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(Article begins on next page)

L^p SPECTRAL MULTIPLIERS ON THE FREE GROUP $N_{3,2}$

ALESSIO MARTINI AND DETLEF MÜLLER

ABSTRACT. Let L be a homogeneous sublaplacian on the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators. We prove a theorem of Mihlin-Hörmander type for the functional calculus of L, where the order of differentiability s > 6/2 is required on the multiplier.

1. INTRODUCTION

The free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators is the simply connected, connected nilpotent Lie group defined by the relations

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2,$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ is a basis of its Lie algebra (that is, the Lie algebra of the left-invariant vector fields on $N_{3,2}$). In exponential coordinates, $N_{3,2}$ can be identified with $\mathbb{R}^3_x \times \mathbb{R}^3_y$, where the group law is given by

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \land x'/2)$$

and $x \wedge x'$ denotes the usual vector product of $x, x' \in \mathbb{R}^3$. The family $(\delta_t)_{t>0}$ of automorphic dilations of $N_{3,2}$, defined by

$$\delta_t(x,y) = (tx, t^2y),$$

turns $N_{3,2}$ into a stratified group of homogeneous dimension Q = 9.

Let L be a homogeneous sublaplacian on $N_{3,2}$; without loss of generality, we may assume that $L = -(X_1^2 + X_2^2 + X_3^2)$. L is a self-adjoint operator on $L^2(N_{3,2})$, hence a functional calculus for L is defined via spectral integration and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator F(L) is bounded on $L^2(N_{3,2})$ whenever the "spectral multiplier" F is a bounded function. Here we are interested in giving a sufficient condition for the L^p -boundedness (for $p \neq 2$) of the operator F(L), in terms of smoothness properties of the multiplier F.

Let $W_2^s(\mathbb{R})$ denote the L^2 Sobolev space of (fractional) order s. Then our main result reads as follows.

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Theorem 1.1. Suppose that a function $F : \mathbb{R} \to \mathbb{C}$ satisfies

$$\sup_{t>0} \|\eta F(t\cdot)\|_{W_2^s} < \infty$$

for some s > 6/2 and some nonzero $\eta \in C_c^{\infty}(]0, \infty[)$. Then the operator F(L) is of weak type (1, 1) and bounded on $L^p(N_{3,2})$ for all $p \in]1, \infty[$.

Observe that the general multiplier theorem for homogeneous sublaplacians on stratified Lie groups by Christ [3] and Mauceri and Meda [16] requires the stronger regularity condition s > Q/2 = 9/2. To the best of our knowledge, in the case of $N_{3,2}$ none of the results and techniques known so far allowed one to go below the condition s > Q/2. Our result pushes the regularity assumption down to s > d/2 = 6/2, where d = 6 is the topological dimension of $N_{3,2}$. We conjecture that this condition is sharp.

The problem of L^p -boundedness for spectral multipliers on nilpotent Lie groups has a long history, and the theorem by Christ and Mauceri and Meda is itself an improvement of a series of previous results (see, e.g., [4, 8, 5]). Nevertheless it is still an open question, whether the homogeneous dimension in the smoothness condition may always be replaced by the topological dimension.

It has been known for a long time [10, 17] that such an improvement of the multiplier theorem holds true in the case of the Heisenberg and related groups (more precisely, for direct products of Métivier and abelian groups; see also [11, 14]). This class of groups, however, does not include $N_{3,2}$, nor any free 2-step nilpotent group $N_{n,2}$ on n generators (see [20, §3] for a definition), except for the smallest one, $N_{2,2}$, which is the 3-dimensional Heisenberg group. The free groups $N_{n,2}$ have in a sense the maximal structural complexity among 2-step groups, since every 2-step nilpotent Lie group is a quotient of a free one. Our result should then hopefully shed some new light and contribute to the understanding of the problem for general 2-step nilpotent Lie groups.

2. Strategy of the proof

The sublaplacian L is a left-invariant operator on $N_{3,2}$, hence any operator of the form F(L) is left-invariant too. Let $\mathcal{K}_{F(L)}$ then denote the convolution kernel of F(L). As shown, e.g., in [14, Theorem 4.6], the previous Theorem 1.1 is a consequence of the following L^1 -estimate.

Proposition 2.1. For all s > 6/2, for all compact sets $K \subseteq [0, \infty[$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that supp $F \subseteq K$,

(2.1)
$$\|\mathcal{K}_{F(L)}\|_{1} \leq C_{K,s} \|F\|_{W_{2}^{s}}.$$

Let $|\cdot|_{\delta}$ be any δ_t -homogeneous norm on $N_{3,2}$; take, e.g., $|(x,y)|_{\delta} = |x| + |y|^{1/2}$. The crucial estimate in the proof of [16] of the general theorem for stratified groups, that is,

(2.2)
$$\|(1+|\cdot|_{\delta})^{\alpha} \mathcal{K}_{F(L)}\|_{2} \leq C_{K,\alpha,\beta} \|F\|_{W_{2}^{\beta}}$$

for all $\alpha \ge 0$ and $\beta > \alpha$, implies (2.1) when s > 9/2, by Hölder's inequality. In order to push the condition down to s > 6/2, here we prove an enhanced version of (2.2), that is,

(2.3)
$$\|(1+|\cdot|_{\delta})^{\alpha} w^{r} \mathcal{K}_{F(L)}\|_{2} \leq C_{K,\alpha,\beta,r} \|F\|_{W_{\alpha}^{\beta}},$$

for some "extra weight" function w on $N_{3,2}$, and suitable constraints on the exponents α, β, r .

A similar approach is adopted in the mentioned works on the Heisenberg and related groups. However, in [17] the extra weight w is the full weight $1 + |\cdot|_{\delta}$, while [10] employs the weight w(x, y) = 1 + |x|. Here instead the weight w(x, y) = 1 + |y| is used, and (2.3) is proved under the conditions $\alpha \ge 0, 0 \le r < 3/2, \beta > \alpha + r$ (see Proposition 4.6 below).

The proof of (2.3) when $\alpha = 0$ is based on a careful analysis exploiting identities for Laguerre polynomials, somehow in the spirit of [4, 17, 19], but with additional complexity due, inter alia, to the simultaneous use of generalized Laguerre polynomials of different types. The estimate for arbitrary α is then recovered by interpolation with (2.2). An analogous strategy is followed in [15], where identities for Hermite polynomials are used in order to prove a sharp spectral multiplier theorem for Grushin operators.

3. A joint functional calculus

It is convenient for us to embed the functional calculus for the sublaplacian L in a larger functional calculus for a system of commuting leftinvariant differential operators on $N_{3,2}$. Specifically, the operators

$$(3.1) L, -iY_1, -iY_2, -iY_3$$

are essentially self-adjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [13]).

If **Y** denotes the "vector of operators" $(-iY_1, -iY_2, -iY_3)$, then we can express the convolution kernel $\mathcal{K}_{G(L,\mathbf{Y})}$ of the operator $G(L,\mathbf{Y})$ in terms of Laguerre functions (cf. [7]). Namely, for all $n, k \in \mathbb{N}$, let

$$L_{n}^{(k)}(u) = \frac{u^{-k}e^{u}}{n!} \left(\frac{d}{du}\right)^{n} (u^{k+n}e^{-u})$$

be the n-th Laguerre polynomial of type k, and define

$$\mathcal{L}_{n}^{(k)}(t) = (-1)^{n} e^{-t} L_{n}^{(k)}(2t).$$

Further, for all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\xi \in \mathbb{R}^3$, define ξ_{\parallel}^{η} and ξ_{\perp}^{η} by

$$\xi^{\eta}_{\parallel} = \langle \xi, \eta / |\eta| \rangle, \qquad \xi^{\eta}_{\perp} = \xi - \xi^{\eta}_{\parallel} \eta / |\eta|$$

Proposition 3.1. Let $G : \mathbb{R}^4 \to \mathbb{C}$ be in the Schwartz class, and set

(3.2)
$$m(n,\mu,\eta) = G((2n+1)|\eta| + \mu^2,\eta),$$

for all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^3$ with $\eta \neq 0$. Then

$$\mathcal{K}_{G(L,\mathbf{Y})}(x,y) = \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n,\xi_{\parallel}^{\eta},\eta) \,\mathcal{L}_n^{(0)}(|\xi_{\perp}^{\eta}|^2/|\eta|) \,e^{i\langle\xi,x\rangle} \,e^{i\langle\eta,y\rangle} \,d\xi \,d\eta$$

Proof. For all $\eta \in \mathbb{R}^3 \setminus \{0\}$, choose a unit vector $E_\eta \in \eta^{\perp}$, and set $\overline{E}_\eta = (\eta/|\eta|) \wedge E_\eta$; moreover, for all $x \in \mathbb{R}^3$, denote by $x_1^{\eta}, x_2^{\eta}, x_{\parallel}^{\eta}$ the components of x with respect to the positive orthonormal basis $E^{\eta}, \overline{E}^{\eta}, \eta/|\eta|$ of \mathbb{R}^3 .

For all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and all $\mu \in \mathbb{R}$, an irreducible unitary representation $\pi_{\eta,\mu}$ of $N_{3,2}$ on $L^2(\mathbb{R})$ is defined by

$$\pi_{\eta,\mu}(x,y)\phi(u) = e^{i\langle\eta,y\rangle} e^{i|\eta|(u+x_1^{\eta}/2)x_2^{\eta}} e^{i\mu x_{\parallel}^{\eta}} \phi(x_1^{\eta}+u)$$

for all $(x, y) \in N_{3,2}$, $u \in \mathbb{R}$, $\phi \in L^2(\mathbb{R})$. Following, e.g., [1, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of $N_{3,2}$, and the corresponding Fourier inversion formula:

(3.3)
$$f(x,y) = (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \operatorname{tr}(\pi_{\eta,\mu}(x,y) \, \pi_{\eta,\mu}(f)) \, |\eta| \, d\mu \, d\eta$$

for all $f : N_{3,2} \to \mathbb{C}$ in the Schwartz class and all $(x, y) \in N_{3,2}$, where $\pi_{\eta,\mu}(f) = \int_{N_{3,2}} f(z) \pi_{\eta,\mu}(z^{-1}) dz$.

Fix $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\mu \in \mathbb{R}$. The operators (3.1) are represented in $\pi_{\eta,\mu}$ as

(3.4)
$$d\pi_{\eta,\mu}(L) = -\partial_u^2 + |\eta|^2 u^2 + \mu^2, \quad d\pi_{\eta,\mu}(-iY_j) = \eta_j.$$

If h_n is the *n*-th Hermite function, that is,

$$h_n(t) = (-1)^n (n! \, 2^n \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt}\right)^n e^{-t^2},$$

and $\tilde{h}_{\eta,n}$ is defined by

$$\tilde{h}_{\eta,n}(u) = |\eta|^{1/4} h_n(|\eta|^{1/2}u),$$

then $\{\tilde{h}_{\eta,n}\}_{n\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R})$, made of joint eigenfunctions of the operators (3.4); in fact,

(3.5)
$$d\pi_{\eta,\mu}(L)\tilde{h}_{\eta,n} = (|\eta|(2n+1) + \mu^2)\tilde{h}_{\eta,n}, \\ d\pi_{\eta,\mu}(-iY_j)\tilde{h}_{\eta,n} = \eta_j\tilde{h}_{\eta,n}.$$

Moreover the corresponding diagonal matrix coefficients $\varphi_{\eta,\mu,n}$ of $\pi_{\eta,\mu}$ are given by

$$\begin{aligned} \varphi_{\eta,\mu,n}(x,y) &= \langle \pi_{\eta,\mu}(x,y)\tilde{h}_{\eta,n}, \tilde{h}_{\eta,n} \rangle \\ &= e^{i\langle \eta,y \rangle} e^{i\mu x_{\parallel}^{\eta}} |\eta|^{1/2} \int_{\mathbb{R}} e^{i|\eta|ux_{2}^{\eta}} h_{n}(|\eta|^{1/2}(u+x_{1}^{\eta}/2)) h_{n}(|\eta|^{1/2}(u-x_{1}^{\eta}/2)) \, du. \end{aligned}$$

The last integral is essentially the Fourier-Wigner transform of the pair (h_n, h_n) , whose Fourier transform has a particularly simple expression (cf. [9, formula (1.90)]); the parity of the Hermite functions then yields

$$\varphi_{\eta,\mu,n}(x,y) = e^{i\langle\eta,y\rangle} e^{i\mu x_{\parallel}^{\eta}} \frac{(-1)^n}{\pi|\eta|} \int_{\mathbb{R}^2} e^{iv_2 x_2^{\eta}} e^{iv_1 x_1^{\eta}} \\ \times \int_{\mathbb{R}} e^{-it(2v_1/|\eta|^{1/2})} h_n(t+v_2/|\eta|^{1/2}) h_n(t-v_2/|\eta|^{1/2}) dt dv,$$

that is,

(3.6)
$$\varphi_{\eta,\mu,n}(x,y) = \frac{1}{\pi |\eta|} e^{i\langle \eta,y \rangle} e^{i\mu x_{\parallel}^{\eta}} \int_{\mathbb{R}^2} e^{iv_1 x_1^{\eta}} e^{iv_2 x_2^{\eta}} \mathcal{L}_n^{(0)}(|v|^2/|\eta|) \, dv$$

(see [21, Theorem 1.3.4] or [9, Theorem 1.104]).

Note that $\mathcal{K}_{G(L,\mathbf{Y})} \in \mathcal{S}(N_{3,2})$ since $G \in \mathcal{S}(\mathbb{R}^4)$ (see [2, Theorem 5.2] or [12, §4.2]). Moreover

$$\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde{h}_{\eta,n} = G(|\eta|(2n+1) + \mu^2, \eta)\tilde{h}_{\eta,n}$$

by (3.5) and [18, Proposition 1.1], hence

$$\langle \pi_{\eta,\mu}(x,y)\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde{h}_{\eta,n},\tilde{h}_{\eta,n}\rangle = m(n,\mu,\eta)\,\varphi_{\eta,\mu,n}(x,y).$$

Therefore, by (3.3) and (3.6),

$$\begin{aligned} \mathcal{K}_{G(L,\mathbf{Y})}(x,y) &= (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \sum_{n \in \mathbb{N}} m(n,\mu,\eta) \,\varphi_{\eta,\mu,n}(x,y) \,|\eta| \,d\mu \,d\eta \\ &= \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n,\xi_3,\eta) \,e^{i\langle\eta,y\rangle} e^{i\langle\xi,(x_1^\eta,x_2^\eta,x_\|)\rangle} \mathcal{L}_n^{(0)}((\xi_1^2 + \xi_2^2)/|\eta|) \,d\xi \,d\eta. \end{aligned}$$

The conclusion follows by a change of variable in the inner integral. \Box

4. Weighted estimates

For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all n < 0. The following identities are easily obtained from the properties of Laguerre polynomials (see, e.g., [6, §10.12]).

Lemma 4.1. For all $k, n, n' \in \mathbb{N}$ and $t \in \mathbb{R}$,

(4.1)
$$\mathcal{L}_{n}^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_{n}^{(k+1)}(t),$$

(4.2)
$$\frac{d}{dt}\mathcal{L}_{n}^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_{n}^{(k+1)}(t),$$

(4.3)
$$\int_0^\infty \mathcal{L}_n^{(k)}(t) \, \mathcal{L}_{n'}^{(k)}(t) \, t^k \, dt = \begin{cases} \frac{(n+k)!}{2^{k+1}n!} & \text{if } n = n', \\ 0 & \text{otherwise} \end{cases}$$

We introduce some operators on functions $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$:

$$\tau f(n, \mu, \eta) = f(n + 1, \mu, \eta),$$

$$\delta f(n, \mu, \eta) = f(n + 1, \mu, \eta) - f(n, \mu, \eta),$$

$$\partial_{\mu} f(n, \mu, \eta) = \frac{\partial}{\partial \mu} f(n, \mu, \eta),$$

$$\partial_{\eta}^{\alpha} f(n, \mu, \eta) = \left(\frac{\partial}{\partial \eta}\right)^{\alpha} f(n, \mu, \eta),$$

for all $\alpha \in \mathbb{N}^3$. For all multiindices $\alpha \in \mathbb{N}^3$, we denote by $|\alpha|$ its length $\alpha_1 + \alpha_2 + \alpha_3$. We set moreover $\langle t \rangle = 2|t| + 1$ for all $t \in \mathbb{R}$.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}, \tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$; hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1+\tau)^{-1}f = \sum_{l\in\mathbb{N}} (-1)^l \tau^l f,$$

and therefore the operator $(1 + \tau)^q$ is well-defined for all $q \in \mathbb{Z}$.

Proposition 4.2. Let $G : \mathbb{R}^4 \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and let $m(n, \mu, \eta)$ be defined by (3.2). For all $\alpha \in \mathbb{N}^3$,

$$(4.4) \quad \int_{N_{3,2}} |y^{\alpha} \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^{2} dx dy$$

$$\leq C_{\alpha} \sum_{\iota \in I_{\alpha}} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^{3}} \int_{\mathbb{R}} |\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}} (1+\tau)^{|\beta^{\iota}|-k_{\iota}} m(n,\mu,\eta)|^{2}$$

$$\times \mu^{2b_{\iota}} |\eta|^{2|\gamma^{\iota}|-2|\alpha|-2k_{\iota}+|\beta^{\iota}|+1} \langle n \rangle^{|\beta^{\iota}|} d\mu d\eta$$

where I_{α} is a finite set and, for all $\iota \in I_{\alpha}$,

• $\gamma^{\iota} \in \mathbb{N}^3$, $l_{\iota}, k_{\iota} \in \mathbb{N}$, $\gamma^{\iota} \le \alpha$, $\min\{1, |\alpha|\} \le |\gamma^{\iota}| + l_{\iota} + k_{\iota} \le |\alpha|$,

• $b_{\iota} \in \mathbb{N}, \ \beta^{\iota} \in \mathbb{N}^3, \ b_{\iota} + |\beta^{\iota}| = l_{\iota} + 2k_{\iota}, \ |\gamma^{\iota}| + l_{\iota} + b_{\iota} \le |\alpha|.$

Proof. Proposition 3.1 and integration by parts allow us to write

$$(4.5) \quad y^{\alpha} \mathcal{K}_{G(L,\mathbf{Y})}(x,y) = \frac{2i^{|\alpha|}}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\left(\frac{\partial}{\partial \eta} \right)^{\alpha} \sum_{n \in \mathbb{N}} m(n,\xi_{\parallel}^{\eta},\eta) \mathcal{L}_n^{(0)}(|\xi_{\perp}^{\eta}|^2/|\eta|) \right] e^{i\langle\xi,x\rangle} e^{i\langle\eta,y\rangle} d\xi d\eta$$

From the definition of ξ_{\parallel}^{η} and ξ_{\perp}^{η} , the following identities are not difficult to obtain:

(4.6)
$$\frac{\partial}{\partial \eta_j} \xi_{\parallel}^{\eta} = (\xi_{\perp}^{\eta})_j \frac{1}{|\eta|}, \qquad \frac{\partial}{\partial \eta_j} (\xi_{\perp}^{\eta})_k = -\xi_{\parallel}^{\eta} \frac{\partial}{\partial \eta_j} \frac{\eta_k}{|\eta|} - (\xi_{\perp}^{\eta})_j \frac{\eta_k}{|\eta|^2},$$
$$\frac{\partial}{\partial \eta_j} \frac{|\xi_{\perp}^{\eta}|^2}{|\eta|} = -\xi_{\parallel}^{\eta} (\xi_{\perp}^{\eta})_j \frac{2}{|\eta|^2} - |\xi_{\perp}^{\eta}|^2 \frac{\eta_j}{|\eta|^3}.$$

The multiindex notation will also be used as follows:

$$(\xi_{\perp}^{\eta})^{\beta} = (\xi_{\perp}^{\eta})_{1}^{\beta_{1}} (\xi_{\perp}^{\eta})_{2}^{\beta_{2}} (\xi_{\perp}^{\eta})_{3}^{\beta_{3}}$$

for all $\xi, \eta \in \mathbb{R}$, with $\eta \neq 0$, and all $\beta \in \mathbb{N}^3$; consequently

$$|\xi_{\perp}^{\eta}|^{2} = (\xi_{\perp}^{\eta})^{(2,0,0)} + (\xi_{\perp}^{\eta})^{(0,2,0)} + (\xi_{\perp}^{\eta})^{(0,0,2)}.$$

Via these identities, one can prove inductively that, for all $\alpha \in \mathbb{N}^3$,

(4.7)
$$\begin{pmatrix} \frac{\partial}{\partial \eta} \end{pmatrix}^{\alpha} \sum_{n \in \mathbb{N}} m(n, \xi_{\parallel}^{\eta}, \eta) \mathcal{L}_{n}^{(0)}(|\xi_{\perp}^{\eta}|^{2}/|\eta|)$$
$$= \sum_{\iota \in I_{\alpha}} \sum_{n \in \mathbb{N}} \partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}} m(n, \xi_{\parallel}^{\eta}, \eta) (\xi_{\parallel}^{\eta})^{b_{\iota}} (\xi_{\perp}^{\eta})^{\beta^{\iota}} \Theta_{\iota}(\eta) \mathcal{L}_{n}^{(k_{\iota})}(|\xi_{\perp}^{\eta}|^{2}/|\eta|),$$

where $I_{\alpha}, \gamma^{\iota}, l_{\iota}, k_{\iota}, b_{\iota}, \beta^{\iota}$ are as in the statement above, while $\Theta_{\iota} : \mathbb{R}^{3} \setminus \{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $|\gamma^{\iota}| - |\alpha| - k_{\iota}$. For the inductive step, one employs Leibniz' rule, and when a derivative hits a Laguerre function, the identity (4.2) together with summation by parts is used.

Note that, for all compactly supported $f: \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$,

$$\sum_{n \in \mathbb{N}} f(n, \mu, \eta) \,\mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1+\tau) f(n, \mu, \eta) \,\mathcal{L}_n^{(k+1)}(t),$$

by (4.1). Since $1 + \tau$ is invertible, simple manipulations and iteration yield the more general identity

$$\sum_{n \in \mathbb{N}} f(n,\mu,\eta) \,\mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1+\tau)^{k'-k} f(n,\mu,\eta) \,\mathcal{L}_n^{(k')}(t),$$

for all $k, k' \in \mathbb{N}$. This formula allows us to adjust in (4.7) the type of the Laguerre functions to the exponent of ξ_{\perp} , and to obtain that

$$\left(\frac{\partial}{\partial\eta}\right)^{\alpha} \sum_{n\in\mathbb{N}} m(n,\xi_{\parallel}^{\eta},\eta) \,\mathcal{L}_{n}^{(0)}(|\xi_{\perp}^{\eta}|^{2}/|\eta|)$$

$$= \sum_{\iota\in I_{\alpha}} \sum_{n\in\mathbb{N}} \partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}}(1+\tau)^{|\beta^{\iota}|-k_{\iota}} m(n,\xi_{\parallel}^{\eta},\eta)$$

$$\times (\xi_{\parallel}^{\eta})^{b_{\iota}} \,(\xi_{\perp}^{\eta})^{\beta^{\iota}} \,\Theta_{\iota}(\eta) \,\mathcal{L}_{n}^{(|\beta^{\iota}|)}(|\xi_{\perp}^{\eta}|^{2}/|\eta|),$$

By plugging this identity into (4.5) and exploiting Plancherel's formula for the Fourier transform, the finiteness of I_{α} and the triangular inequality, we get that

A passage to polar coordinates in the ζ -integral and a rescaling then give that

and the conclusion follows by applying the orthogonality relations (4.3) for the Laguerre functions to the inner integral. \Box

Note that $\tau f(\cdot, \mu, \eta)$, $\delta f(\cdot, \mu, \eta)$ depend only on $f(\cdot, \mu, \eta)$; in other words, τ and δ can be considered as operators on functions $\mathbb{N} \to \mathbb{C}$. The next lemma will be useful in converting finite differences into continuous derivatives.

Lemma 4.3. Let $f : \mathbb{N} \to \mathbb{C}$ have a smooth extension $\tilde{f} : [0, \infty[\to \mathbb{C}, and let k \in \mathbb{N}$. Then

$$\delta^k f(n) = \int_{J_k} \tilde{f}^{(k)}(n+s) \, d\nu_k(s)$$

for all $n \in \mathbb{N}$, where $J_k = [0, k]$ and ν_k is a Borel probability measure on J_k . In particular

$$|\delta^k f(n)|^2 \le \int_{J_k} |\tilde{f}^{(k)}(n+s)|^2 \, d\nu_k(s)$$

for all $n \in \mathbb{N}$.

Proof. Iterated application of the fundamental theorem of integral calculus gives

$$\delta^k f(n) = \int_{[0,1]^k} \tilde{f}^{(k)}(n+s_1+\cdots+s_k) \, ds.$$

The conclusion follows by taking as ν_k the push-forward of the uniform distribution on $[0,1]^k$ via the map $(s_1,\ldots,s_k) \mapsto s_1 + \cdots + s_k$, and by Hölder's inequality.

We give now a simplified version of the right-hand side of (4.4), in the case where we restrict to the functional calculus for the sublaplacian L alone. In order to avoid divergent series, however, it is convenient at first to truncate the multiplier along the spectrum of **Y**.

Lemma 4.4. Let $\chi \in C_c^{\infty}(\mathbb{R})$ be supported in [1/2, 2], $K \subseteq [0, \infty[$ be compact and $M \in [0, \infty[$. If $F : \mathbb{R} \to \mathbb{C}$ is smooth and supported in K, and $F_M : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is given by

$$F_M(\lambda, \eta) = F(\lambda) \chi(|\eta|/M),$$

then, for all $r \in [0, \infty[$,

$$\int_{N_{3,2}} ||y|^r \, \mathcal{K}_{F_M(L,\mathbf{Y})}(x,y)|^2 \, dx \, dy \le C_{K,\chi,r} \, M^{3-2r} ||F||_{W_2^r}^2.$$

Proof. We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence we need to prove that

(4.8)
$$\int_{N_{3,2}} |y^{\alpha} \mathcal{K}_{F_M(L,\mathbf{Y})}(x,y)|^2 dx dy \le C_{K,\chi,\alpha} M^{3-2|\alpha|} ||F||^2_{W_2^{|\alpha|}}$$

for all $\alpha \in \mathbb{N}^3$. On the other hand, if

$$m(n,\mu,\eta) = F(|\eta|\langle n\rangle + \mu^2) \,\chi(|\eta|/M),$$

then the left-hand side of (4.8) can be majorized by (4.4), and we are reduced to proving that

$$(4.9) \quad \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}} (1+\tau)^{|\beta^{\iota}|-k_{\iota}} m(n,\mu,\eta)|^2 \, \mu^{2b_{\iota}} \, |\eta|^{2|\gamma^{\iota}|-2|\alpha|-2k_{\iota}+|\beta^{\iota}|+1} \\ \times \langle n \rangle^{|\beta^{\iota}|} \, d\mu \, d\eta \leq C_{K,\chi,\alpha} \, M^{3-2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2$$

for all $\iota \in I_{\alpha}$.

Consider first the case $|\beta^{\iota}| \geq k_{\iota}$. A smooth extension $\tilde{m} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ of m is defined by

$$\tilde{m}(t,\mu,\eta) = F(|\eta|(2t+1) + \mu^2) \,\chi(|\eta|/M).$$

Then, by Lemma 4.3,

$$\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}} (1+\tau)^{|\beta^{\iota}|-k_{\iota}} m(n,\mu,\eta) \\ = \sum_{j=0}^{|\beta^{\iota}|-k_{\iota}} \binom{|\beta^{\iota}|-k_{\iota}}{j} \int_{J_{\iota}} \partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \partial_{t}^{k_{\iota}} \tilde{m}(n+j+s,\mu,\eta) \, d\nu_{\iota}(s),$$

where $J_{\iota} = [0, k_{\iota}]$ and ν_{ι} is a suitable probability measure on J_{ι} ; consequently (4.9) will be proved if we show that

(4.10)
$$\sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{k_{\iota}} \tilde{m}(n+s,\mu,\eta)|^2 \,\mu^{2b_{\iota}} \,|\eta|^{2|\gamma^{\iota}|-2|\alpha|-2k_{\iota}+|\beta^{\iota}|+1} \\ \times \langle n \rangle^{|\beta^{\iota}|} \,d\mu \,d\eta \leq C_{K,\chi,\alpha} \,M^{3-2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2$$

for all $s \in [0, |\beta^{\iota}|]$. On the other hand, it is easily proved inductively that

$$\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \partial_{t}^{k_{\iota}} \tilde{m}(t,\mu,\eta) = \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \sum_{q=0}^{|\gamma^{\iota}|-v} \Psi_{\iota,v,q}(\eta) \langle t \rangle^{v} \mu^{2r-l_{\iota}} M^{-q} F^{(k_{\iota}+v+r)}(|\eta|\langle t \rangle + \mu^{2}) \chi^{(q)}(|\eta|/M)$$

for all $t \geq 0$, where $\Psi_{\iota,v,q} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $k_{\iota} + v + q - |\gamma^{\iota}|$; hence

$$(4.11) \quad |\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \partial_{t}^{k_{\iota}} \tilde{m}(t,\mu,\eta)|^{2} \leq C_{\chi,\alpha} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} M^{2k_{\iota}+2v-2|\gamma^{\iota}|} \langle t \rangle^{2v} \mu^{4r-2l_{\iota}} \\ \times |F^{(k_{\iota}+v+r)}(|\eta|\langle t \rangle+\mu^{2})|^{2} \tilde{\chi}(|\eta|/M),$$

where $\tilde{\chi}$ is the characteristic function of [1/2, 2], and we are using the fact that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta|/M) \neq 0$. Consequently the left-hand

side of (4.10) is majorized by

$$\begin{split} C_{\chi,\alpha} & \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} M^{2v-2|\alpha|+|\beta^{\iota}|+1} \sum_{n\in\mathbb{N}} \langle n \rangle^{|\beta^{\iota}|} \langle n+s \rangle^{2v} \\ & \times \int_{\mathbb{R}^{3}} \int_{\mathbb{R}} |F^{(k_{\iota}+v+r)}(|\eta| \langle n+s \rangle + \mu^{2})|^{2} \, \mu^{2b_{\iota}+4r-2l_{\iota}} \, \tilde{\chi}(|\eta|/M) \, d\mu \, d\eta \\ & \leq C_{\chi,\alpha} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} M^{2v-2|\alpha|+|\beta^{\iota}|+3} \sum_{n\in\mathbb{N}} \langle n+s \rangle^{|\beta^{\iota}|+2v} \\ & \times \int_{0}^{\infty} \int_{0}^{\infty} |F^{(k_{\iota}+v+r)}(\rho \langle n+s \rangle + \mu^{2})|^{2} \, \mu^{2b_{\iota}+4r-2l_{\iota}} \, \tilde{\chi}(\rho/M) \, d\mu \, d\rho \\ & \leq C_{\chi,\alpha} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} M^{2v-2|\alpha|+|\beta^{\iota}|+3} \int_{0}^{\infty} \int_{0}^{\infty} |F^{(k_{\iota}+v+r)}(\rho+\mu^{2})|^{2} \\ & \times \mu^{2b_{\iota}+4r-2l_{\iota}} \, \sum_{n\in\mathbb{N}} \langle n+s \rangle^{|\beta^{\iota}|+2v-1} \tilde{\chi}(\rho/(\langle n+s \rangle M)) \, d\mu \, d\rho, \end{split}$$

by passing to polar coordinates and rescaling. The last sum in n is easily controlled by $(\rho/M)^{|\beta^{\iota}|+2v}$, hence the left-hand side of (4.10) is majorized by

$$C_{\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \int_{0}^{\infty} \int_{0}^{\infty} |F^{(k_{\iota}+v+r)}(\rho+\mu^{2})|^{2} \mu^{2b_{\iota}+4r-2l_{\iota}} \rho^{|\beta^{\iota}|+2v} d\mu d\rho$$

$$\leq C_{K,\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \sup_{u\in[0,\max K]} \int_{0}^{\infty} |F^{(k_{\iota}+v+r)}(\rho+u)|^{2} d\rho,$$

because $2b_{\iota} + 4r - 2l_{\iota} \ge 0$ and $|\beta^{\iota}| + 2v \ge 0$ if r and v are in the range of summation, and supp $F \subseteq K$. Since moreover $k_{\iota} + v + r \le k_{\iota} + |\gamma^{\iota}| + l_{\iota} \le |\alpha|$, the last integral is dominated by $||F||_{W_{2}^{|\alpha|}}^{2}$ uniformly in r, v, u, and (4.10) follows.

Consider now the case $|\beta^{\iota}| < k_{\iota}$. Via the identity

$$(1+\tau)^{-1} = (1-\tau)(1-\tau^2)^{-1} = -\delta(1-\tau^2)^{-1} = -\delta\sum_{j=0}^{\infty} \tau^{2j},$$

together with Lemma 4.3, we obtain that

$$(4.12) \quad \partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \delta^{k_{\iota}} (1+\tau)^{|\beta^{\iota}|-k_{\iota}} m(n,\mu,\eta) \\ = (-1)^{k_{\iota}-|\beta^{\iota}|} \sum_{j=0}^{\infty} {j+k_{\iota}-|\beta^{\iota}|-1 \choose k_{\iota}-|\beta^{\iota}|-1} \int_{J_{\iota}} \partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{l_{\iota}} \partial_{t}^{2k_{\iota}-|\beta^{\iota}|} \tilde{m}(n+2j+s,\mu,\eta) \, d\nu_{\iota}(s),$$

where $J_{\iota} = [0, 2k_{\iota} - |\beta^{\iota}|]$ and ν_{ι} is a suitable probability measure on J_{ι} . Note that, because of the assumptions on the supports of F and χ , the sum on j

in the right-hand side of (4.12) is a finite sum, that is, the *j*-th summand is nonzero only if $\langle n+2j\rangle \leq 2M^{-1} \max K$; consequently, by applying the Cauchy-Schwarz inequality to the sum in *j*, and by (4.11),

$$\begin{split} |\partial_{\eta}^{\gamma^{\iota}}\partial_{\mu}^{l_{\iota}}\delta^{k_{\iota}}(1+\tau)^{|\beta^{\iota}|-k_{\iota}}m(n,\mu,\eta)|^{2} \\ &\leq C_{K,\alpha}\,M^{1+2|\beta^{\iota}|-2k_{\iota}}\sum_{j=0}^{\infty}\int_{J_{\iota}}|\partial_{\eta}^{\gamma^{\iota}}\partial_{\mu}^{l_{\iota}}\partial_{t}^{2k_{\iota}-|\beta^{\iota}|}\tilde{m}(n+2j+s,\mu,\eta)|^{2}\,d\nu_{\iota}(s) \\ &\leq C_{K,\chi,\alpha}\sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}}\sum_{v=0}^{|\gamma^{\iota}|}M^{1+2k_{\iota}+2v-2|\gamma^{\iota}|}\sum_{j=0}^{\infty}\int_{J_{\iota}}\langle n+2j+s\rangle^{2v}\mu^{4r-2l_{\iota}} \\ &\times|F^{(2k_{\iota}-|\beta^{\iota}|+v+r)}(|\eta|\langle n+2j+s\rangle+\mu^{2})|^{2}\,\tilde{\chi}(|\eta|/M)\,d\nu_{\iota}(s). \end{split}$$

Remember that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta|/M) \neq 0$. Hence the left-hand side of (4.9) is majorized by

$$\begin{split} C_{K,\chi,\alpha} &\sum_{r=[l_{\iota}/2]}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \int_{J_{\iota}} \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} \langle n+2j+s \rangle^{2v} \langle n \rangle^{|\beta^{\iota}|} \int_{\mathbb{R}^{3}} \int_{\mathbb{R}} M^{2+2v-2|\alpha|+|\beta^{\iota}|} \\ &\times \mu^{2b_{\iota}+4r-2l_{\iota}} |F^{(2k_{\iota}-|\beta^{\iota}|+v+r)}(|\eta| \langle n+2j+s \rangle + \mu^{2})|^{2} \tilde{\chi}(|\eta|/M) \, d\mu \, d\eta \, d\nu_{\iota}(s) \\ &\leq C_{K,\chi,\alpha} \sum_{r=[l_{\iota}/2]}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \int_{J_{\iota}} \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} \langle n+2j+s \rangle^{2v+|\beta^{\iota}|} \int_{0}^{\infty} \int_{0}^{\infty} M^{4+2v-2|\alpha|+|\beta^{\iota}|} \\ &\times \mu^{2b_{\iota}+4r-2l_{\iota}} |F^{(2k_{\iota}-|\beta^{\iota}|+v+r)}(\rho \langle n+2j+s \rangle + \mu^{2})|^{2} \tilde{\chi}(\rho/M) \, d\mu \, d\rho \, d\nu_{\iota}(s) \\ &\leq C_{K,\chi,\alpha} \sum_{r=[l_{\iota}/2]}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} M^{4+2v-2|\alpha|+|\beta^{\iota}|} \int_{0}^{\infty} \int_{0}^{\infty} |F^{(2k_{\iota}-|\beta^{\iota}|+v+r)}(\rho + \mu^{2})|^{2} \\ &\times \mu^{2b_{\iota}+4r-2l_{\iota}} \int_{J_{\iota}} \sum_{(n,j)\in\mathbb{N}^{2}} \frac{\tilde{\chi}(\rho/(\langle n+2j+s\rangle M))}{\langle n+2j+s \rangle^{1-2v-|\beta^{\iota}|}} \, d\nu_{\iota}(s) \, d\mu \, d\rho, \end{split}$$

by passing to polar coordinates and rescaling. The sum in (n, j) is dominated by $(\rho/M)^{2v+|\beta^{\iota}|+1}$, uniformly in $s \in J_{\iota}$, and moreover supp $F \subseteq K$. Therefore the left-hand side of (4.9) is majorized by

$$C_{K,\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_{\iota}/2\rceil}^{l_{\iota}} \sum_{v=0}^{|\gamma^{\iota}|} \sup_{u \in [0,\max K]} \int_{0}^{\infty} |F^{(2k_{\iota}-|\beta^{\iota}|+v+r)}(\rho+u)|^{2} d\rho.$$

On the other hand, $b_{\iota} + |\beta^{\iota}| = l_{\iota} + 2k_{\iota}$, hence $2k_{\iota} - |\beta^{\iota}| + v + r \leq 2k_{\iota} - |\beta^{\iota}| + |\gamma^{\iota}| + l_{\iota} = b_{\iota} + |\gamma^{\iota}| \leq |\alpha|$ if r and v are in the range of summation, therefore the last integral is dominated by $||F||_{W_{2}^{|\alpha|}}^{2}$ uniformly in r, v, u, and (4.9) follows.

Proposition 4.5. Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that supp $F \subseteq K$ for some compact set $K \subseteq [0, \infty[$. For all $r \in [0, 3/2[$,

$$\int_{N_{3,2}} \left| (1+|y|)^r \, \mathcal{K}_{F(L)}(x,y) \right|^2 \, dx \, dy \le C_{K,r} \|F\|_{W_2^r}^2.$$

Proof. Take $\chi \in C_c^{\infty}(]0, \infty[)$ such that supp $\chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t \in]0, \infty[$. Note that, if (λ, η) belongs to the joint spectrum of L, \mathbf{Y} , then $|\eta| \leq \lambda$. Therefore, if $k_K \in \mathbb{Z}$ is sufficiently large so that $2^{k_K-1} > \max K$, and if F_M is defined for all $M \in]0, \infty[$ as in Lemma 4.4, then

$$F(L) = \sum_{k \in \mathbb{Z}, \, k \leq k_K} F_{2^k}(L, \mathbf{Y})$$

(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}(L,\mathbf{Y})}$ given by Lemma 4.4. If r < 3/2, then the series $\sum_{k \leq k_K} (2^k)^{3/2-r}$ converges, thus

$$\int_{N_{3,2}} \left| |y|^r \ \mathcal{K}_{F(L)}(x,y) \right|^2 \, dx \, dy \le C_{K,r} \|F\|_{W_2^r}^2.$$

The conclusion follows by combining the last inequality with the corresponding one for r = 0.

Recall that $|\cdot|_{\delta}$ denotes a δ_t -homogeneous norm on $N_{3,2}$, thus $|(x, y)|_{\delta} \sim |x| + |y|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

Proposition 4.6. Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that supp $F \subseteq K$ for some compact set $K \subseteq [0, \infty[$. For all $r \in [0, 3/2[$, $\alpha \ge 0$ and $\beta > \alpha + r$, (4.13)

$$\int_{N_{3,2}} \left| (1+|(x,y)|_{\delta})^{\alpha} (1+|y|)^r \mathcal{K}_{F(L)}(x,y) \right|^2 dx \, dy \le C_{K,\alpha,\beta,r} \|F\|_{W_2^{\beta}}^2.$$

Proof. Note that $1 + |y| \leq C(1 + |(x, y)|_{\delta})^2$. Hence, in the case $\alpha \geq 0$, $\beta > \alpha + 2r$, the inequality (4.13) follows by the standard estimate [16, Lemma 1.2]. On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (4.13) is given by Proposition 4.5. The full range of α and β is then obtained by interpolation (cf. the proof of [16, Lemma 1.2]).

We can finally prove the fundamental L^1 -estimate, and consequently Theorem 1.1.

Proof of Proposition 2.1. Take $r \in [9/2 - s, 3/2[$. Then s - r > 3/2 + 3 - 2r, hence we can find $\alpha_1 > 3/2$ and $\alpha_2 > 3 - 2r$ such that $s - r > \alpha_1 + \alpha_2$.

Therefore, by Proposition 4.6 and Hölder's inequality,

$$\|\mathcal{K}_{F(L)}\|_{1}^{2} \leq C_{k,s} \|F\|_{W_{2}^{s}}^{2} \int_{N_{3,2}} (1+|(x,y)|_{\delta})^{-2\alpha_{1}-2\alpha_{2}} (1+|y|)^{-2r} \, dx \, dy.$$

The integral on the right-hand side is finite, because $2\alpha_1 > 3$, $\alpha_2 + 2r > 3$, and

$$(1+|(x,y)|_{\delta})^{-2\alpha_1-2\alpha_2}(1+|y|)^{-2r} \le C_s(1+|x|)^{-2\alpha_1}(1+|y|)^{-\alpha_2-2r},$$

and we are done.

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