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Statistical distributions of Transition Fatigue Strength and Transition Fatigue Life in duplex S–N fatigue curves

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ABSTRACT

In recent years, Very-High-Cycle Fatigue (VHCF) behavior of metallic materials has become a major point of interest for researchers and industries. The needs of specific industrial fields (aerospace, mechanical and energy industry) for structural components with increasingly large fatigue lives, up to 10^{10} cycles (gigacycle fatigue), requested for a more detailed investigation on the experimental properties of materials in the VHCF regime.

Gigacycle fatigue tests are commonly performed using resonance fatigue testing machines with a loading frequency of 20 kHz (ultrasonic tests). Experimental results showed that failure is due to cracks which nucleate at the specimen surface if the stress amplitude is above the conventional fatigue limit (surface nucleation) and that failure is generally due to cracks which nucleate from inclusions or internal defects (internal nucleation) when specimens are subjected to stress amplitudes below the conventional fatigue limit. Following the experimental evidence, the Authors recently proposed a new probabilistic model for the complete description of S–N curves both in the High-Cycle Fatigue (HCF) and in the VHCF fatigue regions (duplex S–N curves). The model differentiates between the two failure modes (surface and internal nucleation), according to the estimated distribution of the random transition stress (corresponding to the conventional fatigue limit). No assumption is made about the statistical distribution of the number of cycles at which the transition between surface and internal nucleation occurs (i.e., the Transition Fatigue Life TFL).

In the present paper, the TFL distribution is obtained. The resulting distribution depends on the distance between the HCF and the VHCF regions and on the distribution of the random transition stress. It is also shown that the statistical distribution of the fatigue strength at the median TFL (i.e., the Transition Fatigue Strength TFS) has median which corresponds to the mean transition stress. Finally, a procedure for computing Likelihood Ratio Confidence Intervals (LRCIs) for both the median TFL and the median TFS is given in the paper.

The estimated TFL and TFS distributions can be effectively used for properly choosing the duration of HCF tests in terms of number of cycles and the stress amplitude below which VHCF failures more probably occur. LRCIs for the median TFL and TFS can be usefully computed for assessing uncertainty in the estimation procedure when a limited number of experimental data is available.

A numerical example based on an experimental dataset taken from the literature is provided.

1. Introduction

In recent years, Very-High-Cycle Fatigue (VHCF) test results showed that specimens may also fail at stress amplitudes below the conventional fatigue limit and, therefore, drastically affected the way of modelling fatigue data and designing machine components under VHCF loading conditions [1].

Two distinct failure mechanisms are generally visible in VHCF data plots and, at a stress value near the conventional fatigue limit, plots show a plateau separating the two failure modes. For this reason, the conventional fatigue limit can be considered as a transition stress that differentiates between the two failure modes [2]. In particular, the plateau separating different failure mechanisms represents a transition stress, while the plateau separating finite lives from infinite lives can be considered as a real fatigue limit, if it

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Nomenclature

cdf	cumulative distribution function
HCF	High-Cycle Fatigue
LRCI	Likelihood Ratio Confidence Interval
ML	Maximum Likelihood
pdf	probability density function
PL	Profile Likelihood
rv	random variable
TFL	Transition Fatigue Life
TFS	Transition Fatigue Strength
VHCF	Very-High-Cycle Fatigue
$a_{Y\|int}, b_{Y\|int}, a_{Y\|surf}, b_{Y\|surf}$	parameters involved in the Basquin's laws
$F_{X\|Y=y}, F_{X_l}, F_{X_t}, F_{X_{t_0}}, F_{X_{t_m}}, F_{Y\|int}, F_{Y\|surf}, F_{Y\|X=x}$	cdfs
$f_{X\|Y=y}, f_{Y\|X=x}$	pdfs
int	internal-nucleated failure
$L[\cdot]$	Likelihood function
$PL[\cdot]$	Profile Likelihood function
$surf$	surface-nucleated failure
$TFL_{0.5}$	median TFL
$TFS_{0.5}$	median TFS
x	logarithm of the stress amplitude (value)
$x_{t,\alpha}$	α-th quantile of the transition stress
$X\|Y = y$	logarithm of the random fatigue strength at a given fatigue life
X_l	random fatigue limit
X_t	random transition stress
y	fatigue life (value)
$y_{t,\alpha}$	α-th quantile of the TFL
$Y\|X = x$	random fatigue life at a given stress amplitude
$Y\|int$	random fatigue life given that failure is internally-nucleated
$Y\|surf$	random fatigue life given that failure is superficially-nucleated
$z_{TFS_{0.5},int}, z_{TFS_{0.5},surf}$	quantiles used for computing the LRCI of $TFS_{0.5}$
α	probability value
$\chi^2_{(1;1-\alpha)}$	$(1-\alpha)$-th quantile of the Chi-square distribution with 1 degree of freedom
$\Phi[\cdot], \Phi_{X\|int}, \Phi_{X\|surf}, \Phi_{X_l}, \Phi_{X_t}, \Phi_{Y\|int}, \Phi_{Y\|surf}$	standardized Normal cdfs
$\varphi[\cdot], \varphi_{X\|int}, \varphi_{X\|surf}, \varphi_{X_l}, \varphi_{X_t}, \varphi_{Y\|int}, \varphi_{Y\|surf}$	standardized Normal pdfs
$\boldsymbol{\theta} = (\theta_1, \boldsymbol{\theta_2})$	parameter set
θ_1	parameter of interest for the PL function
μ_{X_l}, μ_{X_t}	mean values
$\sigma_{X_l}, \sigma_{X_t}, \sigma_{Y\|int}, \sigma_{Y\|surf}$	standard deviations
$\cdot\|\cdot$	conditional event
$\|\cdot\|$	absolute value
$\tilde{\cdot}$	parameter estimate

exists [3,4]. Following the experimental evidence, new fatigue life models [2,5–7] were proposed in the literature for the description of S–N curves characterized by two failure modes.

A novel general statistical model, which can take into consideration the two failure modes (duplex S–N curve) and the possible presence of a fatigue limit is described in [8]. The model differentiates between the two failure modes (surface and internal nucleation), according to the estimated distribution of the random transition stress (corresponding to the conventional fatigue limit). No assumption is made about the statistical distribution of the number of cycles at which the transition between surface and internal nucleation occurs (i.e., the Transition Fatigue Life TFL).

In the present paper, the TFL distribution is obtained, according to the statistical model proposed in [8]. The statistical distribution of the fatigue strength at the median TFL (i.e., the Transition Fatigue Strength TFS) is also estimated. Finally, a procedure for computing Likelihood Ratio Confidence Intervals (LRCIs) for both the median TFL and the median TFS is given.

A numerical example, based on an experimental dataset taken from the literature, is provided. The paper shows results obtained in case of a duplex S–N curve with fatigue limit.

2. Methods

In [8], a unified statistical model for various types of S–N curve was defined. In Section 2.1, the particular case of duplex S–N curves is recalled. The model is able to take into account the possible presence of a fatigue limit. According to the approach proposed in [9] and commonly adopted in the literature (see e.g., [10–12]), the fatigue strength distribution for a given number of cycles is also derived in Section 2.1.

Given the fatigue life and the fatigue strength distributions, the procedure for the estimation of the TFL and TFS distributions is presented in Section 2.2. Finally, Section 2.3 shows the steps for computing the LRCIs for both the median TFL and the median TFS.

2.1. Duplex S–N curve: statistical distributions of fatigue life and fatigue strength

In case of duplex S–N curve with fatigue limit, the cumulative distribution function (cdf) of the fatigue life Y (i.e., logarithm of the number of cycles to failure) for a given logarithm of the stress amplitude x can be expressed as [8]:

$$F_{Y|X=x} = F_{Y|surf}F_{X_t} + F_{Y|int}F_{X_l}(1 - F_{X_t}), \tag{1}$$

where $F_{Y|surf}$ is the cdf of the fatigue life if crack nucleates superficially (i.e., of the random variable (rv) $Y|surf$), $F_{Y|int}$ is the cdf of the fatigue life if crack nucleates internally (i.e., of the rv $Y|int$), F_{X_t} is the cdf of the logarithm of the transition stress (i.e., of the rv X_t) and F_{X_l} is the cdf of the logarithm of the fatigue limit (i.e., of the rv X_l).

$F_{Y|X=x}$ given in Eq. (1) depends on the cdfs of the continuous rvs $X_l, X_t, Y|int$ and $Y|surf$. According to the literature [13–16] on the fatigue strength, both X_l and X_t can be assumed as Normal distributed (i.e., the fatigue limit and the transition stress are Log-Normal distributed). In particular, let X_l have mean value μ_{X_l} and standard deviation σ_{X_l}, and X_t have mean value μ_{X_t} and standard deviation σ_{X_t}, then:

$$F_{X_l} = \Phi\left[\frac{x - \mu_{X_l}}{\sigma_{X_l}}\right], \tag{2}$$

and

$$F_{X_t} = \Phi\left[\frac{x - \mu_{X_t}}{\sigma_{X_t}}\right], \tag{3}$$

where $\Phi[\cdot]$ is the standardized Normal cdf.

In the literature [13–15], different types of continuous distribution have been proposed for the number of cycles to failure. Usually, either a 2-parameter Weibull distribution or a Log-Normal distribution are used for the cycles to failure. Without loss of generality, the conditional fatigue life is supposed to be Normal distributed (i.e., the conditional number of cycles to failure is Log-Normal distributed). Therefore, suppose that the mean values of $Y|int$ and $Y|surf$ follow the Basquin's law and that the standard deviations are constant for any value of x, then:

$$F_{Y|int} = \Phi\left[\frac{y-(a_{Y|int}+x\cdot b_{Y|int})}{\sigma_{Y|int}}\right], \quad (4)$$

and

$$F_{Y|surf} = \Phi\left[\frac{y-(a_{Y|surf}+x\cdot b_{Y|surf})}{\sigma_{Y|surf}}\right], \quad (5)$$

where $a_{Y|int}, b_{Y|int}, a_{Y|surf}$ and $b_{Y|surf}$ are four constant coefficients related to the Basquin's law and $\sigma_{Y|int}$ and $\sigma_{Y|surf}$ denote the standard deviations of $Y|int$ and $Y|surf$, respectively.

Fig. 1 shows a schematic of a duplex S–N curve together with the statistical distributions assumed in each characteristic region: the surface-nucleation and the internal-nucleation regions are described by a randomly variable fatigue life (Eqs. (4) and (5)), while the transition and fatigue limit regions are described by a randomly variable stress amplitude (Eqs. (2) and (3)).

By taking into account Eqs. (2)–(5), $F_{Y|X=x}$ finally becomes:

$$F_{Y|X=x} = \Phi_{Y|surf}\Phi_{X_t} + \Phi_{Y|int}\Phi_{X_l}(1-\Phi_{X_t}), \quad (6)$$

where $\Phi_{X_t} = \Phi\left[\frac{x-\mu_{X_t}}{\sigma_{X_t}}\right]$, $\Phi_{X_l} = \Phi\left[\frac{x-\mu_{X_l}}{\sigma_{X_l}}\right]$, $\Phi_{Y|surf} = \Phi\left[\frac{y-(a_{Y|surf}+x\cdot b_{Y|surf})}{\sigma_{Y|surf}}\right]$ and $\Phi_{Y|int} = \Phi\left[\frac{y-(a_{Y|int}+x\cdot b_{Y|int})}{\sigma_{Y|int}}\right]$. As a result, $F_{Y|X=x}$ depends on the set of 10 parameters $(a_{Y|surf}, b_{Y|surf}, \sigma_{Y|surf}, \mu_{X_t}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu_{X_l}, \sigma_{X_l})$.

Derivation of $F_{Y|X=x}$ with respect to y yields:

$$f_{Y|X=x} = \varphi_{Y|surf}\Phi_{X_t} + \varphi_{Y|int}\Phi_{X_l}(1-\Phi_{X_t}), \quad (7)$$

where $f_{Y|X=x}$ represents the probability density function (pdf) of Y given x, $\varphi_{Y|surf} = \frac{\varphi\left[\frac{y-(a_{Y|surf}+x\cdot b_{Y|surf})}{\sigma_{Y|surf}}\right]}{\sigma_{Y|surf}}$ and $\varphi_{Y|int} = \frac{\varphi\left[\frac{y-(a_{Y|int}+x\cdot b_{Y|int})}{\sigma_{Y|int}}\right]}{\sigma_{Y|int}}$, being $\varphi[\cdot]$ the standardized Normal pdf.

According to the approach proposed in [9] and commonly adopted in the literature (see e.g., [10–12]), the fatigue strength distribution for a given number of cycles can be directly obtained from the cdf of $Y|X=x$ given in Eq. (6), by assuming that $F_{X|Y=y} = F_{Y|X=x}$. Therefore, by rearranging Eq. (6), the cdf of the fatigue strength (logarithm of the stress amplitude) for a given fatigue life y is as follows:

$$F_{X|Y=y} = \Phi_{X|surf}\Phi_{X_t} + \Phi_{X|int}\Phi_{X_l}(1-\Phi_{X_t}), \quad (8)$$

where $\Phi_{X|surf} = \Phi\left[\frac{x-\frac{a_{Y|surf}-y}{|b_{Y|surf}|}}{\sigma_{Y|surf}/|b_{Y|surf}|}\right]$ and $\Phi_{X|int} = \Phi\left[\frac{x-\frac{a_{Y|int}-y}{|b_{Y|int}|}}{\sigma_{Y|int}/|b_{Y|int}|}\right]$, being $b_{Y|surf}$ and $b_{Y|int}$ negative parameters, according to the Basquin's law.

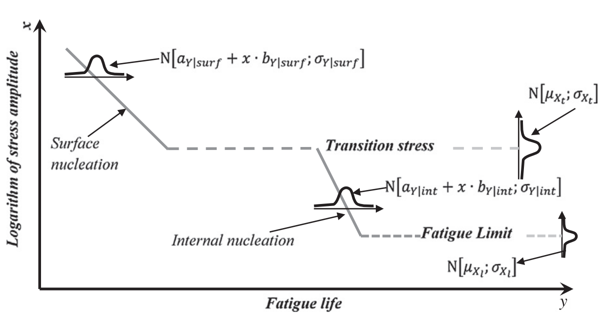

Fig. 1. Schematic of a statistical duplex S–N curve with fatigue limit.

Derivation of $F_{X|Y=y}$ with respect to x yields:

$$f_{X|Y=y} = \varphi_{X|surf}\Phi_{X_t} + \Phi_{X|surf}\varphi_{X_t} + \varphi_{X|int}\Phi_{X_l}(1-\Phi_{X_t}) + \Phi_{X|int}\left(\varphi_{X_l}(1-\Phi_{X_t}) - \Phi_{X_l}\varphi_{X_t}\right), \quad (9)$$

where $f_{X|Y=y}$ represents the pdf of X given y, $\varphi_{X_t} = \frac{\varphi\left[\frac{x-\mu_{X_t}}{\sigma_{X_t}}\right]}{\sigma_{X_t}}$, $\varphi_{X_l} = \frac{\varphi\left[\frac{x-\mu_{X_l}}{\sigma_{X_t}}\right]}{\sigma_{X_l}}$, $\varphi_{X|surf} = \frac{\varphi\left[\frac{x-\frac{a_{Y|surf}-y}{|b_{Y|surf}|}}{\sigma_{Y|surf}/|b_{Y|surf}|}\right]}{\sigma_{Y|surf}/|b_{Y|surf}|}$ and $\varphi_{X|int} = \frac{\varphi\left[\frac{x-\frac{a_{Y|int}-y}{|b_{Y|int}|}}{\sigma_{Y|int}/|b_{Y|int}|}\right]}{\sigma_{Y|int}/|b_{Y|int}|}$.

2.2. Transition Fatigue Life and Transition Fatigue Strength: statistical distributions

Statistical estimation of the 10 parameters in Eq. (6) permits to compute the S–N curves corresponding to different probabilities of failure (α-th quantile S–N curves): if $F_Y = \alpha$ and the 10 parameters are substituted with their estimates, Eq. (6) provides the expression which correlates x and y when the probability of failure equals α, which is the definition of the α-th quantile S–N curve.

In a statistical framework, the transition stress may vary from one specimen to another and each specimen can be considered as representative of a particular quantile of the transition stress distribution. Similarly, a particular quantile S–N curve hides out each specimen. Therefore, for a given specimen, both the quantile S–N curve and the quantile of the transition stress distribution are uniquely determined. In particular, let the specimen be representative of the α-th quantile S–N curve (i.e., $F_Y = \alpha$) and of the α-th quantile of the transition stress distribution (i.e., $X_t = x_{t,\alpha}$). If the stress amplitude equals the transition stress of the specimen (i.e., if $x = x_{t,\alpha}$), then the fatigue life of the specimen can be referred to as the Transition Fatigue Life (TFL) of the specimen (i.e., then $y = y_{t,\alpha}$). Thus, Eq. (6) becomes:

$$\alpha = \Phi\left[\frac{y_{t,\alpha}-(\tilde{a}_{Y|surf}+x_{t,\alpha}\cdot\tilde{b}_{Y|surf})}{\tilde{\sigma}_{Y|surf}}\right]\alpha + \Phi\left[\frac{y_{t,\alpha}-(\tilde{a}_{Y|int}+x_{t,\alpha}\cdot\tilde{b}_{Y|int})}{\tilde{\sigma}_{Y|int}}\right]\Phi\left[\frac{x_{t,\alpha}-\tilde{\mu}_{X_l}}{\tilde{\sigma}_{X_l}}\right](1-\alpha), \quad (10)$$

where $\tilde{}$ denotes a parameter estimate, $y_{t,\alpha}$ is the α-th quantile of the TFL and $x_{t,\alpha}$ is the α-th quantile of the transition stress distribution (i.e., $x_{t,\alpha} = \tilde{\mu}_{X_t} + \Phi^{-1}[\alpha]\cdot\tilde{\sigma}_{X_t}$). It must be noted that the term $\Phi\left[\frac{x_{t,\alpha}-\tilde{\mu}_{X_l}}{\tilde{\sigma}_{X_l}}\right]$ in Eq. (10) is almost equal to 1 since, according to the hypotheses stated in the definition of the unified statistical model [8], X_t must be larger than X_l (i.e., $x_{t,\alpha} \gg \tilde{\mu}_{X_l}$). Therefore, by taking into account that $\Phi\left[\frac{x_{t,\alpha}-\tilde{\mu}_{X_l}}{\tilde{\sigma}_{X_l}}\right] \to 1$, Eq. (10) can be reformulated as follows:

$$\alpha = \frac{\Phi\left[\frac{y_{t,\alpha}-\left(\tilde{a}_{Y|int}+\left(\tilde{\mu}_{X_t}+\Phi^{-1}[\alpha]\cdot\tilde{\sigma}_{X_t}\right)\cdot\tilde{b}_{Y|int}\right)}{\tilde{\sigma}_{Y|int}}\right]}{1+\Phi\left[\frac{y_{t,\alpha}-\left(\tilde{a}_{Y|int}+\left(\tilde{\mu}_{X_t}+\Phi^{-1}[\alpha]\cdot\tilde{\sigma}_{X_t}\right)\cdot\tilde{b}_{Y|int}\right)}{\tilde{\sigma}_{Y|int}}\right]-\Phi\left[\frac{y_{t,\alpha}-\left(\tilde{a}_{Y|surf}+\left(\tilde{\mu}_{X_t}+\Phi^{-1}[\alpha]\cdot\tilde{\sigma}_{X_t}\right)\cdot\tilde{b}_{Y|surf}\right)}{\tilde{\sigma}_{Y|surf}}\right]}. \quad (11)$$

Eq. (11) provides an implicit relationship between $y_{t,\alpha}$ and α and, consequently, permits the numerical computation of the TFL distribution. In particular, the median TFL, which can be considered as representative of the TFL of the material, can be computed from Eq. (11) with $\alpha = 0.5$ by taking into consideration the properties of the Normal distribution:

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$$\widetilde{TFL}_{0.5} = y_{t,0.5} = \frac{\tilde{\sigma}_{Y|surf} \cdot \left(\tilde{a}_{Y|int} + \tilde{\mu}_{X_t} \cdot \tilde{b}_{Y|int}\right) + \tilde{\sigma}_{Y|int} \cdot \left(\tilde{a}_{Y|surf} + \tilde{\mu}_{X_t} \cdot \tilde{b}_{Y|surf}\right)}{\tilde{\sigma}_{Y|surf} + \tilde{\sigma}_{Y|int}}. \quad (12)$$

The fatigue strength corresponding to the median TFL given in Eq. (12) can be referred to as the Transition Fatigue Strength (TFS) of the material. According to the plug-in estimation method [17], the TFS distribution can be computed from Eqs. (8) and (9) with $y = \widetilde{TFL}_{0.5}$.

2.3. Median TFL and TFS: Likelihood Ratio Confidence Intervals

Likelihood Ratio Confidence Intervals (LRCIs) for the median TFL and TFS can be obtained by exploiting the asymptotic distribution of the Profile Likelihood (PL). Several studies (see e.g., [13,14,18,19]) have shown that LRCIs are more accurate than the Normal-based confidence intervals and thus they are adopted in the present paper.

Section 2.3.1 shows details regarding the definition and the computation of the PL and the LRCIs. Sections 2.3.2 and 2.3.2 apply the computational procedure for the estimation of the LRCIs of the median TFL and median TFS, respectively.

2.3.1. Profile Likelihood and Likelihood Ratio Confidence Intervals: definitions

The PL approach is based on the Maximum Likelihood (ML) Principle, which is commonly adopted as an estimation method, since it allows for censoring and truncation of experimental data and it gives raise to estimators with good asymptotic properties (consistency, unbiasedness, efficiency and normality [20]).

In order for the ML Principle to be applied, the Likelihood function must be firstly defined. Let (x_i, y_i), with $i = 1, \ldots, n_f$, be the set of failure data and (x_j, y_j^*), with $j = 1, \ldots, n_r$, be the set of runout specimens in the experimental dataset, the Likelihood function, L, takes the form:

$$L[\boldsymbol{\theta}] = \prod_{i=1}^{n_f} f_{Y|X=x}[y_i; x_i, \boldsymbol{\theta}] \cdot \prod_{j=1}^{n_r} \left(1 - F_{Y|X=x}\left[y_j^*; x_j, \boldsymbol{\theta}\right]\right), \quad (13)$$

where $\boldsymbol{\theta}$ denotes the set of 10 parameters in the statistical model (i.e., $a_{Y|surf}, b_{Y|surf}, \sigma_{Y|surf}, \mu_{X_t}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu_{X_l}, \sigma_{X_l}$).

According to the ML Principle, the ML estimate $\tilde{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is the set of parameter values that maximizes $L[\boldsymbol{\theta}]$ in Eq. (13). Maximization can be obtained by applying an optimization algorithm (e.g., the function `fminsearch` of Matlab® adopts the Nelder–Mead simplex algorithm) to the experimental data [21].

Computation of the PL takes into account the ML, $L[\tilde{\boldsymbol{\theta}}]$. Let $\boldsymbol{\theta} = (\theta_1, \boldsymbol{\theta}_2)$ be a partition of $\boldsymbol{\theta}$, where θ_1 is a parameter of interest, the PL for θ_1 is defined as:

$$PL[\theta_1] = \frac{\max_{\boldsymbol{\theta}_2}[L[\theta_1, \boldsymbol{\theta}_2]]}{L[\tilde{\boldsymbol{\theta}}]}. \quad (14)$$

Exploitation of the asymptotic behavior of the Likelihood estimators permits to show that the statistics $-2\ln[PL[\theta_1]]$ is asymptotically Chi-squared with 1 degree of freedom. As a result, an approximate $(1-\alpha)\%$ confidence interval for θ_1 is given by the set of all θ_1 such that:

$$PL[\theta_1] \geqslant e^{-\frac{\chi^2_{(1;1-\alpha)}}{2}}, \quad (15)$$

where $\chi^2_{(1;1-\alpha)}$ is the $(1-\alpha)$-th quantile of a Chi-square distribution with 1 degree of freedom.

LRCIs can be obtained by substituting Eq. (14) in Eq. (15):

$$\frac{\max_{\boldsymbol{\theta}_2}[L[\theta_1, \boldsymbol{\theta}_2]]}{L[\tilde{\boldsymbol{\theta}}]} \geqslant e^{-\frac{\chi^2_{(1;1-\alpha)}}{2}}. \quad (16)$$

The range of θ_1 values which satisfies Eq. (16) represents the LRCI of parameter θ_1 [17].

2.3.2. LRCI for the median TFL: estimation procedure

In order to estimate the LRCI for $TFL_{0.5}$, the PL must be a function of $TFL_{0.5}$. Taking into account Eq. (12) with parameter estimates substituted by parameters (i.e., the ˜ symbol is eliminated everywhere in Eq. (12)), the parameter $\sigma_{Y|surf}$ can be expressed in terms of $TFL_{0.5}$ as follows:

$$\sigma_{Y|surf} = \sigma_{Y|int} \frac{(a_{Y|surf} + \mu_{X_t} \cdot b_{Y|surf}) - TFL_{0.5}}{TFL_{0.5} - (a_{Y|int} + \mu_{X_t} \cdot b_{Y|int})}. \quad (17)$$

If Eq. (17) is plugged in Eqs. (6) and (7) (i.e., if parameter $\sigma_{Y|surf}$ is substituted with the expression given by Eq. (17) in the cdf and pdf of $Y|X = x$), the set of parameters involved in the Likelihood Function (Eq. (13)) becomes

$\boldsymbol{\theta} = (a_{Y|surf}, b_{Y|surf}, TFL_{0.5}, \mu_{X_t}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu_{X_l}, \sigma_{X_l})$.

Let $\theta_1 = TFL_{0.5}$ and $\boldsymbol{\theta_2} = (a_{Y|surf}, b_{Y|surf}, \mu_{X_t}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu_{X_l}, \sigma_{X_l})$, the PL in Eq. (14) can be computed and a LRCI for $TFL_{0.5}$ can be finally estimated according to Eq. (15).

2.3.3. LRCI for the median TFS: estimation procedure

In order to estimate the LRCI for the median TFS, $TFS_{0.5}$, the PL must be a function of $TFS_{0.5}$. Taking into account Eq. (8) with $F_{X|Y=TFL_{0.5}} = 0.5$ (i.e., the median fatigue strength evaluated at the median TFL) and Eq. (12) with parameter estimates substituted by parameters, Eq. (8) becomes:

$$0.5 = \Phi[z_{TFS_{0.5},surf}]\Phi\left[\frac{TFS_{0.5} - \mu_{X_t}}{\sigma_{X_t}}\right] + \Phi[z_{TFS_{0.5},int}]\Phi\left[\frac{TFS_{0.5} - \mu_{X_l}}{\sigma_{X_l}}\right]\left(1 - \Phi\left[\frac{TFS_{0.5} - \mu_{X_t}}{\sigma_{X_t}}\right]\right), \quad (18)$$

where $z_{TFS_{0.5},int} = \frac{TFS_{0.5} - \frac{a_{Y|int} - TFL_{0.5}}{|b_{Y|int}|}}{\sigma_{Y|int}/|b_{Y|int}|}$ and $z_{TFS_{0.5},surf} = \frac{TFS_{0.5} - \frac{a_{Y|surf} - TFL_{0.5}}{|b_{Y|surf}|}}{\sigma_{Y|surf}/|b_{Y|surf}|}$, being $TFL_{0.5} = \frac{\sigma_{Y|surf}(a_{Y|int} + \mu_{X_t} b_{Y|int}) + \sigma_{Y|int}(a_{Y|surf} + \mu_{X_t} b_{Y|surf})}{\sigma_{Y|surf} + \sigma_{Y|int}}$, according to Eq. (12). It must be noted that the term $\Phi\left[\frac{TFS_{0.5} - \mu_{X_l}}{\sigma_{X_l}}\right]$ in Eq. (18) is almost equal to 1 since, according to the hypotheses stated in the definition of the unified statistical model [8], $TFS_{0.5} \gg \tilde{\mu}_{X_l}$. Therefore, by taking into account that $\Phi\left[\frac{TFS_{0.5} - \mu_{X_l}}{\sigma_{X_l}}\right] \to 1$, Eq. (18) can be further simplified as follows:

$$0.5 = \Phi[z_{TFS_{0.5},surf}]\Phi\left[\frac{TFS_{0.5} - \mu_{X_t}}{\sigma_{X_t}}\right] + \Phi[z_{TFS_{0.5},int}]\left(1 - \Phi\left[\frac{TFS_{0.5} - \mu_{X_t}}{\sigma_{X_t}}\right]\right). \quad (19)$$

It can be shown that Eq. (19) is fulfilled only if:

$$TFS_{0.5} = \mu_{X_t}. \quad (20)$$

Indeed, if $TFS_{0.5} = \mu_{X_t}$ then $\Phi\left[\frac{TFS_{0.5} - \mu_{X_t}}{\sigma_{X_t}}\right] = 0.5$ and Eq. (19) yields $1 = \Phi[z_{TFS_{0.5},surf}] + \Phi[z_{TFS_{0.5},int}]$, which, in agreement with the initial assumption (i.e., $TFS_{0.5} = \mu_{X_t}$), is fulfilled only if Eq. (20) holds.

If Eq. (20) is plugged in Eqs. (6) and (7) (i.e., if parameter μ_{X_t} is substituted with the $TFS_{0.5}$ in the cdf and pdf of $Y|X = x$), the set of parameters involved in the Likelihood Function (Eq. (13)) becomes $\boldsymbol{\theta} = \left(a_{Y|surf}, b_{Y|surf}, \sigma_{Y|surf}, TFS_{0.5}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu_{X_l}, \sigma_{X_l}\right)$. Let $\theta_1 = TFS_{0.5}$ and $\boldsymbol{\theta_2} = (a_{Y|surf}, b_{Y|surf}, \sigma_{Y|surf}, \sigma_{X_t}, a_{Y|int}, b_{Y|int}, \sigma_{Y|int}, \mu$

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X_l, σ_{X_l}), the PL in Eq. (14) can be computed and a LRCI for $TFS_{0.5}$ can be finally estimated according to Eq. (15).

3. Numerical example

An experimental dataset taken from the literature [22] is analyzed in order to show the main characteristics of the statistical distributions of TFL and TFS. The selected experimental data [22] are obtained by testing Ti–6Al–4V titanium alloy specimens and are shown in Fig. 2. Ref. [22] provides the crack originating failure (surface or internal nucleation) for each specimen.

Estimates of the parameter involved in the model given in Eq. (6) can be computed by applying the ML Principle to the experimental data. Results, obtained with a code developed in Matlab® [21], are given in the following list:

$$\begin{cases} \tilde{a}_{Y|surf} = 100.21 & \tilde{b}_{Y|surf} = -33.26 & \tilde{\sigma}_{Y|surf} = 0.4639, \\ \tilde{a}_{Y|int} = 40.34 & \tilde{b}_{Y|int} = -11.67 & \tilde{\sigma}_{Y|int} = 0.3280, \\ \tilde{\mu}_{X_t} = 2.8190 & - & \tilde{\sigma}_{X_t} = 0.0025, \\ \tilde{\mu}_{X_l} = 2.7200 & - & \tilde{\sigma}_{X_l} = 0.0059. \end{cases} \tag{21}$$

3.1. Quantile S–N curves

Parameter estimates given in Eq. (21) can be used for computing quantile S–N curves. In particular, if the α-th quantile S–N curve is of interest, the following equation:

$$\alpha = \Phi\left[\frac{y - \left(\tilde{a}_{Y|surf} + x \cdot \tilde{b}_{Y|surf}\right)}{\tilde{\sigma}_{Y|surf}}\right] \Phi\left[\frac{x - \tilde{\mu}_{X_t}}{\tilde{\sigma}_{X_t}}\right] + \Phi\left[\frac{y - \left(\tilde{a}_{Y|int} + x \cdot \tilde{b}_{Y|int}\right)}{\tilde{\sigma}_{Y|int}}\right] \Phi\left[\frac{x - \tilde{\mu}_{X_l}}{\tilde{\sigma}_{X_l}}\right] \left(1 - \Phi\left[\frac{x - \tilde{\mu}_{X_t}}{\tilde{\sigma}_{X_t}}\right]\right), \tag{22}$$

must be solved with respect to y for different values of x. Fig. 3 shows the S–N plot together with the 10%, 50% and 90% quantile S–N curves.

As shown in Fig. 3 the region between the 10% and 90% quantile S–N curves includes about the 86% (which is close to the expected 80%) of the failure data; while the 50% quantile S–N curve is almost median between failure data at each stress amplitude.

3.2. TFL: statistical distribution

If parameters are substituted by their estimates, Eq. (11) can be used for numerically computing the TFL distribution. To this aim, Eq. (11) must be solved with respect to $y_{t,\alpha}$ for different values of α ranging from zero to one. Fig. 4 shows the computed TFL distribution.

As shown in Fig. 4, the median TFL ($TFL_{0.5}$) can be used to discriminate between the two fatigue regions of HCF and VHCF: failures that occur at a number of cycles smaller than the median more probabilistically belong to the HCF region; while failures that occur at a number of cycles larger than the median more probabilistically belong to the VHCF region. For the analyzed case, the $TFL_{0.5}$ estimate, computed through Eq. (12), is equal to 7.031, which results in a median transition fatigue cycle equal to $1.075 \cdot 10^7$. As visible in Fig. 5, the $TFL_{0.5}$ estimate properly differentiates between the two fatigue regions: each internally nucleated failure is above the median value, while each superficially nucleated failure is below the median value.

It must be pointed out that, even though the $TFL_{0.5}$ estimate is able to properly differentiate between the HCF and VHCF regions, uncertainty in the estimation should also be considered from an engineering point of view. If a large uncertainty (i.e., a wide LRCI for $TFL_{0.5}$) descends from the set of the experimental data, then the statistical significance of the computed $TFL_{0.5}$ estimate is reduced and a set of $TFL_{0.5}$ estimates (i.e., the LRCI for $TFL_{0.5}$) should be more properly considered for engineering purposes.

3.3. TFS: statistical distribution

If parameters are substituted by their estimates, Eq. (8) can be used for numerically computing the TFS distribution. In particular, the cdf of TFS can be numerically computed from Eq. (8) with $y = \widetilde{TFL}_{0.5}$, according to the definition of TFS.

Fig. 6 shows the computed TFS distribution, together with the cdf of the transition stress X_t, for comparison. According to what obtained in Eq. (6) the median TFS, $TFS_{0.5}$, and the mean transition stress, μ_{X_t}, are equal. The distance between the two distributions increases when approaching the queues. It must be pointed out that, even though the two distributions refer to the same fatigue region (i.e., the transition region), they pertain to different aspects of the transition region and, consequently, they differ in shape. In particular, the TFS distribution describes the strength of the material in the transition region, while the transition stress distribution is a theoretical assumption adopted to discriminate between the HCF and the VHCF fatigue regions in the unified statistical model proposed in [8]. According to Eq. (8), the two distributions tend to overlap (i.e., $F_{TFS} \to \Phi_{X_t}$) when the distance between the HCF and the VHCF regions increases: if $a_{Y|surf} - x_{transition}|b_{Y|surf}| \ll TFL$ $0.5 \ll a_{Y|int} - x_{transition}|b_{Y|int}|$ (being $x_{transition}$ the range of stress values

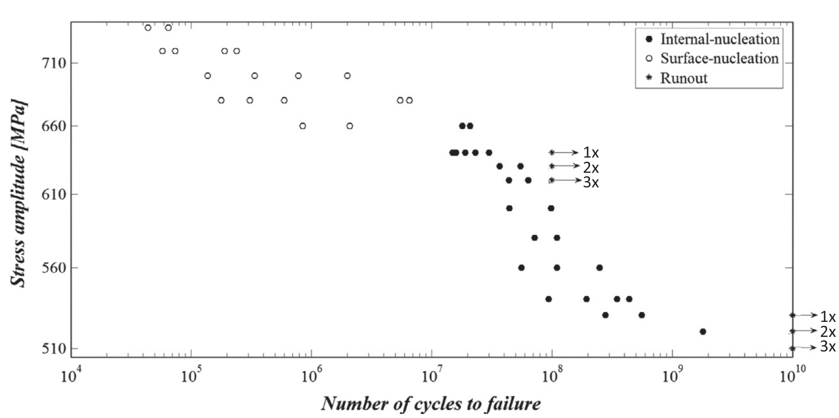

Fig. 2. Experimental fatigue data plot [22].

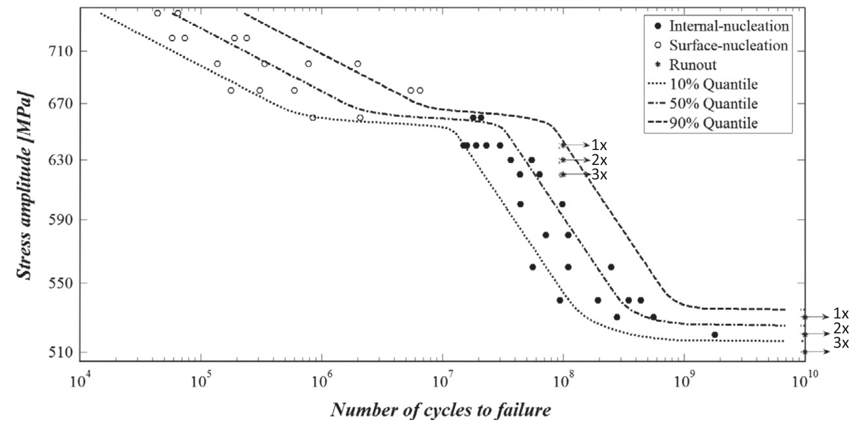

Fig. 3. Quantile S–N curves.

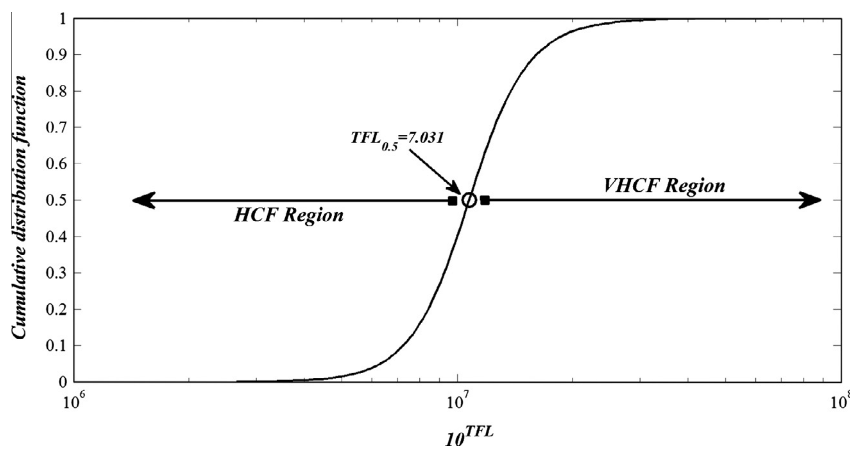

Fig. 4. TFL distribution together with the probable fatigue regions.

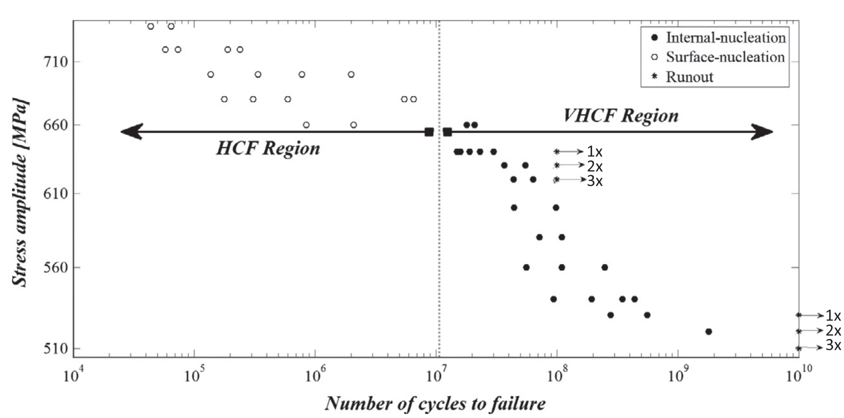

Fig. 5. Experimental data and probable fatigue regions as discriminated by the median TFL.

in the transition region), then $\Phi_{X|surf} = \Phi\left[\frac{x_{transition} - \frac{a_{Y|surf} - TFL_{0.5}}{|b_{Y|surf}|}}{\sigma_{Y|surf}/|b_{Y|surf}|}\right] \to 1$ and $\Phi_{X|int} = \Phi\left[\frac{x_{transition} - \frac{a_{Y|int} - TFL_{0.5}}{|b_{Y|int}|}}{\sigma_{Y|int}/|b_{Y|int}|}\right] \to 0$, which finally yields $F_{TFS} = F_{X|Y=TFL_{0.5}} \to \Phi_{X_t}$.

Similarly to the median TFL, the median TFS ($TFS_{0.5}$) can be usefully considered to discriminate between the two fatigue regions of HCF and VHCF: stress amplitudes larger than the median more probabilistically lead to surface-nucleated failures (i.e., HCF region); while stress amplitudes smaller than the median more probabilistically lead to internal-nucleated failures (i.e., VHCF region). For the analyzed case, the $TFS_{0.5}$ estimate, computed through Eq. (20), is equal to $\tilde{\mu}_{X_t} = 2.8190 = \log_{10}[659]$. As visible in Fig. 7, the $TFS_{0.5}$ estimate properly differentiates between the two fatigue regions: only 2 out of 25 internal-nucleated failures

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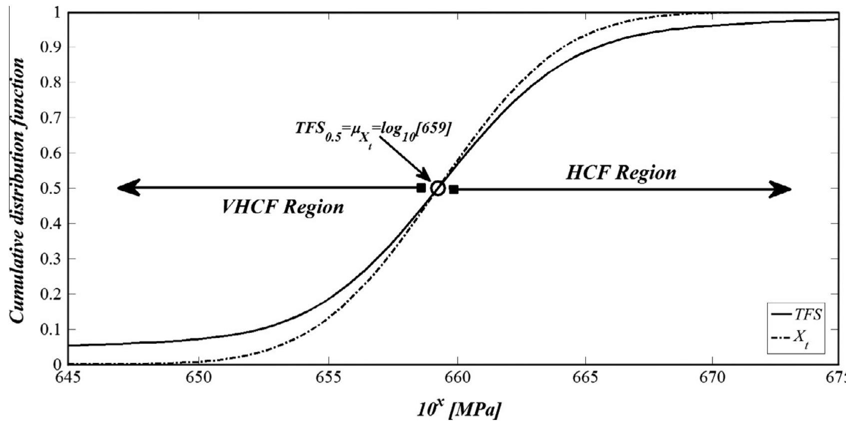

Fig. 6. TFS and transition stress distributions together with the probable fatigue regions.

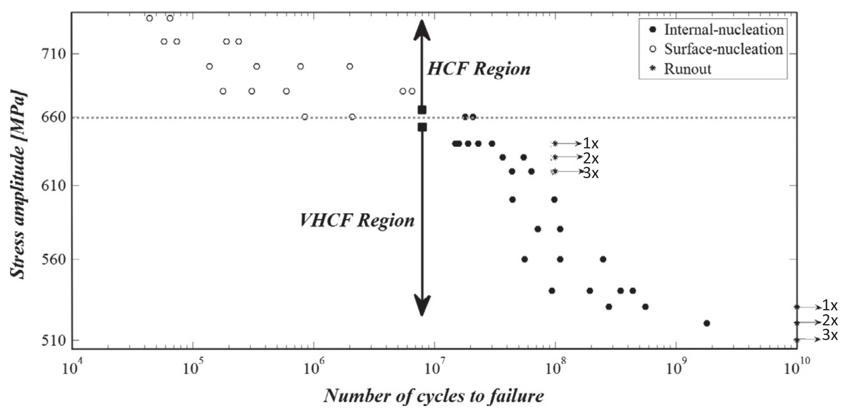

Fig. 7. Experimental data and probable fatigue regions as discriminated by the median TFS.

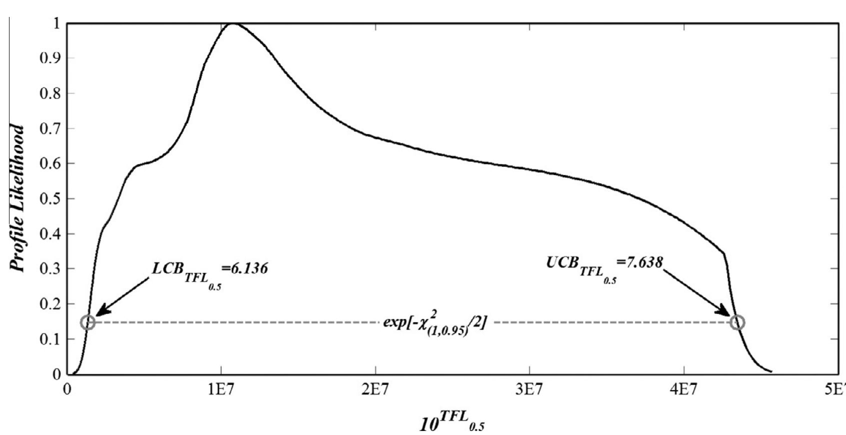

Fig. 8. Profile Likelihood of the median TFL together with 95% Lower and Upper Confidence Bounds (LCB and UCB in the graph) for the median TFL.

are erroneously considered as surface-nucleated failures (i.e., only two filled circles are above the $TFS_{0.5}$ estimate).

Again, it must be pointed out that, even though the $TFS_{0.5}$ estimate is able to properly differentiate between the HCF and VHCF regions, uncertainty in the estimation should also be considered from an engineering point of view. If a large uncertainty (i.e., a wide LRCI for $TFS_{0.5}$) descends from the set of the experimental data, then the statistical significance of the computed $TFS_{0.5}$ estimate is reduced and a set of $TFL_{0.5}$ estimates (i.e., the LRCI for $TFS_{0.5}$) should be more properly considered for engineering purposes.

3.4. Median TFL: LRCI

The LRCI for $TFL_{0.5}$ can be computed from the PL of $TFL_{0.5}$, according to the procedure given in Section 2.3.2. Fig. 8 shows the computed PL of $TFL_{0.5}$ together with the 95% Lower and Upper Confidence Bounds for $TFL_{0.5}$ (i.e., $LCB_{TFL_{0.5}}$ and $UCB_{TFL_{0.5}}$ in Fig. 8).

As shown in Fig. 8, the LRCI for $TFL_{0.5}$ is not symmetric with respect to the $TFL_{0.5}$ estimate and is quite wide if compared with the TFL cdf shown in Fig. 4. As a result, the statistical significance of the computed $TFL_{0.5}$ estimate is reduced and the LRCI for $TFL_{0.5}$

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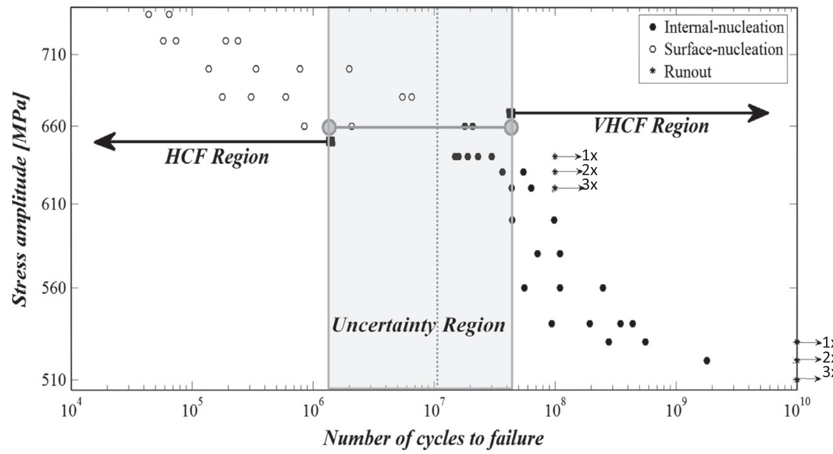

Fig. 9. Experimental data and fatigue regions as discriminated by the 95% LRCI of the median TFL.

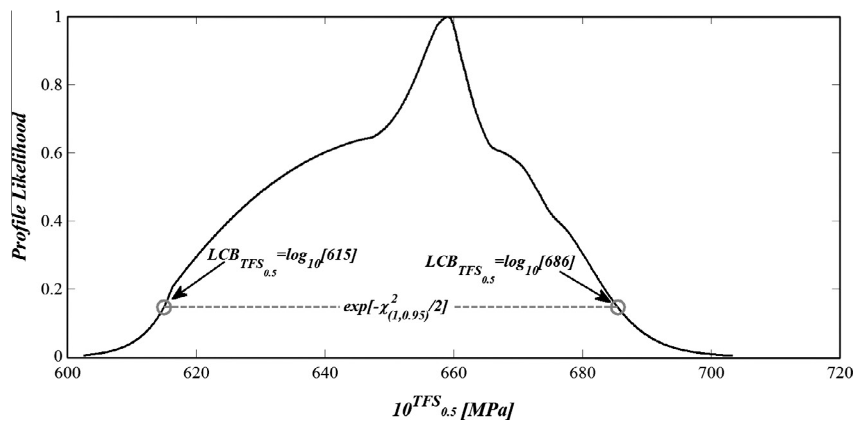

Fig. 10. Profile Likelihood of the median TFS together with 95% Lower and Upper Confidence Bounds (LCB and UCB in the graph) for the median TFS.

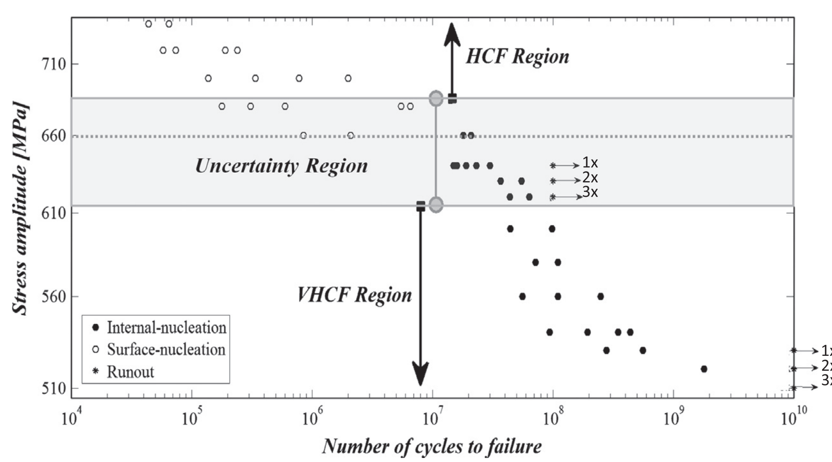

Fig. 11. Experimental data and probable fatigue regions as discriminated by the 95% LRCI of the median TFS.

should be more properly considered from an engineering point of view. In particular, the $LCB_{TFL_{0.5}}$ should be used as an upper limit for the HCF region; while the $UCB_{TFL_{0.5}}$ should be used as a lower limit for the VHCF region. The region enclosed by the LRCI (i.e., from $LCB_{TFL_{0.5}}$ to $UCB_{TFL_{0.5}}$) cannot be statistically determined and can be referred to as the Uncertainty Region for fatigue failures (Fig. 9).

3.5. Median TFS: LRCI

The LRCI for $TFS_{0.5}$ can be computed from the PL of $TFS_{0.5}$, according to the procedure given in Section 2.3.3. Fig. 10 shows the computed PL of $TFS_{0.5}$ together with the 95% Lower and Upper Confidence Bounds for $TFS_{0.5}$ (i.e., $LCB_{TFS_{0.5}}$ and $UCB_{TFS_{0.5}}$ in Fig. 10).

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Again, as shown in Fig. 10, the LRCI for $TFS_{0.5}$ is not symmetric with respect to the $TFS_{0.5}$ estimate and is quite wide if compared with the TFS cdf shown in Fig. 6. Therefore, according to what found with $TFL_{0.5}$, the statistical significance of the computed $TFS_{0.5}$ estimate is reduced and the LRCI for $TFS_{0.5}$ should be more properly considered from an engineering point of view. In particular, the $LCB_{TFS_{0.5}}$ should be used as an upper limit for the VHCF region; while the $UCB_{TFS_{0.5}}$ should be used as a lower limit for the HCF region. The region enclosed by the LRCI (i.e., from $LCB_{TFS_{0.5}}$ to $UCB_{TFS_{0.5}}$) cannot be statistically determined and, again, can be referred to as the Uncertainty Region for the fatigue strength (Fig. 11).

4. Conclusions

A procedure for the estimation of the statistical distribution of the Transition Fatigue Life (TFL) and Transition Fatigue Strength (TFS) in a duplex S–N curve was shown. The statistical model allows fitting the experimental results obtained on specimens tested within the same testing conditions (e.g., material type, specimen shape and size, loading type and frequency).

The TFL distribution was estimated by numerically solving an equation, which correlates the cumulative distribution function (cdf) to the quantile of the distribution. The TFS distribution was estimated from the cdf of the fatigue life for a fatigue life equal to median TFL. The median TFL was found to be a weighted average of the median HCF and VHCF lives computed at the transition stress, while the median TFS corresponds to the mean transition stress.

A procedure for computing Likelihood Ratio Confidence Intervals (LRCIs) for both the median TFL and the median TFS was also shown.

As shown with a numerical example taken from literature data, the estimated TFL and TFS distributions can be effectively used for differentiating the two fatigue regions of HCF and VHCF for design purposes. The LRCIs for the median TFL and TFS showed that, when a limited number of experimental data is available, uncertainty in the estimation can be large and a wide Uncertainty Region is present in a fatigue plot.

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