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Hardy spaces on homogeneous trees
with flow measures

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Abstract

We consider a homogeneous tree endowed with a nondoubling flow measure μ of exponential growth and a probabilistic Laplacian $\mathcal{L}$ self-adjoint with respect to μ. We prove that the maximal characterization in terms of the heat and the Poisson semigroup of $\mathcal{L}$ and the Riesz transform characterization of the atomic Hardy space introduced in a previous work fail.

1 Introduction

Let $H^1(\mathbb{R}^n)$ be the Hardy space defined by

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) \; : \; |\nabla \Delta^{-1/2} f| \in L^1(\mathbb{R}^n)\},$$

where ∇ denotes the standard Euclidean gradient and Δ denotes the standard positive Euclidean Laplacian. It is a well-known fact that $H^1(\mathbb{R}^n)$ can be defined in several equivalent ways. Indeed, a celebrated result of C. Fefferman and E.M. Stein [8, 20] states the equivalence between $H^1(\mathbb{R}^n)$ and the maximal Hardy spaces defined via the heat semigroup and the Poisson semigroup of the Euclidean Laplacian. This result deeply depends on the doubling property of the Euclidean setting. Moreover, R. Coifman proved in [4] that $H^1(\mathbb{R}^n)$ admits an atomic characterization. Subsequently, Coifman and G. Weiss [5] introduced an atomic Hardy space in the setting of spaces of homogeneous type; we refer to [10, 11, 22, 24] for various maximal characterizations of the Hardy space in the context of such spaces. It is worth mentioning that, on $\mathbb{R}^n$ endowed with a nondoubling measure of polynomial growth, X. Tolsa [21] introduced an atomic Hardy space and proved that it can be characterized by a maximal operator as in the doubling setting. Furthermore, G. Mauceri and S. Meda defined an atomic Hardy space in the context of a Gaussian measure and the Ornstein–Uhlenbeck operator in $\mathbb{R}^n$ and, in the one-dimensional case, a maximal characterization of this space was proved in [16]. Many efforts have been made in order to study nondoubling (both continuous and discrete) settings on which these characterizations fail. See for example [15, 18, 19] for a contribution on a Lie group of exponential growth and on locally doubling manifolds and [3] for similar results in the context of a homogeneous tree and the combinatorial Laplacian.

In [13] the authors define an atomic Hardy space on a tree endowed with a nondoubling, locally doubling flow measure and they prove some classical results such as the duality between H^1 and BMO

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and good interpolation properties. In this paper, we focus on the homogeneous tree $\mathbb{T}_{q+1} = (V, E)$ of order $q + 1$, i.e., a tree in which every vertex has exactly $q + 1$ neighbours. We consider the metric measure space (V, d, μ) where d is the usual discrete distance on a graph and the measure μ is the canonical flow measure on a homogeneous tree (see Section 2.3 for a precise definition). It is worth recalling that (V, d, μ) is of exponential growth and does not satisfy the Cheeger isoperimetric inequality (we refer to [13] Section 2).

Inspired by [12], in Section 2.3 we introduce a Laplacian $\mathcal{L}$ self-adjoint on $L^2(\mu)$ that can be thought of as the natural Laplacian in this setting. We define the heat semigroup $(\mathcal{H}_t)_{t>0}$ and the Poisson semigroup $(\mathcal{P}_t)_{t>0}$ associated with $\mathcal{L}$, given respectively by $\mathcal{H}_t = e^{-t\mathcal{L}}$ and $\mathcal{P}_t = e^{-t\sqrt{\mathcal{L}}}$. It is a natural task to investigate whether the Hardy spaces defined in terms of the heat semigroup and the Poisson semigroup are equivalent to the atomic Hardy space $H^1_{at}(\mu)$ defined in [13] or the equivalent Hardy space defined in [2] (see Subsection 2.2 for its definition). We define the heat maximal operator and the Poisson maximal operator as

$$\mathcal{M}_h f = \sup_{t>0} |\mathcal{H}_t f|, \tag{1}$$

$$\mathcal{M}_P f = \sup_{t>0} |\mathcal{P}_t f|, \tag{2}$$

respectively. The aim of the first part of this work is to establish that the spaces

$$H^1_h(\mu) = \{f \in L^1(\mu) \; : \mathcal{M}_h f \in L^1(\mu)\}, \quad \|f\|_{H^1_h} = \|f\|_1 + \|\mathcal{M}_h f\|_1,$$
$$H^1_P(\mu) = \{f \in L^1(\mu) \; : \mathcal{M}_P f \in L^1(\mu)\}, \quad \|f\|_{H^1_P} = \|f\|_1 + \|\mathcal{M}_P f\|_1,$$

do not coincide with the atomic Hardy spaces $H^1_{at}(\mu)$ defined in [13] (see Section 2.2 for its precise definition). The following theorem is one of the main results of this work. It states that, although the inclusions $H^1_{at}(\mu) \subset H^1_h(\mu), H^1_{at}(\mu) \subset H^1_P(\mu)$ are valid, the maximal characterizations of the atomic Hardy space fail in our setting.

Theorem 1.1. *i) There exists a positive constant C such that*

$$\|\mathcal{M}_h f\|_1 \le C\|f\|_{H^1_{at}} \qquad \forall f \in H^1_{at}(\mu);$$

ii) there exists a positive constant C such that

$$\|\mathcal{M}_P f\|_1 \le C\|f\|_{H^1_{at}} \qquad \forall f \in H^1_{at}(\mu);$$

iii) there exists a function $g \in H^1_h(\mu) \cap H^1_P(\mu)$ which does not belong to $H^1_{at}(\mu)$.

It is possible to define the analogue of the Riesz transform in our setting, which we shall denote by $\mathcal{R}$ (see Section 5 for its precise definition). We introduce the Riesz Hardy space $H^1_R(\mu)$ defined by

$$H^1_R(\mu) = \{f \in L^1(\mu) \; : \mathcal{R}f \in L^1(\mu)\}, \tag{3}$$

which we endow with the natural norm $\quad \|f\|_{H^1_R} = \|f\|_1 + \|\mathcal{R}f\|_1.$

The following theorem establishes that the Riesz characterization of the atomic Hardy space fails.

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Theorem 1.2. *i) There exists a positive constant C such that*

$$\|\mathcal{R}f\|_1 \le C\|f\|_{H^1_{at}} \qquad \forall f \in H^1_{at}(\mu);$$

ii) there exists a function $g \in H^1_R(\mu)$ which does not belong to $H^1_{at}(\mu)$.

We point out that the function g in the above statement coincides with the function which appears in the statement of Theorem 1.1 *iii*).

This paper is organized as follows. In Section 2, we introduce the notation, preliminary notions and we provide useful estimates concerning the heat kernel and its gradient. Section 3 is devoted to the proof of Theorem 1.1 *i*) and *ii*), while in Section 4 we construct the function g of Theorem 1.1 *iii*). Finally, in Section 5, we prove Theorem 1.2.

Along the paper, C denotes a positive constant which may vary from line to line. However, when the exact values are unimportant for us, we use the standard notation $f_1(x) \lesssim f_2(x)$ to indicate that there exists a positive constant C, independent of the variable x but possibly depending on some involved parameters, such that $f_1(x) \le Cf_2(x)$ for every x. When both $f_1(x) \lesssim f_2(x)$ and $f_2(x) \lesssim f_1(x)$ are valid, we will write $f_1(x) \approx f_2(x)$.

2 Setting

2.1 Homogeneous trees and canonical flow

Let T be an unoriented tree, i.e., an unoriented connected graph with no cycles. We denote by V the set of vertices and by E the set of edges of T and we write $x \sim y$ when $(x, y) \in E$. If $x \sim y$ we say that x is a neighbour of y. Consider a sequence of vertices $\{x_j\}$ such that $x_j \sim x_{j+1}$. This naturally identifies an associated sequence of edges $\{e_j\}$, where e_j is the edge connecting x_j to x_{j+1}. We say that $\{x_j\}$ is a *path* if $\{e_j\}$ does not contain repeated edges. If the path $\gamma = \{x_j\}_{j=0}^n$ is finite, x_0 and x_n are called the endpoints of γ. The *discrete distance* $d(x, y)$ counts the minimum number of edges one has to cross while moving from x to y along a path. In a tree, for every couple of vertices (x, y), there exists a unique path (which we call geodesic) realizing such a distance. In this case, we denote by $[x, y]$ the geodesic connecting x to y. We denote by Γ the family of geodesics and by $S_r(x)$ and $B_r(x)$ the metric sphere and ball of center $x \in V$ and radius $r \ge 0$ respectively. Let $\mathbb{T}_{q+1}$ denote the *homogeneous tree* of order $q + 1$, namely, a tree such that each vertex has exactly $q + 1$ neighbours, where $q \in \mathbb{N} \setminus \{0\}$. Hereinafter, we assume $T = \mathbb{T}_{q+1} = (V, E)$ with $q \ge 2$.

We fix a distinguished point $o \in V$ which we call the *origin* of the tree. When $x \in V$ we denote the distance between x and o by $|x|$. We write Γ_0 for the family of half-infinite geodesics having an endpoint in the origin, $\Gamma_0 = \{\gamma = \{x_j\}_{j=0}^\infty \in \Gamma, x_0 = o\}$. The *boundary* of the tree ∂T is classically identified with the set of labels corresponding to elements of Γ_0,

$$\partial T = \{\zeta_\gamma : \ \gamma \in \Gamma_0\}.$$

It is standard to denote a half-infinite geodesic starting at the vertex x and ending at $\xi \in \partial T$ by $[x, \xi)$. A point $\xi_0 \in \partial T$ can be chosen to play the role of *root* of the tree. The role of such a point is to induce a partial order relation on V. We say that $x \ge y$ if and only if $x \in [y, \xi_0)$. We define the projection of x on the half-infinite geodesic $[o, \xi_0)$ as

$$\Pi_{\xi_0}(x) = \operatorname{argmin}_{y \in [o, \xi_0)} d(x, y),$$

and the *level* of x as

$$\ell(x) = d(o, \Pi_{\xi_0}(x)) - d(\Pi_{\xi_0}(x), x).$$

The fixed point $\xi_0 \in \partial T$ is called *mythical ancestor*. Note that $x \geq y$ if and only if $\ell(x) - \ell(y) = d(x, y)$.

The *predecessor* of x is the unique vertex $p(x)$ such that $x \sim p(x)$ and $\ell(p(x)) = \ell(x) + 1$, while y is a *son* of x if it belongs to the set $s(x) = \{y \sim x : \ell(y) = \ell(x) - 1\}$. More generally, for any integer $m \geq 2$ we denote by p^m the composition $p \circ p^{m-1}$, where $p^1 = p$. We define the *confluent* of $x, y \in V$ as the point

$$x \wedge y = \operatorname{argmax}\{\ell(z) : z \in [x, y]\} = \operatorname{argmin}\{\ell(z) : z \geq x, z \geq y\}.$$

We denote by $C(V)$ the set of complex-valued functions on V. If $A \subset V$ we write $|A|$ to denote the cardinality of A. We endow V with the measure μ defined as

$$\mu(A) = \sum_{x \in A} q^{\ell(x)},$$

where $A \subset V$. We recall that μ is a flow measure in the sense that

$$\mu(x) = q^{\ell(x)} = q q^{\ell(x)-1} = \sum_{y \in s(x)} \mu(y) \qquad \forall x \in V,$$

(see [13] for more information about flows). The measure μ was introduced by W. Hebisch and T. Steger in [12] and it represents the canonical flow measure on T, since it equally distributes the mass of a vertex among its sons.

For $p \in [1, \infty)$ we define $L^p(\mu) = \left\{ f \in C(V) : \|f\|_p = \left(\sum_{x \in V} |f(x)|^p \mu(x) \right)^{1/p} < +\infty \right\}$. We also define $L^\infty(\mu) = \{f \in C(V) : \|f\|_\infty = \sup_{x \in V} |f(x)| < +\infty\}$ and denote by $\#$ the counting measure on V. Finally, if $f \in C(V)$ we define the gradient of f as

$$\nabla f(x) = f(x) - f(p(x)) \qquad \forall x \in V.$$

2.2 Atomic Hardy space

In [13] the authors develop a Calderón–Zygmund theory with respect to locally doubling flow measures and a family of sets $\mathcal{F}$ which are called admissible trapezoids. Hereinafter, we say that a set R belongs to $\mathcal{F}$ if either $R = \{y\}$ for some $y \in V$ or there exist a vertex y_R and two positive integers h', h'', such that $R = \{y \leq y_R : h' \leq d(y, y_R) < h''\} = R_{h'}^{h''}(y_R)$ and $2 \leq \frac{h''}{h'} \leq 12$.

It is worth noticing that $\mu(R_{h'}^{h''}(y_R)) = q^{\ell(y_R)}(h'' - h')$. We introduce the atomic Hardy space $H_{at}^1(\mu)$ and its dual (we refer to [13] and [1] for more details). A function a is a $(1, \infty)$-atom if the following hold:

(i) a is supported in a set $R \in \mathcal{F}$;

(ii) $\|a\|_\infty \leq \frac{1}{\mu(R)}$;

(iii) $\sum_{x \in R} a(x) \mu(x) = 0$.

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The atomic Hardy space is defined as

$$H_{at}^1(\mu) = \left\{ f \in L^1(\mu) \ : f = \sum_j \lambda_j a_j, \ \{\lambda_j\}_j \in \ell^1(\mathbb{N}), \ a_j \ (1,\infty) - atom \right\},$$

endowed with the norm $\|f\|_{H_{at}^1} = \inf\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j, \ a_j \ (1,\infty) - atom\}$.
The space of functions of bounded mean oscillation is

$$BMO(\mu) = \left\{ f \in C(V) \ : \ \sup_{R \in \mathcal{F}} \frac{1}{\mu(R)} \sum_{x \in R} |f(x) - f_R|\mu(x) < +\infty \right\},$$

where f_R denotes the average of f on R.
The dual of $H_{at}^1(\mu)$ can be identified with $BMO(\mu)$, see [13, Th. 4.10]. In particular, if $f \in BMO(\mu)$ and a is a $(1,\infty)$-atom, then

$$\left| \sum_{x \in V} f(x)a(x)\mu(x) \right| \lesssim \|f\|_{BMO}\|a\|_{H_{at}^1}, \tag{4}$$

where $\|f\|_{BMO} = \sup_{R \in \mathcal{F}} \frac{1}{\mu(R)} \sum_{x \in R} |f(x) - f_R|\mu(x)$.

2.3 Laplacians and Heat kernel

Let Δ denote the combinatorial Laplacian, namely the operator defined on every $f \in C(V)$ by

$$\Delta f(x) = \frac{1}{q+1} \sum_{y \sim x} (f(x) - f(y)) \qquad \forall x \in V.$$

The Laplacian Δ is bounded on $L^p(\#)$ for any $p \in [1,\infty]$. Moreover, the $L^2(\#)$ spectrum of Δ is $[b, 2-b]$, where $b = \frac{(\sqrt{q}-1)^2}{q+1}$ (see [6]). We refer to [9] for more information about Δ and the spherical analysis on T.

Consider the operator $A : C(V) \to C(V)$ defined on $f \in C(V)$ by

$$Af(x) = \frac{1}{2}\left(\frac{1}{q} \sum_{y \in s(x)} f(y) + f(p(x)) \right) \qquad \forall x \in V. \tag{5}$$

Observe that we can associate to A a probabilistic transition matrix, in the sense that

$$Af(x) = \sum_{y \in V} A(x,y)f(y) \ \text{ and } \ \sum_{y \in V} A(x,y) = 1, \tag{6}$$

where $A(x,y) = \begin{cases} \frac{1}{2q} & y \in s(x), \\ \frac{1}{2} & y = p(x), \\ 0 & \text{otherwise.} \end{cases}$

We define the operator

$$\mathcal{L} = I - A, \tag{7}$$

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which is the natural Laplacian in our setting. By (6), it is clear that $\mathcal{L}$ is a Laplacian from the probabilistic viewpoint (for more information about random walks and Laplacians on graphs we refer to [23]). It is also easy to see that $\mathcal{L}$ is self-adjoint on $L^2(\mu)$. Such operator was originally introduced in [12]. It is worth noticing that

$$\mathcal{L} = \frac{1}{1-b}\mu^{-1/2}(\Delta - bI)\mu^{1/2}. \tag{8}$$

Using the fact that the pointwise multiplication by $\mu^{1/2}$ is a surjective isometry between $L^2(\#)$ and $L^2(\mu)$ and the pointwise multiplication by $\mu^{-1/2}$ is its inverse, the previous identity implies that $L^2(\mu)$-spectrum of $\mathcal{L}$ is [0,2].

Next, we shall define the heat kernel associated to $\mathcal{L}$ and provide some useful estimates. We denote by $\mathcal{H}_t$ the operator $e^{-t\mathcal{L}}$, $t > 0$. Its integral kernel with respect to the measure μ is the function $H_t(\cdot, \cdot)$ such that for $f \in C(V)$

$$\mathcal{H}_t f(x) = \sum_{y \in V} H_t(x,y) f(y)\mu(y) \qquad \forall x \in V.$$

By (8) we can explicitly write H_t in terms of the heat kernel associated to Δ on T, which we shall denote by h_t. By the Spectral Theorem

$$H_t(x,y) = e^{\frac{bt}{1-b}} q^{(-\ell(y)-\ell(x))/2} h_{\frac{t}{1-b}}(x,y) \qquad \forall t > 0, x, y \in V. \tag{9}$$

Notice that, since A is a transition matrix

$$\sum_{y \in V} H_t(x,y)\mu(y) = 1, \qquad \forall t \in \mathbb{R}^+, \ x \in V; \tag{10}$$

moreover, since $h_t(x,y) = h_t(y,x)$ we deduce that

$$H_t(x,y) = H_t(y,x) \qquad \forall t > 0, x, y \in V.$$

In the following, we denote by $h_t^{\mathbb{Z}}$ the heat kernel associated to the combinatorial Laplacian on $\mathbb{Z}$ and, with a slight abuse of notation, we denote by $h_t^{\mathbb{Z}}(j)$ the function $h_t^{\mathbb{Z}}(j,0)$.

In the next proposition, we collect some results of M. Cowling, S. Meda, and A.G. Setti (see [6, Lemma 2.4., Prop. 2.5]) which provide an explicit expression and a sharp approximation of h_t that will be useful in the sequel.

Proposition 2.1 ([6]). *The following hold for all $t > 0$, $x \in V$ and $j \in \mathbb{N}$:*

i) $h_t(x,y) = \dfrac{2e^{-bt}}{(1-b)t} q^{-d(x,y)/2} \displaystyle\sum_{k=0}^{\infty} q^{-k}(d(x,y) + 2k + 1) h_{t(1-b)}^{\mathbb{Z}}(d(x,y) + 2k + 1),$

ii) $h_t^{\mathbb{Z}}(j) \approx \dfrac{e^{-t+\sqrt{j^2+t^2}}}{(1+j^2+t^2)^{1/4}} \left(\dfrac{t}{j + \sqrt{j^2+t^2}} \right)^j,$

iii) $h_t^{\mathbb{Z}}(j) - h_t^{\mathbb{Z}}(j+2) = \dfrac{2(j+1)}{t} h_t^{\mathbb{Z}}(j+1).$

Using i) and (9), we easily get

$$H_t(x, y) = q^{-\ell(x)/2 - \ell(y)/2} e^{bt/(1-b)} h_{t/(1-b)}(x, y) = Q(x, y) J_t(x, y),$$

where

$$Q(x, y) = q^{[-\ell(x)/2 - \ell(y) - d(x,y)]/2} \tag{11}$$

and

$$J_t(x, y) = \frac{2}{t} \sum_{k=0}^{\infty} q^{-k} (d(x, y) + 2k + 1) h_t^{\mathbb{Z}}(d(x, y) + 1). \tag{12}$$

Then, by means of i), we obtain the following estimate for H_t

$$H_t(x, y) \approx \frac{Q(x, y)}{t} (d(x, y) + 1) h_t^{\mathbb{Z}}(d(x, y) + 1). \tag{13}$$

We now introduce some notation. For every $n \in \mathbb{N}$ we define the function $s_n : \mathbb{R}^+ \to \mathbb{R}$ by

$$s_n(t) = (n+1) \frac{e^{-t} e^{\sqrt{(n+1)^2 + t^2}} \left(\frac{t}{n+1+\sqrt{(n+1)^2 + t^2}} \right)^{n+1}}{t(1 + (n+1)^2 + t^2)^{1/4}} \qquad \forall t > 0. \tag{14}$$

Observe that by (13) and Proposition 2.1 ii)

$$H_t(x, y) \approx Q(x, y) s_{d(x,y)}(t). \tag{15}$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\varphi(t) = -t + \sqrt{1 + t^2} + \log t - \log(1 + \sqrt{1 + t^2}) \qquad \forall t > 0. \tag{16}$$

We have that

$$s_n(t(n+1)) = \frac{e^{(n+1)\varphi(t)}}{t(1 + (n+1)^2 + t^2(n+1)^2)^{1/4}}.$$

It is easy to verify that φ is negative, increasing and

$$\varphi(t) \leq \frac{1}{2t} - \log\left(1 + \frac{1}{t}\right) \qquad \forall t > 0. \tag{17}$$

We now state a technical lemma involving the function s_n defined in (14).

Lemma 2.2. *The following hold*

i) $\sup_{t>0} s_n(t) \lesssim \frac{1}{(n+1)^2}$.

ii) $\sup_{t>0} \frac{n}{t} s_n(t) \lesssim \frac{1}{(n+1)^3}$.

Proof. We distinguish three different cases, namely, we estimate the supremum of the above functions when $t \geq (n+1)^2$, $n+1 \leq t < (n+1)^2$ and $0 < t < n+1$.

Case 1. Observe that

$$\sup_{t \geq (n+1)^2} s_n(t) = \sup_{t > n+1} s_n(t(n+1)) = \sup_{t > n+1} \frac{e^{(n+1)\varphi(t)}}{t[1 + (n+1)^2(1+t^2)]^{1/4}}.$$

Since φ is negative on $\mathbb{R}^+$ it follows

$$\sup_{t \geq (n+1)^2} s_n(t) \leq \frac{1}{(n+1)^2} \quad \text{and} \quad \sup_{t \geq (n+1)^2} \frac{n}{t} s_n(t) \leq \frac{1}{(n+1)^3}.$$

Case 2. When $t \in [n+1, (n+1)^2)$ we can write $t = (n+1)\alpha$ with $\alpha \in [1, n+1)$ and

$$\sup_{n+1 \leq t < (n+1)^2} s_n(t) = \sup_{1 \leq \alpha < n+1} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1 + (n+1)^2(1+\alpha^2)]^{1/4}}.$$

By using (17) and the fact that $(1 + 1/\alpha)^\alpha \geq 2$ for all $\alpha \geq 1$, we get

$$\frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1 + (n+1)^2(1+\alpha^2)]^{1/4}} \leq \frac{\left(\frac{e^{1/2}}{(1+1/\alpha)^\alpha}\right)^{(n+1)/\alpha}}{\alpha^{3/2}(n+1)^{1/2}} \leq \frac{\left(\frac{e^{1/2}}{2}\right)^{(n+1)/\alpha}}{\alpha^{3/2}(n+1)^{1/2}}.$$

Next, we use that $\left(\frac{e^{1/2}}{2}\right)^{(n+1)/\alpha} \lesssim \frac{\alpha^3}{(n+1)^3}$ to obtain

$$\sup_{1 \leq \alpha < n+1} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1 + (n+1)^2(1+\alpha^2)]^{1/4}} \lesssim \sup_{1 \leq \alpha < n+1} \frac{\alpha^{3/2}}{(n+1)^{7/2}} \leq \frac{1}{(n+1)^2}$$

and

$$\sup_{1 \leq \alpha < n+1} \frac{n}{(n+1)\alpha} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1 + (n+1)^2(1+\alpha^2)]^{1/4}} \lesssim \sup_{1 \leq \alpha < n+1} \frac{\alpha^{1/2}}{(n+1)^{7/2}} \leq \frac{1}{(n+1)^3}.$$

Case 3. In this last case $t \in (0, n+1)$ thus we can write $t = (n+1)\alpha$ with $\alpha \in (0,1)$. By using the fact that φ is increasing and negative, we get

$$s_n(\alpha(n+1)) = \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1 + (n+1)^2(1+\alpha^2)]^{1/4}} \leq e^{n\varphi(\alpha)} \frac{e^{\varphi(\alpha)}}{\alpha}$$

$$\lesssim e^{n\varphi(1)} \lesssim \frac{1}{(n+1)^2},$$

where we have used that $\frac{e^{\varphi(\alpha)}}{\alpha} \lesssim 1$ when $\alpha \in (0,1)$. If $n = 0$, then *ii)* follows trivially. Assume $n \geq 1$ and by repeating the same argument

$$\frac{n}{(n+1)\alpha} s_n(\alpha(n+1)) \lesssim e^{(n-1)\varphi(\alpha)} \frac{e^{2\varphi(\alpha)}}{\alpha^2} \lesssim \frac{1}{(n+1)^3}.$$

This concludes the proof. $\qquad\square$

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Combining the above lemma with (15), we obtain that

$$\sup_{t>0} H_t(x,y) \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2} \qquad \forall x,y \in V, \tag{18}$$

and

$$\sup_{t>0} \frac{d(x,y)}{t} H_t(x,y) \lesssim \frac{Q(x,y)}{(d(x,y)+1)^3} \qquad \forall x,y \in V. \tag{19}$$

In the next results we recall some pointwise and integral estimates concerning the gradient of the heat kernel which were proved in [14].

Lemma 2.3. *Assume* $x \not\leq y$ *where* $x,y \in V$*. Then,*

$$i) \qquad |H_t(x,y) - H_t(x,p(y))| \lesssim \max\left\{ \frac{d(x,y)H_t(x,p(y))}{t}, \frac{H_t(x,y)}{d(x,y)+1} \right\},$$

$$ii) \qquad \sup_{t>0} |H_t(x,y) - H_t(x,p(y))| \lesssim \frac{Q(x,y)}{(d(x,y)+1)^3}.$$

Proof. i) is proved in [14, Lemma 3.2]. Combining i) with (19), we obtain ii). $\qquad\square$

Lemma 2.4. *The following estimates hold*

$$i) \int_1^\infty t^{-1/2} |H_t(x,y) - H_t(p(x),y)| \, dt \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2} \qquad \forall y \not\leq x,$$

$$ii) \int_1^\infty t^{-1/2} \frac{H_t(x,y)}{(d(x,y)+1)} \, dt \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2} \qquad \forall x,y \in V.$$

Proof. We refer to [14, Lemmas 3.4, 3.5] for a detailed proof. $\qquad\square$

We conclude this section with a technical lemma that provides an algorithm that we will apply to integrate a certain class of functions.

Lemma 2.5. *Let* $f_{x,n}$ *be the function in* $C(V)$ *defined by*

$$f_{x,n}(y) = \frac{q^{-(\ell(x)+d(x,y))/2}}{(d(x,y)+n)^2} \qquad y \in V,$$

for some fixed $x \in V$ *and* $n \in \mathbb{N} \setminus \{0\}$*. Then, for any* $m \in \mathbb{N} \setminus \{0\}$

$$\sum_{y \in S_m(x)} q^{\ell(y)/2} f_{x,n}(y) = \frac{1}{(m+n)^2}\left(2 + (m-1)\frac{q-1}{q}\right).$$

Proof. We introduce the family of sets $\{E_m^j\}_{j=1}^m, F_m$ defined by

$$E_m^j = S_m(x) \cap \{y \ : \ \ell(y) = \ell(x) + 2j - m\} = S_m(x) \cap \{y \leq p^j(x), y \not\leq p^{j-1}(x)\}, \quad j = 1, ..., m,$$

$$F_m = S_m(x) \cap \{y \ : \ \ell(y) = \ell(x) - m\} = S_m(x) \cap \{y \ : \ y \leq x\}.$$

Clearly $\left\{ \{E_m^j\}_{j=1}^m, F_m \right\}$ is a partition of $S_m(x)$. Moreover, $|E_m^j| = (q-1)q^{m-j-1}$ if $j < m$, $|E_m^m| = 1$ and $|F_m| = q^m$. Thus,

$$\sum_{y \in S_m(x)} q^{\ell(y)/2} f_{x,n}(y) = \sum_{j=1}^m \sum_{y \in E_m^j} q^{(\ell(x)+2j-m)/2} f_{x,n}(y) + \sum_{y \in F_m} q^{(\ell(x)-m)/2} f_{x,n}(y)$$

$$= \frac{1}{(m+n)^2} \left(\sum_{j=1}^{m-1} \frac{q-1}{q} + 2 \right).$$

$\square$

Remark 2.6. The above proof illustrates the algorithm on which the computation of most of the sums throughout this paper relies. Unfortunately, although the functions we will integrate are usually of the form $f_{x,n}$, the domain of integration might not coincide with the whole sphere $S_m(x)$. Thus, in each specific case, we will adapt the above idea to the particular geometry of the domain.

3 Proof of Theorem 1.1 i)-ii)

In this section, we shall prove that the L^1-norm of the heat maximal operator $\mathcal{M}_h$ defined in (1) is uniformly bounded on atoms and deduce that $H^1_{at}(\mu) \subset H^1_h(\mu)$. By using the well-known subordination formula for the Poisson semigroup, a standard argument shows that $H^1_h(\mu) \subset H^1_P(\mu)$. Thus, Theorem 1.1 *ii)* will follow immediately by Theorem 1.1 *i)*.

We preliminarily need to show that $\mathcal{M}_h$ is of weak type $(1,1)$. It is worth recalling that the weak type (1,1) boundedness of the heat maximal operator associated to the combinatorial Laplacian Δ is a well-known fact proved by M. Pagliacci and M. Picardello in [17].

Before establishing the abovementioned properties, we define the local maximal heat operator by

$$\mathcal{M}_{\mathrm{loc}} f(x) = \sup_{0 < t < 1} |\mathcal{H}_t f(x)| \qquad \forall f \in C(V), x \in V.$$

Proposition 3.1. *The operator $\mathcal{M}_{loc}$ is bounded on $L^1(\mu)$.*

Proof. Let $f \in C(V)$. By (15)

$$\|\mathcal{M}_{\mathrm{loc}} f\|_1 \leq \sum_{y \in V} |f(y)| \sum_{x \in V} \sup_{0 < t < 1} H_t(x,y) \mu(x) \mu(y)$$

$$\lesssim \sum_{y \in V} |f(y)| \mu(y) \sum_{x \in V} \sup_{0 < t < 1} Q(x,y) s_{d(x,y)}(t) \mu(x).$$

It is easy to see that the term inside the second sum can be dominated as follows

$$Q(x,y) s_{d(x,y)}(t) \mu(x) \lesssim q^{\frac{-d(x,y)-\ell(y)+\ell(x)}{2}} \left(\frac{et}{d(x,y)+1} \right)^{d(x,y)}$$

$$\leq q^{\frac{-d(x,y)-\ell(y)+\ell(x)}{2}} \left(\frac{e}{d(x,y)+1} \right)^{d(x,y)} \qquad 0 < t < 1.$$

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Recalling that $\ell(x) - \ell(y) \leq d(x,y)$, it suffices to notice that

$$\sum_{x \in V} Q(x,y) \sup_{0 < t < 1} s_{d(x,y)}(t)\mu(x) \lesssim \sum_{x \in V} \left(\frac{e}{d(x,y)+1}\right)^{d(x,y)} = \sum_{d=0}^{\infty} \left(\frac{qe}{d+1}\right)^d < +\infty.$$

$\square$

Proposition 3.2. *The operator $\mathcal{M}_h$ is of weak type (1,1) and bounded on $L^p(\mu)$ for all $p \in (1,\infty]$.*

Proof. It suffices to prove the weak type (1,1) boundedness of $\mathcal{M}_h$ and then use interpolation. Pick $f \in L^1(\mu)$ and assume without loss of generality $f \geq 0$. Then, for every $t > 0$ we have

$$\frac{1}{2t} \int_0^{2t} \mathcal{H}_z f(x) \, dz \geq \frac{1}{2t} \int_t^{2t} \mathcal{H}_z f(x) \, dz = \frac{1}{2t} \sum_{y \in V} f(y) \int_t^{2t} H_z(x,y) \, dz\mu(y)$$

$$\gtrsim \frac{1}{2t} \sum_{y \in V} f(y) \int_t^{2t} Q(x,y) s_d(z) \, dz\mu(y),$$

where $d = d(x,y)$. Recall that $s_d(z) = (d+1)\frac{e^{(d+1)\varphi(z/(d+1))}}{z[1+(d+1)^2+z^2]^{1/4}}$ where φ is defined in (16), and

$$\mathbb{R}^+ \ni z \mapsto e^{(d+1)\varphi(z/(d+1))} \quad \text{is increasing,}$$

$$\mathbb{R}^+ \ni z \mapsto \frac{1}{z[1+(d+1)^2+z^2]^{1/4}} \quad \text{is decreasing,}$$

thus

$$\frac{1}{2t} \int_0^{2t} \mathcal{H}_z f(x) \, dz \gtrsim \sum_{y \in V} f(y)Q(x,y)\frac{(d+1)e^{(d+1)\varphi(t/(d+1))}}{2t[1+(d+1)^2+(2t)^2]^{1/4}}\mu(y)$$

$$\gtrsim \sum_{y \in V} f(y)H_t(x,y)\mu(y) = \mathcal{H}_t f(x), \tag{20}$$

where in the last line we have used (15). Observe that, by (10), $(\mathcal{H}_t)_t$ is a strongly measurable semigroup which satisfies the contraction property, namely, if $f \in L^1(\mu)$

$$\|\mathcal{H}_t f\|_1 \leq \sum_{x \in V} \sum_{y \in V} |f(y)|H_t(x,y)\mu(y)\mu(x) = \sum_{y \in V} |f(y)| \sum_{x \in V} H_t(x,y) \, \mu(x) \, \mu(y) = \|f\|_1.$$

Thus, by the Hopf-Dunford-Schwartz Theorem (see [7]), the ergodic operator associated to the heat semigroup is of weak type (1,1). We conclude passing to the supremum in (20). $\square$

Proposition 3.3. *There exists a positive constant $C > 0$ such that $\|\mathcal{M}_h a\|_1 \leq C$ for any $(1,\infty)$-atom a.*

Proof. Let a be a $(1,\infty)$-atom. If $\mathcal{F} \ni R = R_{h'}^{h''}(y_R)$ is the support of a, then we define its enlargement $R^* = \{x \in V : d(x,R) \leq h'\}$. By the Cauchy-Schwarz inequality and the $L^2(\mu)$-boundedness of $\mathcal{M}_h$

$$\|\mathcal{M}_h a\|_{L^1(R^*)} \leq \|\mathcal{M}_h a\|_2 \mu(R^*)^{1/2} \leq C'\|\mathcal{M}_h\|_{2 \to 2}\left(\frac{\mu(R^*)}{\mu(R)}\right)^{1/2} \leq C,$$

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where we have used the fact that $\mu(R^*) \lesssim \mu(R)$, see [13].

We now split $(R^*)^c$ in two regions, namely,

$$\Gamma_1 = \{x \in (R^*)^c \ : \ x \le y_R\},$$
$$\Gamma_2 = (R^*)^c \setminus \Gamma_1 = \{x : \ x \nleq y_R\}.$$

We start with

$$\sum_{x \in \Gamma_1} \mathcal{M}_h a(x)\mu(x) \lesssim \sum_{x \in \Gamma_1} \sup_{t>0} \sum_{y \in R} Q(x,y)s_{d(x,y)}(t)|a(y)|\mu(y)\mu(x).$$

By exploiting (18) and the size condition of the atom, we get

$$\sum_{x \in \Gamma_1} \mathcal{M}_h a(x)\mu(x) \lesssim \sum_{x \in \Gamma_1} \sum_{y \in R} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} \frac{1}{\mu(R)}\mu(x).$$

If $x \in \Gamma_1$, then

$$\frac{1}{\mu(R)}\sum_{y \in R} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} = \sum_{l=\ell(y_R)-h''+1}^{\ell(y_R)-h'} \frac{1}{\mu(R)} \sum_{y \in R \cap \{\ell(y)=l\}} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2}.$$

We briefly explain how to compute the above sum. Fix $x \in \Gamma_1$ and an integer $l \in [\ell(y_R) - h'' + 1, \ell(y_R) - h']$. Then, there exist
• one vertex $y_l \ge x$ in R at level $\ell(y_l) = l$. In this case $d(x, y_l) = \ell(y_l) - \ell(x)$;
• $q-1$ vertices which lie at the same level as y_l which belong to $U_{l,1} = \{y \ : \ell(y) = \ell(y_l), y \le p(y_l), y \ne y_l\}$. In this case, for any $y \in U_{l,1}$, $d(y,x) = d(y_l,x) + 2$;
• $(q-1)q$ vertices which lie at the same level as y_l which belong to $U_{l,2} = \{y \ : \ell(y) = \ell(y_l), y \le p^2(y_l), y \nleq p(y_l)\}$. In this case, for any $y \in U_{l,2}$, $d(y,x) = d(y_l,x) + 4$;
⋮
• $(q-1)q^{d(y_l,y_R)-1}$ vertices which lie at the same level as y_l which belong to $U_{l,d(y_l,y_R)} = \{y \ : \ell(y) = \ell(y_l), y \le y_R, y \nleq p^{d(y_l,y_R)-1}(y_l)\}$. In this case, for any $y \in U_{l,d(y_l,y_R)}$, $d(y,x) = d(y_l,x) + 2d(y_l,y_R)$.

We can rewrite the previous sum as

$$\sum_{y \in R \cap \{\ell(y)=l\}} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} = 1 \cdot \frac{1}{(d(x,y_l)+1)^2} + \sum_{j=1}^{d(y_l,y_R)} (q-1)q^{j-1} \cdot \frac{q^{(d(y_l,x)-d(y_l,x)-2j)/2}}{(d(x,y_l)+2j+1)^2}$$

$$= 1 \cdot \frac{1}{(d(x,y_l)+1)^2} + \sum_{j=1}^{d(y_l,y_R)} (q-1)q^{-1} \cdot \frac{1}{(d(x,y_l)+2j+1)^2}$$

$$\lesssim \frac{h''+h'}{(d(x,y_R)-h'')^2},$$

since $d(x,y_l) = d(x,y_R) - d(y_l,y_R) \ge d(x,y_R) - h''$. Summing up over the $h'' - h'$ level which intersects R, we get

$$\frac{1}{\mu(R)} \sum_{y \in R} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} \lesssim \frac{h''-h'}{q^{\ell(y_R)}(h''-h')} \cdot \frac{h''+h'}{(d(x,y_R)-h'')^2} \lesssim \frac{h'}{q^{\ell(y_R)}(d(x,y_R)-h'')^2}.$$

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We conclude that

$$\sum_{x\in\Gamma_1}\frac{1}{q^{\ell(y_R)}}\frac{h'}{(d(x,y_R)-h'')^2}\mu(x) = \sum_{x\in\Gamma_1}\frac{q^{\ell(y_R)-d(x,y_R)}}{q^{\ell(y_R)}}\frac{h'}{(d(x,y_R)-h'')^2}$$
$$\leq \sum_{j\geq h'}\frac{h'}{j^2}\lesssim 1.$$

Now we shall integrate on Γ_2. In this case we need to use the cancellation condition of the atom.

It is worth noticing that the function $R\ni y\mapsto H_t(x,y)$ with $x\in\Gamma_2$ fixed, is radial (namely, it depends only on $d(x,y)$ or equivalently, in this particular case, it depends only on $\ell(y)$). Let y^L denote a vertex of maximum level in R. We have $d(x,y^L)=d(x,y_R)+h'$ for any $x\in\Gamma_2$. Given a vertex $y\in R$, let $\overline{y}$ denote the predecessor of y of maximum level in R. An easy application of Lemma 2.3 and the fact that $\ell(p^j(y))+d(x,p^j(y))=\ell(y_R)+d(x,y_R)$ for every $1\leq j\leq d(y,\overline{y})$, $x\in\Gamma_2$ and $y\in R$, yield

$$\sup_{t>0}|H_t(x,y)-H_t(x,y^L)| \leq \sum_{j=0}^{d(y,\overline{y})}\sup_{t>0}|H_t(x,p^j(y))-H_t(x,p^{j+1}(y))|$$

$$\lesssim \sum_{j=0}^{d(y,\overline{y})}\frac{q^{-(\ell(x)+\ell(p^j(y))+d(x,p^j(y)))/2}}{(d(x,p^j(y))+1)^3}$$

$$\leq \sum_{j=0}^{d(y,\overline{y})}\frac{q^{-(\ell(x)+\ell(p^j(y))+d(x,p^j(y)))/2}}{(d(x,y_R)+h')^3}$$

$$\leq \frac{(h''-h')q^{-(\ell(x)+\ell(y_R)+d(x,y_R))/2}}{(d(x,y_R)+h')^3}, \tag{21}$$

where in the second line we have used Lemma 2.3 ii) and $p^0(y)=y$. By the cancellation and the size condition of the atom and (21)

$$\sup_{t>0}\left|\sum_{y\in R}H_t(x,y)a(y)\mu(y)\right| = \sup_{t>0}\left|\sum_{y\in R}(H_t(x,y)-H_t(x,y^L))a(y)\mu(y)\right|$$

$$\leq \sum_{y\in R}\sup_{t>0}|H_t(x,y)-H_t(x,y^L)|\frac{\mu(y)}{\mu(R)} \lesssim \frac{(h''-h')q^{-(\ell(x)+\ell(y_R)+d(x,y_R))/2}}{(d(x,y_R)+h')^3}.$$

It follows that

$$\|\mathcal{M}_h a\|_{L^1(\Gamma_2)} = \sum_{x\in\Gamma_2}q^{\ell(x)}\sup_{t>0}\left|\sum_{y\in R}H_t(x,y)a(y)\mu(y)\right|$$

$$= \sum_{x\in\Gamma_2}q^{\ell(x)}\sup_{t>0}\left|\sum_{y\in R}\left(H_t(x,y)-H_t(x,y^L)\right)a(y)\mu(y)\right|$$

$$\lesssim \sum_{x\in\Gamma_2}q^{\ell(x)/2-\ell(y_R)/2-d(x,y_R)/2}\frac{(h''-h')}{(d(x,y_R)+h')^3}.$$

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We can integrate over the intersection of the spheres $S_m(y_R)$ and Γ_2 for $m \geq 1$. Arguing as in Lemma 2.5 we get

$$\sum_{x \in \Gamma_2 \cap S_m(y_R)} q^{\ell(x)} \sup_{t>0} \left| \sum_{y \in R} H_t(x,y) a(y) \mu(y) \right|$$

$$\lesssim \frac{(h''-h') q^{-\ell(y_R)/2-m/2}}{(m+h')^3} \left[(q-1) \sum_{j=1}^{m-1} \left(q^{m-(j+1)} q^{(\ell(y_R)+2j-m)/2} \right) + q^{(m+\ell(y_R))/2} \right]$$

$$\lesssim \frac{(h''-h')m}{(m+h')^3} \leq \frac{(h''-h')}{(m+h')^2}.$$

Summing up over $m \geq 1$, we obtain

$$\sum_{m=1}^{\infty} \sum_{x \in \Gamma_2 \cap S_m(y_R)} q^{\ell(x)} \sup_{t>0} \left| \sum_{y \in R} H_t(x,y) a(y) \mu(y) \right| \lesssim \sum_{m=1}^{\infty} \frac{(h''-h')}{(m+h')^2} \lesssim 1.$$

This concludes the proof. $\qquad\square$

Using the weak type (1,1) boundedness of $\mathcal{M}_h$, it is easy to prove that the uniform boundedness of $\|\mathcal{M}_h a\|_1$ where a ranges over $(1,\infty)$-atoms, implies the boundedness of $\mathcal{M}_h$ from $H_{at}^1(\mu)$ to $L^1(\mu)$. Indeed, the following can be proved by a standard argument.

Lemma 3.4. *Let $\mathcal{K} : H_{at}^1(\mu) \to L^1(\mu)$ be a positive sublinear operator, i.e., $\mathcal{K}f \geq 0$, $\mathcal{K}(\alpha f) = |\alpha| \mathcal{K}(f)$ and*

$$\mathcal{K}(f+g)(x) \leq \mathcal{K}(f)(x) + \mathcal{K}(g)(x), \qquad \forall x \in V,$$

where $\alpha \in \mathbb{C}, f, g \in H_{at}^1(\mu)$. Suppose that there exists a positive constant C such that

$$\|\mathcal{K}a\|_1 \leq C,$$

for all $(1,\infty)$-atoms a. If $\mathcal{K}$ is of weak type (1,1), then

$$\|\mathcal{K}f\|_1 \lesssim \|f\|_{H_{at}^1} \qquad \forall f \in H_{at}^1(\mu).$$

Theorem 1.1 *i)* now follows combining Proposition 3.3 with Lemma 3.4.

We end this section with the proof of Theorem 1.1 *ii)*. The kernel $P_t(\cdot,\cdot)$ of the Poisson semigroup $(\mathcal{P}_t)_t$ is given by the following well-known subordination formula

$$P_t(\cdot,\cdot) = t \int_0^{\infty} (4\pi z)^{-1/2} e^{-t^2/(4z)} H_z(\cdot,\cdot) \, \frac{dz}{z}.$$

We recall that the Poisson maximal operator $\mathcal{M}_P$ is defined by (2). By a change of variables and an application of Fubini-Tonelli's Theorem, it is easily seen that $\mathcal{M}_P f \leq \mathcal{M}_h f$ for any $f \in C(V)$, thus $H_h^1(\mu) \subset H_P^1(\mu)$ and Theorem 1.1 *ii)* is proved.

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4 Proof of Theorem 1.1 iii)

In this section we introduce a sequence of functions $\{g_n\}_n$ and we provide estimates of their norms in $H^1_{at}(\mu)$ and $H^1_h(\mu)$. In particular, we shall obtain that $\|\cdot\|_{H^1_h}$ and $\|\cdot\|_{H^1_{at}}$ are not equivalent norms. By means of the abovementioned estimates, we construct a function g which belongs to $H^1_h(\mu)$ but which does not belong to $H^1_{at}(\mu)$. Exploiting the inclusion $H^1_h(\mu) \subset H^1_P(\mu)$, we will obtain also that $g \in H^1_P(\mu)$.

We introduce a numeration on the set of vertices of level 0 as follows. For all $n \geq 2$ if $\ell(x) = 0$, $x \leq p^n(o)$ and $x \not\leq p^{n-1}(o)$ we assign to x a unique label x_i with $i \in [q^{n-1}, q^n - 1]$. If $x \leq p(o)$, then we define $x_0 = o$ and the remaining $q - 1$ vertices x_i with $i = 1, ..., q - 1$.
Define

$$g_n(x) = \delta_{x_n}(x) - \delta_o(x) \qquad \forall n \geq 2. \tag{22}$$

Since g_n is supported in $\{x_n\} \cup \{o\}$ and has zero average for every $n \geq 2$, it follows that $g_n \in H^1_{at}(\mu)$. In order to estimate $\|g_n\|_{H^1_{at}}$ from below, we shall construct a function $f \in BMO(\mu)$ and apply (4). Consider the function $f : V \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} n \log q & \text{if } x \leq p^n(o), \ x \not\leq p^{n-1}(o), \text{ and } n \geq 2, \\ \log q & \text{if } x \leq p^1(o). \end{cases} \tag{23}$$

Proposition 4.1. *The function f defined by (23) belongs to $BMO(\mu)$.*

Proof. It is easy to see that f is constant on every admissible trapezoid with root not in $[p^2(o), \xi_0)$. Hence, to prove that $f \in BMO(\mu)$ we have to control the average of f on an admissible trapezoid R with root in $[p^2(o), \xi_0)$. We claim that it suffices to prove the uniform boundedness of

$$\frac{1}{\mu(R)} \sum_{x \in R} |f(x) - C_R| \mu(x),$$

where C_R is a suitable constant depending only on R. Indeed, for any $y \in R$

$$|f(y) - f_R| \leq |f(y) - C_R| + |C_R - f_R| \leq |f(y) - C_R| + \frac{1}{\mu(R)} \sum_{x \in R} |f(x) - C_R| \ \mu(x),$$

and it follows

$$\frac{1}{\mu(R)} \sum_{y \in R} |f(y) - f_R| \mu(y) \leq \frac{2}{\mu(R)} \sum_{y \in R} |f(y) - C_R| \mu(y),$$

and the last inequality proves the claim.
Next, we distinguish two cases.
Case 1. Let $R = R^{h''}_{h'}(p^{(n)}(o))$ with $n \geq h''$. We shall estimate from above

$$\frac{1}{\mu(R)} \sum_{x \in R} |f(x) - n \log q| \mu(x).$$

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Using the definition of f, it is convenient to compute the above sum on each level. Indeed, fix a positive integer $l \in [n - h'' + 1, n - h']$. Then,

$$
\frac{1}{\mu(R)} \sum_{x \in R \cap \ell(x) = l} |f(x) - n \log q| \mu(x)
$$

$$
= \frac{q^l}{\mu(R)} \left[\left((q - 1) \sum_{j=l+1}^{n} q^{j-1-l} |j \log q - n \log q| \right) + 1 \cdot |l \log q - n \log q| \right]
$$

$$
\leq \sum_{j=l}^{n-1} \frac{q^l}{\mu(R)} q^{j-l} (n - j) \log q
$$

$$
= \sum_{j=l}^{n-1} q^{j-n} \frac{(n - j)}{(h'' - h')} \log q
$$

$$
\leq \sum_{m=1}^{\infty} q^{-m} \frac{m}{h'' - h'} \log q.
$$

We get an estimate independent of l. Summing over the $h'' - h'$ levels which intersect R, we conclude that

$$
\frac{1}{\mu(R)} \sum_{x \in R} |f(x) - n \log q| \mu(x) = \sum_{l = n - h'' + 1}^{n - h'} \frac{1}{\mu(R)} \sum_{x \in R \cap \ell(x) = l} |f(x) - n \log q| \mu(x)
$$

$$
\leq (h'' - h') \sum_{m=1}^{\infty} q^{-m} \frac{m}{(h'' - h')} \log q
$$

$$
\lesssim 1.
$$

Case 2. Let $R = R_{h'}^{h''}(p^n(o))$ with $2 \leq n < h''$. We can follow the previous argument except for the levels $l \leq 0$. Thus, if $0 \geq l \in [n - h'' + 1, n - h']$ is a fixed level,

$$
\frac{1}{\mu(R)} \sum_{x \in R \cap \ell(x) = l} |f - n \log q| \mu(x)
$$

$$
= \frac{q^l}{\mu(R)} \left[\left((q - 1) \sum_{j=2}^{n} q^{j-1-l} |j \log q - n \log q| \right) + q^{1-l} |\log q - n \log q| \right]
$$

$$
\leq \sum_{j=1}^{n-1} \frac{q^l}{\mu(R)} q^{j-l} (n - j) \log q,
$$

and we conclude as above.

This proves that $f \in BMO(\mu)$. $\qquad\square$

Remark 4.2. If we take n such that $q^{m-1} \leq n \leq q^m - 1$ for $m \geq 2$, then it is easily seen that $|x_n| = 2m \leq 2 \frac{\log n}{\log q} + 2 \lesssim \log n$, while $f(x_n) = m \log q \geq \log n$.
We also underline that $x_n \wedge o = p^{|x_n|/2}(o) = p^{|x_n|/2}(x_n)$ for all $n \geq 2$.

Since g_n is a multiple of a $(1, \infty)-$atom, by (4) we get

$$\|f\|_{BMO}\|g_n\|_{H_{at}^1} \gtrsim \left| \sum_{x \in V} f(x)g_n(x)\mu(x) \right| = |f(x_n) - f(o)| \gtrsim \log n,$$

which implies that

$$\log n \lesssim \|g_n\|_{H_{at}^1}. \tag{24}$$

Moreover, it is clear that $\|g_n\|_1 \approx 1$. Combining the previous inequalities with the following proposition we conclude that the norms on $H_h^1(\mu)$ and $H_{at}^1(\mu)$ are not equivalent.

Proposition 4.3. *Let $\{g_n\}_n$ be the sequence defined in (22). Then, the following holds:*

$$\|\mathcal{M}_h g_n\|_1 \lesssim \log \log n \qquad \forall n \geq 2.$$

Proof. We split the proof into three steps.
Step 1.
Define $B = B(o, |x_n|)$. Our goal is to show that

$$\sum_{x \in B} \mathcal{M}_h \delta_{x_j}(x)\mu(x) \lesssim \log \log n$$

for $j = 0$ and $j = n$.
   Notice that for all $x \in V$, by (18)

$$\mathcal{M}_h(\delta_{x_j})(x)\mu(x) = \mu(x) \sup_{t>0} H_t(x, x_j) \lesssim \frac{Q(x, x_j)\mu(x)}{(d(x, x_j) + 1)^2}. \tag{25}$$

By (25)

$$\sum_{x \in B} \mathcal{M}_h(\delta_o)(x)\mu(x) \lesssim \sum_{x \in B} q^{\ell(x)/2} \frac{q^{-|x|/2}}{(|x| + 1)^2}.$$

We write $B = \cup_{m=0}^{|x_n|} S_m(o)$ and apply Lemma 2.5 to obtain

$$\sum_{x \in B} \mathcal{M}_h(\delta_o)(x)\mu(x) \lesssim \sum_{m=0}^{|x_n|} \frac{1}{m + 1} \lesssim \log |x_n| \lesssim \log \log n, \tag{26}$$

where we refer to Remark 4.2 for the last estimate.
It remains to prove the same inequality which involves $\mathcal{M}_h(\delta_{x_n})$. Again by (25)

$$\sum_{x \in B} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) = \sum_{x \in B} \mu(x) \sup_{t>0} |H_t(x, x_n)| \lesssim \sum_{x \in B} \frac{Q(x, x_n)\mu(x)}{(d(x, x_n) + 1)^2}.$$

Denote by B^* the ball $B(x_n, 2|x_n|)$. Clearly, $B \subset B^*$. Hence

$$\sum_{x \in B} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \sum_{x \in B^*} \frac{q^{\ell(x)/2} q^{-d(x, x_n)/2}}{(d(x, x_n) + 1)^2}.$$

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Exactly as in (26) we get

$$\sum_{x \in B} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \sum_{m=0}^{2|x_n|} \frac{1}{m+1} \lesssim \log 2|x_n| \lesssim \log\log n. \tag{27}$$

This is the desired conclusion.

Step 2.
We divide the complement of $B(o, |x_n|)$ in two regions.

$$\Gamma_1 = \{x \in B(o, |x_n|)^c \ : \ x \le p^{|x_n|}(o)\},$$
$$\Gamma_2 = \{x \in B(o, |x_n|)^c \ : x \notin \Gamma_1\}.$$

We claim that

$$\sum_{x \in \Gamma_1} \mathcal{M}_h(\delta_o)(x)\mu(x) \lesssim 1. \tag{28}$$

The claim follows by a direct computation. Indeed, we estimate the above sum on $S_m(o) \cap \Gamma_1$ for every $m > |x_n|$ as follows

$$\sum_{x \in S_m(o) \cap \Gamma_1} \mathcal{M}_h(\delta_o)(x)\mu(x) \lesssim \sum_{x \in S_m(o) \cap \Gamma_1} \frac{q^{\ell(x)/2 - d(x,o)/2}}{(d(x,o)+1)^2}$$

$$= \frac{q^{-m/2}}{m^2}\left[(q-1)\sum_{j=1}^{|x_n|}\left(q^{m-(j+1)}q^{(2j-m)/2}\right) + q^{m/2}\right] \lesssim \frac{|x_n|}{m^2} \tag{29}$$

where we integrate adapting Lemma 2.5. We conclude by observing that

$$\sum_{x \in \Gamma_1} \mathcal{M}_h(\delta_o)(x)\mu(x) = \sum_{m=|x_n|+1}^{\infty} \sum_{x \in S_m(o) \cap \Gamma_1} q^{\ell(x)}\mathcal{M}_h(\delta_o)(x) \lesssim \sum_{m=|x_n|+1}^{\infty} \frac{|x_n|}{m^2} \lesssim 1,$$

and (28) is proved.
We now claim that

$$\sum_{x \in \Gamma_1} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \log\log n. \tag{30}$$

For establishing it, in order to exploit the symmetries of $\mathcal{M}_h(\delta_{x_n})$, it is convenient to integrate on a larger set than Γ_1. Define $\Gamma_1^* = \{y \in V \ : \ y \not\le p^{|x_n|/2}(o)\}$ and observe that if $x \in \Gamma_1 \cap \Gamma_1^*$ then $d(x_n, x) = d(o, x)$, (because $x_n \wedge o = p^{|x_n|/2}(x_n) = p^{|x_n|/2}(o)$), thus $\mathcal{M}_h(\delta_{x_n})(x) = \mathcal{M}_h(\delta_o)(x)$. Obviously $\Gamma_1 \cap (\Gamma_1^*)^c \subset (\Gamma_1^*)^c = \{y \in : y \le p^{|x_n|/2}(o)\}$.
It suffices to check that

$$\sum_{x \in (\Gamma_1^*)^c} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \log\log n.$$

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It is convenient to think of the above sum as the sum over the disjoint sets $\{S_m(x_n) \cap (\Gamma_1^*)^c\}_{m \geq 0}$. Fix $m \geq 0$ and by applying (18) we obtain

$$\sum_{x \in S_m(x_n) \cap (\Gamma_1^*)^c} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \sum_{x \in S_m(x_n) \cap (\Gamma_1^*)^c} q^{\ell(x)/2} \frac{q^{-d(x,x_n)/2}}{(d(x,x_n)+1)^2}.$$

Assume $m > |x_n|/2$. In the same fashion as we computed in Lemma 2.5, we obtain

$$\sum_{x \in S_m(x_n) \cap (\Gamma_1^*)^c} \mathcal{M}_h(\delta_{x_n})(x)\mu(x)$$

$$\lesssim \frac{q^{-m/2}}{m^2} \left[(q-1) \sum_{j=1}^{|x_n|/2} \left(q^{m-(j+1)} q^{(2j-m)/2} \right) + q^{m/2} \right]$$

$$\lesssim \frac{|x_n|/2}{m^2}.$$

If $m < |x_n|/2$, the same computation still works with a slight modification,

$$\sum_{x \in S_m(x_n) \cap (\Gamma_1^*)^c} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \frac{q^{-m/2}}{m^2} \left[q^{m/2} + (q-1) \sum_{j=1}^{m-1} \left(q^{m-(j+1)} q^{(2j-m)/2} \right) + q^{m/2} \right] \lesssim \frac{1}{m},$$

where the first term inside the square brackets is the contribution due to $p^m(x_n) \in (\Gamma_1^*)^c$. Summing up over the positive integers, we conclude

$$\sum_{m=1}^{\infty} \sum_{x \in S_m(x_n) \cap (\Gamma_1^*)^c} \mathcal{M}_h(\delta_{x_n})(x)\mu(x) \lesssim \sum_{m=1}^{|x_n|/2-1} \frac{1}{m} + \sum_{m=|x_n|/2}^{\infty} \frac{|x_n|/2}{m^2}$$

$$\lesssim \log(|x_n|) + 1 \lesssim \log \log n,$$

which proves (30).

Step 3.
Notice that, if $x \not\leq x_n \wedge o = p^{|x_n|/2}(o)$, then $d(x_n, x) = d(x, o)$. This is true because, for such a vertex x

$$d(x,o) = d(x, x_n \wedge o) + d(x_n \wedge o, o) = d(x, x_n \wedge o) + d(x_n \wedge o, x_n) = d(x, x_n).$$

Observe that this together with (9) imply

$$\sum_{x \in \Gamma_2} \mathcal{M}_h(\delta_{x_n} - \delta_o)(x)\mu(x)$$

$$= \frac{1}{1-b} \sum_{x \in \Gamma_2} q^{\ell(x)/2} \sup_{t>0} e^{bt/(1-b)} |q^{-\ell(x_n)/2} h_{t/(1-b)}(x, x_n) - q^{-\ell(o)/2} h_{t/(1-b)}(x, o)| \, dx = 0, \quad (31)$$

since $q^{\ell(x_n)} = q^{\ell(o)} = 1$ and $h_{t/(1-b)}(x, y) = h_{t/(1-b)}(d(x, y))$.
In conclusion, (26), (27), (28), (30) and (31) yield

$$\|\mathcal{M}_h g_n\|_1 \lesssim \log \log n.$$

$\square$

It follows that

$$\lim_{n \to \infty} \frac{\|g_n\|_{H^1_h}}{\|g_n\|_{H^1_{at}}} = 0,$$

and in particular, $\| \cdot \|_{H^1_h}$, $\| \cdot \|_{H^1_{at}}$ are not equivalent.
We are now ready to prove Theorem 1.1 iii).

Proof of Theorem 1.1 iii). Define the function g on the set of vertices at level 0 as $g(o) = c_0$, $g(x) = 0$ if $x \leq p^1(o) \setminus \{o\}$ and $g(x_n) = \frac{1}{n(\log n)^{3/2}}$ for every $n \geq q$. Then we extend $g = 0$ outside the level zero. Choose c_0 such that $\sum_{x \in V} g(x)\mu(x) = 0$. Clearly,

$$\|g\|_1 = |c_0| + \sum_{n=q}^{\infty} \frac{1}{n(\log n)^{3/2}} < +\infty.$$

We now show that $\|\mathcal{M}_h g\|_1$ is finite. Indeed, we observe that

$$g = \sum_{k=q}^{\infty} c_k g_k,$$

where $\{g_k\}_k$ is defined in (22) and c_k is the value of g at x_k. Then, by using Proposition 4.3

$$\|\mathcal{M}_h g\|_1 \lesssim \sum_{k=q}^{\infty} c_k \log \log k \lesssim \sum_k \frac{\log \log k}{k(\log k)^{3/2}} < +\infty.$$

This implies that $g \in H^1_h(\mu)$.
We now prove that $g \notin H^1_{at}(\mu)$. Indeed, suppose the converse by contradiction. Then it would be

$$\sum_{x \in V} g(x)f(x)\mu(x) < +\infty, \tag{32}$$

where f is the BMO function defined in (23). But using the estimate $f(x_n) \geq \log n$ (see Remark 4.2), (32) would imply

$$\sum_{n=q}^{\infty} \frac{1}{n(\log n)^{1/2}} < +\infty,$$

which is clearly false. Then $g \notin H^1_{at}(\mu)$. $\qquad\square$

5   Proof of Theorem 1.2

This last section is devoted to the proof of Theorem 1.2. We briefly recall some preliminary notion.
We define the discrete Riesz transform $\mathcal{R} = \nabla \mathcal{L}^{-1/2}$, which corresponds to the integral operator with integral kernel with respect to μ

$$R(x,y) = \int_0^{+\infty} t^{-1/2}(H_t(x,y) - H_t(p(x),y)) \, dt.$$

Recall that the Riesz Hardy space is defined by (3). It is a well-known fact that $\mathcal{R}$ maps $H^1_{at}(\mu)$ to $L^1(\mu)$, indeed, it is an easy consequence of the discrete version of Hörmander's condition for singular operators (see [2, Th. 3] or [13] and [12]). Thus, the inclusion $H^1_{at}(\mu) \subset H^1_R(\mu)$ is trivial. In order to show that such inequality is strict, we need the following result.

Proposition 5.1. *The following holds*

$$\|\mathcal{R}g_n\|_1 \lesssim \log\log n \qquad \forall n \geq 2,$$

where $\{g_n\}_n$ is the sequence defined in (22).

Proof. We write

$$R(x,y) = \int_0^1 t^{-1/2}(H_t(x,y) - H_t(p(x),y)) \, dt + \int_1^\infty t^{-1/2}(H_t(x,y) - H_t(p(x),y)) \, dt$$
$$= R^{(0)}(x,y) + R^{(\infty)}(x,y)$$

and consequently $\mathcal{R} = \mathcal{R}^{(0)} + \mathcal{R}^{(\infty)}$. It follows from Proposition 3.1 that $\mathcal{R}^{(0)}$ is bounded on $L^1(\mu)$, hence $\|\mathcal{R}^{(0)}g_n\|_1 \lesssim 1$. We now consider $\|\mathcal{R}^{(\infty)}g_n\|_1$. We recall that

$$\|\mathcal{R}^{(\infty)}g_n\|_1 = \sum_{x \in V} \left| \sum_{y \in V} \int_1^\infty t^{-1/2}(H_t(x,y) - H_t(p(x),y)) \, dt g_n(y)\mu(y) \right| \mu(x)$$
$$= \sum_{x \in V} \left| \int_1^\infty t^{-1/2}(H_t(x,x_n) - H_t(x,o) + H_t(p(x),o) - H_t(p(x),x_n)) \, dt \right| \mu(x).$$

Arguing as in Step 3 of Proposition 4.3, we get that, if $x \not\leq x_n \wedge o$, the first difference inside the integral in the last line vanishes. The same happens for the second difference if $p(x) \not\leq x_n \wedge o$. Since

$$\{x \in V : x \not\leq x_n \wedge o\} \subset \{x \in V : p(x) \not\leq x_n \wedge o\},$$

we can estimate the previous sum as follows

$$\|\mathcal{R}^{(\infty)}g_n\|_1$$
$$\leq \sum_{x \in E_n} \int_1^\infty \frac{|H_t(x,x_n) - H_t(p(x),x_n)|}{t^{1/2}} \, dt \, \mu(x) + \sum_{x \in E_n} \int_1^\infty \frac{|H_t(x,o) - H_t(p(x),o))|}{t^{1/2}} \, dt \, \mu(x)$$
$$= I_1 + I_2,$$

where $E_n = \{x \in V : x \leq x_n \wedge o\}$. Observe that $E_n = \Gamma_1 \cup \Gamma_2 = \Sigma_1 \cup \Sigma_2$, where

$$\Gamma_1 = \{x \in E_n : x_n \not\leq x\},$$
$$\Gamma_2 = \{x \in E_n : x_n \leq x\},$$
$$\Sigma_1 = \{x \in E_n : o \not\leq x\},$$
$$\Sigma_2 = \{x \in E_n : o \leq x\}.$$

We start studying I_1. Exploiting the symmetry of the problem, the same computations are valid for I_2. It can be useful to split the sum which defines I_1 as

$$I_1 = \sum_{i=1}^2 \sum_{x \in \Gamma_i} \int_1^\infty \frac{|H_t(x,x_n) - H_t(p(x),x_n))|}{t^{1/2}} \, dt \, \mu(x) = I_1^1 + I_1^2.$$

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By Lemma 2.4 i),

$$I_1^1 \lesssim \sum_{x \in \Gamma_1} \frac{Q(x, x_n)}{(d(x, x_n) + 1)^2} \mu(x).$$

Since $x_n \wedge o = p^{|x_n|/2}(x_n)$, we can think of the sum on Γ_1 as the sum on the sequence of disjoint sets $\{\Gamma_1^j\}_{j=0}^{|x_n|/2}$, where Γ_1^j is defined by

$$\Gamma_1^j = \begin{cases} \{x \le x_n\} & \text{if } j = 0, \\ \{x \le p^j(x_n) \text{ and } x \not\le p^{j-1}(x_n)\} & \text{if } 1 \le j \le |x_n|/2, \end{cases}$$

with $p^0(x_n) = x_n$. Observe that, for any $j = 1, ..., |x_n|/2$, $x \in \Gamma_1^j$ implies that

$$d(x, x_n) = 2j - \ell(x),$$

where we have used that $\ell(p^j(x_n)) = j$. Then, for any $1 \le j \le |x_n|/2$

$$\sum_{x \in \Gamma_1^j} \frac{q^{\ell(x)/2 - d(x,x_n)/2}}{(d(x, x_n) + 1)^2} \le \sum_{l=-\infty}^{j} q^{l-j} \frac{1}{(2j-l)^2}(q-1)q^{j-l-1} \le \frac{2}{j},$$

where $(q-1)q^{j-l-1}$ corresponds to the cardinality of vertices in Γ_1^j at the level l. The sum over Γ_1^0 contributes to the sum as a constant independent of n. Summing up

$$I_1^1 \lesssim \sum_{j=1}^{|x_n|/2} \frac{1}{j} \lesssim \log \log n.$$

It remains to estimate I_1^2. By Lemma 2.4 ii) and the fact that if $x \in \Gamma_2$, then $\ell(x) = d(x, x_n)$ and

$$Q(x, x_n) = qQ(p(x), x_n) = q^{-d(x,x_n)},$$

we get

$$I_1^2 \le \sum_{x \in \Gamma_2} \int_1^\infty t^{-1/2} \max\{H_t(x, x_n), H_t(p(x), x_n)\} \, dt \, \mu(x)$$

$$\lesssim \sum_{x \in \Gamma_2} \frac{q^{-d(x,x_n)}}{d(x, x_n) + 1} \mu(x) = \sum_{d=1}^{|x_n|/2} \frac{1}{d} \lesssim \log \log n.$$

Similar computations can be repeated to estimate I_2 if we replace Γ_i by Σ_i. In conclusion

$$\|\mathcal{R}g_n\|_1 \lesssim \log \log n,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude the proof of Theorem 1.2.

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Proof of Theorem 1.2 ii). Let g be the function constructed in the proof of Theorem 1.1 *iii)*. Then,

$$\|\mathcal{R}g\|_1 \lesssim \sum_{k=q}^{\infty} c_k \|\mathcal{R}g_k\|_1 \lesssim \sum_{k=q}^{\infty} \frac{\log\log k}{k(\log k)^{3/2}} < +\infty.$$

Hence $g \in H^1_R(\mu)$ but $g \notin H^1_{at}(\mu)$. □

Remark 5.2. It is not clear whether $H^1_h(\mu), H^1_P(\mu)$ and $H^1_R(\mu)$ are the same space or not. This is an interesting open problem that we have not been able to answer and leave for further work.

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