dot\alpha'}$ as follows:

$$
\Lambda^{\alpha\alpha'\dot\alpha'}_I = \mathrm{i}\,(\gamma^k)^{\alpha\alpha'}\hat\chi^{\dot\alpha'}_{Ik} + \epsilon^{\alpha\alpha'}\chi^{\dot\alpha'}_I\,,
\tag{3.3.19}
$$

This twist is suggested by the ansatz (1.3.3), in which the field Ψ^α_{iI} is expressed in terms of spin 1/2 fields $\chi^{\dot\alpha'}_I$, to be related to the components on the right hand side of (3.3.19) in the following.

Equations (3.3.4) and (3.3.19) amount to writing the spinors in two different bases. The relation between the corresponding components reads

$$
\begin{cases} \chi_I \;= \frac{1}{2}(\mathrm{i}\,\not{\Psi}_I - \eta_I), \\ \hat{\chi}_{Ii} = \Psi_{Ii} + \frac{\mathrm{i}}{2}\,\gamma_i\,(\mathrm{i}\,\not{\Psi}_I + \eta_I)\,, \end{cases} \qquad \begin{cases} \eta_I \;= -\frac{1}{2}\left[\mathrm{i}\,\not{\hat{\chi}}_I + \chi_I\right], \\ \Psi_{iI} = \hat{\chi}_{iI} - \frac{1}{2}\,\gamma_i\,\left(\not{\hat{\chi}}_I + \mathrm{i}\,\chi_I\right)\,, \end{cases} \tag{3.3.20}
$$

where the spinor indices have been suppressed[13]. The spinor equations (3.3.12), written in terms of the new fields, yield

$$
(\boldsymbol{\alpha}+1)\epsilon^{\ell ik}\,\nabla_i\left(\hat{\chi}_{kI} - \frac{\mathrm{i}}{2}\,\gamma_k\,(\chi_I - \mathrm{i}\,\not{\hat{\chi}}_I)\right) + \frac{1-\boldsymbol{\alpha}}{2}\,\nabla^\ell(\chi_I + \mathrm{i}\,\not{\hat{\chi}}_I) = 0\,, \tag{3.3.21}
$$

$$
\frac{1}{2}\left(-\mathrm{i}\,\not{\nabla}\chi_I - \frac{2}{L}\chi_I\right) - \frac{\mathrm{i}}{2}\left(-\mathrm{i}\,\not{\nabla}\not{\hat{\chi}}_I + \frac{2}{L}\not{\hat{\chi}}_I\right) + \nabla^i\hat{\chi}_{iI} = 0\,, \tag{3.3.22}
$$

and the field variations read

$$
\begin{aligned}
\delta_\varepsilon\phi_I^{\alpha A} &= \mathrm{i}\left(\gamma^k\right)^{\alpha}{}_{\alpha'}\,\varepsilon^{\alpha'\dot{\alpha}'A}\hat{\chi}_{Ik\dot{\alpha}'} + \varepsilon^{\alpha\dot{\alpha}'A}\chi_{I\dot{\alpha}'}\,, \\
\delta_\varepsilon\hat{\chi}_{Il}^{\dot{\alpha}'} &= \frac{\mathrm{i}}{8}\nabla_i\phi_I^{\alpha A}\left[\left(\gamma_l\gamma^i\right)_{\alpha\beta'}\varepsilon_A^{\beta'\dot{\alpha}'} - \boldsymbol{\alpha}\,(\gamma_l)_{\alpha\beta'}\left(\gamma^i\right)^{\dot{\alpha}'}{}_{\dot{\beta}'}\varepsilon_A^{\beta'\dot{\beta}'}\right] + \frac{\mathrm{i}}{2}\mathbb{N}_{IA}^{\alpha}\,(\gamma_l)_{\alpha\alpha'}\,\varepsilon^{\alpha'\dot{\alpha}'A}\,, \\
\delta_\varepsilon\chi_I^{\dot{\alpha}'} &= \frac{1}{8}\nabla_i\phi_I^{\alpha A}\left[\left(\gamma^i\right)_{\alpha\beta'}\varepsilon_A^{\beta'\dot{\alpha}'} - \boldsymbol{\alpha}\epsilon_{\alpha\beta'}\left(\gamma^i\right)^{\dot{\alpha}'}{}_{\dot{\beta}'}\varepsilon_A^{\beta'\dot{\beta}'}\right] + \frac{1}{2}\mathbb{N}_{IA}^{\alpha}\epsilon_{\alpha\alpha'}\varepsilon^{\alpha'\dot{\alpha}'A}\,.
\end{aligned} \tag{3.3.23}
$$

In terms of the fields $\hat{\chi}_{Ik}^{\dot{\alpha}'}$, $\chi_I^{\dot{\alpha}'}$, the spacetime Lagrangian (3.2.4) takes the form

$$
\begin{aligned}
\mathcal{L}_{\text{spacetime}} =& \frac{1}{2}\nabla_i\phi_I^{\alpha A}\,\nabla^i\phi^{\beta BI}\epsilon_{\alpha\beta}\epsilon_{AB} - \mathcal{V}(\phi) + \\
& - \frac{4}{\boldsymbol{\alpha}^2-1}\left[\mathrm{i}\epsilon^{ijk}\hat{\chi}_{iI}^t\epsilon\nabla_j\hat{\chi}_k^I + \boldsymbol{\alpha}\hat{\chi}_{iI}^t\epsilon\gamma^j\nabla_j\hat{\chi}_i^I + \alpha\chi_I^t\epsilon\gamma^i\nabla_i\chi^I - 2\mathrm{i}\chi_I^t\epsilon\nabla^i\hat{\chi}_i^I\right] + \\
& + \frac{2\mathrm{i}}{L(\boldsymbol{\alpha}+1)}\left[\hat{\chi}_{kI}^t\epsilon\hat{\chi}^{kI} - \mathrm{i}\epsilon^{ijk}\hat{\chi}_{iI}^t\epsilon\gamma_j\hat{\chi}_k^I + \chi_I^t\epsilon\chi^I - 2\mathrm{i}\hat{\chi}_{iI}^t\epsilon\gamma^i\chi^I\right]\,,
\end{aligned} \tag{3.3.24}
$$

where we have used the matrix notation for the bispinor index: $\xi^t\epsilon\zeta \equiv \xi^\alpha\zeta^\beta\,\epsilon_{\alpha\beta}$.
The above results can be further rewritten by decomposing the $\hat{\chi}_{Ii}$ field into its spin 3/2 and spin 1/2 components, $\mathring{\chi}_{Ii}$ and $\hat{\not{\chi}}_I$, as follows:

$$
\hat{\chi}_{Ii} = \mathring{\chi}_{Ii} + \frac{1}{3}\,\gamma_i\,\hat{\not{\chi}}_I\,, \tag{3.3.25}
$$

where $\gamma^i\,\mathring{\chi}_{Ii} = 0$. In this way, we can rewrite the expressions of Ψ_{Ii} and η_I in (3.3.20) in the form

$$
\Psi_{Ii} = \mathring{\chi}_{Ii} - \frac{\mathrm{i}}{2}\,\gamma_i\,\chi_{1I}\,, \qquad \eta_I = -\frac{1}{2}\,\chi_{2I}\,, \tag{3.3.26}
$$

where χ_{1I} and χ_{2I} are

$$
\chi_{1I} \equiv -\frac{\mathrm{i}}{3}\,\hat{\not{\chi}}_I + \chi_I\,, \qquad \chi_{2I} \equiv \mathrm{i}\,\hat{\not{\chi}}_I + \chi_I\,. \tag{3.3.27}
$$

[13]Note that the above manipulations require that the only manifest symmetry effectively acting on the odd sector is the diagonal subgroup $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{R})_D \times \mathrm{SL}(2,\mathbb{R})'_2$, so that the three indices α, α', $\dot{\alpha}'$ are treated on an equal footing.

The inverse relations read

$$\chi_I = \frac{1}{4}\left(3\chi_{1I} + \chi_{2I}\right)\,, \qquad \hat{\chi}_I = \frac{3}{4}\mathrm{i}\left(\chi_{1I} - \chi_{2I}\right) \tag{3.3.28}$$

and the Lagrangian (3.3.24), when expressed in terms of the fields $\mathring{\chi}_{Ii}$, χ_{1I}, χ_{2I}, takes the simpler form

$$\begin{aligned}\mathcal{L}_{\text{spacetime}} =& \frac{1}{2}\nabla_i\phi_I^{\alpha A}\,\nabla^i\phi^{\beta BI}\epsilon_{\alpha\beta}\epsilon_{AB} - \mathcal{V}(\phi) + \\ &+ \frac{4\mathrm{i}}{1-\boldsymbol{\alpha}}\left[\epsilon^{ijk}\mathring{\chi}^t_{Ii}\epsilon\nabla_j\mathring{\chi}^I_k - \frac{\mathrm{i}}{2}\chi^t_{1I}\epsilon\nabla\!\!\!\!/\,\chi^I_1 + \mathring{\chi}^t_{Ii}\epsilon\nabla^i\,\chi^I_1\right] + \\ &+ \frac{2\mathrm{i}}{\boldsymbol{\alpha}+1}\left[2\,\mathring{\chi}^t_{Ii}\epsilon\,\nabla^i\,\chi^I_2 + \mathrm{i}\,\chi^t_{1I}\,\epsilon\,\nabla\!\!\!\!/\chi^I_2 + \frac{1}{L}\,\chi^t_{2I}\epsilon\,\chi^I_2\right]\,,\end{aligned} \tag{3.3.29}$$

where, for the spinor bilinears, we have used the notation illustrated in Appendix B. The field equations are readily written as

$$\delta\mathring{\chi}:\quad \mathbb{P}_{\left(\frac{3}{2}\right)}{}^l{}_i\left((\boldsymbol{\alpha}+1)\,\epsilon^{ijk}\,\nabla_j\left(\mathring{\chi}_{Ik} - \frac{\mathrm{i}}{2}\,\gamma_k\,\chi_{1I}\right) - \frac{(\boldsymbol{\alpha}-1)}{2}\,\nabla^i\,\chi_{2I}\right) = 0\,, \tag{3.3.30}$$

$$\delta\chi_1:\quad \nabla^i\mathring{\chi}_{iI} + \mathrm{i}\nabla\!\!\!\!/\chi_{1I} + \mathrm{i}\frac{\alpha-1}{2(1+\boldsymbol{\alpha})}\nabla\!\!\!\!/\chi_{2I} = 0\,, \tag{3.3.31}$$

$$\delta\chi_2:\quad -2\nabla^i\mathring{\chi}_{iI} + \mathrm{i}\nabla\!\!\!\!/\chi_{1I} + \frac{2}{L}\chi_{2I} = 0\,, \tag{3.3.32}$$

where $\mathbb{P}_{\left(\frac{3}{2}\right)}{}^j{}_i \equiv \mathbf{1}\,\delta^j_i - \frac{1}{3}\,\gamma^j\gamma_i$ is the projector on the spin 3/2 representation. Combining the three equations above, we get the conditions:

$$\begin{aligned}&\nabla^i\mathring{\chi}_{iI} = \frac{1}{6}\frac{1-\boldsymbol{\alpha}}{1+\boldsymbol{\alpha}}\left[\mathrm{i}\nabla\!\!\!\!/\chi_{2I} - \frac{4(1+\boldsymbol{\alpha})}{L(\boldsymbol{\alpha}-1)}\chi_{2I}\right]\,,\\ &2\,\epsilon^{ijk}\nabla_j\mathring{\chi}_{Ik} + \left(\nabla^i - \gamma^i\nabla\!\!\!\!/\right)\,\chi_{1I} - \frac{\boldsymbol{\alpha}-1}{\boldsymbol{\alpha}+1}\,\nabla^i\chi_{2I} = 0\,.\end{aligned} \tag{3.3.33}$$

Let us show that the solutions to equations (3.3.30), (3.3.31) and (3.3.32) comprise a massive Dirac spinor, which can be related to the unconventional supersymmetry ansatz (1.3.3). To this end it suffices to restrict to solutions satisfying the further condition

$$\nabla^i\mathring{\chi}_{Ii} = 0\,. \tag{3.3.34}$$

In fact, equations (3.3.31) and (3.3.32) take the form

$$\nabla\!\!\!\!/\chi_{1I} = \frac{2\mathrm{i}}{L}\,\chi_{2I}\,, \tag{3.3.35}$$

$$\mathrm{i}\,\nabla\!\!\!\!/\chi_{2I} = m\,\chi_{2I}\,, \tag{3.3.36}$$

where

$$m = \frac{4(\boldsymbol{\alpha}+1)}{L(\boldsymbol{\alpha}-1)}\,. \tag{3.3.37}$$

The fields χ_{2I} are now massive Dirac spinors of mass m. Note that, as expected, this mass depends on the parameter $g = (\boldsymbol{\alpha} + 1)$, namely on the gauging of the R-symmetry SU(2). If the configuration χ_{2I} is taken as solution of (3.3.36), it is straightforward to verify that χ_{1I} is given by the general expression

$$\chi_{1I} = -\frac{2}{Lm}\,\chi_{2I} + \sigma_I = \frac{1-\boldsymbol{\alpha}}{2(\boldsymbol{\alpha}+1)}\,\chi_{2I} + \sigma_I\,, \tag{3.3.38}$$

where σ_I are massless spinor fields: $\mathrm{i}\,\not{\nabla}\sigma_I = 0$.
Now, we impose a stronger condition on the solutions to equations (3.3.30), (3.3.31), (3.3.32) and set the spin 3/2 field to zero:

$$\mathring{\chi}_{Ii} = 0\,. \tag{3.3.39}$$

Finally, this allows to make contact with unconventional supersymmetry, where the fields Ψ^α_{Ii} have a vanishing spin 3/2 component. Indeed, from (3.3.26), we obtain

$$\Psi_{Ii} = -\frac{\mathrm{i}}{2}\,\gamma_i\,\chi_{1I}\ , \quad \eta_I = -\frac{1}{2}\,\chi_{2I}\,, \tag{3.3.40}$$

where Ψ_{Ii} only has a spin 1/2 component χ_{1I}, which is expressed in terms of the massive spinor field χ_{2I} through (3.3.38). The propagating spinor $\chi_I^{(\mathrm{AVZ})}$ of [54], appearing in (1.3.3), has to be identified, using (3.3.26), with

$$\chi_I^{(\mathrm{AVZ})} = \mathrm{i}\gamma^i\Psi_{iI} = \frac{3}{2}\chi_{1I}\,.$$

We have discussed above only those solutions for which either (3.3.34) or the stronger equation (3.3.39) holds. Our supersymmetric model, however, features more general solutions, which non trivially involve the spin 3/2 fields and whose physical applications deserve investigation. We postpone this analysis to future developments.
Eventually, we notice that the condition for unconventional supersymmetry $\mathring{\chi}_{iI} = 0$ breaks, in general, all supersymmetries of our superspace. Indeed, we can use the supersymmetry variations of $\mathring{\chi}_{iI}$ in the twisted form to write

$$\delta_{\varepsilon^B\mathcal{S}_B}\mathring{\chi}_{iI} = -\frac{\mathrm{i}(1+\boldsymbol{\alpha})}{8}(\mathbb{P}_{ij}\nabla^j\phi^A_I)\varepsilon_A\,, \tag{3.3.41}$$

$$\begin{aligned}\delta_{\varepsilon^{lB}\mathcal{S}_{lB}}\mathring{\chi}_{iI} &= \mathbb{P}_{\left(\frac{3}{2}\right)ij}\,\delta_{\varepsilon^{lB}\mathcal{S}_{lB}}\Psi^j_I\\ &= \frac{\mathrm{i}}{8}\left((1-\boldsymbol{\alpha})\epsilon_{ijk}\nabla^j\phi^A_I + \frac{1+\boldsymbol{\alpha}}{L}\phi^A_I\eta_{ik}\right)\varepsilon^k_A +\\ &\quad - \frac{\mathrm{i}}{24}\gamma_i\left(3\mathrm{i}(\boldsymbol{\alpha}-1)\mathbb{P}_{kj}\nabla^j\phi^A_I - 2\mathrm{i}(\boldsymbol{\alpha}-1)\nabla_k\phi^A_I + \frac{\boldsymbol{\alpha}+1}{L}\gamma_k\phi^A_I\right)\varepsilon^k_A\,,\end{aligned} \tag{3.3.42}$$

from which it follows that, in general, the vanishing of $\mathring{\chi}_{iI}$ is not preserved by supersymmetry transformations in $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ superspace. It is important to emphasise, however, that this supersymmetry is not related to the *unconventional* one exhibited by the model in [54], which originated from a target space symmetry.

3.4 Conclusions and Outlook

In this final Section, we review the outcome of our analysis and we conclude with some comments on future developments and perspectives.

Our results can be summarised as follows:

1) We have constructed a three dimensional model of rigid supersymmetry featuring eight supercharges on a curved AdS_3 worldvolume background whose superspace is based on the supergroup $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$. The resulting model describes the dynamics of a set of hypermultiplets $(\Lambda_I^{\alpha\alpha'\dot\alpha'}, \phi_I^{\alpha A})$.

   A peculiarity of the chosen superalgebra is the presence of a parameter $\boldsymbol{\alpha}$, independent of the cosmological constant, which defines, through the combination $g = \boldsymbol{\alpha} + 1$, the gauging of an internal SU(2), in the absence of dynamical gauge fields. To clarify the meaning of the word "gauging" in the present context, let us notice that the "coupling constant" g generates a fermion shift $\mathbb{N}_I^{\alpha A}$ in the supersymmetry transformation of the hyperini (3.1.32), together with mass terms for the latter fields and non trivial scalar dynamics. This feature is not fully apparent from the Lagrangian, since the structure of $\mathrm{D}^2(2,1;\,\boldsymbol{\alpha})$ naturally leads to a redundant (4-component) description of the spinorial degrees of freedom and to a generalised definition of gamma matrices which include a dependence on the parameter $\boldsymbol{\alpha}$. Consequently, the dependence of the Lagrangian on the parameter $\boldsymbol{\alpha}$ is somewhat concealed in the matrices $\mathbb{M}^i_\pm$. The proper definition of the hyperini mass requires a formulation of the above fields as ordinary 2-component spinors, and this in turn implies a choice of the Lorentz symmetry in the superspace. This reformulation is intimately related to the issue of the twists.

2) Two inequivalent twists have indeed been performed, corresponding to two different identifications of the Lorentz group.

   In the first one, the Lorentz group is identified with the diagonal subgroup of the two $\mathrm{SL}(2,\mathbb{R})$ factors in the AdS_3 isometry group. This is the counterpart, in our setting, of the topological twist discussed in [79, 80]. After performing this twist, the hyperini decompose into an abelian gauge connection $\mathit{\Psi}^\alpha_{iI}$ associated with odd symmetry generators, transforming as a vector with respect to the Lorentz group and in a Grassmann-valued field, η^α_I, singlet of the Lorentz group, as can be seen from (3.3.4). Correspondingly, the supersymmetry generators decompose into vector-like and scalar-like odd generators $\mathcal{S}^A_i, \mathcal{S}^A$. Our spacetime Lagrangian takes the form of a Chern-Simons Lagrangian for $\mathit{\Psi}^\alpha_{iI}$ plus a term in the image of $\mathcal{S}^A$, containing the interaction with the other fields.

   An alternative twist can be performed, involving the $\mathrm{SL}(2,\mathbb{R})$ flavour group. More precisely, in this case the Lorentz group is defined to be the diagonal of the previously chosen Lorentz group with the flavour $\mathrm{SL}(2,\mathbb{R})$ acting on the index α carried by the dynamical fields of the model. This corresponds to identifying the Lorentz group as the diagonal of the three groups $\mathrm{SL}(2,\mathbb{R})'_1$, $\mathrm{SL}(2,\mathbb{R})'_2$, $\mathrm{SL}(2,\mathbb{R})_{\text{flavour}}$. In particular, the anticommuting fields $\mathit{\Psi}^\alpha_{iI}$ and η^α_I now transform in half-integer Lorentz representations, which are appropriate to their spin statistics, while $\phi_I^{\alpha A}$ transform as commuting spin

1/2 fields, fully decoupled from the rest. In this new setting, the superspace structure is not manifest. This identification, which in fact describes a subsector of the first twist, unveils the interesting structure described by the Lagrangian (3.3.29). Indeed, Ψ^{α}_{iI} and η^{α}_{I} acquire a natural interpretation as spinorial fields, in particular as a purely spin 3/2 field $\mathring{\chi}_{iI}$ coupled to two spin 1/2 particles χ_{1I}, χ_{2I}. A subset of the solutions of the field equations, defined by the condition $\nabla_i \mathring{\chi}^i_I = 0$, describes a massive spin 1/2 field, χ_{2I}, which acts as a source for the field χ_{1I}. The latter can therefore be expressed as a combination of χ_{2I} and an arbitrary massless spin 1/2 field σ_I.
Imposing the stronger condition $\mathring{\chi}_{iI} = 0$, we recover the most general solution of the model with unconventional supersymmetry of [54, 57, 58]. It is worth emphasising, however, that the complete set of solutions of the field equations of our model is richer and describes non trivial dynamics involving $\mathring{\chi}_{iI}$, χ_{1I}, χ_{2I}, yet to be explored.

A few questions, however, remain open. We have found that the action, though perfectly supersymmetric in spacetime, is only *quasi-supersymmetric* when extended as a 3-form in the full superspace, its invariance requiring the addition of boundary terms. This is possibly related to the presence, for $\boldsymbol{\alpha} + 1 \neq 0$, of an SU(2)-gauging involving non dynamical vector fields A^x_μ which, in our model, are frozen as background fields. As $\boldsymbol{\alpha} + 1$ is set to zero, indeed, the full supersymmetry of the Lagrangian is restored in superspace.
In this same limit, in the context of the first twist, the interpretation of the $\boldsymbol{\mathcal{S}}^A$ generators as anticommuting BRST operators is recovered.
As $\boldsymbol{\alpha}$ is set to this singular value, the fermion masses, which would vanish in the model considered here, could instead be obtained through the gauging of the flavour group $\mathcal{G} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(2)$, according to the analysis of [55]. This gauging, which we did not consider here, can be regarded as more *conventional* in that it will involve gauge fields sitting in vector multiplets and thus that are not background fields. The flavour group, however, can also be gauged for a generic value of $\boldsymbol{\alpha}$.

Let us now turn to the discussion of perspectives and future developments. Given the peculiar structure of the supergroup considered here and the chosen dynamical supermultiplets, there are multiple possible routes.
A first choice would be, as mentioned above, to introduce a proper gauging of the flavour symmetry group: we expect this idea to lead to a structure similar to the one of [79], where the spacetime Lagrangian truly is a Chern-Simons theory, having both even and odd connections. Further insight could be found by including in our analysis also the interaction with a set of twisted hypermultiplets [87].
Another possibility for extending the present analysis is given by the choice of a more general, curved scalar manifold of hyper-Kähler type. In this case the hypermultiplet Lagrangian should be modified by the addition of terms accounting for the curvature of the hyper-Kähler geometry, as sketched in Section 3.2.
Finally, it would be appealing to consider the $\mathrm{D}^2(2, 1; \boldsymbol{\alpha})$ superalgebra as a framework to derive new models of interacting massive Dirac particles, whose application to the description, for instance, of graphene-like materials [103], similarly to what has been done for models with unconventional supersymmetry, is an interesting task to be pursued.

Appendix A

Supplementary Material of Chapter 2

A.1 Conventions

In this Appendix we summarise our conventions for Chapter 2.

Curvatures

The bulk local coordinates are denoted by $x^{\hat{\mu}} = (x^\mu, z)$ and the boundary coordinates by x^μ ($\mu = 0, 1, 2$). In general, the hatted quantities always refer to the bulk and the non hatted ones to the boundary placed at $z = 0$.

With regards to connections and curvatures conventions, besides the hatted (bulk) ones $\{\hat{\omega}, \hat{\Gamma}, \hat{\mathcal{R}}, \hat{R}, \hat{\rho}\}$ and the non hatted (boundary) ones $\{\omega, \Gamma, \mathcal{R}, R, \rho\}$, a circle above quantities, $\{\mathring{\omega}, \mathring{\Gamma}, \mathring{\mathcal{R}}\}$, denote the torsion-free condition, whereas the bold symbol, $\{\hat{\mathbf{R}}, \mathbf{R}, \hat{\boldsymbol{\rho}}, \boldsymbol{\rho}\}$, corresponds to the fact that it is super-covariant.
In our case, $\{\hat{\rho}, \hat{\boldsymbol{\rho}}, \rho, \boldsymbol{\rho}\}$ correspond to the fermionic supercurvatures. A similar notation applies for the Abelian supercurvatures $\{\hat{F}, \hat{\mathbf{F}}, F, \mathbf{F}\}$ and, moreover, the Maxwell field strength on the boundary is denoted by $\mathcal{F}$.

Explicitly, in the bulk we have the Lorentz curvature 2-form $\hat{\mathcal{R}}^{ab} = \frac{1}{2}\,\hat{\mathcal{R}}^{ab}{}_{\hat{\mu}\hat{\nu}}\, \mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}}$ defined in terms of the bulk spin connection $\hat{\omega}^{ab}_{\hat{\mu}}$. By using the first vielbein postulate,

$$\partial_{\hat{\mu}} V^a_{\ \hat{\nu}} + \hat{\omega}^{ab}_{\hat{\mu}} V_{b\hat{\nu}} = \hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}} V^a_{\ \hat{\lambda}}\,, \tag{A.1.1}$$

$\hat{\mathcal{R}}^{ab}$ is mapped to the bulk curvature tensor,

$$\hat{\mathcal{R}}^{\hat{\lambda}}{}_{\hat{\sigma}\hat{\mu}\hat{\nu}}(\hat{\Gamma}) = \hat{\mathcal{R}}^{ab}{}_{\hat{\mu}\hat{\nu}}(\hat{\omega}) V^{\hat{\lambda}}{}_a V_{\hat{\sigma} b}\,, \tag{A.1.2}$$

expressed in terms of the bulk affine connection $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}}$. The bulk AdS curvature 2-form is denoted by $\hat{R}^{ab}$ and the super AdS curvature by $\hat{\mathbf{R}}^{ab}$.

On the other hand, on the boundary, the Lorentz curvature 2-form is $\mathcal{R}^{ij} = \frac{1}{2}\,\mathcal{R}^{ij}{}_{\mu\nu}$ $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, from which we can obtain $\mathcal{R}^\lambda{}_{\sigma\mu\nu}(\Gamma) = \mathcal{R}^{ij}{}_{\mu\nu}(\omega) E^\lambda{}_i E_{\sigma j}$, where $\Gamma^\lambda_{\nu\mu}$ and ω^{ij}_μ are the torsionful affine and spin connection, respectively. The boundary AdS curvature 2-form is R^{ij} and the super AdS curvarure $\mathbf{R}^{ij}$. Similarly, the torsionless quantities on the boundary are $\mathring{\mathcal{R}}^{\lambda\sigma}{}_{\mu\nu} = \mathring{\mathcal{R}}^{ij}{}_{\mu\nu} E^{\lambda}{}_{i} E^{\sigma}{}_{j}$, and the corresponding Levi-Civita connections are $\mathring{\Gamma}^{\lambda}_{\nu\mu}$ and $\mathring{\omega}^{ij}_{\mu}$.

Gamma matrices and spinor conventions

We follow the notation of [58].
The four dimensional 4×4 gamma matrices Γ^a $(a = 0, 1, 2, 3)$ satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\,, \quad \eta^{ab} = \mathrm{diag}(+, -, -, -)\,, \tag{A.1.3}$$

and the fifth matrix is defined by

$$\Gamma_5 = \mathrm{i}\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3\,. \tag{A.1.4}$$

They have the properties

$$(\Gamma^i)^\dagger = \Gamma^0\Gamma^i\Gamma^0\,, \qquad (\Gamma_5)^\dagger = \Gamma_5, \tag{A.1.5}$$

and they satisfy the identity

$$\frac{1}{2}\,\epsilon_{abcd}\,\Gamma^{cd} = \mathrm{i}\Gamma_{ab}\Gamma_5\,, \tag{A.1.6}$$

where

$$\Gamma^{a_1\cdots a_n} = \Gamma^{[a_1\cdots a_n]} \equiv \begin{cases} \frac{1}{2}\Big[\Gamma^{a_1}, \Gamma^{a_2\cdots a_n}\Big]\,, & \text{for even } n\,, \\ \frac{1}{2}\Big\{\Gamma^{a_1}, \Gamma^{a_2\cdots a_n}\Big\}\,, & \text{for odd } n\,. \end{cases} \tag{A.1.7}$$

We can also define the charge conjugation matrix C that determines the symmetry properties of the gamma matrices,

$$C = \Gamma^0\,, \qquad C\Gamma^a C^{-1} = -(\Gamma^a)^t\,, \tag{A.1.8}$$

the upper t denoting transposition. From the latter condition, we can derive a general property for the antisymmetric product of k gamma matrices as

$$(C\Gamma^{a_1\ldots a_k})^t = -(-1)^{\frac{k(k+1)}{2}}\, C\Gamma^{a_1\ldots a_k}\,. \tag{A.1.9}$$

Furthermore, the following identity holds for the gamma matrices in any D dimension [10]

$$\Gamma^{a_1\ldots a_n c_1\ldots c_q}\Gamma_{c_1\ldots c_q b_1\ldots b_m} = \sum_{k=0}^{\inf(n,m)} c_k(q,n,m)\,\delta^{[a_1}_{[b_1}\ldots\delta^{a_k}_{b_k}\,\Gamma^{a_{k+1}\ldots a_n]}{}_{b_{k+1}\ldots b_m]}\,, \tag{A.1.10}$$

where the coefficient read

$$c_k(q,n,m) = (-1)^{\frac{1}{2}q(q-1)+\frac{k}{2}\left[k-(-1)^{n-1}\right]}\binom{n}{k}\binom{m}{k}\,q!\,k!\binom{D-n-m+k}{q}\,. \tag{A.1.11}$$

It is convenient to introduce the 2×2 gamma matrices γ^i $(i = 0, 1, 2)$, which are the elements of the $d = 3$ Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij}\,, \qquad \eta^{ij} = \mathrm{diag}(+,-,-)\,. \tag{A.1.12}$$

The $D = 4$ gamma matrices can be represented in terms of these $d = 3$ gamma matrices as

$$\begin{aligned} \Gamma^i &= \sigma_1 \otimes \gamma^i\,, \quad \gamma^0 = \sigma_2\,, \quad \gamma^1 = \mathrm{i}\sigma_1\,, \quad \gamma^2 = \mathrm{i}\sigma_3\,, \\ \Gamma^3 &= \mathrm{i}\sigma_3 \otimes \mathbf{1}\,, \quad \Gamma_5 = \mathrm{i}\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = -\sigma_2 \otimes \mathbf{1} = \begin{pmatrix} \mathbb{0} & \mathrm{i}\mathbb{1}_2 \\ -\mathrm{i}\mathbb{1}_2 & \mathbb{0} \end{pmatrix}. \end{aligned} \tag{A.1.13}$$

An identity often used in the text is

$$\gamma^i\gamma^j = \eta^{ij} + \mathrm{i}\,\epsilon^{ijk}\,\gamma_k\,, \qquad \epsilon^{012} = 1\,, \tag{A.1.14}$$

which implies

$$\gamma^{ij} = \mathrm{i}\,\epsilon^{ijk}\,\gamma_k\,, \qquad \gamma^{ij} \equiv \frac{1}{2}\,[\gamma^i, \gamma^j]\,. \tag{A.1.15}$$

Let us now focus on the spinor conventions.
The Majorana 4-spinor 1-form $\Psi = \Psi_{\hat{\mu}}\,\mathrm{d}x^{\hat{\mu}}$ has Grassmannian components $\Psi_{\hat{\mu}}$. By using the symmetry properties of the gamma matrices (A.1.8), we obtain the following relations for the fermionic bilinears,

$$\begin{aligned} \overline{\Psi}_{A\hat{\mu}}\Psi_{B\hat{\nu}} &= \overline{\Psi}_{B\hat{\nu}}\Psi_{A\hat{\mu}}\,, & \overline{\Psi}_{A\hat{\mu}}\Gamma_5\Psi_{B\hat{\nu}} &= \overline{\Psi}_{B\hat{\nu}}\Gamma_5\Psi_{A\hat{\mu}}\,, \\ \overline{\Psi}_{A\hat{\mu}}\Gamma^a\Psi_{B\hat{\nu}} &= -\overline{\Psi}_{B\hat{\nu}}\Gamma^a\Psi_{A\hat{\mu}}\,, & \overline{\Psi}_{A\hat{\mu}}\Gamma^a\Gamma_5\Psi_{B\hat{\nu}} &= \overline{\Psi}_{B\hat{\nu}}\Gamma^a\Gamma_5\Psi_{A\hat{\mu}}\,, \\ \overline{\Psi}_{A\hat{\mu}}\Gamma^{ab}\Psi_{B\hat{\nu}} &= -\overline{\Psi}_{B\hat{\nu}}\Gamma^{ab}\Psi_{A\hat{\mu}}\,, & \overline{\Psi}_{A\hat{\mu}}\Gamma^{ab}\Gamma_5\Psi_{B\hat{\nu}} &= -\overline{\Psi}_{B\hat{\nu}}\Gamma^{ab}\Gamma_5\Psi_{A\hat{\mu}}\,. \end{aligned} \tag{A.1.16}$$

In view of the application to the holographic duality, it is convenient to choose a gamma matrices basis where only Lorentz invariance in $d = 3$ dimensions is manifest, where the radial matrix Γ^3 is associated with the generator T_0 of the $\mathrm{SO}(1,1)$ group given by (2.5.2). Then, for our purposes, it is useful to decompose the four-spinor Ψ in eigenmodes $\Psi_\pm$ of the matrix Γ^3,

$$\Gamma^3\Psi_\pm = \pm\mathrm{i}\Psi_\pm\,, \tag{A.1.17}$$

where the projectors and the corresponding projections are given by

$$\mathbb{P}_\pm = \frac{\mathbb{1} \mp \mathrm{i}\Gamma^3}{2} \quad \Rightarrow \quad \mathbb{P}_\pm\Psi_\pm = \Psi_\pm\,, \quad \overline{\Psi}_\pm = \overline{\Psi}_\pm\mathbb{P}_\mp\,. \tag{A.1.18}$$

Furthermore, in order to find chiral components of the fermionic expressions, we list the following useful identities,

$$\begin{aligned} \mathbb{P}_\pm\Gamma^3 &= \pm\mathrm{i}\mathbb{P}_\pm\,, & \mathbb{P}_\pm\Gamma_{ij} &= \Gamma_{ij}\mathbb{P}_\pm\,, \\ \mathbb{P}_\pm\Gamma_i &= \Gamma_i\mathbb{P}_\mp\,, & \mathbb{P}_\pm\Gamma_{i3} &= \pm\mathrm{i}\Gamma_i\mathbb{P}_\mp\,, \end{aligned} \tag{A.1.19}$$

as well as

$$\mathbb{P}_\pm\Gamma_5 = \Gamma_5\mathbb{P}_\mp\,. \tag{A.1.20}$$

When the chiral spinors are involved, the fermionic bilinears have only the following non vanishing terms

$$\begin{aligned}
\overline{\Psi}_{\hat\mu}\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Psi_{\hat\nu-}+\overline{\Psi}_{\hat\mu-}\Psi_{\hat\nu+}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu-}\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu+}\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^i\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Gamma^i\Psi_{\hat\nu+}+\overline{\Psi}_{\hat\mu-}\Gamma^i\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^i\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu+}\Gamma^i\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu-}\Gamma^i\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^{ij}\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Gamma^{ij}\Psi_{\hat\nu-}+\overline{\Psi}_{\hat\mu-}\Gamma^{ij}\Psi_{\hat\nu+}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^{ij}\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu-}\Gamma^{ij}\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu+}\Gamma^{ij}\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma_5\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Gamma_5\Psi_{\hat\nu+}+\overline{\Psi}_{\hat\mu-}\Gamma_5\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma_5\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu+}\Gamma_5\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu-}\Gamma_5\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^i\Gamma_5\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Gamma^i\Gamma_5\Psi_{\hat\nu-}+\overline{\Psi}_{\hat\mu-}\Gamma^i\Gamma_5\Psi_{\hat\nu+}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^i\Gamma_5\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu-}\Gamma^i\Gamma_5\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu+}\Gamma^i\Gamma_5\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^{ij}\Gamma_5\Psi_{\hat\nu} &= \overline{\Psi}_{\hat\mu+}\Gamma^{ij}\Gamma_5\Psi_{\hat\nu+}+\overline{\Psi}_{\hat\mu-}\Gamma^{ij}\Gamma_5\Psi_{\hat\nu-}\,,\\
\overline{\Psi}_{\hat\mu}\Gamma^{ij}\Gamma_5\Gamma^3\Psi_{\hat\nu} &= \mathrm{i}\overline{\Psi}_{\hat\mu+}\Gamma^{ij}\Gamma_5\Psi_{\hat\nu+}-\mathrm{i}\overline{\Psi}_{\hat\mu-}\Gamma^{ij}\Gamma_5\Psi_{\hat\nu-}\,.
\end{aligned}\tag{A.1.21}$$

In the context of holography, only the radial decomposition (with respect to Γ^3) is relevant and used to define the chiral componets. We do not use the Weyl decomposition of the 4-spinor with respect to Γ_5.

Finally, let us list the three dimensional Fierz identities used in the main text,

$$\begin{aligned}
\psi_{A+}\overline{\zeta}_{B+} =& -\frac{1}{4}\,\delta_{AB}\left(\overline{\psi}^C_+\zeta_{C+}\right)-\frac{1}{4}\,\epsilon_{AB}\epsilon^{CD}\left(\overline{\psi}_{C+}\zeta_{D+}\right)\\
&+\frac{1}{4}\,\delta_{AB}\,\gamma_i\left(\overline{\psi}^C_+\gamma^i\zeta_{C+}\right)+\frac{1}{4}\,\epsilon_{AB}\epsilon^{CD}\,\gamma_i\left(\overline{\psi}_{C+}\gamma^i\zeta_{D+}\right)\,,\\
\psi_{A+}\overline{\psi}_{B+} =& -\frac{1}{4}\,\epsilon_{AB}\epsilon^{CD}\left(\overline{\psi}_{C+}\psi_{D+}\right)+\frac{1}{4}\,\delta_{AB}\,\gamma_i\left(\overline{\psi}^C_+\gamma^i\psi_{C+}\right)\,,
\end{aligned}\tag{A.1.22}$$

with the following convention for the SO(2) invariant tensor

$$\epsilon_{AB}=\epsilon^{AB}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad A,B,\ldots=1,2\,.\tag{A.1.23}$$

A.2 Asymptotic expansions

Spin connection

In pure AdS_4 gravity, the spin connection $\mathring{\omega}^{ab}_{\hat\mu}(x,z)$ satisfies the torsion constraint $\hat{T}^a_{\hat\mu\hat\nu}=\mathring{\mathcal{D}}_{\hat\mu}V^a_{\ \hat\nu}-\mathring{\mathcal{D}}_{\hat\nu}V^a_{\ \hat\mu}=0$, see (2.1.21). If we use $\mathring{\omega}^{ab}_{\hat\mu}$ as a reference spin connection on spacetime also in the supersymmetric case, where the vielbein satisfies, on the contrary, the supertorsion constraint $\hat{\mathbf{R}}^a_{\hat\mu\hat\nu}=\hat{\mathcal{D}}_{\hat\mu}V^a_{\ \hat\nu}-\hat{\mathcal{D}}_{\hat\nu}V^a_{\ \hat\mu}-\mathrm{i}\overline{\Psi}_{A[\hat\mu}\Gamma^a\Psi_{A\hat\nu]}=0$, the contribution of fermions (gravitini and conformini) can be taken into account as contorsion on spacetime,

$$\hat\omega^{ab}=\mathring{\omega}^{ab}+C^{ab}\,,\quad C^{ab}=C^{ab}{}_{\hat\mu}\,dx^{\hat\mu}\,.\tag{A.2.1}$$

We now evaluate how fermions contribute to the contorsion by using the condition of vanishing supertorsion. From the decomposition $\hat{\mathcal{D}}_{\hat{\mu}} V^a_{\ \hat{\nu}} = \mathring{\mathcal{D}}_{\hat{\mu}} V^a_{\ \hat{\nu}} + C^a_{\ \hat{\nu}\hat{\mu}}$, we find

$$\hat{\mathbf{R}}^a_{\hat{\mu}\hat{\nu}} = 0 \quad \Leftrightarrow \quad C_{\hat{\lambda}[\hat{\mu}\hat{\nu}]} = -\frac{\mathrm{i}}{2}\, \overline{\Psi}^A_{\hat{\mu}} \Gamma_{\hat{\lambda}} \Psi_{A\hat{\nu}} \,. \tag{A.2.2}$$

The solution is

$$C_{\hat{\lambda}\hat{\mu}\hat{\nu}} = \frac{\mathrm{i}}{2}\, \overline{\Psi}^A_{\hat{\lambda}} \Gamma_{\hat{\mu}} \Psi_{A\hat{\nu}} - \frac{\mathrm{i}}{2}\, \overline{\Psi}^A_{\hat{\mu}} \Gamma_{\hat{\lambda}} \Psi_{A\hat{\nu}} + \frac{\mathrm{i}}{2}\, \overline{\Psi}^A_{\hat{\lambda}} \Gamma_{\hat{\nu}} \Psi_{A\hat{\mu}} \,, \tag{A.2.3}$$

which can be restated in the following way

$$C^{ab}_{\ \ \hat{\mu}} = \frac{\mathrm{i}}{2}\, V^{\hat{\nu}a} \overline{\Psi}^A_{\hat{\nu}} \Gamma^b \Psi_{A\hat{\mu}} - \frac{\mathrm{i}}{2}\, V^{\hat{\nu}b} \overline{\Psi}^A_{\hat{\nu}} \Gamma^a \Psi_{A\hat{\mu}} + \frac{\mathrm{i}}{2}\, V^{\hat{\nu}a} V^{\hat{\lambda}b} V_{c\hat{\mu}}\, \overline{\Psi}^A_{\hat{\nu}} \Gamma^c \Psi_{A\hat{\lambda}} \,. \tag{A.2.4}$$

Note that, since $\overline{\Psi}^A_{\hat{\nu}} \Gamma^c \Psi_{A\hat{\lambda}} = -\overline{\Psi}^A_{\hat{\lambda}} \Gamma^c \Psi_{A\hat{\nu}}$, the tensor $C^{ab}_{\ \ \hat{\mu}}$ is explicitly antisymmetric in $[ab]$. In order to determine the radial dependence of the spin connection as it approaches the boundary, we express each component of the contorsion in terms of the fermionic fields regular on $\partial\mathcal{M}$

$$\begin{aligned}
C^{i3}_{\ \ z} &= \hat{E}^{\mu i} \left(\overline{\varphi}^A_{+\mu} \varphi_{A-z} - \frac{z^2}{\ell^2}\, \overline{\varphi}^A_{-\mu} \varphi_{A+z} \right) + \frac{\mathrm{i}}{2} \left(\overline{\varphi}^A_{-z} \Gamma^i \varphi_{A-z} + \frac{z^2}{\ell^2}\, \overline{\varphi}^A_{+z} \Gamma^i \varphi_{A+z} \right) , \\
C^{ij}_{\ \ z} &= \frac{\mathrm{i}z}{\ell}\, \hat{E}^{\mu[i} \left(\overline{\varphi}^A_{+\mu} \Gamma^{j]} \varphi_{A+z} + \overline{\varphi}^A_{-\mu} \Gamma^{j]} \varphi_{A-z} \right) + \frac{z}{2\ell}\, \hat{E}^{\mu i}\, \hat{E}^{\nu j} \left(\overline{\varphi}^A_{-\mu} \varphi_{A+\nu} - \overline{\varphi}^A_{+\mu} \varphi_{A-\nu} \right) , \\
C^{i3}_{\ \ \mu} &= \frac{z}{2\ell}\, \hat{E}^{\nu i} \left(\overline{\varphi}^A_{+\nu} \varphi_{A-\mu} - \overline{\varphi}^A_{-\nu} \varphi_{A+\mu} \right) + \frac{\mathrm{i}z}{2\ell} \left(\overline{\varphi}^A_{+z} \Gamma^i \varphi_{A+\mu} + \overline{\varphi}^A_{-z} \Gamma^i \varphi_{A-\mu} \right) \\
&\quad - \frac{\mathrm{i}z}{2\ell}\, \hat{E}^{\nu i} \hat{E}_{j\mu} \left(\overline{\varphi}^A_{+\nu} \Gamma^j \varphi_{A+z} + \overline{\varphi}^A_{-\nu} \Gamma^j \varphi_{A-z} \right) , \qquad (\text{A.2.5}) \\
C^{ij}_{\ \ \mu} &= \mathrm{i}\hat{E}^{\nu[i} \left(\overline{\varphi}^A_{+\nu} \Gamma^{j]} \varphi_{A+\mu} + \frac{z^2}{\ell^2}\, \overline{\varphi}^A_{-\nu} \Gamma^{j]} \varphi_{A-\mu} \right) + \frac{\mathrm{i}}{2}\, \hat{E}^{\nu i} \hat{E}^{\lambda j} \hat{E}_{k\mu} \left(\overline{\varphi}^A_{+\nu} \Gamma^k \varphi_{A+\lambda} + \frac{z^2}{\ell^2}\, \overline{\varphi}^A_{-\nu} \Gamma^k \varphi_{A-\lambda} \right) .
\end{aligned}$$

From (2.1.21), we find for the full spin connection

$$\begin{aligned}
\hat{\omega}^{i3}_z &= \left(\overline{\varphi}^{Ai}_+ + \frac{\mathrm{i}}{2}\, \overline{\varphi}^A_{-z} \Gamma^i \right) \varphi_{A-z} + \frac{z^2}{\ell^2} \left(-\overline{\varphi}^{Ai}_- + \frac{\mathrm{i}}{2} \overline{\varphi}^A_{+z} \Gamma^i \right) \varphi_{A+z} \,, \\
\hat{\omega}^{ij}_z &= \frac{z}{\ell} \left(\mathrm{i}\overline{\varphi}^{A[i}_+ \Gamma^{j]} \varphi_{A+z} + \mathrm{i}\overline{\varphi}^{A[i}_- \Gamma^{j]} \varphi_{A-z} + \overline{\varphi}^{A[i}_- \varphi^{j]}_{A+} \right) , \\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\, \hat{E}^i_{\ \mu} - \frac{1}{2}\, k_{\mu\nu} \hat{E}^{\nu i} + \frac{z}{2\ell} \Big(\overline{\varphi}^{Ai}_+ \varphi_{A-\mu} - \overline{\varphi}^{Ai}_- \varphi_{A+\mu} + \mathrm{i}\overline{\varphi}^A_{+z} \Gamma^i \varphi_{A+\mu} \\
&\qquad - \mathrm{i}\overline{\varphi}^{Ai}_+ \Gamma_\mu \varphi_{A+z} + \mathrm{i}\overline{\varphi}^A_{-z} \Gamma^i \varphi_{A-\mu} - \mathrm{i}\overline{\varphi}^{Ai}_- \Gamma_\mu \varphi_{A-z} \Big) , \qquad (\text{A.2.6}) \\
\hat{\omega}^{ij}_\mu &= \mathring{\omega}^{ij}_\mu + \mathrm{i}\overline{\varphi}^{A[i}_+ \Gamma^{j]} \varphi_{A+\mu} + \frac{\mathrm{i}}{2}\, \overline{\varphi}^{Ai}_+ \Gamma_\mu \varphi^j_{A+} + \frac{z^2}{\ell^2} \left(\mathrm{i}\overline{\varphi}^{A[i}_- \Gamma^{j]} \varphi_{A-\mu} + \frac{\mathrm{i}}{2}\, \overline{\varphi}^{Ai}_- \Gamma_\mu \varphi^j_{A-} \right) .
\end{aligned}$$

Therefore, the $\mathcal{O}(1/z)$ term of the connection is not modified by the spinors. This is consistent with the asymptotically AdS behaviour of the extrinsic curvature, being proportional to the induced metric.

The most general gauge fixing, with $\Psi_{\pm z} \neq 0$, is

$$\hat{\omega}^{i3}_z = w^i(x,z) \,,$$

$$\hat{\omega}^{ij}_{z} \;=\; \frac{z}{\ell}\, w^{ij}(x,z)\,, \tag{A.2.7}$$

where $w^i, w^{ij} = \mathcal{O}(1)$ and the boundary fields are

$$\begin{aligned}
\hat{\omega}^{i3}_{\mu} &= \frac{1}{z}\, E^i{}_{\mu} - \frac{z}{\ell^2}\, \tilde{S}^i{}_{\mu} - \frac{2z^2}{\ell^3}\, \tilde{\tau}^i{}_{\mu} + \mathcal{O}(z^3)\,,\\
\hat{\omega}^{ij}_{\mu} &= \omega^{ij}_{\mu} + \frac{z}{\ell}\omega^{ij}_{\mu(1)} + \frac{z^2}{\ell^2}\, \omega^{ij}_{(2)\mu} + \frac{z^3}{\ell^3}\, \omega^{ij}_{(3)\mu} + \mathcal{O}(z^4)\,,
\end{aligned} \tag{A.2.8}$$

where now $S^i{}_{\mu} \neq \tilde{S}^i{}_{\mu}$, $\tau^i{}_{\mu} \neq \tilde{\tau}^i{}_{\mu}$ and $\omega^{ij}_{\mu} \neq \mathring{\omega}^{ij}_{\mu}$.
As particular cases, let us notice that, when $\Psi^A_{-z} = 0$ and $\Psi^A_{+z} \neq 0$, the behaviour (A.2.6) yields $w^i = \mathcal{O}(z^2)$, while all the other components behave in the same way. Furthermore, if we set to zero both $\Psi^A_{\pm z} = 0$, we obtain $w^i = 0$ exactly.
The w^i and w^{ij} behaviour is summarised in the table (2.3.12).

The supercurvatures

In this Section we evaluate, for the most general gauge fixings, the first contributions in the asymptotic expansion of the super field strengths, decomposing them with respect to a worldvolume basis on the four dimensional spacetime. Let us generically denote the 2-form supercurvatures by $\hat{\mathbf{R}}^{\Lambda} = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^{a}, \hat{\boldsymbol{\rho}}^{A}, \hat{\mathbf{F}}\}$,

$$\hat{\mathbf{R}}^{\Lambda} = \frac{1}{2}\, \hat{\mathbf{R}}^{\Lambda}_{\hat{\mu}\hat{\nu}}\, \mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}} = \frac{1}{2}\, \hat{\mathbf{R}}^{\Lambda}_{\mu\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} + \hat{\mathbf{R}}^{\Lambda}_{\mu z}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}z\,. \tag{A.2.9}$$

We use the following notation for the supercurvature expansion,

$$\hat{\mathbf{R}}^{\Lambda}_{\hat{\mu}\hat{\nu}} = \sum_{n=n_{\min}}^{\infty} \left(\frac{z}{\ell}\right)^n \hat{\mathbf{R}}^{\Lambda}_{(n)\hat{\mu}\hat{\nu}}\,, \tag{A.2.10}$$

where $n_{\min}$ denotes the minimal power of $\frac{z}{\ell}$ in the expansion, that is the order of the most divergent term. Our covariant derivatives $\hat{\mathcal{D}}$ and $\mathcal{D}$, acting as exterior covariant derivatives, include only the spin connection.

From the supertorsion constraint $\hat{\mathbf{R}}^{a}_{\hat{\mu}\hat{\nu}} = 2\,\hat{\mathcal{D}}_{[\hat{\mu}} V^a{}_{\hat{\nu}]} - \mathrm{i}\,\overline{\Psi}^A_{\hat{\mu}} \Gamma^a \Psi^A_{\hat{\nu}} = 0$, we get

$$\hat{\mathbf{R}}^{a}_{\hat{\mu}\hat{\nu}} = \sum_{n=n_{\min}}^{\infty} \left(\frac{z}{\ell}\right)^n \hat{\mathbf{R}}^{a}_{(n)\hat{\mu}\hat{\nu}}(x) = 0\,, \tag{A.2.11}$$

and find the following expansion coefficients in terms of the boundary quantities,

$$\begin{aligned}
\hat{\mathbf{R}}^{i}_{(-1)\mu\nu} &= \mathbf{R}^{i}_{\mu\nu} \;=\; 2\,\mathcal{D}_{[\mu} E^i{}_{\nu]} - \mathrm{i}\,\overline{\psi}^A_{+[\mu} \gamma^i \psi^A_{\nu]+} = 0\,,\\
\hat{\mathbf{R}}^{i}_{(0)\mu\nu} &= 2\,\omega^{ij}_{(1)[\mu} E_{j|\nu]} - 2\,\mathrm{i}\,\overline{\zeta}^A_{+[\mu} \gamma^i \psi^A_{\nu]+} = 0\,,\\
\hat{\mathbf{R}}^{i}_{(1)\mu\nu} &= 2\,\mathcal{D}_{[\mu} S^i{}_{\nu]} + 2\,\omega^{ij}_{(2)[\mu} E_{j|\nu]}\\
&\quad -\mathrm{i}\left(\overline{\zeta}^A_{+[\mu} \gamma^i \zeta^A_{\nu]+} + 2\,\overline{\Pi}^A_{+[\mu} \gamma^i \psi^A_{\nu]+} + \overline{\psi}^A_{-[\mu} \gamma^i \psi^A_{\nu]-}\right) = 0\,,\\
\hat{\mathbf{R}}^{i}_{(2)\mu\nu} &= 2\,\mathcal{D}_{[\mu} \tau^i{}_{\nu]} + 2\,\omega^{ij}_{(1)[\mu} S_{j|\nu]} + 2\,\omega^{ij}_{(3)[\mu} E_{j|\nu]}
\end{aligned} \tag{A.2.12}$$

$$-2\,\mathrm{i}\left(\overline{\zeta}^A_{+[\mu}\gamma^i\Pi^A_{\nu]+}+\overline{\mho}^A_{+[\mu}\gamma^i\psi^A_{\nu]+}+\overline{\zeta}^A_{-[\mu}\gamma^i\psi^A_{\nu]-}\right)=0\,,$$

where we identified $\mho^A_{+\mu}=\psi^A_{(3)+\mu}$. Note that the last equation gives the expression for $\omega^{ij}_{(3)\mu}$ in the supersymmetric case. The next supertorsion components to be expanded are $\hat{\mathbf{R}}^i_{\mu z}$, for which we obtain

$$\begin{aligned}\hat{\mathbf{R}}^i_{(0)\mu z} &= \frac{1}{2\ell}\left(\tilde{S}^i{}_\mu-S^i{}_\mu\right)-\frac{1}{2}\,w^{ij}_{(0)}E_{j\mu}\\ &\quad -\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\gamma^i\psi_{A+z}+\overline{\psi}_{A-\mu}\gamma^i\psi_{A-z}\right)=0\,, \qquad\qquad (A.2.13)\\ \hat{\mathbf{R}}^i_{(1)\mu z} &= \frac{1}{\ell}\left(\tilde{\tau}^i{}_\mu-\tau^i{}_\mu\right)-\frac{1}{2}\,w^{ij}_{(1)}E_{j\mu}-\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\gamma^i\zeta_{A+z}\right.\\ &\quad \left.+\,\overline{\psi}_{A-\mu}\gamma^i\zeta_{A-z}+\overline{\zeta}_{A+\mu}\gamma^i\psi_{A+z}+\overline{\zeta}_{A-\mu}\gamma^i\psi_{A-z}\right)=0\,. \qquad\qquad (A.2.14)\end{aligned}$$

On the other hand, for the $\hat{\mathbf{R}}^3$ components restricted to $\partial\mathcal{M}$ we find

$$\begin{aligned}\hat{\mathbf{R}}^3_{(0)\mu\nu} &= -\frac{2}{\ell}\left(S_{[\mu\nu]}-\tilde{S}_{[\nu\mu]}\right)-2\,\mathrm{i}\,\overline{\psi}_{A+[\mu}\psi_{A-\nu]}=0\,,\\ \hat{\mathbf{R}}^3_{(1)\mu\nu} &= -\frac{2}{\ell}\left(\tau_{[\mu\nu]}-2\,\tilde{\tau}_{[\nu\mu]}\right)-2\,\mathrm{i}\left(\overline{\psi}_{A+[\mu}\zeta_{A-\nu]}+\overline{\zeta}_{A+[\mu}\psi_{A-\nu]}\right)=0\,, \qquad (A.2.15)\end{aligned}$$

and projected to $\mathrm{d}x^\mu\wedge\mathrm{d}z$ we have

$$\begin{aligned}\hat{\mathbf{R}}^3_{(-1)\mu z} &= \frac{1}{2}\,w^i_{(0)}E_{i\mu}-\frac{\mathrm{i}}{2}\,\overline{\psi}_{A+\mu}\psi_{A-z}=0\,,\\ \hat{\mathbf{R}}^3_{(0)\mu z} &= \frac{1}{2}\,w^i_{(1)}E_{i\mu}-\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\zeta_{A-z}+\overline{\zeta}_{A+\mu}\psi_{A-z}\right)=0\,,\\ \hat{\mathbf{R}}^3_{(1)\mu z} &= \frac{1}{2}\,w^i_{(0)}S_{i\mu}+\frac{1}{2}\,w^i_{(2)}E_{i\mu}-\frac{\mathrm{i}}{2}\,\overline{\psi}_{A-\mu}\psi_{A+z}\\ &\quad -\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\Pi_{A-z}+\overline{\zeta}_{A+\mu}\zeta_{A-z}+\overline{\Pi}_{A+\mu}\psi_{A-z}\right)=0\,, \qquad (A.2.16)\end{aligned}$$

where $\Pi^A_{-z}=\psi^A_{(2)-z}$. The latter equation gives the expression for $w^i_{(2)}$.

Focusing on the AdS supercurvature, $\hat{\mathbf{R}}^{ij}=\mathcal{R}^{ij}+\frac{4}{\ell^2}\,V^{[i}_+V^{j]}_- -\frac{1}{\ell}\left(\overline{\Psi}^A_+\Gamma^{ij}\Psi^A_-+\overline{\Psi}^A_-\Gamma^{ij}\Psi^A_+\right)$ yields

$$\begin{aligned}\hat{\mathbf{R}}^{ij}_{(0)\mu\nu} &= \mathbf{R}^{ij}_{\mu\nu} = 2\,\mathcal{R}^{ij}_{\mu\nu}-4\,E^{[i}{}_{[\mu}\mathcal{S}^{j]}{}_{\nu]}-\frac{2}{\ell}\,\overline{\psi}^A_{-\mu}\gamma^{ij}\psi^A_{+\nu}=0\,, \qquad (A.2.17)\\ \hat{\mathbf{R}}^{ij}_{(1)\mu\nu} &= 2\,\mathcal{D}_{[\mu}\omega^{ij}_{(1)|\nu]}-\frac{4}{\ell^2}\,E^{[i}{}_{[\mu}(\tau^{j]}{}_{\nu]}+2\tilde{\tau}^{j]}{}_{\nu]})\\ &\quad -\frac{2}{\ell}\left(\overline{\psi}^A_{-[\mu}\gamma^{ij}\zeta^A_{+\nu]}+\overline{\psi}^A_{+[\mu}\gamma^{ij}\zeta^A_{-\nu]}\right)\,,\\ \hat{\mathbf{R}}^{ij}_{(-1)\mu z} &= E^{[i}{}_\mu w^{j]}_{(0)}-\frac{1}{2\ell}\,\overline{\psi}^A_{+\mu}\gamma^{ij}\psi^A_{-z}\,,\\ \hat{\mathbf{R}}^{ij}_{(0)\mu z} &= -\frac{1}{2\ell}\left(-2\ell\,E^{[i}{}_\mu w^{j]}_{(1)}+\omega^{ij}_{(1)\mu}+\overline{\psi}^A_{+\mu}\gamma^{ij}\zeta^A_{-z}\right)\,,\\ \hat{\mathbf{R}}^{ij}_{(1)\mu z} &= \frac{1}{2}\,\mathcal{D}_\mu w^{ij}_{(0)}-\frac{1}{\ell}\,\omega^{ij}_{(2)\mu}-\tilde{S}^{[i}{}_\mu w^{j]}_{(0)}\end{aligned}$$

$$-\frac{1}{2\ell}\left(\overline{\psi}^A_{-\mu}\gamma^{ij}\psi^A_{+z}+\overline{\psi}^A_{+\mu}\gamma^{ij}\Pi^A_{-z}+\overline{\Pi}^A_{+\mu}\gamma^{ij}\psi^A_{-z}\right).$$

Then, from $\hat{\mathbf{R}}^{i3}=\hat{\mathcal{D}}\hat{\omega}^{i3}-\frac{1}{\ell^2}V^iV^3-\frac{\mathrm{i}}{2\ell}\left(\overline{\Psi}^A_+\Gamma^i\Psi^A_+-\overline{\Psi}^A_-\Gamma^i\Psi^A_-\right)$, we find

$$\begin{aligned}
\hat{\mathbf{R}}^{i3}_{(-1)\mu\nu} &= \hat{\mathbf{R}}^{i}_{(-1)\mu\nu} = 0\,,\\
\hat{\mathbf{R}}^{i3}_{(0)\mu\nu} &= \hat{\mathbf{R}}^{i}_{(0)\mu\nu} = 0\,,\\
\hat{\mathbf{R}}^{i3}_{(1)\mu\nu} &= -\ell\,\mathcal{C}^i_{\mu\nu} = -\frac{2}{\ell}\mathcal{D}_{[\mu}\tilde{S}^i{}_{\nu]}+\frac{2}{\ell}\omega^{ij}_{(2)[\mu}E_{j|\nu]}+\frac{\mathrm{i}}{\ell}\left(\overline{\psi}^A_{-[\mu}\gamma^i\psi^A_{-\nu]}-\overline{\zeta}^A_{+[\mu}\gamma^i\zeta^A_{+\nu]}\right.\\
&\quad\left.-2\,\overline{\Pi}^A_{+[\mu}\gamma^i\psi^A_{+\nu]}\right),\\
\hat{\mathbf{R}}^{i3}_{(0)\mu z} &= \frac{1}{2}\mathcal{D}_\mu w^i_{(0)}+\frac{\mathrm{i}}{\ell}\overline{\psi}^A_{-\mu}\gamma^i\psi^A_{-z}\,,\\
\hat{\mathbf{R}}^{i3}_{(1)\mu z} &= \frac{1}{2}\mathcal{D}_\mu w^i_{(1)}+\omega^{(1)|ij}_\mu w_{(0)|j}+\frac{1}{2\,\ell^2}\left(2\tilde{\tau}^i{}_\mu+\tau^i{}_\mu\right)+\frac{1}{2\ell}w^{ij}_{(0)}S_{j|\mu}\\
&\quad+\frac{\mathrm{i}}{\ell}\left(\overline{\psi}^A_{-\mu}\gamma^i\zeta^A_{-z}+\overline{\zeta}^A_{-\mu}\gamma^i\psi^A_{-z}\right),
\end{aligned}\tag{A.2.18}$$

where we have also exploited the vanishing supertorsion equations (A.2.13) and (A.2.14).

As regards to the graviphoton super field strength $\hat{\mathbf{F}}=\mathrm{d}\hat{A}-2\,\epsilon^{AB}\overline{\Psi}_{+A}\Psi_{-B}$, we obtain

$$\begin{aligned}
\hat{\mathbf{F}}_{(0)\mu\nu} &= \mathbf{F}_{\mu\nu} = 2\,\partial_{[\mu}A_{\nu]}-4\,\epsilon_{AB}\overline{\psi}^A_{+[\mu}\psi^B_{-\nu]}=0\,, &\text{(A.2.19)}\\
\hat{\mathbf{F}}_{(1)\mu\nu} &= 2\,\partial_{[\mu}A_{(1)\nu]}-4\left(\overline{\psi}^A_{+[\mu}\zeta^B_{-\nu]}+\overline{\zeta}^A_{+[\mu}\psi^B_{-\nu]}\right)\epsilon_{AB}\,,\\
\hat{\mathbf{F}}_{(-1)\mu z} &= \frac{1}{2}\partial_\mu A_{(-1)z}-\overline{\psi}^A_{+\mu}\psi^B_{-z}\epsilon_{AB}\,,\\
\hat{\mathbf{F}}_{(0)\mu z} &= \frac{1}{2}\partial_\mu A_{(0)z}-\frac{1}{2\ell}A_{(1)\mu}-\overline{\psi}^A_{+\mu}\zeta^B_{-z}\epsilon_{AB}\,,\\
\hat{\mathbf{F}}_{(1)\mu z} &= \frac{1}{2}\partial_\mu A_{(1)z}-\frac{1}{\ell}A_{(2)\mu}-\left(\overline{\psi}^A_{-\mu}\psi^B_{+z}+\overline{\psi}^A_{+A\mu}\Pi^B_{-z}\right)\epsilon_{AB}\,.
\end{aligned}$$

Furthermore, the gravitini supercurvature $\hat{\boldsymbol{\rho}}_{+A}=\mathrm{d}\Psi_{+A}+\frac{1}{4}\hat{\omega}^{ij}\Gamma_{ij}\Psi_{+A}-\frac{1}{2\ell}\epsilon_{AB}\hat{A}\Psi_{+B}+\frac{\mathrm{i}}{\ell}V^i_+\Gamma_i\Psi_{-A}-\frac{1}{2\ell}\Psi_{+A}V^3$ leads to

$$\begin{aligned}
\hat{\boldsymbol{\rho}}_{(-1/2)+A\mu\nu} &= \boldsymbol{\rho}_{+A\mu\nu} = 2\,\nabla_{[\mu}\psi^A_{+\nu]}+\frac{2\mathrm{i}}{\ell}\gamma_{[\mu}\psi^A_{-\nu]}=0\,, &\text{(A.2.20)}\\
\hat{\boldsymbol{\rho}}_{(1/2)+A\mu\nu} &= 2\,\nabla_{[\mu}\zeta_{+A\nu]}+\frac{2}{\ell}\gamma_{[\mu}\zeta_{-A\nu]}+\frac{1}{2}\gamma_{ij}\omega^{ij}_{(1)[\mu}\psi_{+A\nu]}-\frac{1}{\ell}A_{(1)[\mu}\psi_{+B\nu]}\epsilon_{AB}\,,\\
\hat{\boldsymbol{\rho}}_{(-3/2)+A\mu z} &= \frac{\mathrm{i}}{2\ell}\left(\gamma_\mu\psi_{-Az}-\frac{\mathrm{i}}{2}A_{(-1)z}\psi_{+B\mu}\epsilon_{AB}\right),\\
\hat{\boldsymbol{\rho}}_{(-1/2)+A\mu z} &= \frac{\mathrm{i}}{2\ell}\gamma_\mu\zeta_{-Az}+\frac{1}{4\ell}\left(A_{(0)z}\psi_{+B\mu}+A_{(-1)z}\zeta_{+B\mu}\right)\epsilon_{AB}-\frac{1}{2\ell}\zeta_{+A\mu}\,,\\
\hat{\boldsymbol{\rho}}_{(1/2)+A\mu z} &= \frac{1}{2}\nabla_\mu\psi_{+Az}-\frac{1}{8}w^{ij}_{(0)}\gamma_{ij}\psi_{+A\mu}+\frac{\mathrm{i}}{4\ell}\left(S^i{}_\mu-\tilde{S}^i{}_\mu\right)\gamma_i\psi_{-Az}-\frac{1}{\ell}\Pi_{+A\mu}\\
&\quad-\frac{\mathrm{i}}{4}w^i_{(0)}\gamma_i\psi_{-A\mu}+\frac{\mathrm{i}}{2\ell}\gamma_\mu\Pi_{-Az}+\frac{1}{4\ell}\left(A_{(1)z}\psi_{+B\mu}+A_{(-1)z}\Pi_{+B\mu}+A_{(0)z}\zeta_{+B\mu}\right)\epsilon_{AB}\,.
\end{aligned}$$

Eventually, by exploiting the negatively graded fermionic supercurvature $\hat{\boldsymbol{\rho}}_{-A}=\mathrm{d}\Psi_{-A}+\frac{1}{4}\hat{\omega}^{ij}\Gamma_{ij}\Psi_{-A}-\frac{1}{2\ell}\epsilon_{AB}\hat{A}\Psi_{-B}-\frac{\mathrm{i}}{\ell}V^i_-\Gamma_i\Psi_{+A}+\frac{1}{2\ell}\Psi_{-A}V^3$, we are left with

$$\hat{\boldsymbol{\rho}}_{(1/2)-A\mu\nu} = \Omega_{A\mu\nu} = 2\,\nabla_{[\mu}\psi_{-A\nu]}-\mathrm{i}\,\ell\,\gamma_i\psi_{+A[\mu}\mathcal{S}^i{}_{\nu]}\,,$$

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}_{(3/2)-A\mu\nu} &= \nabla_{[\mu}\zeta_{-A\nu]} - \frac{\mathrm{i}}{2}\,\ell\,\gamma_i\zeta_{+A[\mu}\mathcal{S}^i{}_{\nu]} + \frac{1}{4}\,\omega^{ij}_{(1)|[\mu}\gamma_{ij}\psi_{-A\nu]} \\
&\quad -\frac{1}{2\ell}\,A_{(1)[\mu}\psi_{-B\nu]}\epsilon_{AB} + \frac{\mathrm{i}}{2\ell}\,\left(\tau^i{}_{[\mu} + 2\,\tilde{\tau}^i{}_{[\mu}\right)\gamma_i\psi_{+A\nu]}\,, \\
\hat{\boldsymbol{\rho}}_{(-1/2)-A\mu z} &= \frac{1}{2}\,\nabla_\mu\psi_{-Az} + \frac{1}{4\ell}\,A_{(-1)z}\,\epsilon_{AB}\psi_{-B\mu} + \frac{\mathrm{i}}{4}\gamma_i w^i_{(0)}\psi_{+A\mu}\,, \\
\hat{\boldsymbol{\rho}}_{(1/2)-A\mu z} &= \frac{1}{2}\,\nabla_\mu\zeta_{-Az} + \frac{1}{4\ell}\,A_{(0)z}\,\epsilon_{AB}\psi_{-B\mu} + \frac{\mathrm{i}}{4}\gamma_i w^i_{(1)}\psi_{+A\mu} \\
&\quad -\frac{1}{2\ell}\,\zeta_{-A\mu} + \frac{1}{4\ell}\,A_{(-1)z}\zeta_{-B\mu}\epsilon_{AB}\,.
\end{aligned}
\tag{A.2.21}
$$

We observe that the $\hat{\mathbf{R}}^\Lambda_{(n_{\min})\mu\nu}$ components of $\hat{\mathbf{R}}^\Lambda = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^a, \hat{\boldsymbol{\rho}}^A, \hat{\mathbf{F}}\}$ define the curvatures $\{\mathbf{R}^{ij}, \mathbf{R}^i, \boldsymbol{\rho}^A, \mathbf{F}, \mathcal{C}^i, \Omega^A\}$ of the $\mathcal{N}=2$ superconformal group OSp(2|4) discussed in Section 2.5 and given by (2.5.6). One can naively expect that they all vanish in the vacuum with the OSp(2|4) isometries in a superconformal theory on the three dimensional boundary. However, we obtain $\hat{\mathbf{R}}^\Lambda_{(n_{\min})\mu\nu} = 0$ for all the curvatures except the ones with negative grading, $\hat{\mathbf{R}}^{i3}_{\mu\nu}$ and $\hat{\boldsymbol{\rho}}_{-A\mu\nu}$, where we find instead that the equations (2.2.15) lead to the weaker condition (2.4.37).

Equations of motion of the graviphoton

Here, we analyse the relation between the gauge fixing and the asymptotic behaviour of the fields, by using the radial field equations. In the previous Section, a similar problem was discussed for the spin connection by exploiting the vanishing supertorsion.

The radial evolution of the graviphoton is given by the respective field equation (2.2.16), which, in components, with the definition of Hodge star dual (2.2.11), has the form

$$
\hat{\mathcal{D}}_{\hat\nu}\hat{\mathbf{F}}^{\hat\nu\hat\mu} = \frac{\mathrm{i}}{e}\,\epsilon^{\hat\mu\hat\nu\hat\lambda\hat\rho}\,\overline{\Psi}^A_{\hat\nu}\Gamma^5\hat{\boldsymbol{\rho}}^B_{\hat\lambda\hat\rho}\,\epsilon_{AB}\,. \tag{A.2.22}
$$

By using the conventions (2.1.9) and (2.1.13), the component $\hat\mu = \mu$ acquires the form

$$
\hat{\mathcal{D}}_\nu\hat{\mathbf{F}}^{\nu\mu} + \hat{\mathcal{D}}_z\hat{\mathbf{F}}^{z\mu} = -\frac{\mathrm{i}}{e}\,\epsilon^{\mu\nu\lambda}\left(2\overline{\Psi}^A_\nu\Gamma_5\hat{\boldsymbol{\rho}}^B_{\lambda z} + \overline{\Psi}^A_z\Gamma_5\hat{\boldsymbol{\rho}}^B_{\nu\lambda}\right)\epsilon_{AB}\,. \tag{A.2.23}
$$

For convenience, we factorise the relevant field strength components as

$$
\begin{aligned}
\hat{\mathbf{F}}^{z\mu} &= -\left(\tfrac{z}{\ell}\right)^4 g^{\mu\nu}\hat{\mathbf{F}}_{z\nu}\,, & \hat{\boldsymbol{\rho}}^A_{\mu z\pm} &= \left(\tfrac{z}{\ell}\right)^{\pm\frac{1}{2}}\Xi^A_{\mu\pm}\,, \\
\hat{\mathbf{F}}^{\mu\nu} &= \left(\frac{z}{\ell}\right)^4 F^{\mu\nu}\,, & \hat{\boldsymbol{\rho}}^A_{\mu\nu\pm} &= \left(\tfrac{z}{\ell}\right)^{\mp\frac{1}{2}}\Xi^A_{\mu\nu\pm}\,,
\end{aligned}
\tag{A.2.24}
$$

where $\hat{\mathbf{F}}_{\mu\nu} = F_{\mu\nu}$ and the tensors $\hat{\mathbf{F}}_{z\mu}$, $F_{\mu\nu}$, $\Xi^A_{\mu\pm}$ and $\Xi^A_{\mu\nu\pm}$ have to be expanded in power series in z. The metric $g_{\mu\nu}(x,z)$ and its inverse $g^{\mu\nu}$ rise and lower the spacetime indices on $\partial\mathcal{M}$. Recalling the FG metric (2.1.1) and the auxiliary tensor $k_{\mu\nu} = \partial_z g_{\mu\nu}$, we get

$$
\begin{aligned}
\hat{\Gamma}^\mu_{\nu z} &= -\frac{1}{z}\,\delta^\mu_\nu + \frac{1}{2}\,k^\mu_\nu\,, & \hat{\Gamma}^\mu_{zz} &= 0 = \hat{\Gamma}^z_{z\mu}\,, \\
\hat{\Gamma}^z_{\mu\nu} &= -\frac{1}{z}\,g_{\mu\nu} + \frac{1}{2}\,k_{\mu\nu}\,, & \hat{\Gamma}^z_{zz} &= -\frac{1}{z}\,,
\end{aligned}
\tag{A.2.25}
$$

the radial graviphoton equation becoming

$$\begin{aligned} &\mathcal{D}_\nu F^{\nu\mu} - \left(k^{\mu\nu} - \frac{k}{2}\, g^{\mu\nu}\right)\hat{\mathbf{F}}_{\nu z} + g^{\mu\nu}\partial_z \hat{\mathbf{F}}_{\nu z} \\ = &\; -\frac{\mathrm{i}}{\hat{e}_3}\,\epsilon^{\mu\nu\lambda}\left(2\,\overline{\varphi}^A_{+\nu}\Gamma_5\Xi^B_{\lambda+} + 2\,\overline{\varphi}^A_{-\nu}\Gamma_5\Xi^B_{\lambda-} + \overline{\varphi}^A_{+z}\Gamma_5\Xi^B_{\nu\lambda+} + \overline{\varphi}^A_{-z}\Gamma_5\Xi^B_{\nu\lambda-}\right)\epsilon_{AB}\,. \end{aligned} \tag{A.2.26}$$

Now we compute $\hat{\mathbf{F}}_{\mu z}$, $\Xi^A_{\mu\pm}$ and $\Xi^A_{\mu\nu\pm}$, defined in (A.2.24). Evaluation of the components

$$\begin{aligned} \hat{\mathbf{F}}^{\hat\mu\hat\nu} &= \hat{g}^{\hat\mu\hat\alpha}\hat{g}^{\hat\nu\hat\beta}\left(\partial_{\hat\alpha}\hat{A}_{\hat\beta} - \partial_{\hat\beta}\hat{A}_{\hat\alpha} - 2\epsilon_{AB}\,\overline{\Psi}^A_{\hat\alpha}\Psi^B_{\hat\beta}\right), \\ \hat{\boldsymbol{\rho}}^A_{\hat\mu\hat\nu} &= 2\hat{\mathcal{D}}_{[\hat\mu}\Psi^A_{\hat\nu]} - \frac{1}{\ell}\,\epsilon_{AB}\hat{A}_{[\hat\mu}\Psi^B_{\hat\nu]} - \frac{\mathrm{i}}{\ell}\,\Gamma_a\Psi^A_{[\hat\mu}V^a{}_{\hat\nu]} \end{aligned} \tag{A.2.27}$$

leads to

$$\begin{aligned} \hat{\mathbf{F}}_{\mu z} &= \partial_\mu\hat{A}_z - \partial_z A_\mu - \frac{2\ell}{z}\,\epsilon_{AB}\,\overline{\varphi}^A_{+\mu}\varphi^B_{z-} - \frac{2z}{\ell}\,\epsilon_{AB}\,\overline{\varphi}^A_{-\mu}\varphi^B_{z+}\,, \\ F^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}\left(\mathcal{F}_{\alpha\beta} - 4\epsilon_{AB}\,\overline{\varphi}^A_{+\alpha}\varphi^B_{-\beta}\right) = 0 \end{aligned} \tag{A.2.28}$$

and, by means of the rescalings (2.3.6), we get

$$\begin{aligned} \Xi^A_{\mu\pm} &= \mathcal{D}_\mu\varphi^A_{\pm z} - \frac{1}{4}\left(\frac{z}{\ell}\right)^{1\mp1} w^{ij}\Gamma_{ij}\varphi^A_{\pm\mu} - \left(\frac{z}{\ell}\right)^{\mp1}\partial_z\varphi^A_{\pm\mu} - \frac{1}{2\ell}\,\epsilon_{AB}\,A_\mu\varphi^B_{\pm z} \\ &\quad \mp\frac{\mathrm{i}}{2}\,w^i\Gamma_i\varphi^A_{\mp\mu} + \frac{1}{2\ell}\left(\frac{z}{\ell}\right)^{\mp1}\epsilon_{AB}\,\hat{A}_z\varphi^B_{\pm\mu} \pm \frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\mp2} E^i_{\pm\mu}\Gamma_i\varphi^A_{\mp z}\,, \\ \Xi^A_{\mu\nu\pm} &= 2\mathcal{D}_{[\mu}\varphi^A_{\nu]\pm} \pm \frac{2\mathrm{i}}{\ell}\,E^i_{\pm[\mu}\Gamma_i\varphi^A_{\nu]\mp} - \frac{1}{\ell}\,\epsilon_{AB}\,A_{[\mu}\varphi^B_{\nu]\pm}\,. \end{aligned} \tag{A.2.29}$$

We also assume that the gauge fixing functions are

$$\begin{aligned} \hat{A}_z &= \frac{\ell}{z}\,A_{(-1)z} + A_{(0)z} + \frac{z}{\ell}\,A_{(1)z} + \mathcal{O}(z^3)\,, \\ \hat{A}_\mu &= \frac{\ell}{z}\,A_{(-1)\mu} + A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,, \\ \varphi^A_{+\mu} &= \varphi^A_{(0)+\mu} + \frac{z}{\ell}\,\varphi^A_{(1)+\mu} + \mathcal{O}(z^2)\,, \end{aligned} \tag{A.2.30}$$

allowing in general for linear terms, and we find

$$\begin{aligned} \hat{\mathbf{F}}_{\mu z} &= \frac{\ell}{z^2}\,A_{(-1)\mu} + \frac{\ell}{z}\left(\partial_\mu A_{(-1)z} - 2\epsilon_{AB}\,\overline{\varphi}^A_{(0)+\mu}\varphi^B_{(0)z-}\right) + \mathcal{O}(1)\,, \\ \Xi^A_{\mu+} &= \frac{\ell}{2z^2}\left(A_{(-1)z}\,\epsilon^{AB}\,\varphi_{B+\mu} + 2\mathrm{i}E^i{}_\mu\Gamma_i\varphi^A_{(0)-z}\right) + \frac{1}{z}\left(\frac{1}{2}\,\epsilon_{AB}\,A_{(0)z}\varphi^B_{(0)+\mu} - \varphi^A_{(1)+\mu}\right) + \mathcal{O}(1)\,, \\ &F^{\mu\nu},\ \Xi^A_{\mu-},\ \Xi^A_{\mu\nu\pm} = \mathcal{O}(1)\,. \end{aligned} \tag{A.2.31}$$

Remembering that $k_{\mu\nu} = \mathcal{O}(z)$, the graviphoton equation (A.2.26) yields

$$\frac{\ell}{z^3}\ :\qquad A_{(-1)\mu} = 0\,,$$

$$
\begin{aligned}
\frac{\ell}{z^2} \quad &: \qquad \partial_\mu A_{(-1)z} = \left(2\,\overline{\varphi}^A_{(0)+\mu} - \frac{1}{e_3}\, g_{(0)\mu\sigma}\, \epsilon^{\sigma\nu\lambda} E^i_\lambda \overline{\varphi}^A_{(0)+\nu} \Gamma_5 \Gamma_i\right) \epsilon_{AB} \varphi^B_{(0)-z}\,, \\
\frac{1}{z} \quad &: \qquad 0 = \epsilon^{\mu\nu\lambda}\, \overline{\varphi}^A_{(0)+\nu} \Gamma_5 \left(\frac{1}{2}\, A_{(0)z} \varphi^A_{(0)+\mu} + \varphi^B_{(1)+\mu} \epsilon_{AB}\right)\,, \qquad \text{(A.2.32)}
\end{aligned}
$$

and all other terms are finite. In order to obtain (A.2.32), we used the fact that the term $\overline{\varphi}^A_{+\nu}\Gamma_5 \varphi_{A+\lambda}$ is symmetric in $(\nu\lambda)$, which makes it vanish when contracted with $\epsilon^{\sigma\nu\lambda}$.
From the last equation in (A.2.32), when $\varphi^A_{(0)+\mu} \neq 0$, we can choose a particular solution $A_{(0)z} = 0$, $\varphi^A_{(1)+\mu} \equiv \binom{\zeta^A_{\mu+}}{0} = 0$, which is in agreement with (2.4.11) obtained in Section 2.4. This choice was also considered in [26], in the context of $\mathcal{N} = 1$ supergravity. We will show below (see (A.2.44)) that, in fact, this is the only solution if we assume the stronger condition (A.2.47) to hold. Then (A.2.30) implies

$$
\begin{aligned}
\hat{A}_z &= \frac{\ell}{z}\, A_{(-1)z} + \frac{z}{\ell}\, A_{(1)z} + \mathcal{O}(z^3)\,, \\
\hat{A}_\mu &= A_\mu + \frac{z}{\ell}\, A_{(1)\mu} + \frac{z^2}{\ell^2}\, A_{(2)\mu} + \mathcal{O}(z^3)\,, \\
\varphi^A_{+\mu} &= \varphi^A_{(0)+\mu} + \mathcal{O}(z^2)\,. \qquad \text{(A.2.33)}
\end{aligned}
$$

We also conclude that the gauge fixing functions $A_{(-1)z}$ and $\varphi^B_{(0)-z}$ are correlated, which is consistent with the table (2.3.12). Moreover, the boundary graviphoton does not acquire divergent terms of the form $1/z$, even when $\varphi^A_{(0)z-} \neq 0$. We have not considered the logarithmic terms here.

The graviphoton curvature behaves in the following way on the boundary,

$$
\begin{aligned}
\hat{\mathbf{F}}_{\mu z} &= \frac{\ell}{z}\left(\partial_\mu A_{(-1)z} - 2\epsilon_{AB}\, \overline{\varphi}^A_{(0)+\mu} \varphi^B_{(0)-z}\right) - \frac{1}{\ell}\, A_{(1)\mu} + \mathcal{O}(z)\,, \\
\hat{\mathbf{F}}_{\mu\nu} &= \mathcal{F}_{\mu\nu} - 4\epsilon_{AB}\, \overline{\varphi}^A_{+[\mu} \varphi^B_{-\nu]} = 0\,. \qquad \text{(A.2.34)}
\end{aligned}
$$

This shows the possibility to have the components $\hat{\mathbf{F}}_{\mu z} \neq 0$ on the boundary $z = 0, \mathrm{d}z = 0$, with a suitable gauge choice which changes the asymptotics.

Equations of motion of the gravitini

The equation of motion that describes the dynamics of gravitini (2.2.16) in components has the form

$$
0 = \epsilon^{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\tau}} \left(V^a{}_{\hat{\mu}} \Gamma_a \Gamma_5 \hat{\boldsymbol{\rho}}_{A\hat{\nu}\hat{\lambda}} + \frac{\mathrm{i}}{2}\, \epsilon_{AB} \hat{\mathbf{F}}_{\hat{\mu}\hat{\nu}} \Gamma_5 \Psi^B_{\hat{\lambda}}\right) + e\, \epsilon_{AB} \Psi^B_{\hat{\lambda}}\, \hat{\mathbf{F}}^{\hat{\lambda}\hat{\tau}}\,, \qquad \text{(A.2.35)}
$$

where the formula (2.2.11) was applied. The radial expansion of the gravitini is given by the components $\hat{\tau} = \mu$, which, with the conventions (2.1.9) and (2.1.13), leads to

$$
\begin{aligned}
0 \;=\; & \epsilon^{\mu\nu\lambda} \left(-V^3{}_z \Gamma^3 \Gamma_5 \hat{\boldsymbol{\rho}}_{A\nu\lambda} - 2V^i{}_\nu \Gamma_i \Gamma_5 \hat{\boldsymbol{\rho}}_{Az\lambda} + \frac{\mathrm{i}}{2}\, \epsilon_{AB} \hat{\mathbf{F}}_{\nu\lambda} \Gamma_5 \Psi^B_z + \mathrm{i}\, \epsilon_{AB} \hat{\mathbf{F}}_{z\nu} \Gamma_5 \Psi^B_\lambda\right) \\
& + e\, \epsilon_{AB} \left(\Psi^B_z\, \hat{\mathbf{F}}^{z\mu} + \Psi^B_\nu\, \hat{\mathbf{F}}^{\nu\mu}\right)\,. \qquad \text{(A.2.36)}
\end{aligned}
$$

Projecting it by means of $\mathbb{P}_\pm$, defined in (A.1.18), and applying the identities (A.1.19) and (A.1.20) from Appendix A.1, we find

$$\begin{aligned} 0 &= \epsilon^{\mu\nu\lambda}\left(\mp \mathrm{i} V^3{}_z \Gamma_5 \hat{\boldsymbol{\rho}}_{\mp A\nu\lambda} - 2V^i{}_\nu \Gamma_i \Gamma_5 \hat{\boldsymbol{\rho}}_{\pm Az\lambda} + \frac{\mathrm{i}}{2}\,\epsilon_{AB}\hat{\mathbf{F}}_{\nu\lambda}\Gamma_5\Psi^B_{\mp z} + \mathrm{i}\,\epsilon_{AB}\hat{\mathbf{F}}_{z\nu}\Gamma_5\Psi^B_{\mp\lambda}\right) \\ &\quad + e\,\epsilon_{AB}\left(\Psi^B_{\pm z}\,\hat{\mathbf{F}}^{z\mu} + \Psi^B_{\pm\nu}\,\hat{\mathbf{F}}^{\nu\mu}\right) . \end{aligned} \tag{A.2.37}$$

Now, we can use (A.2.24), (2.1.13), (2.3.11) and (2.4.27) to obtain the equation expressed in terms of the auxiliary quantities with known asymptotic behaviour,

$$\begin{aligned} 0 &= \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}-1} \epsilon^{\mu\nu\lambda}\left(\mp \mathrm{i}\,\Gamma_5 \Xi^A_{\nu\lambda\mp} + 2\hat{E}^i{}_\nu \Gamma_i\Gamma_5 \Xi^A_{\lambda\pm}\right) \\ &\quad + \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}} \epsilon_{AB}\left(-\mathrm{i}\,\epsilon^{\mu\nu\lambda}\Gamma_5\varphi^B_{\mp\lambda} + e_3\, g^{\mu\nu}\varphi^B_{\pm z}\right)\hat{\mathbf{F}}_{\nu z} \\ &\quad + \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}} \epsilon_{AB}\left(\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\lambda}F_{\nu\lambda}\Gamma_5\varphi^B_{\mp z} + e_3\, F^{\nu\mu}\varphi^B_{\pm\nu}\right) . \end{aligned} \tag{A.2.38}$$

All tensors appearing above are finite, except $\hat{\mathbf{F}}_{\mu z}$ and $\Xi^A_{\mu+}$. With this at hand and looking separately at the two projections, we identify the leading orders of the gravitini equations of motion. By requiring the most divergent terms to vanish (which are $(\ell/z)^{5/2}$ and $(\ell/z)^{3/2}$ in the two chiralities), we get

$$\begin{aligned} 0 &= \epsilon^{ijk}\left(A_{(-1)z}\,\epsilon_{AB}\,\Gamma_i\varphi^B_{(0)+\mu}E^\mu_j + 2\mathrm{i}\Gamma_{ij}\varphi_{A(0)-z}\right) , \\ 0 &= \epsilon^{\mu\nu\lambda}\left(\mathrm{i}\,\Xi^A_{(0)\nu\lambda+} - 2E^i{}_\nu\Gamma_i\Xi^A_{(0)\lambda-}\right) \\ &\quad + \epsilon_{AB}\left(-\mathrm{i}\,\epsilon^{\mu\nu\lambda}\varphi^B_{(0)+\lambda} + e_{3(0)}\, g^{\mu\nu}_{(0)}\Gamma_5\varphi^B_{(0)-z}\right)\left(\partial_\nu A_{(-1)z} - 2\epsilon_{AC}\,\overline{\varphi}^A_{(0)+\nu}\varphi^C_{(0)-z}\right) , \end{aligned} \tag{A.2.39}$$

where we multiplied the equations by Γ_5. Since $\partial_\nu A_{(-1)z}$ is related with $\varphi^A_{(0)-z}$ through the condition (A.2.32), it can be eliminated in the second equation.
It turns out that we can solve the gauge fixing functions from the first equation in terms of the dynamical fields. Contracting it by $\epsilon_{ki'j'}$, it acquires the equivalent form

$$0 = -A_{(-1)z}\,\epsilon_{AB}\,E^\mu_{[i}\Gamma_{j]}\varphi^B_{(0)+\mu} + 2\mathrm{i}\,\Gamma_{ij}\varphi_{A(0)-z} . \tag{A.2.40}$$

We can further contract the above equation by Γ^{ij} and use the properties of contractions of gamma matrices (A.1.10). As a result, we obtain a solution which relates the gauge fixings $\varphi^A_{(0)-z}$ and $A_{(-1)z}$,

$$\varphi^A_{(0)-z} = \frac{\mathrm{i}}{6}\,A_{(-1)z}\epsilon^{AB}\,\Gamma^i\varphi_{B(0)+\mu}E^\mu_i . \tag{A.2.41}$$

Then, the second equation in (A.2.32) becomes a linear differential equation in $A_{(-1)z}$. One possible solution is $A_{(-1)z} = 0$ which, from (A.2.41), yields $\varphi^A_{(0)-z} = 0$. On the other hand, when $A_{(-1)z} \neq 0$, we can solve $\varphi^A_{(0)+\mu}$ from the first equation in (A.2.39) as

$$A_{(-1)z}\varphi^A_{(0)+\mu} = 2\mathrm{i}E^i{}_\mu\Gamma_i\varphi^B_{(0)-z}\epsilon_{AB} , \tag{A.2.42}$$

and the differential equation becomes

$$A_{(-1)z}\partial_\mu A_{(-1)z} = 2\mathrm{i}\, E_{k\mu}\overline{\varphi}^A_{(0)-z}\left(2\Gamma^k + \epsilon^{ijk}\Gamma_5\Gamma_{ij}\right)\varphi^A_{(0)-z} = 0\,. \tag{A.2.43}$$

The only solution to the above equation is $A_{(-1)z} = \text{constant}$.

Moreover, as previously shown in the main text, we can choose a particular solution with

$$\varphi_{A(1)+\mu} = 0\,. \tag{A.2.44}$$

Consequently, taking $A_{(-1)z} = 0$ and plugging (A.2.44) into the last equation in (A.2.32), we are left with $A_{(0)z} = 0$. On the other hand, if we take $A_{(-1)z} \neq 0$ and use (A.2.42) and (A.2.44) into the last equation of (A.2.32), we obtain

$$A_{(0)z}E^\lambda{}_k\,\overline{\varphi}^A_{(0)-z}\Gamma^k\varphi^A_{(0)-z} = 0\,, \tag{A.2.45}$$

which is identically satisfied since $\overline{\varphi}^A_{(0)-z}\Gamma^k\varphi^A_{(0)-z} = 0$. In particular, this means that, in this case, the last equation in (A.2.32) is solved by (A.2.42) and (A.2.44), without forcing $A_{(0)z}$ to vanish.[1]

Summing up the results, the following gauge fixings for A_z and φ^A_{-z} are allowed:

$$\begin{aligned}
A_{(-1)z} &= 0\,, \quad A_{(0)z} = 0\,, \quad \varphi^A_{(1)+\mu} = 0\,, \quad \varphi^A_{(0)-z} = 0\,,\\
A_{(-1)z} &= 0\,, \quad A_{(0)z} \neq 0\,, \quad \varphi^A_{(1)+\mu} = \frac{1}{2}\,A_{(0)z}\varphi^B_{(0)+\mu}\epsilon_{AB}\,, \quad \varphi^A_{(0)-z} = 0\,,\\
A_{(-1)z} &= \text{constant}\,, \quad A_{(0)z} \neq 0\,, \quad \varphi^A_{(1)+\mu} = 0\,, \quad \varphi^A_{(0)-z} = \frac{\mathrm{i}}{6}\,A_{(-1)z}\Gamma^\mu\varphi^B_{(0)+\mu}\epsilon_{AB}\,,
\end{aligned} \tag{A.2.49}$$

where the first line can be seen as a special case of the general solution given in the second line. If one imposes the condition $\Gamma^{\hat\mu}\Psi_{\hat\mu} = 0$, as in [26], then (A.2.41) implies $\psi_{-z} = 0$ and, therefore, $A_{(-1)z} = 0$ as the only solution.

In this text, we mostly focus on the case $\varphi^A_{(0)-z} = 0$. Then, the gauge fixing function Ψ^A_{-z} becomes subleading and can be safely set to zero at all orders, as suggested by (A.2.46).

Eventually, let us recall that, in our approach, the gauge fixing functions are invariant under the gauge transformations. Thus, since $A_{(-1)z}$ is constant, the above solutions are consistent, because it also implies $\delta A_{(-1)z} = 0$ for the asymptotic transformations.

---

[1] Let us notice that the relation of proportionality between φ^A_{-z} and A_z, given by (A.2.41), can be consistently assumed to hold at all orders, in the neighborhood of the boundary, imposing the stronger condition

$$\varphi^A_{(n)-z} = \frac{\mathrm{i}}{6}\,A_{(n-1)z}\epsilon^{AB}\,\Gamma^i\,E^\mu_i\,\varphi_{B(0)+\mu}\,, \quad \forall\, n\,, \tag{A.2.46}$$

that is equivalent to

$$\varphi^A_{-z} = \frac{\mathrm{i}}{6}\,A_z\epsilon^{AB}\,\Gamma^i\,E^\mu_i\,\varphi_{B(0)+\mu}\,. \tag{A.2.47}$$

One can then prove that, considering the divergent terms in the z/ℓ expansion of the outer components ($\hat\tau = z$) of the gravitini equations (A.2.35), that is $E^i{}_{[\mu}\Gamma_i\hat{\boldsymbol{\rho}}_{(-1/2)+A\nu]z} = 0$, and, in particular, by using (A.2.46) in the equation for $\hat{\boldsymbol{\rho}}_{(-1/2)+A\mu z}$ in (A.2.20), one obtains

$$\Gamma_i E^i{}_{[\mu}\left(A_{(-1)z}\epsilon_{AB} - 2\,\delta_{AB}\right)\varphi_{B(1)+\nu]} = 0\,, \tag{A.2.48}$$

which enforces the condition (A.2.44) to hold also in the case $\hat A_z \neq 0$, $\Psi_{z-} \neq 0$. If we now take $A_{(-1)z} = 0$ and plug (A.2.44) into the last equation of (A.2.32), we can see that, in this case, $A_{(0)z} = 0$, $\varphi^A_{(1)+\mu} = 0$ is actually the only solution to the aforementioned equation.

A.3 The rheonomic parametrisations

We present the asymptotic expansion of the rheonomic parametrisations $\tilde{R}^{ab}{}_{cd}$, $\tilde{\rho}^A_{ab}$ and $\tilde{F}_{ab}$ for the considered gauge fixing $A_{(-1)z} = 0$ and $\Psi^A_{z-} = 0$. The procedure to compute them was already partially described in the introductory Section 1.1.

We start from the graviphoton field strength

$$\hat{\mathbf{F}} = \mathrm{d}\hat{A} - \overline{\Psi}_A \Psi_B \epsilon^{AB} = \tilde{F}_{ab} V^a V^b \,. \tag{A.3.1}$$

By expanding both sides of this equation onto the basis $\mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}}$, one can derive the explicit expression of the rheonomic parametrisations

$$\begin{aligned}
\tilde{F}_{ij} &= \left(\frac{z}{\ell}\right)^3 E^\mu_{[i} E^\nu_{j]} \left(\partial_\mu A_{(1)\nu} - 2\epsilon_{AB} \overline{\psi}^A_{\mu+} \zeta^B_{\nu-} - 2\epsilon_{AB} \overline{\zeta}^A_{\mu+} \psi^B_{\nu-}\right) + \mathcal{O}(z^4)\,,\\
2\tilde{F}_{i3} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^2 A_{(1)\mu} E^\mu_i + \left(\frac{z}{\ell}\right)^3 \left(\partial_\mu A_{(1)z} - \frac{2}{\ell} A_{(2)\mu} + 2\epsilon_{AB} \overline{\psi}^A_{z+} \psi^B_{\mu-}\right) E^\mu_i + \mathcal{O}(z^4)\,,
\end{aligned} \tag{A.3.2}$$

where we have used that $\hat{\mathbf{F}}_{\mu\nu} = \mathcal{O}(z)$.

We now focus on the supercurvature of the gravitino and conformino,

$$\begin{aligned}
\hat{\boldsymbol{\rho}}^A &= \mathrm{d}\Psi^A + \frac{1}{4}\Gamma_{ab}\hat{\omega}^{ab}\Psi^A - \frac{1}{2\ell}\hat{A}\epsilon^{AB}\Psi_B - \frac{\mathrm{i}}{2\ell}\Gamma_a \Psi^A V^a\\
&= \tilde{\rho}^A_{ab} V^a V^b - \frac{\mathrm{i}}{2}\Gamma^a \Psi_B V^b \tilde{F}_{ab}\epsilon^{AB} - \frac{1}{4}\Gamma_5 \Gamma^a \Psi_B V^b \tilde{F}^{cd} \epsilon^{AB} \epsilon_{abcd}
\end{aligned} \tag{A.3.3}$$

and expand this relation onto the basis $\mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}}$ to obtain

$$\begin{aligned}
\tilde{\rho}^A_{ij+} &= \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_{[i} E^\nu_{j]} \Bigg(\nabla_\mu \zeta^A_{\nu+} + \frac{\mathrm{i}}{\ell} E^k_\mu \gamma_k \zeta^A_{\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi^A_{\nu+} - \frac{1}{4\ell} A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB}\\
&\quad + \frac{\mathrm{i}}{4\ell}\epsilon_{lmn}\gamma^l \psi_{B\mu+} E^m_\nu E^{\rho n} A_{(1)\rho}\epsilon^{AB}\Bigg) + \mathcal{O}(z^{7/2})\,,\\
2\tilde{\rho}^A_{i3+} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}} E^\mu_i \zeta^A_{\mu+} + \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_i \Bigg(\nabla_\mu \psi^A_{z+} - \frac{1}{4} w^{jk}_{(0)}\gamma_{jk}\psi^A_{\mu+} + \frac{1}{2\ell}\epsilon^{AB} A_{(1)z}\psi_{B\mu+}\\
&\quad - \frac{2}{\ell}\Pi^A_{\mu+}\Bigg) + \mathcal{O}(z^{7/2})\,,\\
\tilde{\rho}^A_{ij-} &= \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_{[i} E^\nu_{j]} \left(\nabla_\mu \psi^A_{\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu \gamma_k \psi^A_{\nu+}\right) + \mathcal{O}(z^{7/2})\,,\\
2\tilde{\rho}^A_{i3-} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_i \left(\zeta^A_{\mu-} + \frac{1}{4}\epsilon^{AB}\gamma^j \psi_{B\mu+} A_{(1)\nu} E^\nu_j\right) + \mathcal{O}(z^{7/2})\,,
\end{aligned} \tag{A.3.4}$$

where we used $\hat{\boldsymbol{\rho}}^A_{\mu\nu} = \mathcal{O}(z^{1/2})$. This result allows to compute the spinor-tensor

$$\Theta^{ab|c}_A = -2\mathrm{i}\Gamma^{[a}\tilde{\rho}^{b]c}_A + \mathrm{i}\Gamma^c \tilde{\rho}^{ab}_A \tag{A.3.5}$$

as an intermediate step necessary to find the remaining parametrisations. In particular, we obtain

$$
\begin{aligned}
\Theta^{ij|k}_{A+} &= \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\left(-\gamma^i E^{[j\mu}E^{k]\nu} + \gamma^j E^{[i\mu}E^{k]\nu} + \gamma^k E^{[i\mu}E^{j]\nu}\right)\left(\nabla_\mu\psi_{A\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^l{}_\mu\gamma_l\psi_{A\nu+}\right)\\
&\quad + \mathcal{O}(z^{7/2})\,,\\
\Theta^{ij|3}_{A+} &= -\frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{[i}E^{j]\mu}\left(\zeta_{A\mu-} + \frac{\mathrm{i}}{4}\epsilon_{AB}\gamma^k\psi^B_{\mu+}A_{(1)\rho}E^\rho_k\right) - \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\Bigg(\nabla_\mu\zeta_{A\nu+}\\
&\quad + \frac{\mathrm{i}}{\ell}E^k_\mu\gamma_k\zeta_{A\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{klm}\gamma^k\psi^B_{\mu+}E^l_\nu E^{\rho m}A_{(1)\rho}\epsilon_{AB}\Bigg)\\
&\quad + \mathcal{O}(z^{7/2})\,,\\
\Theta^{i3|j}_{A+} &= \frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{(i}E^{j)\mu}\left(\zeta_{A\mu-} + \frac{\mathrm{i}}{4}\epsilon_{AB}\gamma^k\psi^B_{\mu+}A_{(1)\nu}E^\nu_k\right) - \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\Bigg(\nabla_\mu\zeta_{A\nu+}\\
&\quad + \frac{\mathrm{i}}{\ell}E^k_\mu\gamma_k\zeta_{A\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{klm}\gamma^k\psi^B_{\mu+}E^l_\nu E^{\rho m}A_{(1)\rho}\epsilon_{AB}\Bigg)\\
&\quad + \mathcal{O}(z^{7/2})\,,\\
\Theta^{i3|3}_{A+} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\zeta_{A\mu+}E^{\mu i} + \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{i\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{jk}_{(0)}\gamma_{jk}\psi_{A\mu+} + \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+}\\
&\quad - \frac{2}{\ell}\Pi_{A\mu+}\Bigg) + \mathcal{O}(z^{7/2})\,,\\
\Theta^{ij|k}_{A-} &= \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\left(-\gamma^i E^{[j\mu}E^{k]\nu} + \gamma^j E^{[i\mu}E^{k]\nu} + \gamma^k E^{[i\mu}E^{j]\nu}\right)\Bigg(\nabla_\mu\zeta_{A\nu+} + \frac{\mathrm{i}}{\ell}E^l_\mu\gamma_l\zeta_{A\nu-}\\
&\quad + \frac{1}{4}\omega^{lm}_{(1)\mu}\gamma_{lm}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{lmn}\gamma^l\psi^B_{\mu+}E^m_\nu E^{\rho n}A_{(1)\rho}\epsilon_{AB}\Bigg) + \mathcal{O}(z^{7/2})\,,\\
\Theta^{ij|3}_{A-} &= -\frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\gamma^{[i}E^{j]\mu}\zeta_{A\mu+} + \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{[i}E^{j]\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{kl}_{(0)}\gamma_{kl}\psi_{A\mu+}\\
&\quad + \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+} - \frac{2}{\ell}\Pi_{A\mu+}\Bigg) + \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\left(\nabla_\mu\psi_{A\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu\gamma_k\psi_{A\nu+}\right) + \mathcal{O}(z^{7/2})\,,\\
\Theta^{i3|j}_{A-} &= \frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\gamma^{(i}E^{j)\mu}\zeta_{A\mu+} - \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{(i}E^{j)\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{kl}_{(0)}\gamma_{kl}\psi_{A\mu+} + \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+}\\
&\quad - \frac{2}{\ell}\Pi_{A\mu+}\Bigg) + \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\left(\nabla_\mu\psi_{A\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu\gamma_k\psi_{A\nu+}\right) + \mathcal{O}(z^{7/2})\,,\\
\Theta^{i3|3}_{A-} &= \frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{i\mu}\left(\zeta_{A\mu-} + \frac{\mathrm{i}}{4}\epsilon_{AB}\gamma^j\psi^B_{\mu+}A_{(1)\rho}E^\rho_j\right) + \mathcal{O}(z^{7/2})\,.
\end{aligned}
$$

We are now ready to compute the rheonomic parametrisation of the supercurvature $\hat{\boldsymbol{R}}^{ab}$. Since

$$
\hat{\boldsymbol{R}}^{ab} = \mathrm{d}\hat{\omega}^{ab} + \hat{\omega}^{ac}\hat{\omega}_c{}^b - \frac{1}{\ell^2}V^aV^b - \frac{1}{2\ell}\overline{\Psi}^A\Gamma^{ab}\Psi_A
$$

$$= \tilde{R}^{ab}{}_{cd} V^c V^d - \overline{\Theta}^{ab}_{A|c} \Psi_A V^c - \frac{1}{2} \overline{\Psi}_A \Psi_B \epsilon_{AB} \tilde{F}^{ab} - \frac{\mathrm{i}}{4} \epsilon^{abcd} \overline{\Psi}_A \Gamma_5 \Psi_B \epsilon_{AB} \tilde{F}_{cd} \,, \tag{A.3.6}$$

applying the usual procedure yields

$$\begin{aligned}
\tilde{R}^{i3}{}_{jk} &= \frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2 E^\mu_{[j} E^\nu_{k]} \overline{\psi}^A_{\mu+} \gamma^i \zeta_{A\nu+} + \frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2 E^\mu_{[j} E^\nu_{k]} \overline{\psi}^A_{\mu+} \gamma^l \zeta_{A\rho+} E_{l\nu} E^{i\rho} \\
&+ \frac{1}{\ell}\left(\frac{z}{\ell}\right)^3 E^\mu_{[j} E^\nu_{k]} \Bigg\{ -\mathcal{D}_\mu \tilde{S}^i_\nu + \omega^i_{(2)l\mu} E^l_\nu - \mathrm{i}\overline{\Pi}^A_{\mu+} \gamma^i \psi_{A\nu+} - \frac{\mathrm{i}}{2} \overline{\zeta}^A_{\mu+} \gamma^i \zeta_{A\nu+} \\
&+ \frac{\mathrm{i}}{2} \overline{\psi}^A_{\mu-} \gamma^i \psi_{A-\nu} + \overline{\psi}^A_{\mu+} E_{l\nu} \bigg[-\mathrm{i}\gamma^{(i} E^{l)\rho} \bigg(\nabla_\rho \psi_{Az+} - \frac{1}{4} w^{mn}_{(0)} \gamma_{mn} \psi_{A\rho+} \\
&+ \frac{1}{2\ell} \epsilon_{AB} A_{(1)z} \psi^B_{\rho+} - \frac{2}{\ell} \Pi_{A\rho+} \bigg) + E^{[i\rho} E^{l]\sigma} \bigg(\nabla_\rho \psi_{A\sigma-} + \frac{\mathrm{i}\ell}{2} \mathcal{S}^m{}_\rho \gamma_m \psi_{A\sigma+} \bigg) \bigg] \Bigg\} + \mathcal{O}(z^4) \,, \\
2\tilde{R}^{i3}{}_{j3} &= \left(\frac{z}{\ell}\right)^3 E^\mu_j \Bigg\{ -\frac{1}{\ell} w^i_{(1)k} E^k{}_\mu + \frac{1}{\ell^2} \left(4\tilde{\tau}^i_\mu - \tau^i_\mu\right) - \frac{\mathrm{i}}{\ell} \overline{\zeta}^A_{\mu+} \gamma^i \psi_{Az+} - \frac{\mathrm{i}}{\ell} \overline{\psi}^A_{\mu+} \gamma^i \zeta_{Az+} \\
&+ \frac{1}{\ell} \overline{\psi}^A_{\mu-} \zeta_{A\nu+} E^{\nu i} - \overline{\psi}^A_{\mu+} E^{i\nu} \left(\frac{1}{\ell} \zeta_{A\nu-} + \frac{\mathrm{i}}{4\ell} \epsilon_{AB} \gamma^l \psi^B_{\nu+} A_{(1)\rho} E^\rho_l \right) \Bigg\} + \mathcal{O}(z^4) \,, \qquad \text{(A.3.7)} \\
\tilde{R}^{ij}{}_{kl} &= \left(\frac{z}{\ell}\right)^3 E^\mu_{[k} E^\nu_{l]} \Bigg\{ \partial_\mu \omega^{ij}_{(1)\nu} + \omega^i_{(1)m\mu} \omega^{mj}{}_\nu + \omega^i{}_{m\mu} \omega^{mj}_{(1)\nu} - \frac{2}{\ell^2} (\tau^{[i}_\mu + 2\tilde{\tau}^{[i}_\mu) E^{j]}_\nu \\
&- \frac{1}{\ell} \left(\overline{\psi}^A_{\mu+} \gamma^{ij} \zeta_{A\nu-} + \overline{\zeta}^A_{\mu+} \gamma^{ij} \psi_{A\nu-} \right) + \mathrm{i} E_{m\nu} \overline{\psi}^A_{\mu+} \bigg(-\gamma^i E^{[j\rho} E^{m]\sigma} + \gamma^j E^{[i\rho} E^{m]\sigma} \\
&+ \gamma^m E^{[i\rho} E^{j]\sigma} \bigg) \bigg(\nabla_\rho \zeta_{A\sigma+} + \frac{\mathrm{i}}{\ell} E^n_\rho \gamma_n \zeta_{A\sigma-} + \frac{1}{4} \omega^{np}_{(1)\rho} \gamma_{np} \psi_{A\sigma+} \\
&- \frac{1}{4\ell} A_{(1)\rho} \psi^B_{\sigma+} \epsilon_{AB} + \frac{\mathrm{i}}{4\ell} \epsilon_{npq} \gamma^n \psi^B_{\rho+} E^p_\sigma E^{\lambda q} A_{(1)\lambda} \epsilon_{AB} \bigg) \Bigg\} + \mathcal{O}(z^4) \,, \\
2\tilde{R}^{ij}{}_{k3} &= -\left(\frac{z}{\ell}\right)^2 E^\mu_k \left(\frac{1}{\ell} \omega^{ij}_{(1)\mu} - \frac{\mathrm{i}}{\ell} \overline{\psi}^A_{\mu+} \gamma^{[i} E^{j]\nu} \zeta_{A\nu+} \right) \\
&+ \left(\frac{z}{\ell}\right)^3 E^\mu_k \Bigg\{ \partial_\mu w^{ij} - \frac{2}{\ell} \omega^{ij}_{(2)\mu} + \omega^i{}_{l\mu} w^{lj}_{(0)} - w^i{}_l \omega^{lj}{}_\mu + \frac{1}{\ell} \left(E^i_\mu w^j_{(0)} - w^i_{(0)} E^j_\mu \right) \\
&+ \frac{1}{\ell} \overline{\psi}^A_{z+} \gamma^{ij} \psi_{A\mu-} - \overline{\psi}^A_{\mu+} \bigg[\mathrm{i}\gamma^{[i} E^{j]\nu} \bigg(\nabla_\nu \psi_{Az+} - \frac{1}{4} w^{lm}_{(0)} \gamma_{lm} \psi_{A\nu+} + \frac{1}{2\ell} \epsilon_{AB} A_{(1)z} \psi^B_{\nu+} \\
&- \frac{2}{\ell} \Pi_{A\nu+} \bigg) + E^{[i\nu} E^{j]\rho} \bigg(\nabla_\nu \psi_{A\rho-} + \frac{\mathrm{i}\ell}{2} \mathcal{S}^l{}_\nu \gamma_l \psi_{A\rho+} \bigg) \bigg] \Bigg\} + \mathcal{O}(z^4) \,.
\end{aligned}$$

To obtain the above formulas, we used $\hat{\boldsymbol{R}}^{ab}_{\mu\nu} = \mathcal{O}(z)$ and the vanishing condition for the supertorsion (see, in particular, (A.2.13)).

Appendix B

Supplementary Material of Chapter 3

B.1 Conventions

In this Appendix we state some of the properties of matrices and the Clifford algebra used throughout Chapter 3. We are particularly interested in the interplay among the $\mathrm{SL}(2,\mathbb{R})$ factors, appearing both in the bosonic subalgebra of $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ and as a flavour group.
For this reason, we will not distinguish between different types of spinorial indices here (e.g. $\alpha, \alpha', \dot{\alpha}'$), unless explicitly stated, and we will identify spacetime indices belonging to different $\mathrm{SL}(2,\mathbb{R})$ factors, since we take the diagonal group $\mathrm{SL}(2,\mathbb{R})_D$ as Lorentz symmetry of our theory.
We adopt the same conventions of Appendix A.1 for gamma matrices in three dimensions, namely

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij}\mathbb{1}_{2\times 2}\,, \quad [\gamma^i, \gamma^j] \equiv 2\gamma^{ij} = 2\mathrm{i}\epsilon^{ijk}\gamma_k\,. \tag{B.1.1}$$

Once the $\mathrm{SL}(2,\mathbb{R})$-invariant tensor $\epsilon_{12} = \epsilon^{12} = 1$ is introduced, one can lower and raise the indices of the gamma matrices in the following way:

$$(\gamma^i)_{\alpha\beta} = \epsilon_{\alpha\gamma}(\gamma^i)^{\gamma}{}_{\beta}, \qquad (\gamma^i)^{\alpha\beta} = (\gamma^i)^{\alpha}{}_{\gamma}\epsilon^{\gamma\beta}\,, \tag{B.1.2}$$

where the obtained matrices are symmetric.
The antisymmetric matrix $\epsilon_{\alpha\beta}$ satisfies

$$\epsilon^{\alpha\beta}\epsilon_{\rho\sigma} = \delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho}\,, \qquad \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^{\alpha}_{\gamma}\,,$$

whereas the sum of all gamma matrices with uncontracted indices yields

$$(\gamma^i)^{\alpha\beta}(\gamma_i)_{\rho\sigma} = -(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} + \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho})\,.$$

The conventions used in the text for traces and spinor bilinears are the following:

$$\mathrm{Tr}(\gamma^i\gamma^j) \equiv (\gamma^i)^{\alpha}{}_{\beta}(\gamma^j)^{\beta}{}_{\alpha}\,, \quad \lambda^t\epsilon\chi \equiv \lambda^{\alpha}\epsilon_{\alpha\beta}\chi^{\beta}\,, \quad \lambda^t\epsilon\gamma^i\chi \equiv \lambda^{\alpha}\epsilon_{\alpha\beta}(\gamma^i)^{\beta}{}_{\gamma}\chi^{\gamma}\,, \quad (\gamma_i)^t = \epsilon(\gamma_i)\epsilon\,, \tag{B.1.3}$$

where λ, χ are two generic spinors and the upper t denotes transposition.
Other conventions, needed to justify the form of the $\mathrm{D}^2(2,1;\boldsymbol{\alpha})$ superalgebra, concern the properties of spinors and spinorial forms under complex conjugations,

$$\left(\lambda^\alpha \psi^\beta\right)^* \equiv \psi^{\beta *}\lambda^{\alpha *}\,, \tag{B.1.4}$$

$$\left(\mathrm{d}\theta^\alpha \wedge \mathrm{d}\theta^\beta\right)^* = -\mathrm{d}\theta^\alpha \wedge \mathrm{d}\theta^\beta\,. \tag{B.1.5}$$

We end up with a list of the properties of the 4×4 matrices $\left(\mathbb{T}^i_{(1)}\right)^{(\alpha)}{}_{(\beta)}$ and $\left(\mathbb{T}^i_{(2)}\right)^{(\alpha)}{}_{(\beta)}$ defined in the main text, where $(\alpha)=\alpha'\dot{\alpha}'$:

$$\begin{aligned}
\left(\mathbb{T}^i_{(1)}\right)^{(\alpha)}{}_{(\beta)}\left(\mathbb{T}^j_{(1)}\right)^{(\beta)}{}_{(\gamma)} &= -\frac{1}{4}\eta^{ij}\delta^{(\alpha)}_{(\gamma)} - \frac{1}{2}\epsilon^{ijk}\left(\mathbb{T}_{(1)k}\right)^{(\alpha)}{}_{(\gamma)}\,,\\
\left(\mathbb{T}^i_{(2)}\right)^{(\alpha)}{}_{(\beta)}\left(\mathbb{T}^j_{(2)}\right)^{(\beta)}{}_{(\gamma)} &= -\frac{1}{4}\eta^{ij}\delta^{(\alpha)}_{(\gamma)} - \frac{1}{2}\epsilon^{ijk}\left(\mathbb{T}_{(2)k}\right)^{(\alpha)}{}_{(\gamma)}\,,\\
\left(\mathbb{T}^i_{(1)}\right)^{(\alpha)}{}_{(\beta)}\left(\mathbb{T}^j_{(2)}\right)^{(\beta)}{}_{(\gamma)} &= \left(\mathbb{T}^j_{(2)}\right)^{(\alpha)}{}_{(\beta)}\left(\mathbb{T}^i_{(1)}\right)^{(\beta)}{}_{(\gamma)} = -\frac{1}{4}\left(\gamma^i\right)^{\alpha'}{}_{\gamma'}\otimes\left(\gamma^j\right)^{\dot{\alpha}'}{}_{\dot{\gamma}'}\,,\\
\sum_{a=1}^{2}(\mathbb{T}^i_{(a)})^{(\alpha)(\beta)}(\mathbb{T}_{(a)\,i})_{(\rho)(\sigma)} &= \delta^{(\alpha)}_{(\rho)}\delta^{(\beta)}_{(\sigma)} - \delta^{(\alpha)}_{(\sigma)}\delta^{(\beta)}_{(\rho)}\,,\\
(\mathbb{T}^i_{(1)}\mathbb{T}^j_{(2)})^{(\alpha)(\beta)}(\mathbb{T}_{(1)\,i}\mathbb{T}_{(2)\,j})_{(\rho)(\sigma)} &= \frac{1}{8}\left(\delta^{(\alpha)}_{(\rho)}\delta^{(\beta)}_{(\sigma)} + \delta^{(\alpha)}_{(\sigma)}\delta^{(\beta)}_{(\rho)}\right) - \frac{1}{16}\,\delta^{(\alpha)(\beta)}\,\delta_{(\rho)(\sigma)}\,,
\end{aligned} \tag{B.1.6}$$

from which we see that

$$\mathrm{Tr}\left(\mathbb{T}^i_{(1)}\mathbb{T}^j_{(1)}\right) = \mathrm{Tr}\left(\mathbb{T}^i_{(2)}\mathbb{T}^j_{(2)}\right) = -\eta^{ij}\,;\quad \mathrm{Tr}\left(\mathbb{T}^k_{(1)}\mathbb{T}^i_{(2)}\right) = \mathrm{Tr}\left(\mathbb{T}^k_{(2)}\mathbb{T}^i_{(1)}\right) = 0\,. \tag{B.1.7}$$

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