Quadric surfaces in the Pfaffian hypersurface in 14

QUADRIC SURFACES IN THE PFAFFIAN HYPERSURFACE IN $\mathbb{P}^{14}$

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Abstract. We study smooth quadric surfaces in the Pfaffian hypersurface in $\mathbb{P}^{14}$ parameterising 6×6 skew-symmetric matrices of rank at most 4, not intersecting its singular locus. Such surfaces correspond to quadratic systems of skew-symmetric matrices of size 6 and constant rank 4, and give rise to a globally generated vector bundle E on the quadric. We analyse these bundles and their geometry, relating them to linear congruences in $\mathbb{P}^5$.

1. Introduction

Denote by V_{n+1} a complex vector space of dimension $n+1$. Recall that, after fixing a basis, skew-symmetric tensors in $\wedge^2 V_{n+1}$ can be interpreted as skew-symmetric matrices of size $n+1$, and that the Grassmannian of lines in $\mathbb{P}^n = \mathbb{P}(V_{n+1})$ corresponds to matrices of rank 2. A linear congruence in $\mathbb{P}^n$ is a $(n-1)$-dimensional linear section of $\mathbb{G}(1,n)$, given by the intersection $\mathbb{G}(1,n) \cap \Delta$, where Δ is a linear space of codimension $n-1$. The space Δ is therefore given by the intersection of $n-1$ hyperplanes, that, in turn, correspond to points in the dual space $\check{\mathbb{P}}(\wedge^2 V_{n+1})$, generating a $(n-2)$-space $\check{\Delta}$.

The study and classification of linear congruences in $\mathbb{P}^n$ is a classical topic, that has recently found interesting applications in different areas, such as, for instance, systems of conservation laws of Temple type [1], degree of irrationality [2, 3], foliations [4]. Thus far, a complete classification is known only for values of $n \leq 4$ [5, 6, 7].

In this article we give a contribution to the study of linear congruences in $\mathbb{P}^5$, that amounts to describing all special positions of the 3-space $\check{\Delta}$ with respect to the dual Grassmannian $\check{\mathbb{G}}(1,5)$ and to its singular locus. Let us consider the Grassmannian of lines in $\mathbb{P}^5 = \mathbb{P}(V_6)$:

$$\mathbb{G}(1,5) \hookrightarrow \mathbb{P}(\wedge^2 V_6) = \mathbb{P}^{14}.$$

There is a natural filtration, based on the rank of tensors, namely

$$\mathbb{G}(1,5) \subset \sigma_2(\mathbb{G}(1,5)) \subset \mathbb{P}(\wedge^2 V_6) = \mathbb{P}^{14},$$

corresponding to 6×6 skew-symmetric matrices of

$$\{\mathrm{rk} \leqslant 2\} \subset \{\mathrm{rk} \leqslant 4\} \subset \{\mathrm{rk} \leqslant 6\} = \mathbb{P}(\wedge^2 V_6),$$

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where we denote by $\sigma_2(\mathbb{G}(1,5))$ the variety of secant lines to the Grassmannian.

Inside the dual space $\check{\mathbb{P}}^{14}$ there lives the dual variety $\check{\mathbb{G}}(1,5)$ parameterising hyperplanes tangent to $\mathbb{G}(1,5)$: it is the cubic hypersurface of 6×6 skew-symmetric matrices defined by the equation Pfaff $=0$, so it corresponds to matrices of rk $\leqslant 4$: $\check{\mathbb{G}}(1,5)\simeq\sigma_2(\mathbb{G}(1,5))$. Its singular locus is naturally isomorphic to $\mathbb{G}(3,5)$, and it is formed by hyperplanes tangent to $\mathbb{G}(1,5)$ at the points corresponding to the lines of a $\mathbb{P}^3$, and the associated skew-symmetric matrix has rank 2.

If the 3-space Δ is general, the intersection $\check{\mathbb{G}}(1,5)\cap\check{\Delta}$ is a cubic surface S. Otherwise $\check{\Delta}\subset\check{\mathbb{G}}(1,5)$, but in this second case $\check{\Delta}$ meets the rank 2 locus (see [8]).

An interesting case arises when the intersection $\check{\mathbb{G}}(1,5)\cap\check{\Delta}$ is a reducible cubic surface S not intersecting $\mathbb{G}(3,5)$, having a plane and a smooth quadric surface as irreducible components. Then such a plane (respectively quadric surface) can be interpreted naturally as a linear (respectively quadratic) system of skew-symmetric matrices of constant rank 4, of projective dimension 2.

Planes of this type have been completely classified in [8]: up to the action of PGL_6 there are exactly four different orbits, all of dimension 26; they correspond to the double Veronese embeddings of $\mathbb{P}^2$ in $\mathbb{G}(1,5)$, or, equivalently, to rank 2 globally generated vector bundles on $\mathbb{P}^2$ with first Chern class $c_1=2$ (see [9]). Indeed, given a plane of 6×6 skew-symmetric matrices of constant rank 4, there is an exact sequence of the form

$$0\to E^*(-1)\to 6\mathcal{O}_{\mathbb{P}^2}(-1)\to 6\mathcal{O}_{\mathbb{P}^2}\to E\to 0,$$

where E is a rank two vector bundle on $\mathbb{P}^2$ with $c_1(E)=2$.

The aim of this note is to study the embeddings of smooth quadric surfaces Q contained in the smooth locus of $\check{\mathbb{G}}(1,5)$, that we denote by $\check{\mathbb{G}}(1,5)_{sm}$. The hope of achieving for quadric surfaces the same kind of classification obtained in the case of planes fades immediately, after a quick parameter count shows the existence of an infinite number of orbits.

However, since rank 2 globally generated bundles on a smooth quadric surface are classified (see [10]), we have considered the following problems: first, understanding which of these vector bundles are associated to a quadratic system of skew-symmetric matrices of constant rank 4; second, studying the geometry of the found examples, relating them to linear congruences of lines in $\mathbb{P}^5$.

Our main result is the existence Theorem 3.5, that gives the complete list of rank 2 globally generated vector bundles on Q associated to a quadratic system of skew-symmetric matrices of size 6 and constant rank 4 (see Section 3 for a more precise statement).

Our techniques rely on the already mentioned classification of planes contained in $\check{\mathbb{G}}(1,5)_{sm}$, and on a study of the geometry of the bundles involved. More precisely, we construct examples of such quadric surfaces either by considering directly the case of decomposable bundles (Section 4), or by constructing bigger size matrices and then projecting them to desired size ones with a projection technique (Section 5), or else by extending some known examples on $\mathbb{P}^2$ to a suitable 3-dimensional space (Section 6).

We make frequent use of Macaulay2 software [11] to study details about the geometry of our examples.

We end the article with an open problem that we believe is worthy of consideration, since it would allow to take a step forward in the classification of linear congruences in $\mathbb{P}^5$, begun in [12].

To the best of our knowledge, ours is the first instance of the study of nonlinear spaces contained in these orbits. While some of the ideas and proofs working for the linear case still hold on the quadric surface, there are a few differences to note, such as the fact that the rank 2 vector bundles that we construct are globally generated (so they define morphisms to the Grassmannian $\mathbb{G}(1,5)$), but they do not always define embeddings the way they did in the linear case.

Finally, it is worth mentioning some interesting related work: in the paper [13], Ferapontov-Manivel have considered a problem kindred to ours, that admits an interpretation in terms of integrable systems: there, they are interested in 3-dimensional linear spaces $\mathbb{P}^3 \subset \check{\mathbb{G}}(1,5)$ satisfying some extra condition. In [14], Comaschi studied and classified (stable) $\mathrm{SL}(V_6)$-orbits of linear systems in $\mathbb{P}(\wedge^2 V_6)$, whose generic element is a tensor of rank 4, thus generalising the work of [8] in a different direction with respect to ours.

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2. Quadrics of skew-symmetric matrices of constant rank and vector bundles

Let Q be a smooth quadric surface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and embedded into $\mathbb{P}^3$ through the Segre map, and let us call π_i the projections to $\mathbb{P}^1$. Any line bundle over Q is of the form $\mathcal{O}_Q(a,b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(b))$. For the sake of brevity, we denote $\mathcal{O}_Q(a,a)$ by $\mathcal{O}_Q(a)$. Given a vector bundle E over Q, we write $E(a,b)$ (respectively $E(a)$) for the tensor product $E \otimes \mathcal{O}_Q(a,b)$ (resp. $E \otimes \mathcal{O}_Q(a)$). We also write $c_1(E) = (a,b)$ to mean $c_1(E) = \mathcal{O}_Q(a,b)$.

The existence of a smooth quadric surface $Q \subset \check{\mathbb{G}}(1,5)_{sm}$ of skew-symmetric matrices of size 6 and constant rank 4 entails a long exact sequence of vector bundles on Q, of the form:

$$0 \to K \to 6\mathcal{O}_Q(-1) \xrightarrow{A} 6\mathcal{O}_Q \to E \to 0,$$

where E and K are rank 2 vector bundles on the surface, satisfying some non-degeneracy conditions in cohomology. Notice that E is globally generated. The skew-symmetry of the map A above implies a symmetry of the exact sequence above; in particular there is an isomorphism $K \simeq E^*(-1)$, so in fact the sequence looks like:

$$0 \to E^*(-1) \to 6\mathcal{O}_Q(-1) \xrightarrow{A} 6\mathcal{O}_Q \to E \to 0. \tag{2.1}$$

We refer to [15] for details.

Similarly to what happens in the linear case, the Chern classes of a bundle E fitting in a long exact sequence of type (2.1) must meet some requirements.

Proposition 2.1. *Let E be a vector bundle, fitting in an exact sequence of the form (2.1) as cokernel of a skew-symmetric matrix of size* 6 *and constant rank* 4 *over a quadric surface Q. Then the Chern classes of E must satisfy $c_1(E) = (2, 2)$ and $0 \leqslant c_2(E) \leqslant 6$.*

Proof. We split sequence (2.1) into two short exact sequences:

$$0 \to E^*(-1) \to 6\mathcal{O}_Q(-1) \to F \to 0, \tag{2.2}$$

$$0 \to F \to 6\mathcal{O}_Q \to E \to 0, \tag{2.3}$$

and compute invariants. From (2.3) we deduce $c_1(E) = (a, b) = -c_1(F)$, while from (2.2) we get $c_1(E^*(-1)) + (-a, -b) = (-6, -6)$. Since $\mathrm{rk}(E) = 2$, we have an isomorphism $E^* \simeq E(-a, -b)$, and hence $c_1(E^*(-1)) = (-a-2, -b-2)$. Putting all together we conclude that $(a, b) = (2, 2)$.

Moving on to the bounds on c_2, we first remark that the globally generated bundle E must have $c_2(E) \geqslant 0$. For the upper bound, we tensor sequence (2.3) by $\mathcal{O}_Q(-1, 0)$ and compute cohomology: since the cohomology groups of bundle $6\mathcal{O}_Q(-1, 0)$ are all zero, we deduce a vanishing $\mathrm{H}^2(E(-1, 0)) = 0$, and an isomorphism $\mathrm{H}^1(E(-1, 0)) \simeq \mathrm{H}^2(F(-1, 0))$. Computing cohomology of sequence (2.2), again tensored by $\mathcal{O}_Q(-1, 0)$, we see that the fact that $6\mathcal{O}_Q(-2, -1)$'s cohomology vanishes entirely entails a vanishing $\mathrm{H}^2(F(-1, 0)) = 0$. All in all, $\chi(E(-1, 0)) = \mathrm{h}^0(E(-1, 0)) \geqslant 0$. Computing the same Euler characteristic via Hirzebruch-Riemann-Roch using Chern classes, we get $\chi(E(-1, 0)) = 6 - c_2(E) \geqslant 0$. □

2.1. **The Gauss map.** Consider the rational Gauss map $\gamma : \check{\mathbb{G}}(1, 5) \dashrightarrow \mathbb{G}(1, 5)$, associating to a tangent hyperplane its tangency point when unique. It is defined by the partial derivatives of Pfaff, the generic 6×6 Pfaffian determinant, that is, by the 4×4 principal minors' Pfaffians. Given a quadric surface Q contained in $\check{\mathbb{G}}(1, 5)_{sm}$, the equations defining γ cannot all vanish on Q, since the rank there is constant and equal to 4, therefore the restriction

$$\gamma|_Q : Q \to \gamma(Q) \subset \mathbb{G}(1, 5)$$

is a regular map. For a line $\ell \in \mathbb{G}(1, 5)$, the fibre of γ over ℓ consists of all hyperplanes H tangent to $\mathbb{G}(1, 5)$ at ℓ, i.e. such that $\mathrm{T}_\ell\, \mathbb{G}(1, 5) \subseteq H$, so $\gamma^{-1}(\ell) \simeq (\mathrm{T}_\ell\, \mathbb{G}(1, 5))\check{}$ is a 5-dimensional linear space.

Remark 2.2. The regular map $\gamma|_Q$ has degree 1 or 2. In detail, if we choose a basis $\{e_0, \dots, e_5\}$ of V_6 and $\ell =< e_0, e_1 >$, then the tangent space $\mathrm{T}_\ell\, \mathbb{G}(1, 5)$ is defined by equations $p_{ij} = 0$ for $i \geqslant 2$, so its dual $(\mathrm{T}_\ell\, \mathbb{G}(1, 5))\check{}$ coincides with the space of 6×6 skew-symmetric matrices having all zero entries in the first two rows and columns, that is, the space spanned by a sub-Grassmannian $\mathbb{G}(1, 3)$. Therefore, the general element of $(\mathrm{T}_\ell\, \mathbb{G}(1, 5))\check{}$ has rank 4, and those of rank 2 form a quadric hypersurface. This means that the fibres of $\gamma|_Q$ are of the form $(\mathrm{T}_\ell\, \mathbb{G}(1, 5))\check{} \cap Q$, the intersection of a 5-dimensional linear space with a quadric surface. If $(\mathrm{T}_\ell\, \mathbb{G}(1, 5))\check{} \cap Q$ had positive dimension, the rank on Q would not be constant, hence these fibres must consist of either 1 or 2 points: in other words, $\deg(\gamma|_Q)$ is either 1 or 2.

Recall from [16] (and refer to the excellent notes [17] for details) that, given a globally generated rank 2 vector bundle E on Q, and a fixed $(N+1)$-dimensional vector subspace $V_{N+1} \subset \mathrm{H}^0(E)$ generating E, that is, an epimorphism $V \otimes \mathcal{O}_Q \to E$, one can construct a morphism $\varphi_E : Q \to \mathbb{G}(1, N)$ from Q to the Grassmannian of lines in $\mathbb{P}^N = \mathbb{P}(V_{N+1})$.

Given the globally generated rank 2 vector bundle E from sequence (2.1), we remark that E is generated by a 6-dimensional subspace of its space of global sections, which gives a map $\varphi_E : Q \to \mathbb{G}(1, 5)$. Moreover $E(-2, -2) \simeq E^*$, therefore, up to a twist, E is the kernel of the 6×6 skew-symmetric matrix of linear forms A of constant rank 4. Consider the 6×6 matrix A' whose entries are the 4×4 principal minors' Pfaffians of the matrix A. Using a Laplace expansion for the Pfaffian of A, we see that locally E is generated by the rows of the matrix A'. In other words, the map φ_E coincides exactly with the Gauss map $\gamma|_Q$. According to the remark above, φ_E is therefore a regular map of degree 1 or 2. It is worth noticing that a similar reasoning in [16, Proposition 2.4] entailed that φ_E was an embedding. Here, the fact that our system of constant rank matrices is quadratic makes the difference: in what follows we will find examples where φ_E is not an embedding.

Consider also $\mathbb{P}(E)$, the projective bundle associated to E and let $\pi : \mathbb{P}(E) \to Q$ be the natural projection. The projective bundle $\mathbb{P}(E)$ has a tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ with the property that $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(1)) = E$, and there is an epimorphism

$$\pi^* E \to \mathcal{O}_{\mathbb{P}(E)}(1)$$

that induces an isomorphism $H^0(Q, E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$. Hence an epimorphism $V \otimes \mathcal{O}_Q \to E$ induces an epimorphism $V \otimes \mathcal{O}_{\mathbb{P}(E)} \to \pi^* E \to \mathcal{O}_{\mathbb{P}(E)}(1)$. Viceversa, an epimorphism $V \otimes \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_{\mathbb{P}(E)}(1)$ induces an epimorphism $V \otimes \mathcal{O}_Q \to \pi_*(\mathcal{O}_{\mathbb{P}(E)}(1)) = E$. Thus the map $\varphi_E : Q \to \mathbb{G}(1, N)$ defined by an epimorphism $V \otimes \mathcal{O}_Q \to E$ is equivalent to a map $\bar{\varphi}_E : \mathbb{P}(E) \to \mathbb{P}^N$ of the corresponding ruled variety.

Let Y be the image of $\bar{\varphi}_E$; since Q is a surface, Y is a threefold and the following equality holds:

$$c_2(E) = c_1^2(E) - \deg(\bar{\varphi}_E) \cdot \deg(Y) = 8 - \deg(\bar{\varphi}_E) \cdot \deg(Y). \tag{2.4}$$

Finally, remark that if the morphism $\bar{\varphi}_E$ has degree 1, the same is true for φ_E: if $\deg(\bar{\varphi}_E) = 1$, then for a general $y \in Y$ there exists a unique element $(x, [v]) \in \mathbb{P}(E)$ mapping to y, where $x \in Q$, and $[v] \in \pi^{-1}(x)$. The image $\varphi_E(x)$ represents the only line through x in the family parameterized by $\varphi_E(Q)$: in other words, $\deg(\varphi_E) = 1$.

3. Globally generated vector bundles with $c_1 = (2, 2)$ and main result.

There is a finite list of vector bundles of rank 2 on a smooth quadric surface Q that can appear in an exact sequence of the form (2.1). The first ones that come to mind are of course the ones decomposing as direct sum of two line bundles, that we list below.

Proposition 3.1. *Let E be a* decomposable *globally generated vector bundle of rank 2 on a smooth quadric surface Q, fitting into an exact sequence of type (2.1). Then*

$$E = \mathcal{O}_Q(a, b) \oplus \mathcal{O}_Q(2 - a, 2 - b),$$

with $0 \leqslant a, b \leqslant 2$, and the following cases can occur:

(DEC1) $a = b = 0$, $E = \mathcal{O}_Q \oplus \mathcal{O}_Q(2)$, $c_2(E) = 0$;
(DEC2) $a = b = 1$, $E = \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$, $c_2(E) = 2$;
(DEC3) $a = 2$, $b = 1$ *(and its symmetric)*, $E = \mathcal{O}_Q(2,1) \oplus \mathcal{O}_Q(0,1)$, $c_2(E) = 2$;
(DEC4) $a = 2$, $b = 0$ *(and its symmetric)*, $E = \mathcal{O}_Q(2,0) \oplus \mathcal{O}_Q(0,2)$, $c_2(E) = 4$.

Proof. A decomposable rank 2 bundle on Q is of the form $\mathcal{O}_Q(a,b) \oplus \mathcal{O}_Q(c,d)$; the fact that $c_1 = (2,2)$ implies $a+c = b+d = 2$, while global generation implies $a, b, c, d \geqslant 0$. $\square$

Indecomposable globally generated vector bundles with low first Chern class on a smooth quadric surface have been classified in the paper [10]. The authors prove that there exist such indecomposable and globally generated vector bundles of rank 2 on Q with $c_1 = (2,2)$ if and only if $c_2 = 3, 4, 5, 6, 8$.

In particular, there are no rank 2 globally generated vector bundles on Q satisfying $c_1(E) = (2,2)$ and $c_2(E) = 1$.

One of the tools used in [10] is the notion of index: a pair $(p,q) \in \mathbb{Z}^2$ is an *index* for a globally generated vector bundle E on Q if it is a maximal pair such that the twist $E(-p,-q)$ has global sections: $\mathrm{H}^0(E(-p,-q)) \neq 0$. Here one considers $(p,q) \geqslant (p',q')$ if and only if $p \geqslant p'$ and $q \geqslant q'$. Since the ordering is only partial, a vector bundle can have more than one index.

If (p,q) is an index of our bundle E with $p+q \geqslant 3$, then E decomposes as a direct sum $E = \mathcal{O}_Q(p,q) \oplus \mathcal{O}_Q(2-p,2-q)$. On the other hand, if $p+q \leqslant 1$ then E is (Mumford-Takemoto) stable, simply because E has rank 2, hence its stability is equivalent to the vanishing of the three cohomology groups $\mathrm{H}^0(E(-1))$, $\mathrm{H}^0(E(-2,0))$, and $\mathrm{H}^0(E(0,-2))$.

Lemma 3.2. *Let E be a vector bundle appearing in an exact sequence of type (2.1) as cokernel of a skew-symmetric constant rank matrix over the quadric surface Q, and let (p,q) be an index for E. Then (q,p) is an index for E; if $c_2(E) \leqslant 5$, then $(p,q) > (0,0)$, and if $c_2(E) = 6$, then $p = q = 0$.*

Proof. The first statement is an immediate consequence of the symmetry of the construction with respect to the two rulings. The second statement follows from the equality $\mathrm{h}^0(E(-1,0)) = 6 - c_2(E)$ obtained in the proof of Proposition 2.1. $\square$

Proposition 3.3. *Let E be a globally generated vector bundle of rank 2 on a smooth quadric surface Q, fitting into an exact sequence of type (2.1). If E has $(2,0)$ as index, then it decomposes as a direct sum $\mathcal{O}_Q(0,2) \oplus \mathcal{O}_Q(2,0)$.*

Proof. According to [10], if $(2,0)$ is an index, then E arises in the following extension:

$$0 \to \mathcal{O}_Q(2,0) \xrightarrow{\phi} E \to \mathcal{O}_Q(0,2) \to 0. \tag{3.1}$$

Now if E fits into sequence (2.1), then by Lemma 3.2 $(2,0)$ is an index if and only if $(0,2)$ is also an index, and this, again from [10], is equivalent to an extension of type

$$0 \to \mathcal{O}_Q(0,2) \to E \xrightarrow{\psi} \mathcal{O}_Q(2,0) \to 0. \tag{3.2}$$

The composition $\psi \circ \phi \in \mathrm{Hom}(\mathcal{O}_Q(2,0), \mathcal{O}_Q(2,0))$ can either be the zero map or a scalar multiple of the identity. If it were zero, then it would induce a non-zero map

$\mathcal{O}_Q(2,0) \to \mathrm{Ker}(\psi) = \mathcal{O}_Q(0,2)$, which is impossible. Hence it must be a scalar multiple of the identity, meaning that the extension (3.1) must split. $\square$

We are now ready to list all indecomposable bundles that can appear in the long exact sequence (2.1); the following result, combined with Proposition 3.1, gives a complete picture of all possible cases.

Proposition 3.4. *Let E be an* indecomposable *globally generated vector bundle of rank 2 on a smooth quadric surface Q, fitting into an exact sequence of type (2.1). Then one of the following occurs:*

(IND1) *E has $(1,1)$ as index, $c_2(E) = 3$, and there is a short exact sequence of the form*
$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \to E \to 0;$$
the restriction of E on both rulings of Q is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

(IND2) *E has $(1,1)$ as index, $c_2(E) = 4$, and there is a resolution of type*
$$0 \to \mathcal{O}_Q(-1) \to 2\mathcal{O}_Q \oplus \mathcal{O}_Q(1) \to E \to 0;$$
in this case the restriction of E to both rulings is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

(IND3) *E has indices $(1,0)$ and $(0,1)$ (hence it is a stable bundle), $c_2(E) = 4$, and it fits into the short exact sequence (and its symmetric equivalent)*
$$0 \to \mathcal{O}_Q(1,0) \to E \to \mathcal{I}_Z(1,2) \to 0,$$
where Z is a zero-dimensional scheme of degree 2. E restricts as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on one ruling, and as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on the other one.

(IND4) *E is a stable bundle having indices $(1,0)$ and $(0,1)$, $c_2(E) = 5$, fitting in the exact sequence*
$$0 \to \mathcal{O}_Q \to E \to \mathcal{I}_Z(2) \to 0,$$
where Z is a zero-dimensional scheme of degree 5.

(IND5) *E is a stable bundle having index $(0,0)$ and $c_2(E) = 6$, and it fits in the exact sequence*
$$0 \to \mathcal{O}_Q \to E \to \mathcal{I}_Z(2) \to 0,$$
where Z is a zero-dimensional scheme of degree 6.

Proof. Analysing the classification from [10] in light of Proposition 2.1, Lemma 3.2, and Proposition 3.3, we are able to rule out a few cases, and are left with the ones listed above. $\square$

A very natural question is whether all globally generated bundles appearing in Propositions 3.1 and 3.4 are attained with our construction: a positive answer to this question is our main result.

Theorem 3.5. *Let $X \subset \mathbb{P}^{14}$ be the cubic Pfaffian hypersurface parameterising 6×6 skew-symmetric matrices of rank at most 4. For all cases listed in Propositions 3.1 and 3.4, there exists a smooth quadric surface $Q \subset X$, not intersecting the Grassmannian $\mathbb{G}(1,5)$, giving rise to a long exact sequence of the form*

$$0 \to E^*(-1) \to 6\mathcal{O}_Q(-1) \to 6\mathcal{O}_Q \to E \to 0, \tag{2.1}$$

where the vector bundle E is of the desired type.

We devote the rest of the paper to a constructive proof of Theorem 3.5, that we achieve by giving explicit examples of the vector bundle E in all cases. To this end, we use three different techniques: in Section 4 we use decomposable bundles to settle cases (DEC1) and (DEC2) from Proposition 3.1. Then in Section 5 we introduce and develop a projection technique, that allows us to construct case (DEC4) from Proposition 3.1, as well as all 5 instances of Proposition 3.4. A different technique is needed for the remaining case (DEC3) of Proposition 3.1: this is done in Section 6.

4. Construction techniques, part 1: some decomposable bundles

As anticipated, in this section we construct examples of smooth quadric surfaces contained in $\check{\mathbb{G}}(1,5)_{sm}$ that give rise to the decomposable bundles $\mathcal{O}_Q \oplus \mathcal{O}_Q(2)$ and $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$, that is, cases (DEC1) and (DEC2) from Proposition 3.1.

Example 4.1. Consider the decomposable vector bundle (DEC1) $E = \mathcal{O}_Q \oplus \mathcal{O}_Q(2)$ from Proposition 3.1, having $c_2(E) = 0$. Since $h^0(E) = 10$, taking all the global sections of E we get a map from Q to $\mathbb{G}(1,9)$, whose image represents the lines of a cone over $v_2(Q)$. The map $\varphi_E : Q \to \mathbb{G}(1,5)$, associated to the exact sequence (2.1) as explained in Section 2.1, corresponds to a projection of this cone into $\mathbb{P}^5$, from a linear space disjoint from the vertex, hence the projected variety is a cone again. Therefore, if one has a smooth quadric surface in $\check{\mathbb{G}}(1,5)_{sm}$ corresponding to this bundle, by duality it must be contained in the linear span of a sub-Grassmannian $\mathbb{G}(1,H)$ where $H \subset \mathbb{P}^5$ is a hyperplane, the dual of the vertex of the cone. But $\mathbb{G}(1,H)$ has codimension 3 in its linear span $\mathbb{P}(\wedge^2 V_5) \simeq \mathbb{P}^9$, so a *general* quadric surface contained in this $\mathbb{P}^9$ will be disjoint from $\mathbb{G}(1,H)$. After a linear change of coordinates, one can assume that the matrix representing a constant rank map $6\mathcal{O}_Q(-1) \to 6\mathcal{O}_Q$ as in (2.1) is a general 6×6 skew-symmetric matrix of linear forms in four variables, suitably restricted to Q.

An explicit example is the following:

$$
\left(\begin{array}{ccccc|c}
\cdot & a & b & c & d & \cdot \\
-a & \cdot & a & b & c & \cdot \\
-b & -a & \cdot & d & a & \cdot \\
-c & -b & -d & \cdot & b & \cdot \\
-d & -c & -a & -b & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right), \tag{4.1}
$$

where for the reader's convenience we have adopted the convention to denote a zero in the matrix by a dot.

Matrix (4.1) has Pfaffian vanishing on the reducible cubic surface union of the plane $\Pi : \{a = 0\}$ and the quadric $Q : \{ad - bc = 0\}$. As expected, the vector bundle corresponding to the restriction of (4.1) to the plane Π is $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.

Example 4.2. Let $P \in \mathbb{P}^3$ be a point defined by equations $\ell_1 = \ell_2 = \ell_3 = 0$, with ℓ_i linear form in four variables. The 3×3 skew-symmetric map

$$\begin{pmatrix} \cdot & \ell_1 & \ell_2 \\ -\ell_1 & \cdot & \ell_3 \\ -\ell_2 & -\ell_3 & \cdot \end{pmatrix} \tag{4.2}$$

has constant rank 2 outside the point P, where it becomes the zero matrix. Therefore it has constant rank 2 on every quadric disjoint from P, and on such quadrics defines a long exact sequence of the form

$$0 \to \mathcal{O}_Q(-2) \to 3\mathcal{O}_Q(-1) \to 3\mathcal{O}_Q \to \mathcal{O}_Q(1) \to 0. \tag{4.3}$$

For instance, if $P = [1 : 0 : 0 : 1]$, a possible matrix is

$$\begin{pmatrix} \cdot & a-d & b \\ -(a-d) & \cdot & c \\ -b & -c & \cdot \end{pmatrix}. \tag{4.4}$$

Taking the direct sum of two 3×3 blocks of the type described above, we obtain a 6×6 matrix of constant rank 4 on a quadric Q not containing P, corresponding to the bundle (DEC2) $E = \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$.

There is an interesting difference of behavior depending on whether or not the two points centre of projections coincide.

More in detail, if the two centres of projections are distinct points $P \neq P'$ not on Q, we obtain a 6×6 matrix, which has constant rank 4 on $\mathbb{P}^3 \setminus \{P, P'\}$, and rank 2 at the two points. For instance, taking $P = [1 : 0 : 0 : 1]$, $P' = [0 : 1 : 1 : 0]$, we can construct the matrix

$$\left(\begin{array}{ccc|ccc} \cdot & a-d & b & \cdot & \cdot & \cdot \\ -a+d & \cdot & c & \cdot & \cdot & \cdot \\ -b & -c & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & a & b-c \\ \cdot & \cdot & \cdot & -a & \cdot & d \\ \cdot & \cdot & \cdot & -b+c & -d & \cdot \end{array}\right). \tag{4.5}$$

It is worth noticing that in this example the $\mathbb{P}^3$ having coordinates a, b, c, d is completely contained in $\breve{\mathbb{G}}(1,5)$: indeed, the rank of the matrix (4.5) is at most 4 on all of $\mathbb{P}^3$. Constant rank 4 is achieved on any quadric that does not contain the two points P and P', such as the smooth quadric $Q : \{ad - bc = 0\}$.

With the notation of subsection 2.1, the threefold Y corresponding to the matrix (4.5) turns out to have $\deg(Y) = 6$, as expected, and the morphism $\bar{\varphi}_E : \mathbb{P}(E) \to \mathbb{P}^5$ has degree 1 (see Remark 2.2), so the same is true for the morphism $\varphi_E : Q \to \mathbb{G}(1,5)$. The variety Y can be constructed by taking two isomorphic copies of Q in two disjoint $\mathbb{P}^3$s, then projecting them $2 : 1$ to two disjoint planes, and taking the union of the family of lines joining pairs of points that are images of isomorphic points. Its singular locus is formed by the two planes and a line.

If instead we use the same point P as centre of projection for both 3×3 blocks, the rank of the matrix drops to zero at P. For example, using the point $P = [1 : 0 : 0 : 1]$

as centre of projection, we obtain the matrix

$$(4.6)\qquad \left(\begin{array}{ccc|ccc} \cdot & a-d & b & \cdot & \cdot & \cdot \\ -a+d & \cdot & c & \cdot & \cdot & \cdot \\ -b & -c & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & a-d & b \\ \cdot & \cdot & \cdot & -a+d & \cdot & c \\ \cdot & \cdot & \cdot & -b & -c & \cdot \end{array}\right).$$

Its generic rank is again 4, meaning that again the corresponding $\mathbb{P}^3$ is completely contained in $\check{\mathbb{G}}(1,5)$, and drops to 0 exactly on the point P. Hence we can still consider the smooth quadric $Q : \{ad - bc = 0\}$. This time though, while the associated bundle is still case (DEC2) $E = \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$, the induced threefold Y is the smooth cubic scroll $\mathbb{P}^1 \times \mathbb{P}^2$ and thus the morphism $\bar{\varphi}_E : \mathbb{P}(E) \to \mathbb{P}^5$ has degree 2. Via a direct computation, we see that φ_E also has degree 2.

As we underlined in subsection 2.1, this situation never appears when dealing with linear spaces of dimension two, where φ_E is always an embedding $\mathbb{P}^2 \hookrightarrow \mathbb{G}(1,5)$.

5. Construction techniques, part 2: projection

An efficient method to construct spaces of matrices of constant rank consists in building bigger size matrices of a given rank, and then projecting them to smaller size matrices of the same rank. This technique was introduced in [16] for the case of $\mathbb{P}^2$, and later used in [18], but the results hold in more generality. Indeed, they were already extended to linear spaces of matrices of any size in [19, Proposition 5.1]; here, we wish to apply these results to the case of quadrics. In terms of bundles, this method amounts to expressing the desired rank 2 bundle E as quotient of a bundle of higher rank having the same Chern polynomial.

Let us denote by $\sigma_r(X)$ the r-th secant variety of a projective variety X, that is, the closure of the union of $(r-1)$-planes generated by r independent points of X. Now, assume that we have a surface S contained in the stratum $\sigma_r(\mathbb{G}(1,n)) \setminus \sigma_{r-1}(\mathbb{G}(1,n))$, i.e. S is a surface of skew-symmetric matrices of size $n+1$ and constant rank $2r$. If we project $\mathbb{P}^n = \mathbb{P}(V_{n+1})$ to $\mathbb{P}^{n-1} = \mathbb{P}(V_n)$ from a point O, this projection induces another projection π_O from $\mathbb{P}(\Lambda^2 V_{n+1})$ to $\mathbb{P}(\Lambda^2 V_n)$, whose centre is the subspace $\Lambda_O \subseteq \mathbb{G}(1,n)$ representing all lines through the point O.

It is well known (see [20] for a good reference) that point ω in the stratum $\sigma_r(\mathbb{G}(1,n)) \setminus \sigma_{r-1}(\mathbb{G}(1,n))$ can be written in the form $[v_1 \wedge w_1 + \cdots + v_r \wedge w_r]$, where the v_is and w_is are $2r$ linearly independent vectors; the corresponding points generate a subspace L_ω of $\mathbb{P}^n$ of dimension $2r-1$. The entry locus of ω is the sub-Grassmannian $\mathbb{G}(1, L_\omega)$, namely a point in $\mathbb{P}(\wedge^2 V_{n+1})$ belongs to some $(r-1)$-plane, which is r-secant to $\mathbb{G}(1,n)$ and contains ω, if and only if it belongs to $\mathbb{G}(1, L_\omega)$.

Proposition 5.1. *Let $S \subset \sigma_r(\mathbb{G}(1,n)) \setminus \sigma_{r-1}(\mathbb{G}(1,n))$ be a surface of skew-symmetric matrices of size $n+1$ and constant rank $2r$, and let $O \in \mathbb{P}^n$ be a point such that $S \cap \Lambda_O = \emptyset$. Then the matrices of $\pi_O(S)$ have constant rank $2r$ if and only if the point O does not belong to the union of the spaces L_ω, as ω varies in S. As a consequence, S can be projected to $\sigma_r(\mathbb{G}(1, 2r+1))$ so that its rank remains constant and equal to $2r$.*

Proof. The proofs of [16, Proposition 5.8 and Corollary 5.9] go through step by step; we report them for the reader's convenience.

Since $S \subset \sigma_r(\mathbb{G}(1,n))$, if ω is a point of S, it is the sum of r decomposable skew-symmetric tensors of the form $\omega = [v_1 \wedge w_1 + \cdots + v_r \wedge w_r]$; then $\pi_O(\omega) = [Av_1 \wedge Aw_1 + \cdots + Av_r \wedge Aw_r]$, where A is a matrix representing the projection π_O. Its rank is strictly less than r if and only if the vectors v_is and w_is can be chosen so that some summand $Av_i \wedge Aw_i$ vanishes: this means precisely that $O \in L_\omega$. The last statement follows from the fact that $\dim \bigcup_{\omega \in S} L_\omega \leqslant \dim S + 2r - 1 = 2r + 1$. □

As mentioned above, from the point of view of vector bundles, projecting to a smaller size matrix means that the associated bundle E appearing in sequence (2.1) is a quotient of a higher rank vector bundle F, in the sense that they fit into a short exact sequence of type

$$0 \to (\operatorname{rk} F - 2)\mathcal{O}_Q \to F \to E \to 0. \tag{5.1}$$

A logical way to construct bigger matrices (or higher rank bundles, if one prefers) is using "building blocks", that is, taking the vector bundle F in (5.1) to be a direct sum of two globally generated bundles with first Chern class $(1,1)$. In order to apply this method, we need to recall the classification of such bundles on Q of any rank.

Proposition 5.2. [10] *Let F be a globally generated vector bundle on a smooth quadric surface Q, with $c_1(F) = (1,1)$. Let r be the rank of F, and suppose that F has no trivial summands. Then F is one of the following:*

(i) $\mathcal{O}_Q(1)$, $r = 1$;
(ii) $\mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1)$, $r = 2$, $c_2 = 1$;
(iii) $\mathrm{T}\,\mathbb{P}^3(-1)|_Q$, $r = 3$, $c_2 = 2$;
(iv) $\mathcal{A}_P = \pi_P^*(\mathrm{T}\,\mathbb{P}^2(-1))$, *where* $\pi_P : Q \to \mathbb{P}^2$ *is the projection of centre* $P \notin Q$, $r = 2$, $c_2 = 2$.

The rank 3 bundle of case *(iii)* is the only non-trivial extension of $\mathcal{A}$ by $\mathcal{O}_Q$, as shown in [10, Proposition 5.4].

We now want to study the "building blocks" arising from each of the cases above.

The vector bundle $\mathcal{A}_P$ has a locally free resolution of type

$$0 \to \mathcal{O}_Q(-1) \to 3\mathcal{O}_Q \to \mathcal{A}_P \to 0, \tag{5.2}$$

where the first map is given by the equations of the point $P \notin Q$ (see [10, Proposition 3.5]). Dualising it, we get a short exact sequence of the form

$$0 \to \mathcal{A}_P^* \to 3\mathcal{O}_Q \to \mathcal{O}_Q(1) \to 0,$$

that shows that the building block corresponding to case *(i)* is a 3×3 matrix of the form (4.2) from Example 4.2, namely:

$$\begin{pmatrix} \cdot & \ell_1 & \ell_2 \\ -\ell_1 & \cdot & \ell_3 \\ -\ell_2 & -\ell_3 & \cdot \end{pmatrix},$$

where again ℓ_1, ℓ_2, ℓ_3 are the linear forms defining the point P.

To see what kind of building block corresponds to case *(ii)*, we remark that the decomposable bundle $\mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1)$ gives rise to an exact sequence of the form

$$0 \to \mathcal{O}_Q(-2,-1) \oplus \mathcal{O}_Q(-1,-2) \to 4\mathcal{O}_Q(-1) \to 4\mathcal{O}_Q \to \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \to 0.$$

It can be obtained in the following way: first, compose the short exact sequence

$$0 \to \mathcal{O}_Q(-1,0) \to 2\mathcal{O}_Q \to \mathcal{O}_Q(1,0) \to 0$$

with itself tensored with $\mathcal{O}_Q(-1)$:

$$\begin{tikzcd} \mathcal{O}_Q(-2,-1) \arrow[r, hook] & 2\mathcal{O}_Q(-1) \arrow[rr] \arrow[dr] & & 2\mathcal{O}_Q \arrow[r, two heads] & \mathcal{O}_Q(1,0). \\ & & \mathcal{O}_Q(-1,0) \arrow[ur] & & \end{tikzcd}$$

Then, take the direct sum of the sequence obtained with the symmetric one with respect to the rulings.

A corresponding building block is, for instance, the 4×4 skew-symmetric matrix:

$$\begin{pmatrix} \cdot & \cdot & a & b \\ \cdot & \cdot & c & d \\ -a & -c & \cdot & \cdot \\ -b & -d & \cdot & \cdot \end{pmatrix}. \tag{5.3}$$

It can be interpreted as a quadric surface contained in $\check{\mathbb{G}}(1,3)$, and more precisely it is a linear section of $\check{\mathbb{G}}(1,3)$ cut out by two hyperplanes. It represents a linear congruence of lines in $\mathbb{P}^3$, formed by the lines meeting two fixed skew lines in $\mathbb{P}^3$ (see for instance [6]).

The bundle appearing in case *(iii)* gives rise to the 5×5 skew-symmetric block

$$\left(\begin{array}{c|cccc} \cdot & a & b & c & d \\ \hline -a & \cdot & \cdot & \cdot & \cdot \\ -b & \cdot & \cdot & \cdot & \cdot \\ -c & \cdot & \cdot & \cdot & \cdot \\ -d & \cdot & \cdot & \cdot & \cdot \end{array}\right). \tag{5.4}$$

The map represented by matrix (5.4) is obtained by taking the direct sum of the Euler sequence on $\mathbb{P}^3$ restricted to Q:

$$0 \to \mathcal{O}_Q(-1) \to 4\mathcal{O}_Q \to \mathrm{T}\,\mathbb{P}^3(-1)|_Q \to 0,$$

and its dualised sequence. Alternatively, one can compose the Euler sequence above with its dualised sequence, and then glue an additional isomorphism $\mathcal{O}_Q \to \mathcal{O}_Q$:

$$\begin{tikzcd} (\mathrm{T}\,\mathbb{P}^3(-1)|_Q)^*(-1) \arrow[r, hook] & 5\mathcal{O}_Q(-1) \arrow[rr] \arrow[dr] & & 5\mathcal{O}_Q \arrow[r, two heads] & \mathrm{T}\,\mathbb{P}^3(-1)|_Q. \\ & & \mathcal{O}_Q \oplus \mathcal{O}_Q(-1) \arrow[ur] & & \end{tikzcd}$$

Finally, since the rank two bundle $\mathcal{A}_P$ is a quotient of $\mathrm{T}\,\mathbb{P}^3(-1)|_Q$, a corresponding building block can be obtained by projection from (5.4). For example, if we project from the point $P = [1:0:0:1]$, we can take

$$\begin{pmatrix} \cdot & b & c & d-a \\ -b & \cdot & \cdot & -b \\ -c & \cdot & \cdot & -c \\ a-d & b & c & \cdot \end{pmatrix}. \tag{5.5}$$

Let us now give more details on how we apply the projection technique: first, we consider a direct sum of two of the bundles with $c_1 = (1,1)$ appearing in Proposition 5.2, and the direct sum of two corresponding matrices. Then, we take a quotient of rank two of this bundle and the corresponding projection of the matrix. We compute the Chern class c_2 of the quotient, and we try to identify the rank two bundle so obtained.

The possible values of c_2 that one can obtain are the following:

(1) $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$: $c_2 = 2$, the rank is 2, there is no projection;
(2) $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1)$: $c_2 = 3$, E is of type (IND1);
(3) $\mathcal{O}_Q(1) \oplus \mathrm{T}\,\mathbb{P}^3(-1)|_Q$: $c_2 = 4$;
(4) $2\mathcal{O}_Q(1,0) \oplus 2\mathcal{O}_Q(0,1)$: $c_2 = 4$;
(5) $\mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \oplus \mathrm{T}\,\mathbb{P}^3(-1)|_Q$: $c_2 = 5$, E is of type (IND4);
(6) $2\,\mathrm{T}\,\mathbb{P}^3(-1)|_Q$: $c_2 = 6$, E is of type (IND5).

In the three instances (2), (5), (6), corresponding to second Chern class $3, 5, 6$ respectively, there is only one possible globally generated bundle having these invariants, namely the ones appearing in cases (IND1), (IND4), (IND5) from Propositions 3.1 and 3.4. We start by giving explicit examples for all these three cases.

Example 5.3. The quotient of type

$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \to E \to 0$$

has $c_2(E) = 3$, therefore the vector bundle E corresponds to case (IND1) in Proposition 3.4. A constant rank matrix obtained via the projection technique is:

$$\begin{pmatrix} \cdot & b+c & -a+d & -a+d & \cdot & -a+d \\ -b-c & \cdot & -b & -b & 0 & -b \\ a-d & b & \cdot & \cdot & a & -b \\ a-d & b & \cdot & \cdot & -c & d \\ \cdot & \cdot & -a & c & \cdot & \cdot \\ a-d & b & b & -d & \cdot & \cdot \end{pmatrix}. \tag{5.6}$$

Its Pfaffian defines the cubic surface union of the plane $\Pi : \{b + c = 0\}$ and the quadric surface $Q : \{ad - bc = 0\}$. With the help of Macaulay2, we get that Y is a threefold of degree 5 as expected, and that its singular locus is the union of the line $x_2 = x_3 = x_4 = x_5 = 0$ and the two points $[0:0:1:-1:0:0]$ and $[0:0:0:0:1:0]$.

Example 5.4. The quotient of type

$$0 \to 3\mathcal{O}_Q \to \mathcal{O}_Q(1,0) \oplus \mathcal{O}_Q(0,1) \oplus \mathrm{T}\,\mathbb{P}^3(-1)|_Q \to E \to 0$$

has $c_2(E) = 5$, therefore the bundle E corresponds to case (IND4) in Proposition 3.4. A constant rank matrix obtained via the projection technique is:

$$\begin{pmatrix} \cdot & -b & b & \cdot & \cdot & -a \\ b & \cdot & \cdot & -d & \cdot & (b-c) \\ -b & \cdot & \cdot & a & c & -d \\ \cdot & d & -a & \cdot & \cdot & c \\ \cdot & \cdot & -c & \cdot & \cdot & \cdot \\ a & -(b-c) & d & -c & \cdot & \cdot \end{pmatrix}. \tag{5.7}$$

Its Pfaffian defines the cubic surface union of the plane $\Pi : \{c = 0\}$ and the quadric surface $Q : \{ad - bc = 0\}$. We find that Y is a threefold of degree 3 as expected, and it is singular at four points.

Example 5.5. A quotient of type

$$0 \to 4\mathcal{O}_Q \to 2(\mathrm{T}\,\mathbb{P}^3(-1)|_Q) \to E \to 0 \tag{5.8}$$

has $c_2(E) = 6$, therefore the bundle E corresponds to case (IND5) in Proposition 3.4. A constant rank matrix obtained via the projection technique is the following:

$$\left(\begin{array}{cc|cccc} \cdot & a & b & c & d & \cdot \\ -a & \cdot & a & b & c & d \\ \hline -b & -a & \cdot & \cdot & \cdot & \cdot \\ -c & -b & \cdot & \cdot & \cdot & \cdot \\ -d & -c & \cdot & \cdot & \cdot & \cdot \\ \cdot & -d & \cdot & \cdot & \cdot & \cdot \end{array}\right). \tag{5.9}$$

Matrix (5.9) has generic rank 4, meaning that in this example the $\mathbb{P}^3$ having coordinates a, b, c, d is completely contained in $\check{\mathbb{G}}(1,5)$: the rank drops to 2 exactly on the point $P = [1 : 0 : 0 : 0]$ so we can work on the quadric $Q : \{a^2 - b^2 - c^2 + d^2 = 0\}$. The induced threefold is $Y = \bar{\varphi}_E(\mathbb{P}(E)) = \mathbb{P}^3$, therefore this is another instance where $\bar{\varphi}_E$ has degree 2. With the help of Macaulay2 one can check that the same is true for the map φ_E, that is, $\deg(\varphi_E) = 2$.

Remark 5.6. We note that in the 10×10 matrix, direct sum of two blocks of type (5.4), all the non-zero elements are contained in two rows and columns. This means that the $\mathbb{P}^3$ represented by this matrix is entirely contained in the tangent space to the Grassmannian $\mathbb{G}(1,9)$ at a point ℓ. After projecting and restricting to the quadric, we see that Q is contained in the tangent space to $\mathbb{G}(1,5)$ at the point ℓ' projection of ℓ. Therefore, when we apply the map $\varphi_E = \gamma|_Q$ to Q, the image is contained in $\mathbb{G}(1,H)$, where H is the $\mathbb{P}^3$ dual of ℓ'. Hence $\bar{\varphi}_E(\mathbb{P}(E))$ is contained in a $\mathbb{P}^3$. It follows that $\bar{\varphi}_E$ has degree 2 for any choice of projection.

Remark 5.7. An interesting observation is that the vector bundle E from (5.8), quotient of a direct sum of copies of $\mathrm{T}\,\mathbb{P}^3(-1)|_Q$, attains the maximal possible value of the second Chern class $c_2(E)$, from Proposition 2.1. This can be seen as a "quadratic counterpart" to [18, Proposition 3.2]: there, in the classification of dimension 2 linear spaces of matrices, an upper bound for the second Chern class was found. The bundles whose c_2

attained the maximal values were precisely the ones obtained on $\mathbb{P}^2$ as quotients of a direct sum of copies of $\mathrm{T}\,\mathbb{P}^2(-1)$.

So far we have used the projection technique to construct examples where the value of the second Chern class was associated with a unique vector bundle in Propositions 3.1 and 3.4. We now move on to the trickier case $c_2 = 4$: we will see that, depending on the choice of the centre of projection, we can obtain all three corresponding cases, namely (DEC4), (IND2), and (IND3). Remark that these three cases have different behaviours when restricted to the two rulings of the quadric: the decomposable case (DEC4) restricts as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on both rulings, (IND2) restricts as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on both rulings, and finally (IND3) restricts as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ on one ruling and as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on the other one.

Example 5.8. Taking the 8×8 skew-symmetric matrix direct sum of two blocks of type (5.3), having constant rank 4 on the quadric surface $Q : \{ac - bd = 0\}$, we obtain a quotient of type

$$0 \to 2\mathcal{O}_Q \to 2\mathcal{O}_Q(1,0) \oplus 2\mathcal{O}_Q(0,1) \to E \to 0. \tag{5.10}$$

After computer tests with Macaulay2, we ended up with the following three cases to be considered.

Projecting from the line L of equations $x_0 - x_2 = x_0 + x_1 - x_3 = 2x_2 - x_3 + x_4 = x_3 - x_4 - x_5 = 2x_4 + x_5 - 2x_6 = x_5 - 2x_7 = 0$, we obtain the following 6×6 skew-symmetric matrix whose rank is constant and equal to 4 on Q:

$$\left(\begin{array}{ccc|ccc} \cdot & a-b+c-d & 2a-b+2c-d & a+c & b+d & \cdot \\ -a+b-c+d & \cdot & 2c-d & c & d & \cdot \\ -2a+b-2c+d & -2c+d & \cdot & -a & \cdot & -a+b \\ \hline -a-c & -c & a & \cdot & -a-c & b \\ -b-d & -d & \cdot & a+c & \cdot & a-b+c-d \\ \cdot & \cdot & a-b & -b & -a+b-c+d & \cdot \end{array}\right). \tag{5.11}$$

Its Pfaffian vanishes on the quadric Q and on the plane $\Pi : \{a+b = 0\}$. The threefold Y from subsection 2.1 has degree 4, which is consistent with the fact that the associated bundle E in (5.10) has $c_2(E) = 4$. Its singular locus consists of four points.

One can see from Macaulay2 computations that the restriction of E to both rulings of the quadric is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, hence E is an indecomposable bundle corresponding to case (IND2) in Proposition 3.4.

Projecting from the line of equations $x_0 = x_1 - x_6 = x_2 + x_7 = x_3 - x_6 = x_4 - x_6 = x_5 = 0$ we obtain the following 6×6 skew-symmetric matrix whose rank is constant and equal to 4 on Q:

$$\left(\begin{array}{ccc|ccc} \cdot & \cdot & a & b & \cdot & \cdot \\ \cdot & \cdot & c & d & a & c \\ -a & -c & \cdot & \cdot & -b & -d \\ \hline -b & -d & \cdot & \cdot & a & c \\ \cdot & -a & b & -a & \cdot & c \\ \cdot & -c & d & -c & -c & \cdot \end{array}\right). \tag{5.12}$$

Its Pfaffian vanishes on the quadric Q and on the plane $\{b-c=0\}$. The threefold Y has degree 4, as expected. Its singular locus is the union of a conic and two points. The restriction of E to one of the rulings of the quadric is $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$ and to the other ruling is $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)$, therefore we are dealing with case (IND3) from Proposition 3.4.

Finally, projecting from the line of equation $x_0=x_2=x_3+x_7=x_4+x_1=x_5=x_6=0$ we obtain the following 6×6 skew-symmetric matrix whose rank is constant and equal to 4 on Q:

$$\left(\begin{array}{ccc|ccc} \cdot & a & b & \cdot & \cdot & \cdot \\ -a & \cdot & \cdot & -(b+c) & -d & \cdot \\ -b & \cdot & \cdot & -d & \cdot & \cdot \\ \hline \cdot & b+c & d & \cdot & \cdot & a \\ \cdot & d & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & -a & -c & \cdot \end{array}\right). \tag{5.13}$$

Its Pfaffian vanishes on the quadric Q and on the plane $\{b+c=0\}$. This time again the threefold Y turns out to have degree 4 as expected. The singular locus of Y consists of two disjoint conics.

Since the restriction of E to both rulings of the quadric is $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)$, this means that the matrix (5.13)'s cokernel is the decomposable bundle $\mathcal{O}_Q(2,0)\oplus\mathcal{O}_Q(0,2)$, and that we have constructed an example of case (DEC4) from Proposition 3.1.

Remark 5.9 (Geometrical interpretation of the three cases). The difference among the three cases in Example 5.8 can be explained looking at the position of the line $L\subset\check{\mathbb{P}}^7$, centre of projection, with respect to four 5-spaces we now introduce. The vector bundle $F:=2\mathcal{O}_Q(1,0)\oplus 2\mathcal{O}_Q(0,1)$ defines a map $\psi: Q\to\mathbb{G}(3,7)$ that can be interpreted as follows. Each direct summand defines a map $\pi_i: Q\to\mathbb{P}^1$; we fix 4 general lines ℓ_i, $i=1\dots,4$, in $\mathbb{P}^7$ and identify them with the codomains of the maps π_i. Then ψ sends a point $P\in Q$ to the $\mathbb{P}^3$ generated by the images $\pi_i(P)$. The duals of the lines ℓ_i are the 5-spaces S_{ℓ_i} under consideration.

We have a family of 3-spaces in $\mathbb{P}^7$ parametrized by Q, whose union is $\psi(Q)$; even if two general planes in the family do no intersect, there is a fundamental locus, that results to be union of two disjoint quadric surfaces G_1, G_2. When we consider the projection from the line L giving rise to a rank two bundle E, the threefold $Y=\bar{\varphi}_E(\mathbb{P}(E))$ is precisely the intersection of $\psi(Q)$ with the dual of L, which is a 5-space $\Lambda_L\subset\mathbb{P}^7$. The most special situation is when L meets all the 5-spaces S_{ℓ_i}: dually, Λ_L intersects both G_1 and G_2 along two conics; they are met by all the lines of Y and form the singular locus of Y. This means that E splits as $\mathcal{O}_Q(2,0)\oplus\mathcal{O}_Q(0,2)$ (case (DEC4)).

If L meets two of the 5-spaces S_{ℓ_i}, dually Λ_L intersects the quadrics G_1 and G_2 respectively along a conic and at two points, that form the singular locus of Y. This implies that the splitting type of E on the lines of the two rulings of Q is different, $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$ for one ruling and to $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)$ for the other; we get case (IND3).

Finally, the general case (IND2) is obtained when L is disjoint from all the 5-spaces, and Λ_L meets each quadric at two points, giving rise to the four singularities of Y; the splitting type is the same for the two families of lines, and it is $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$.

6. Construction techniques, part 3: extend & restrict

To conclude the proof of Theorem 3.5 there is only one case left, namely the decomposable bundle $\mathcal{O}_Q(2,1)\oplus\mathcal{O}_Q(0,1)$, case (DEC3) from Proposition 3.1: we could not find a construction with the techniques from the previous sections, and we needed a different approach.

Recall that we are looking at the smooth quadric surface Q as a quadratic system of skew-symmetric matrices of constant rank 4, of projective dimension 2. The spanned $\mathbb{P}^3 =< Q >$ cannot be entirely contained in $\check{\mathbb{G}}(1,5)_{sm}$, therefore two possibilities can occur. The first one is that $\mathbb{P}^3\subset\check{\mathbb{G}}(1,5)$ and $\mathbb{P}^3\cap\mathbb{G}(3,5)\neq\emptyset$: then the general plane $\mathbb{P}^2\subset\mathbb{P}^3$ will be a plane of constant rank matrices, and thus equivalent to one of the four types described in [8]. The other instance that can arise is that $\mathbb{P}^3\not\subseteq\check{\mathbb{G}}(1,5)$: then the intersection $\mathbb{P}^3\cap\check{\mathbb{G}}(1,5)$ will be a cubic surface S, union of a quadric Q and a plane Π. If the intersection of Π with the singular locus $\mathbb{G}(3,5)$ of $\check{\mathbb{G}}(1,5)$ were nonempty, it would contain a singular point for S, which would then necessarily be in Q; since $Q\cap\mathbb{G}(3,5)=\emptyset$, the plane Π is completely contained in $\check{\mathbb{G}}(1,5)_{sm}$, and is therefore a plane of constant rank matrices, again equivalent to one of the types in [8].

Thus, if one considers a plane in one of the 4 orbits of [8], extends the associated 6×6 skew-symmetric matrix to a $\mathbb{P}^3$, and then restricts it to a quadric surface $Q\subset\mathbb{P}^3$ that does not intersect the Grassmannian $\mathbb{G}(3,5)$, one obtains exactly a quadratic system of skew-symmetric matrices of constant rank 4. This should clarify why we call this technique "extend & restrict".

Example 6.1. We extend a plane of type Π_t from [8, Example 3] to a $\mathbb{P}^3$, and then intersect this $\mathbb{P}^3$ with the Pfaffian hypersurface: the intersection is a cubic surface, union of Π_t and a smooth quadric. The corresponding vector bundle on the plane is a Steiner bundle $\mathcal{E}$ on $\mathbb{P}^2$ fitting in a short exact sequence of type

$$0\to 2\mathcal{O}_{\mathbb{P}^2}(-1)\to 2\mathcal{O}_{\mathbb{P}^2}\to\mathcal{E}\to 0. \tag{6.1}$$

Implementing this idea with the help of Macaulay2, we obtain the following example:

$$\left(\begin{array}{cc|cccc} \cdot & \cdot & b & c & d & a \\ \cdot & \cdot & a & b & c & d \\ \hline -b & -a & \cdot & \cdot & a & -a \\ -c & -b & \cdot & \cdot & \cdot & \cdot \\ -d & -c & -a & \cdot & \cdot & \cdot \\ -a & -d & a & \cdot & \cdot & \cdot \end{array}\right), \tag{6.2}$$

whose Pfaffian vanishes, as expected, on the cubic surface in $\mathbb{P}^3$ union of the plane $\Pi:\{a=0\}$ and the quadric $Q:\{ab-c^2+bd-cd=0\}$.

The resulting threefold Y from subsection 2.1 had degree 6; therefore from equation (2.4) we learn that $\deg(\bar{\varphi}_E)=1$ and $c_2(E)=2$, and hence E splits as the direct sum of two line bundles. More in detail, Y is the union of cones having vertices on a given line; its singular locus is the union of the line itself together with a twisted cubic.

Furthermore, the splitting type of E on the two rulings of the quadric is $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)$ on the first ruling and $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$ on the second: we conclude that E is the vector bundle $\mathcal{O}_Q(2,1) \oplus \mathcal{O}_Q(0,1)$ (or its symmetric equivalent $\mathcal{O}_Q(1,2) \oplus \mathcal{O}_Q(1,0)$), that is, we have constructed an example corresponding to case (DEC3).

The proof of the main Theorem 3.5 is now completed.

A natural question arises from this new method: since we saw in Example 6.1 that a plane of type Π_t does "extend & restrict", one would like to show that this holds true for all the planes in the four different orbits.

Of course, a plane of type Π_5 from [8, Example 1], that is, a plane contained in $\mathbb{P}(\wedge^2 V_5) \subset \mathbb{P}(\wedge^2 V_6)$, associated to the split bundle $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, will extend & restrict to the decomposable bundle $\mathcal{O}_Q \oplus \mathcal{O}_Q(2)$, case (DEC1) from Proposition 3.1. Matrix (4.1) is an explicit example.

A plane of type Π_p from [8, Example 4], corresponding to the Null Correlation bundle on $\mathbb{P}^3$ restricted to a hyperplane, extends & restrict to an indecomposable bundle of type (IND1) in Proposition 3.4. An explicit example is the matrix:

$$\left(\begin{array}{ccc|ccc} \cdot & \cdot & d & a & b & c \\ \cdot & \cdot & a & c & d & \cdot \\ -d & -a & \cdot & b & \cdot & d \\ \hline -a & -c & -b & \cdot & \cdot & \cdot \\ -b & -d & \cdot & \cdot & \cdot & \cdot \\ -c & \cdot & -d & \cdot & \cdot & \cdot \end{array}\right). \tag{6.3}$$

Finally, a plane of type Π_g from [8, Example 2], whose corresponding vector bundle is the decomposable bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, extends & restrict to the decomposable bundle $\mathcal{O}_Q(2,0) \oplus \mathcal{O}_Q(0,2)$, case (DEC4) from Proposition 3.1. An explicit example is the matrix:

$$\left(\begin{array}{ccc|ccc} \cdot & a & -b & d & \cdot & \cdot \\ -a & \cdot & c & \cdot & -d & \cdot \\ b & -c & \cdot & \cdot & \cdot & -d \\ \hline -d & \cdot & \cdot & \cdot & a & b \\ \cdot & d & \cdot & -a & \cdot & c \\ \cdot & \cdot & d & -b & -c & \cdot \end{array}\right). \tag{6.4}$$

We conclude our paper with the following open problem.

Question: *is it possible to classify all possible pairs quadric-plane (Q, Π) that arise from an intersection of type $\check{\mathbb{G}}(1,5) \cap \check{\Delta} = Q \cup \Pi$, where $\check{\Delta}$ is the 3-space in $\check{\mathbb{P}}^{14}$ generated from a linear congruence in $\mathbb{P}^5$?*

Such a classification would allow to continue the geometrical description of linear congruences in $\mathbb{P}^5$ that was started in [12], clarifying the case when the cubic surface S is reducible as union of a smooth quadric and a plane meeting transversally.

Below we collected all pairs that we have obtained throughout the article: for each example of quadric Q we considered the position of the spanned $\mathbb{P}^3 =< Q >$ with respect to $\check{\mathbb{G}}(1,5)$, and then analysed either the residual plane, in the cases where $\mathbb{P}^3 \cap \check{\mathbb{G}}(1,5) = Q \cup \Pi$, or a general plane in $\mathbb{P}^3$, in the cases where $\mathbb{P}^3 \subset \check{\mathbb{G}}(1,5)$.

	V.b. on Q	**Residual/general $\mathbb{P}^2$**	**Associated v.b. on $\mathbb{P}^2$**
Matrix (4.1)	(DEC1)	residual, type Π_5	$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$
Matrices (4.5) and (4.6)	(DEC2)	general, type Π_g	$2\mathcal{O}_{\mathbb{P}^2}(1)$
Matrix (5.3)	(IND1)	residual, type Π_p	Null correlation of $\mathbb{P}^3$ restricted to $\mathbb{P}^2$
Matrix (5.4)	(IND4)	residual, type Π_5	$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$
Matrix (5.5)	(IND5)	general, type Π_t	Steiner bundle with resolution (6.1)
Matrix (5.11)	(IND2)	residual, type Π_g	$2\mathcal{O}_{\mathbb{P}^2}(1)$
Matrix (5.12)	(IND3)	residual, type Π_g	$2\mathcal{O}_{\mathbb{P}^2}(1)$
Matrix (5.13)	(DEC4)	residual, type Π_g	$2\mathcal{O}_{\mathbb{P}^2}(1)$
Matrix (6.2)	(DEC3)	residual, type Π_t	Steiner bundle with resolution (6.1)

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