A New Method to Estimate the Contribution of Geometrical Optics in Arbitrary Linear Stratified Planar Structures

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Vito Daniele, and Guido Lombardi

Abstract – The interest of studying Geometrical Optics (GO) contribution in arbitrary linear stratified planar structures is of great importance in practical problems. In this letter we propose a new procedure based on Bresler-Marcuvitz transversalization method and equivalent network modelling that is useful to compute source contributions in Wiener-Hopf formulations of complex scattering problems where angular and/or stratified structures are present. The Generalized Wiener-Hopf technique has demonstrated the capability to handle new complex canonical problems through both exact and semi-analytical factorization methods.

1. Introduction

The Wiener-Hopf technique in its generalized form has been applied effectively in electromagnetic wave scattering problem for angular regions (wedge problems), see the monographies [1-2] and references there in. Following the procedure first proposed in [3], we aim at extend the Wiener-Hopf (WH) technique in angular regions for arbitrary linear wave scattering problems [3-6]. This technique can be also extended to geometries containing angular regions and/or stratified planar regions, see for instance [7]. We start our formulation from electromagnetic applications [3-5] and we extend the procedure to elasticity as reported in [6]. The method is based on two steps: the deduction of the Generalized Wiener-Hopf Equations (GWHEs) for angular region problems [3-6] and the solution of the equations using the semi-analytical procedure of factorization known as Fredholm factorization method, see for instance [8-9]. A key point to implement the solution of GWHEs for arbitrary linear stratified media via Fredholm factorization method is the extraction of source term that is related to GO components. For this reason this letter is dedicated to estimate GO contribution in arbitrary linear stratified planar structures with the help of equivalent network models and Bresler-Marcuvitz transversalization method.

2. Transverse Equations in Stratified Media

In this paper we use only time harmonic fields with a time dependence specified by the factor $e^{j\omega t}$

which is omitted. In absence of sources located at finite, the Maxwell's equations assume the abstract form [5,10] in the Euclidean space of dimension six

$$(\Gamma_\nabla - \mathrm{W})\tilde{\boldsymbol{\psi}} = 0 \qquad (1)$$

where:

$$\tilde{\boldsymbol{\psi}} = \left|\tilde{\mathbf{E}}, \tilde{\mathbf{H}}\right|^t, \Gamma_\nabla = \begin{vmatrix} 0 & 1_3 \times \nabla \\ 1_3 \times \nabla & 0 \end{vmatrix}, \mathrm{W}{=}j\omega \begin{vmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ -\boldsymbol{\zeta} & -\boldsymbol{\mu} \end{vmatrix} \qquad (2)$$

with W containing the dyadic permittivity $\boldsymbol{\varepsilon}$, the dyadic permeability $\boldsymbol{\mu}$, and the additional coupling parameters $\boldsymbol{\xi}, \boldsymbol{\zeta}$ for general bi-anisotropic media.

We introduce Cartesian coordinates (z,x,y) and we consider stratification along the y direction.

The study of the wave motion in stratified media is significantly simplified if we introduce the transverse equations of the fields. These equations involve only the components $\tilde{\psi}_t$ of the field $\tilde{\psi}$ that remain continuous along the stratification according to the boundary conditions on the interfaces. According to the boundary conditions we have

$$\tilde{\boldsymbol{\psi}}_t = \left|\tilde{E}_z, \tilde{E}_x, 0, \tilde{H}_z, \tilde{H}_x, 0\right|^t = \mathbf{I}_t \cdot \tilde{\boldsymbol{\psi}} \qquad (3)$$

where $\mathbf{I}_t = diag[1,1,0,1,1,0]$. The transverse equations are obtained using [10] as reported in [5]

$$-\frac{\partial}{\partial y}\tilde{\boldsymbol{\psi}}_t = \tilde{\mathrm{M}}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) \cdot \tilde{\boldsymbol{\psi}}_t \qquad (4)$$

with matrix differential operator

$$\tilde{\mathrm{M}}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = -\Gamma_y[(W_{ty} - \mathbf{I}_t \cdot \Gamma_{\nabla_t}) \cdot \hat{W}_y(\mathbf{I}_y \cdot \Gamma_{\nabla_t} - W_{yt}) + W_{tt}] \qquad (5)$$

whose terms are explicitly defined and reported in [5]

One of the most important relation in the procedure is

$$\tilde{\boldsymbol{\psi}}_y = \hat{W}_y(\mathbf{I}_y \cdot \Gamma_{\nabla_t} - W_{yt}) \cdot \tilde{\boldsymbol{\psi}}_t \qquad (6)$$

with $\tilde{\boldsymbol{\psi}}_y = \left|0, 0, \tilde{E}_y, 0, 0, \tilde{H}_y\right|^t$ that relates the discontinuous longitudinal component to transverse components without partial derivative along y (only the third and the sixth rows are non null).

Note that the transverse equations (4) are defined in the Euclidean space of dimension four instead of six since the third and the sixth row are null for the definition (3).

In the following we consider invariance of the geometry along z as for stratified media. With this limitation if the sources depend on a $e^{-j\alpha_o z}$ factor, also the total field depends on the same factor, i.e. $\tilde{\boldsymbol{\psi}}_c(y,\boldsymbol{\rho}) = \tilde{f}(y,x)e^{-j\alpha_o z}$. Furthermore, to study the variation of the fields in x, we introduce the Fourier transform

$$f(y,\eta) = \int_{-\infty}^{\infty} \tilde{f}(y,x)e^{j\eta x}dx \qquad (7)$$

That yields from (4) the ordinary differential equations

$$-\frac{d}{dy}\boldsymbol{\psi}_t(\eta,y) = M(\eta)\boldsymbol{\psi}_t(\eta,y) \qquad (8)$$

with $M(\eta) = \tilde{M}(-j\alpha_o, -j\eta)$, see (5).

The definition of explicit form of $M(\eta)$ and related properties need symbolic elaboration that can be performed with the help of software like Wolfram Mathematica [11]. $M(\eta)$ for isotropic and anisotropic cases are reported in [5].

3. Circuit Model of an Arbitrary semi-infinite layer

To introduce a circuital modelling we define as voltage and current the components in the Euclidean space of dimension two related to the Fourier transforms of $\tilde{\boldsymbol{\psi}}_t$. More precisely we define

$$\boldsymbol{\psi}_t(\eta,y) = \left|\mathbf{V}(\eta,y), \mathbf{I}(\eta,y)\right|^t \qquad (9)$$

With

$$\mathbf{V}(\eta,y) = \int_{-\infty}^{\infty} \begin{vmatrix} \tilde{E}_z(x,y) \\ \tilde{E}_x(x,y) \end{vmatrix} e^{j\eta x}dx, \quad \mathbf{I}(\eta,y) = \int_{-\infty}^{\infty} \begin{vmatrix} \tilde{H}_z(x,y) \\ \tilde{H}_x(x,y) \end{vmatrix} e^{j\eta x}dx \qquad (10)$$

Let's consider the plane y=y$_o$ and an arbitrary stratified medium located at y>y$_o$ in presence of arbitrary sources. The linearity of the problem impose that $\mathbf{V}(\eta, y_o)$ and $\mathbf{I}(\eta, y_o)$ satisfy the equation

$$\mathbf{V} = \mathbf{V}_s + Z_e\mathbf{I} \qquad (11)$$

where $\mathbf{V} = \mathbf{V}(\eta, y_o)$ $\quad \mathbf{I} = \mathbf{I}(\eta, y_o)$ and, $\mathbf{V}_s = \mathbf{V}_s(\eta)$ and $Z_e = Z_e(\eta)$ are called the Thevenin voltage and the Thevenin impedance. We estimate the Thevenin voltage $\mathbf{V}_s$ at y=y$_o$ by imposing perfect magnetically conducting (PMC) boundary condition at y=y_o, i.e. $\mathbf{I}=0$. The impedance Z_e is the 2x2 matrix that relates $\mathbf{V}$ and $\mathbf{I}$ when in the region y>y_o the sources are vanishing. The circuital model of the region y>y_o is illustrated in Fig. 1. The Norton representation is the dual circuit of the Thevenin one. Similar consideration apply in the stratified region located at y<y_o.

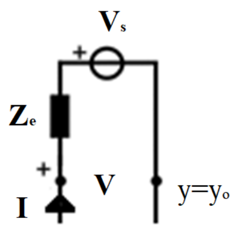


Fig.1. Thevenin representation of the half space y>y_o

In the Thevenin's model of a half-infinite layer $Z_e = Z_e(\eta)$ represents the characteristic impedance of the medium. In particular the matrix $\vec{Z}_c = \vec{Z}_c(\eta)$ is the characteristic impedance that relates $\mathbf{V}$ and $\mathbf{I}$ in the direction y>y_o in absence of sources, while the matrix $\overleftarrow{Z}_c = \overleftarrow{Z}_c(\eta)$ relates $\mathbf{V}$ and $-\mathbf{I}$ in the direction y<y_o.

4. Circuit Model of an Arbitrary Multi-layers

Let us consider an homogeneous slab defined between $y=0$ and y=d. Solution of (8) yields

$$\begin{vmatrix}\mathbf{V}_0\\ \mathbf{I}_0\end{vmatrix} = \Im \begin{vmatrix}\mathbf{V}_d\\ \mathbf{I}_d\end{vmatrix},\quad \mathbf{V}_0 = \mathbf{V}\big|_{y=0}, \mathbf{I}_0 = \mathbf{I}\big|_{y=0}, \mathbf{V}_d = \mathbf{V}\big|_{y=d}, \mathbf{I}_d = \mathbf{I}\big|_{y=d} \tag{12}$$

with the transmission matrix of the slab 0<y<d defined by

$$\Im = \Im(\eta) = e^{M(\eta)d} = \begin{vmatrix}\mathbf{A}(\eta) & \mathbf{B}(\eta)\\ \mathbf{C}(\eta) & \mathbf{D}(\eta)\end{vmatrix} \tag{13}$$

This 4x4 matrix has as elements the 2x2 matrices A,B,C,D defined in terms of *M* matrix of dimension four (see section 2). Fig. 2 reports a convenient representation of (12) in terms of two-port network model. This representation is also valid for a slab constituted by a cascade of s homogeneous consecutive slabs 1,2,3… In this case the transmission matrix of the multi-layer slab is the product of the transmission matrices relevant to each slab:

$$\Im = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \Im_1 \Im_2 \Im_3 \ldots = e^{M_1 d_1} e^{M_2 d_2} e^{M_3 d_3} \ldots \quad (14)$$

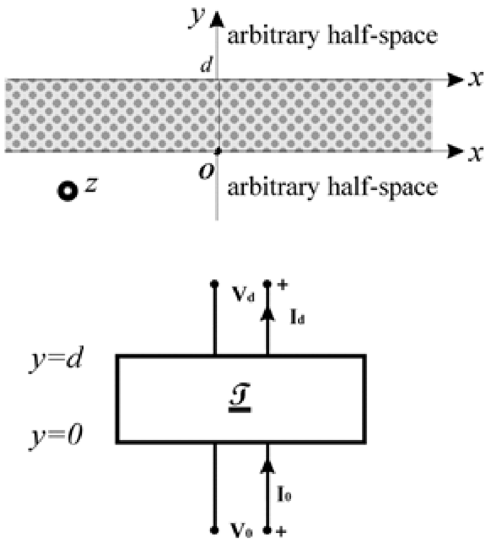


Fig.2: Top: slab filled by arbitrary linear medium. Bottom: network equivalent model.

5. The Eigenvalues and the Eigenvectors

The eigenvalues and the eigenvectors of the matrix *M* reported in (8) are very important in studying the solution of the equation. For example they allow the evaluation of function of *M* such the exponentials that appears in (13)-(14). We study the eigenvalues and the eigenvectors of *M* defined in the Euclidean space of dimension four (see section 2)

$$M(\eta)\boldsymbol{\psi}^{(i)}(\eta) = \gamma_i(\eta)\boldsymbol{\psi}^{(i)}(\eta) \quad (15)$$

where $\gamma_i(\eta)$, $\boldsymbol{\psi}^{(i)}(\eta)$ are respectively the eigenvalues and the eigenvectors. Since M is a semi-simple matrix we have

$$M = UJU^{-1} \quad (16)$$

where $J = diag[\gamma_1(\eta),..,\gamma_4(\eta)]$ and $U = [\boldsymbol{\psi}^{(1)}(\eta),..,\boldsymbol{\psi}^{(4)}(\eta)]$ and $\boldsymbol{\psi}^{(i)}(\eta) = \left|\psi_1^{(i)}(\eta), \psi_2^{(i)}(\eta), \psi_3^{(i)}(\eta), \psi_4^{(i)}(\eta)\right|^t$.

In presence of a passive medium we observe that two eigenvalues (say γ_1, γ_2) present non-negative real part and the other two γ_3, γ_4 present non- positive real part. Whence $\boldsymbol{\psi}^{(i)}(\eta)$ (i=1,2) are called progressive eigenvectors and $\boldsymbol{\psi}^{(i)}(\eta)$ (i=,3,4) are called regressive eigenvectors. The eigenvectors of M provide the characteristic impedance of a medium. In fact [9] we have

$$\vec{\mathbf{Z}}_c = \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \psi_3^{(1)} & \psi_3^{(2)} \\ \psi_4^{(1)} & \psi_4^{(2)} \end{vmatrix}^{-1}, \vec{\mathbf{Z}}_c = \begin{vmatrix} \psi_1^{(3)} & \psi_1^{(4)} \\ \psi_2^{(3)} & \psi_2^{(4)} \end{vmatrix} \cdot \begin{vmatrix} \psi_3^{(3)} & \psi_3^{(4)} \\ \psi_4^{(3)} & \psi_4^{(4)} \end{vmatrix}^{-1} \quad (17)$$

Furthermore the transmission matrix $e^{M(\eta)d}$ (13) is obtained by $e^{M(\eta)d} = Ue^{Jd}U^{-1}$ where $e^{Jd} = diag[e^{\gamma_1(\eta)d},..,e^{\gamma_4(\eta)d}]$.

6. Plane Waves in a Arbitrary Medium

Plane waves in a arbitrary are solutions of the transverse equations (4) with (5) of the form

$$\boldsymbol{\psi}_\mathbf{t} = \boldsymbol{\psi}_\mathbf{o} e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (18)$$

where $\mathbf{r} = |z,x,y|^t$ is the observation point, $\mathbf{k} = |k_y,\alpha_o,\eta_o|^t$ the propagation vector and $\boldsymbol{\psi}_o$ is a constant vector of dimension four. Taking into account that $\frac{\partial}{\partial y} = -jk_y$, (4) becomes

$$jk_y\boldsymbol{\psi}_\mathbf{o} = \mathrm{M}(\eta_o)\boldsymbol{\psi}_\mathbf{o} \quad (19)$$

where k_y are related to the eigenvalues of matrix M as defined in section 5. For a given set of α_o,η_o we have four possible propagation constants k_y and four polarizations $\boldsymbol{\psi}_o$ i.e.

$$jk_{yi} = \gamma_i(\eta), \boldsymbol{\psi}_{\mathbf{o}i} = \boldsymbol{\psi}^{(i)}(\eta), \text{ i=1,2,3,4} \quad (20)$$

We call progressive the plane waves where i=1,2 and regressive the plane waves where i=3,4. Examples of values are given in [12]. For each values of k_y (we omit the subscript i) we define a propagation vector and a propagation constant as

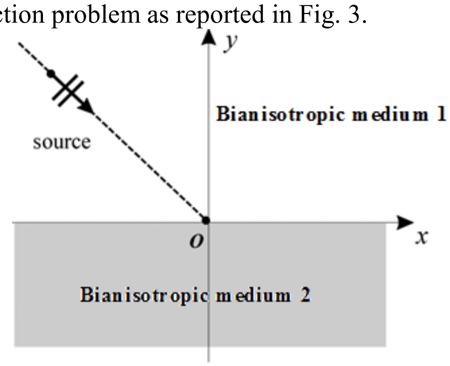
$$k = |\mathbf{k}| = \sqrt{\alpha_o^2 + \eta_o^2 + k_y^2}, \tau_o = \sqrt{k^2 - \alpha_o^2} \quad (21)$$

that identify the direction of the wave in term of zenithal angle β and azimuthal angle φ_o: $\eta_o = -\tau_o \sin\varphi_o$,

$\alpha_o = k\cos\beta$. We recall that the field $\boldsymbol{\psi}_t$ contain only the transverse components of the electromagnetic field. The discontinuous components E_y,H_y, are related to the transverse ones through (6).

6. The Reflection Problem

Considering the network modelling of semi-infinite medium we build the network representation of the reflection problem as reported in Fig. 3.



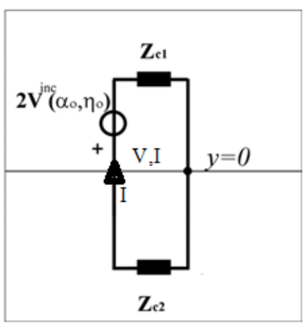


Fig.3. Top: Plane wave reflection between two semi-infinite homogenous bi-anisotropic media. Bottom: network modelling based on the Thevenin's equivalence

In the network representation the impedances Z_{c1} and Z_{c2} are the characteristic ones obtained in (17). We observe that the model is completed considering sources in region 1 (y>0) that yields (by imposing PMC termination at y=0) the Thevenin voltage $2\mathbf{V}^{inc}(\alpha_o,\eta_o)$ where $\mathbf{V}^{inc}(\alpha_o,\eta_o)$ is the incident voltage at y=0. The incident voltage $\mathbf{V}^{inc}(\alpha_o,\eta_o)$ is related to the regressive incident plane wave that constitute the source in region 1. Without loss of generality we suppose the presence of only one kind of incident plane: $\mathbf{V}^{inc}(\alpha_o,\eta_o) = \mathbf{V}^{inc} = V_o\mathbf{V}^{(i)}$ where V_o is the intensity of the plane wave and $\mathbf{V}^{(i)}$ is the voltage part (i.e. the first two components) of one of the regressive eigenvector $\psi_c^{(i)}$ (i=3,4) of the matrix M relevant to medium 1., i.e. $\mathbf{V}^{(i)} = |\psi_1^{(i)} \; \psi_2^{(i)}|'$. Analyzing the circuit model of Fig. 3 we have

$$\begin{aligned} &\mathbf{I} = -(Z_{c1} + Z_{c2})^{-1} 2\mathbf{V}^{\mathbf{inc}} \\ &\mathbf{V} = -Z_{c2}\mathbf{I} = T\mathbf{V}^{\mathbf{inc}} = \mathbf{V}^{\mathbf{inc}} + \Gamma\mathbf{V}^{\mathbf{inc}} = \mathbf{V}^{\mathbf{inc}} + \mathbf{V}^{\mathbf{ref}} \end{aligned} \quad (23)$$

where $\mathbf{V}^{\mathbf{ref}}(0) = \mathbf{V}^{\mathbf{ref}} = \Gamma\mathbf{V}^{\mathbf{inc}} = \Gamma\mathbf{V}^{\mathbf{inc}}(0)$ define the reflected voltage at y=0 and the 2x2 reflection matrix Γ is given by:

$$\Gamma = T - 1_3 = 2Z_{c2}(Z_{c1} + Z_{c2})^{-1} = (Z_{c2} + Z_{c1})(Z_{c1} + Z_{c2})^{-1} \quad (24)$$

While exciting/illuminating the structure with one regressive wave either i=3 or i=4, in general we get as reflected waves all the progressive reflected waves (i=1,2) of the medium 1and all the regressive transmitted waves (i=3,4) of the medium 2. We can evaluate the coupling coefficient in terms of the reflection matrix Γ. Taking into account that the reflected wave

$$\mathbf{V}^{ref}(y) = C_{o1}e^{-\gamma_1 y}\mathbf{V}^{(1)} + C_{o2}e^{-\gamma_2 y}\mathbf{V}^{(2)} \quad (25)$$

contains all the progressive plane wave present at y>0, we obtain the excitation coefficients C_{o1}, C_{o2} by considering that at y=0 we get

$$\mathbf{V}^{ref}(0) = \Gamma\mathbf{V}^{\mathbf{inc}}(0) = C_{o1}\mathbf{V}^{(1)} + C_{o2}\mathbf{V}^{(2)} = \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix} \begin{vmatrix} C_{o1} \\ C_{o2} \end{vmatrix} \quad (25)$$

that yields

$$\begin{vmatrix} C_{o1} \\ C_{o2} \end{vmatrix} = \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix}^{-1} \mathbf{V}^{ref}(0) = \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix}^{-1} \Gamma\mathbf{V}^{\mathbf{inc}}(0) \quad (26)$$

Similar considerations apply for the evaluation of plane waves transmitted in the medium 2 (y<0). By indicating with γ_{ti} and $\mathbf{V}_t^{(i)}$ (i=3,4) the regressive eigenvalues and eigenvectors of the matrix $\mathrm{M_t}(\eta_o)$ defined in the medium 2, we have for y<0:

$$\mathbf{V}(y) = C_{to3}e^{-\gamma_{t3} y}\mathbf{V}_t^{(3)} + C_{to4}e^{-\gamma_{t4} y}\mathbf{V}_t^{(4)} \quad (27)$$

$$\begin{vmatrix} C_{to3} \\ C_{to4} \end{vmatrix} = \begin{vmatrix} \psi_{t1}^{(3)} & \psi_{t1}^{(4)} \\ \psi_{t2}^{(3)} & \psi_{t2}^{(4)} \end{vmatrix}^{-1} T\mathbf{V}^{\mathbf{inc}}(0) \quad (28)$$

7. Conclusions

This work proposes a new effective method to estimate GO contributions in arbitrary linear stratified planar structures, based on equivalent network models and Bresler-Marcuvitz transversalization theory.
The method is in particular useful to start the analysis of novel complex canonical problems constituted of angular and/or stratified structures with the Generalized Wiener Hopf Technique as in [13].

8. Acknowledgment

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