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Stationary distributions of systems with discreteness-induced transitions

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We provide a theoretical analysis of some autocatalytic reaction networks exhibiting the phenomenon of discreteness-induced transitions. The family of networks that we address includes the celebrated Togashi and Kaneko model. We prove positive recurrence, finiteness of all moments and geometric ergodicity of the models in the family. For some parameter values, we find the analytic expression for the stationary distribution and discuss the effect of volume scaling on the stationary behaviour of the chain. We find the exact critical value of the volume for which discreteness-induced transitions disappear.

1. Introduction

In 2001, Togashi & Kaneko [1] described a cycle of stochastic autocatalytic reactions that displays a highly peculiar dynamics in some regions of the parameter space. It is characterized by switches between patterns where one or more reactants are present in small or vanishing numbers of molecules while other reactants are abundant. The switching is triggered by a single molecule of a previously extinct species that drives the system to a different pattern through a sequence of quick reactions. The switches were named discreteness-induced transitions (DITs) since deterministic ODE models are not able to reproduce them [1].

The paper raised much interest and resulted in similar effects being observed in more complicated and realistic models in physics, biology and elsewhere, e.g. in large-scale networks [2], particle systems with finite interaction radius [3], reaction–diffusion systems [4], models of receptor oligomerization by bivalent ligand [5], models of ant foraging [6], chiral autocatalysis [7], tumour growth [8], spatial models [9], viral replication [10] and noise control in synthetic biology [11]. Thus, the phenomomen of DITs is widespread in nature.

Several attempts have been made to underpin the phenomenon theoretically, at least in simplified toy models, through derivation of analytic expressions, without resorting to simulation or approximation. Examples in this direction are [12–17], though many questions remain unsolved. The stationary distributions of autocatalytic systems of a similar nature have been studied in [18], but their theory does not apply to the Togashi–Kaneko (TK) model and similar systems where mass is not conserved.

Despite simulation of the TK model indicating a stationary behaviour after a short transient time, positive recurrence (existence of a unique stationary distribution) of the corresponding continuous-time Markov chain (CTMC) has not been proved. For the original 4-dim TK model, no general result from chemical reaction network theory is applicable. Even if the system is reduced to dimension 2, the problem of finding a stationary distribution remains non-trivial and the curious switching behaviour persists. In dimensions 4 and 2, the switching behaviour causes the *seemingly* stationary distribution emerging from simulation to be multimodal for certain parameter values. When the rates are scaled in the volume V of the container and V is considered large, the multimodality disappears and a distribution with a concentrated peak emerges. In this case, the scaled stochastic model converges to the classical deterministic model (fluid limit).

In this paper, we prove that a family of autocatalytic networks, including the TK model, is positive recurrent in arbitrary dimension (theorem 4.1). For some

parameter values, an explicit expression for the stationary distribution is derived. In 2-dim (see theorem 3.1), the parameter region for which the stationary distribution is known covers the 2-dim TK model. In higher dimension (see theorem 4.2), the parameter region for which the stationary distribution is known does not include the general TK model. However, it includes a large family of TK-like models exhibiting DITs.

The analytic form of the stationary distribution provides a clear theoretical demonstration of the effect of volume scaling on the stationary behaviour of the system. It also allows us to find the exact critical value of the volume from which DITs stop appearing. We also find this value for the TK model. Thus, in particular examples, there is a clear demarcation in terms of the volume between stochastic and deterministic systems with comparable behaviour (large volume) and very different behaviour (small volume).

We finally remark that the stationary distributions we derive in the paper are *not* of product form, which seems quite unusual in the theory of stochastic reaction networks [18–20].

2. Background material

2.1. The original 4-dim Togashi–Kaneko model

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the integers including zero. For any two integers i and n, let $(i)_n$ be the remainder after integer division of i by n (elsewhere denoted by $i \bmod n$). The network proposed by Togashi & Kaneko [1] consists of the following cycle of autocatalytic reactions:

$$A_i + A_{(i+1)_4} \xrightarrow{\kappa} 2A_{(i+1)_4}, \quad i = 1, \ldots, 4, \tag{2.1}$$

together with inflow and outflow reactions

$$A_i \underset{\lambda}{\overset{\delta}{\rightleftharpoons}} \emptyset, \quad i = 1, \ldots, 4.$$

The state of the system is a tuple of four non-negative integers $\mathbf{a} = (a_1, a_2, a_3, a_4)'$. Denote by $\mathbf{e}_j$ the jth unit vector, $j = 1, \ldots, 4$. The transition rates generated by the autocatalytic reactions are

$$q_{\mathbf{a},\mathbf{a}-\mathbf{e}_i+\mathbf{e}_{(i+1)_4}} = \kappa a_i a_{(i+1)_4},$$

while those corresponding to inflow and outflow reactions are

$$q_{\mathbf{a},\mathbf{a}+\mathbf{e}_i} = \lambda \quad \text{and} \quad q_{\mathbf{a},\mathbf{a}-\mathbf{e}_i} = \delta a_i.$$

The qualitative behaviour of the system depends on the parameter values. The classical volume scaling (see [21, ch. 11] or [22]) is adopted in [1], where the initial molecule counts of the species are proportional to the scaling parameter V. This implies that the rate constants are given by

$$\kappa = \frac{\kappa'}{V}, \quad \delta = \delta' \quad \text{and} \quad \lambda = \lambda' V.$$

One parameter can always be set to 1 by linear scaling of time. In [1], $\kappa' = 1$, and further $\lambda' = \delta' = D$ for simplification. According to [23] or [21, ch. 11], when $V \to \infty$, the density process, which is the CTMC rescaled by dividing the molecule numbers by V, converges to the solution of a system of ODEs with stable equilibrium (1, 1, 1, 1). Indeed, when $VD \gg 1$, the reaction rates are large and the trajectories of the density process display only small fluctuations around the deterministic equilibrium.

For $VD \ll 1$, a completely different behaviour appears, triggered by the slow rate of inflow and DITs appear. If the system is initialized at a state where all species counts are large, one of the species at random (say, species 3) is quickly driven to extinction by the fast autocatalytic dynamics. At this point, several molecules of species 2 are produced and not consumed and they catalyse the consumption of all molecules of species 1. We end up with a configuration where species 1 and 3 are both consumed, the count of species 2 is very high and that of species 4 is quite low. We call this pattern 2H4L. In this configuration, only slow inflows and outflows are active, and one needs to wait until a molecule of species 3 or 1 flows in before the autocatalytic dynamics starts again leading to another pattern with two non-contiguous species extinct. The dynamics of the system then proceeds by switching between such patterns in a way that a 2H4L configuration is much more often followed by a 2L4H pattern and only rarely switches to either a 1H3L or 1L3H configuration (see fig. 1 in [1]).

2.2. Lumpability

In the next section, we exploit the notion of lumpability to find the stationary distribution in some cases. We summarize here the meaning of this property.

Let $\{S_I\}_{I \in \mathcal{I}}$ be a partition of a denumerable state space S of a CTMC $X(t)$, $t \geq 0$, with rates q_{ij}, $i, j \in S$. Let moreover ι be the function that maps $x \in S$ to the index of the element of the partition to which x belong (i.e $\iota(x) = K$ if and only if $x \in S_K$). The process $X(t)$, $t \geq 0$, is (strongly) lumpable if the *lumped* process $\overline{X}(t) = \iota(X(t))$, $t \geq 0$, is a CTMC on $\mathcal{I}$ for any choice of initial distribution. Sufficient conditions (see [24]) that guarantee lumpability of a regular, irreducible, positive recurrent CTMC $X(t)$, $t \geq 0$, on the partition $\{S_I\}_{I \in \mathcal{I}}$ are that every subset S_I is finite, and that for any $I, J \in \mathcal{I}$, and any $i, i' \in S_I$,

$$\sum_{j \in S_J} q_{ij} = \sum_{j \in S_J} q_{i'j} = \overline{q}_{IJ}.$$

The rates of the lumped chain $\overline{X}(t)$, $t \geq 0$, are $\overline{q}_{IJ}$, $I, J \in \mathcal{I}$, and for any $s < t$, the lumped variable $\overline{X}(t)$ is independent of X_s given $\overline{X}_s$

3. The 2-dim Togashi–Kaneko model

If the number of species in the TK model is reduced to two, the reaction network becomes

$$\left.\begin{aligned} & 2A_1 \overset{\kappa_1}{\leftarrow} A_1 + A_2 \overset{\kappa_2}{\rightarrow} 2A_2 \\ \text{and} \quad & A_1 \underset{\lambda_1}{\overset{\delta_1}{\rightleftharpoons}} \emptyset \underset{\delta_2}{\overset{\lambda_2}{\rightleftharpoons}} A_2, \end{aligned}\right\} \tag{3.1}$$

where we allow κ_1 and κ_2 to be different. The state of the network is denoted by $\mathbf{a} = (a_1, a_2)' \in \mathbb{N}^2$, the molecule counts of each species. The transition rates owing to the autocatalytic reactions are

$$q_{\mathbf{a},\mathbf{a}-\mathbf{e}_1+\mathbf{e}_2} = \kappa_2 a_1 a_2 \quad \text{and} \quad q_{\mathbf{a},\mathbf{a}-\mathbf{e}_2+\mathbf{e}_1} = \kappa_1 a_1 a_2, \tag{3.2}$$

while those corresponding to inflow and outflow reactions are

$$q_{\mathbf{a},\mathbf{a}+\mathbf{e}_i} = \lambda_i \quad \text{and} \quad q_{\mathbf{a},\mathbf{a}-\mathbf{e}_i} = \delta_i a_i, \quad i = 1, 2. \tag{3.3}$$

The dynamics is simplified, but not too much. When the inflows occur at a much slower rate than the autocatalytic reactions, the system switches between two patterns in a similar way to the original four-dimensional TK system, where one or the other compound is mostly absent. A plot of the two simulated trajectories in this parameter range is shown in figure 1.

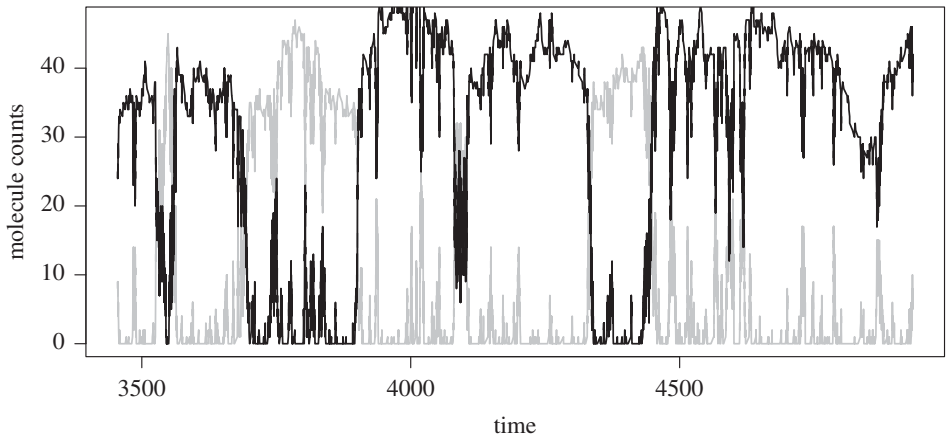

Figure 1. Molecule counts of the two species of network (3.1) along time. Patterns where the grey species is mostly absent alternate with patterns where the mostly absent species is the black one. Parameters are $\lambda_i = 0.2$, $\delta_i = 0.01$ and $\kappa_i = 0.05$ for $i = 1, 2$.

3.1. Positive recurrence and stationary distribution

A proof of positive recurrence in a more general setting is given in §4.1. In this section, we show the sketch of how to derive the stationary distribution of (3.1) by using its lumpability. The subsets

$$E_n = \{\mathbf{a} \in \mathbb{N}^2 \colon a_1 + a_2 = n\}$$

form a partition $\{E_n\}_{n\in\mathbb{N}}$ of the state space. The CTMC model $X(t)$, $t \geq 0$, of (3.1) under stochastic mass-action kinetics is lumpable with respect to this partition if $\delta = \delta_1 = \delta_2$. With this choice, the rate at which the total molecule count n is increased by 1, is equal to the sum of the rates of the inflows,

$$q_{n,n+1} = \sum_{i=1}^{2} q_{\mathbf{a},\mathbf{a}+\mathbf{e}_i} = \lambda_1 + \lambda_2,$$

independently of $\mathbf{a}$. The rate at which n is decreased by 1, is the sum of the rates of the outflows,

$$q_{n,n-1} = \sum_{i=1}^{2} q_{\mathbf{a},\mathbf{a}-\mathbf{e}_i} = \delta(a_1 + a_2) = \delta n,$$

and therefore it does not depend on $\mathbf{a}$ as long as $\mathbf{a} \in E_n$.

The lumped process $\overline{X}(t)$, $t \geq 0$, is described by the following reaction network where a single species B aggregates all molecules of A_1 and A_2:

$$B \underset{\lambda_1+\lambda_2}{\overset{\delta}{\rightleftharpoons}} \emptyset. \tag{3.4}$$

Network (3.4) is weakly reversible and has *deficiency* zero [19]. By [19, theorems 3.6 and 3.7], it admits a unique stationary distribution with Poisson law

$$\nu(n) = \frac{\mu^n}{n!}\exp(-\mu), \quad \mu = \frac{\lambda_1 + \lambda_2}{\delta}, \tag{3.5}$$

where n is the state of the lumped process (i.e. $\overline{X}(t) = n$ if and only if $X(t) \in E_n$). We now aim at factorizing the stationary distribution $\Pi(\mathbf{a})$ of the process $X(t)$, $t \geq 0$, of (3.1) by conditioning on the stationary probability $\nu(n)$ of the lumped process being in state $n = a_1 + a_2$. We write

$$\Pi(\mathbf{a}) = \pi(a_1|n)\nu(n). \tag{3.6}$$

Using (3.6) and (3.5) in the master equation for the stationary distribution $\Pi(\mathbf{a})$ and cancelling out common factors, we get that $\Pi(\mathbf{a})$ is stationary if and only if $\pi(a \mid n)$ fulfils

$$R_n = L_{n-1} + L_n + L_{n+1}, \tag{3.7}$$

where

$$R_n = (\lambda_1 + \lambda_2 + n\delta + (\kappa_1 + \kappa_2)a(n-a))\pi(a|n)$$

sums up all the rates of the reactions that can fire in state $\mathbf{a} = (a, n-a)$, while

$$\begin{aligned} L_{n-1} &= \frac{n\delta\lambda_1}{\lambda_1+\lambda_2}\pi(a-1|n-1) + \frac{n\delta\lambda_2}{\lambda_1+\lambda_2}\pi(a|n-1), \\ L_n &= \kappa_2(a+1)(n-a-1)\pi(a+1|n) \\ &\quad + \kappa_1(a-1)(n-a+1)\pi(a-1|n) \\ \text{and} \quad L_{n+1} &= \frac{\lambda_1+\lambda_2}{n+1}(a+1)\pi(a+1|n+1) \\ &\quad + \frac{\lambda_1+\lambda_2}{n+1}(n-a+1)\pi(a|n+1) \end{aligned}$$

collect the sum of the rates of inflow (L_{n-1}), autocatalytic (L_n) and outflow (L_{n+1}) reactions that lead to state $\mathbf{a} = (a, n-a)$, for $n \geq 0$ and $a = 0, \ldots, n$.

Unfortunately, there is not a simple way to find a closed-form expression of $\pi(\cdot \mid n)$ satisfying equation (3.7). However, simulation of the process for different rate constants, corresponding to different regimes of the volume V (see §3.2 for more details), indicates that the conditional stationary distribution may be unimodal, flat or concentrated at the boundaries (see figure 2). Statistical practice suggests the beta-binomial as a natural candidate for a discrete distribution on the integers $\{0, \ldots, n\}$ that may display these behaviours. The next theorem confirms this, and figure 2 provides a graphical comparison between simulations and theoretical values in different parameter settings.

Theorem 3.1. *Network (3.1), assuming that $\kappa = \kappa_1 = \kappa_2 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\delta = \delta_1 = \delta_2 > 0$, has a unique stationary distribution $\Pi(\mathbf{a})$ that factorizes as (3.6), where $\nu(n)$ is given by (3.5), and $\pi(\cdot \mid n)$ is given by the beta-binomial distribution*

$$\pi(i|n) = \binom{n}{i}\frac{B(i+\alpha, n-i+\beta)}{B(\alpha,\beta)}, \quad i = 0, \ldots, n, \tag{3.8}$$

where

$$\alpha = \frac{\delta\lambda_1}{\kappa(\lambda_1+\lambda_2)}, \quad \beta = \frac{\delta\lambda_2}{\kappa(\lambda_1+\lambda_2)} \tag{3.9}$$

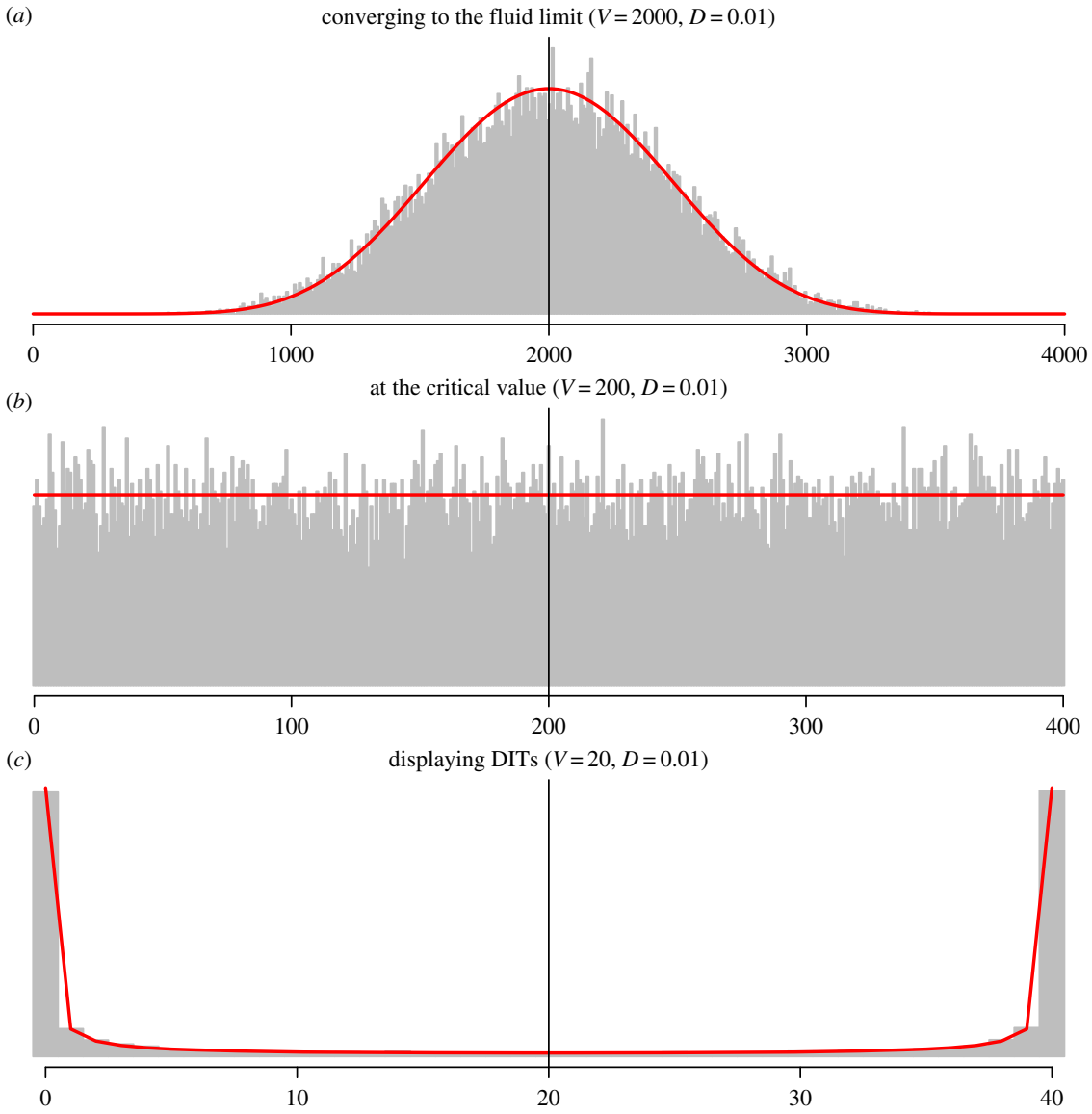

Figure 2. The effect of scaling and DITs. The conditional stationary distribution $\pi(a_1|n)$; see (3.6) from simulation (grey histograms) and from (3.9) in theorem 3.1 (red lines). Parameters are chosen according to (3.11) with $\kappa'_i = 1$, $\delta'_i = \lambda'_i = D = 0.01$ for $i = 1, 2$. The volume parameter V differs in (*a–c*) to illustrate the effect of scaling, and n is chosen as the mean of $\nu(n)$, which is 4000, 400 and 40, respectively; implying that the mean of the scaled process X/V is (1, 1) in all three cases. Simulation set-up: 2.75×10^6, 10^6 and 10^5 simulations (for (*a*), (*b*) and (*c*), respectively) were conducted with fixed time $T = 250$, 50, 50, respectively (the stationary regime already applies). Only values of $\mathbf{a}(T)$ with $a_1(T) + a_2(T) = n$ were kept, and the histogram of $a_1(T)$ was plotted.

and

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0.$$

The proof is by direct verification, substituting expression (3.8) into equation (3.7). Calculations are displayed in appendix B in a more general context.

3.2. Volume scaling

Molecule counts and mass-action rates can be scaled with the volume V in such a way that the scaled stochastic system $X(t)/V$ converges for large V to the solution of the deterministic system on any finite time horizon (see [21, ch. 11, theorem 2.1]).

This is achieved for (3.2) and (3.3), under the hypothesis of theorem 3.1, by setting the constants to

$$\kappa_i = \frac{\kappa'}{V} \quad \delta_i = \delta', \quad \lambda_i = \lambda'_i V, \tag{3.11}$$

for $i = 1, 2$. When V is not sufficiently large the stochastic model differs significantly from the deterministic limit [1] and starts to display the switching behaviour (DITs) illustrated in figure 1.

In [1], the authors set

$$\kappa'_i = 1, \quad \delta'_i = \lambda'_i = D, \quad i = 1, 2. \tag{3.12}$$

With this choice of the rate constants, by theorem 3.1, we know the explicit form of the stationary density, and we can investigate the behaviour of the system at every V without resorting to simulations. The stationary conditional density $\pi(\cdot\,|\,n)$ is beta-binomial with parameters $\alpha = \beta = DV/2$. The beta-binomial density is unimodal when α and β are both greater than 1 (that is, when DV is greater than 2) with the mass concentrated at the equilibrium of the corresponding deterministic model. When both α and β are less than 1 (that is, when DV is small) the density becomes bimodal with most of the mass at the boundaries. The intermediate case is when $\alpha = \beta = 1$ and the conditional distribution reduces to the discrete uniform distribution on $\{0, \ldots, n\}$. In other

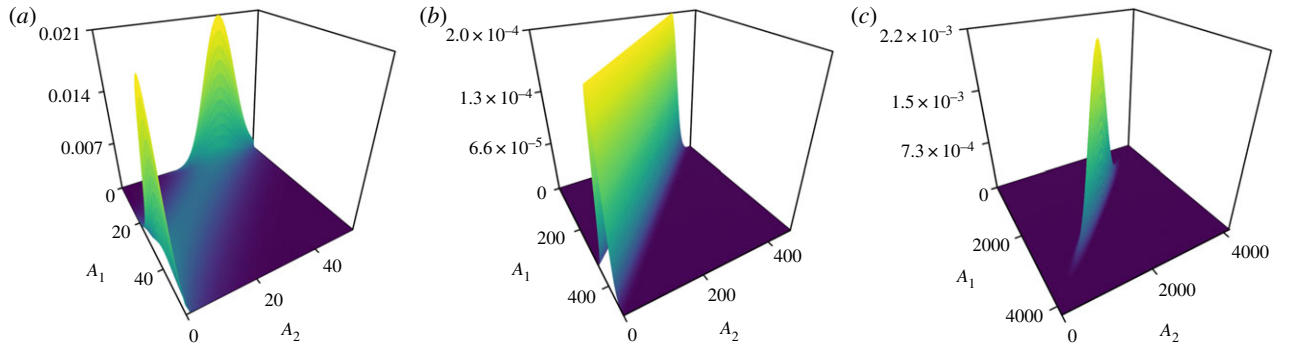

Figure 3. Smoothed representation of the stationary density (3.6). In all panels $D = 0.01$. The volume parameter V is different in the three panels to illustrate the effect of the scaling. In (*a*), $V = 20$ (same range as in figure 1). The density is bimodal and concentrated at the boundaries, since DITs are present. In (*b*), $V = 200$ and the conditional density (3.9) is uniform. In (*c*), $V = 2000$ and the density is concentrated around the deterministic equilibrium.

words, at the critical value $DV = 2$, the conditional density flattens to $\pi(a \mid n) = 1/(n+1)$ for every a. A pictorial representation of the density (3.6), at different values of V with D fixed to the value 0.01, is given in figure 3. The effect of the scaling is apparent. For graphical convenience, the discrete density has been smoothed to a continuous one.

To make this effect quantitatively apparent, in the general setting where (3.11) holds but not necessarily (3.12), we prove that for $V \to 0$ the stationary distribution concentrates at the boundaries by showing that the conditional probability $\pi(0 \mid n) + \pi(n \mid n)$ tends to 1, for any n. Indeed, inserting (3.11) into (3.10), we get $\alpha = \alpha' V$ and $\beta = \beta' V$ with

$$\alpha' = \frac{\delta\lambda_1'}{\kappa(\lambda_1' + \lambda_2')} \quad \text{and} \quad \beta' = \frac{\delta\lambda_2'}{\kappa(\lambda_1' + \lambda_2')}.$$

The sum of the two conditional probabilities reduces to

$$\pi(0|n) + \pi(n|n) = \left[\frac{\Gamma(n+\alpha' V)]}{\Gamma(\alpha' V)} + \frac{\Gamma(n+\beta' V)}{\Gamma(\beta' V)}\right] \times \frac{\Gamma((\alpha'+\beta')V)}{\Gamma(n+(\alpha'+\beta')V)}. \tag{3.13}$$

Whatever n is, since $\Gamma(z) \sim 1/z$ for $z \to 0$, it is easily seen that the sum tends to 1 as $V \to 0$.

For large V, we show that the stationary distribution Π_V of the scaled process $X(t)/V$ concentrates around the deterministic equilibrium (1, 1). The mean μ_V and variance Σ_V of Π_V might easily be computed (by conditioning on n) from the first and second moments of the Poisson distribution and the beta-binomial distribution.

The explicit calculation is reported here for only two components, but can be found for the others as well,

$$(\mu_V)_1 = \frac{\mu}{V}\frac{\alpha}{\alpha+\beta}$$

and

$$(\Sigma_V)_{11} = \frac{1}{V^2}\left[\frac{\alpha\beta}{(\alpha+\beta)^2}\frac{(\alpha+\beta)\mu + \mu^2 + \mu}{\alpha+\beta+1} + \mu\frac{\alpha}{\alpha+\beta}\right],$$

where μ is given in (3.5) and α and β in (3.9). Scaling the parameters as in (3.10), it is easily observed that $(\mu_V)_1 \to 1$ and $(\Sigma_V)_{11} \to 0$ for $V \to \infty$. With a little more effort, the same result extends to the other components, that is, we have

$$\mu_V \to \left(\frac{\lambda_1'}{\delta'}, \frac{\lambda_2'}{\delta'}\right) \quad \text{and} \quad \Sigma_V \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In general, the *agreement* between the stochastic and the deterministic model for large volumes only holds on a finite time horizon. Negative examples where the two modelling paradigms differ asymptotically are discussed in [25,26]. Our result shows that for large V, under the assumptions of theorem 3.1, the stochastic and the deterministic models of (3.1) are in agreement asymptotically.

4. Higher dimensional models

In higher dimension, there exist different models whose 2-dim reduction corresponds to network (3.1). One of them is the 4-dim TK model (2.1), but also the network

$$A_i + A_j \xrightarrow{\kappa_{ij}} 2A_j \quad A_i \underset{\lambda_i}{\overset{\delta_i}{\rightleftharpoons}} \emptyset, \tag{4.1}$$

$i, j = 1, \ldots, d$, $i \neq j$, can be seen as a d-dimensional version of model (3.1). Network (4.1) includes (2.1) as a special case for κ_{ij} equal to κ when $j = (i+1)_d$ and zero otherwise. Reaction rates are the obvious generalizations of (3.2) and (3.3).

4.1. Positive recurrence

In this section, we state the positive recurrence of the Markov process underlying the general d-dimensional model (4.1). To do so, we show that $V(x) = e^{\|x\|_1}$, where $\|x\|_1 = \sum_{i=1}^d |x_i|$ is a Lyapunov function. Non-explosivity and positivity, then, follow by the Foster–Lyapunov criterion [27]. Additionally, as a by-product, all moments of the stationary distribution are positive and convergence to the stationary distribution is exponentially fast. The detailed proof can be found in appendix A.

Theorem 4.1. *For any non-negative values of the parameters κ_{ij}, and for positive λ_i and δ_i, the CTMC associated with the system (4.1) is positive recurrent on $\mathbb{N}^d$ (for any d). Consequently, it has a unique stationary distribution supported on $\mathbb{N}^d$. Moreover, all moments are finite and the convergence to the stationary distribution is exponentially fast.*

If some of the parameters are not positive (hence zero) there might still be a stationary distribution, though it might be either non-unique or not concentrated on all of $\mathbb{N}^d$. For example, if $\lambda_i = 0$ for all $i = 1, \ldots, d$, then the CTMC is absorbed at the origin, and if only some of the λ_is are zero, then the corresponding species will eventually be depleted.

4.2. Stationary distribution

4.2.1. The model

By the same argument as we used in dimension 2, under the assumption of equal outflow rates ($\delta_i = \delta$ for all $i = 1 \cdots d$), the process $X(t)$, $t \geq 0$, that counts the molecules of each species is lumpable on the partition $\{E_n\}$, where $E_n = \{\mathbf{a} \in \mathbb{N}^d : \sum_i a_i = n\}$.

The lumped process $\overline{X}(t) = \sum_i X_i(t)$ represents the total molecule count. It follows a birth and death process (as in (3.4)) with Poisson stationary distribution with intensity

$$\mu = \frac{\sum_{i=1}^{d} \lambda_i}{\delta}. \tag{4.2}$$

Similarly to the 2-dim case, the stationary distribution $\Pi(\mathbf{a})$ factorizes as

$$\Pi(\mathbf{a}) = \pi(\mathbf{a}|n)\nu(n). \tag{4.3}$$

Theorem 4.2. *Assume* $\kappa_{ij} = \kappa > 0$, $i, j = 1, \ldots, d$, $i = j$, $\delta = \delta_1 = \cdots = \delta_d > 0$, *and* $\lambda_i > 0$ *for all* i. *Then, model* (4.1) *has a unique stationary distribution* $\Pi(\mathbf{a})$ *expressed as in* (4.3), *where* $\nu(n)$ *is given as in* (3.5) *and* (4.2), *and* $\pi(\cdot \mid n)$ *is given by the Dirichlet-multinomial distribution*

$$\pi(\mathbf{a}|n) = \binom{n}{\mathbf{a}} \frac{\Gamma(\sum_{i=1}^{d} \alpha_i)}{\Gamma(n + \sum_{i=1}^{d} \alpha_i)} \prod_{i=1}^{d} \frac{\Gamma(a_i + \alpha_i)}{\Gamma(\alpha_i)}, \tag{4.4}$$

where $\mathbf{a}$ is any d-dimensional integer vector with $\|\mathbf{a}\|_1 = n$, and

$$\alpha_i = \frac{\delta \lambda_i}{\kappa \sum_{j=1}^{d} \lambda_j}.$$

The proof is by direct verification, substituting expression (4.4) into equation (4.3) using (3.5). Calculations are displayed in appendix B.

4.3. Volume scaling and other properties

The scaled process $X(t)/V$ in dimension d has similar properties to that of the scaled process in dimension 2. In the case where the stationary distribution is known (theorem 4.2), we might proceed similarly to what was done in dimension 2 and calculate the mean vector and covariance matrix of the molecule counts, now using moment properties of the Poisson and the Dirichlet-multinomial distributions, and assuming the parameters are scaled according to

$$\kappa = \frac{\kappa'}{V}, \quad \delta = \delta', \quad \lambda_i = \lambda_i' V, \tag{4.5}$$

for $i = 1, \ldots, d$.

As the volume V increases towards infinity, it can be shown that the mean vector converges to $(\lambda_1'/\delta', \ldots, \lambda_d'/\delta')$, the equilibrium point of the deterministic process, and the covariance matrix decreases towards the zero matrix. Thus, under the hypothesis of theorem 4.2, the deterministic and the stochastic models of (4.1) are in agreement asymptotically for large volume size in the long run (at stationarity) as well as over the finite time horizon.

At the other extreme, for $V \to 0$, the conditional probability of a corner configuration tends to 1. Indeed, such a

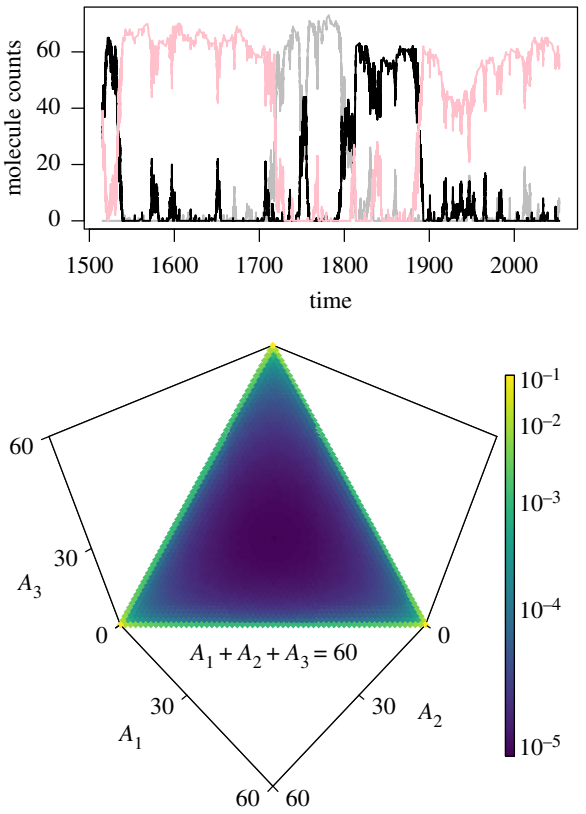

Figure 4. Simulated trajectories and the conditional stationary distribution $\pi(\mathbf{a}|n)$, from model (4.1) in dimension 3. Parameters are chosen according to (4.6) with $D = 0.01$ and $V = 20$. The value of n is fixed to 60 in the lower plot. The presence of DITs is apparent both from the trajectories and from the conditional distribution that is concentrated at the corners of the simplex.

probability, which generalizes (3.13), is equal to

$$\sum_{i=1}^{d} \pi(\mathbf{e}_i|n) = \frac{\Gamma(V \sum_j \alpha_j')}{\Gamma(n + V \sum_j \alpha_j')} \sum_{i=1}^{d} \frac{\Gamma(n + V\alpha_i')}{\Gamma(V\alpha_i')},$$

where

$$\alpha_i' = \frac{\delta \lambda_i'}{\kappa \sum_{j=1}^{d} \lambda_j'}.$$

The convergence to 1 can be easily shown with the same methodology used in dimension 2.

The peaks at the vertexes reflect the presence of DITs that cause the switch between dynamical patterns where only one of the species is present in large quantity at a time, while all the others are almost extinct. A graphical illustration of the presence of DITs in a three-dimensional version of model (4.1) is given in figure 4. In dimension 3, it is no longer possible to plot the stationary distribution $\Pi(\mathbf{a})$. However, we can plot a set of simulated trajectories and the values of the conditional stationary distribution $\pi(\mathbf{a} \mid n)$.

If the parameters are further chosen as

$$\kappa_i' = 1, \quad \delta_i' = \lambda_i' = D, \quad i = 1, \ldots, d, \tag{4.6}$$

in analogy of what was done in [1], the distribution becomes symmetric in the labels of the species and the $\alpha_i = DV/d$, $i = 1, \ldots, d$, are all equal. Moreover, if $V = \frac{d}{D}$, the conditional distribution is flat, providing a transition point from the multimodal case to the unimodal case. If the α_is are not

equal (i.e. the λ'_i are not), the transition will not proceed through a flat conditional distribution. The parameters used for the simulations in figure 4 were $D = 0.01$ and $V = 20$, therefore in the region where DITs are expected.

Other relevant properties of the Dirichlet-multinomial distribution, such as aggregation, marginals and conditional distributions, are discussed in [28,29].

4.4. Back to the d-dim Togashi–Kaneko model

Model (2.1) motivated our interest in autocatalytic networks. Theorem 4.1 guarantees that it is positive recurrent, but an explicit form of the stationary distribution cannot be derived by theorem 4.2. Indeed, it is a special case of (4.1), where some of the κ_{ij} are set to zero (those for which $j \neq (i+1)_d$) and all others are set to the same value κ. However, it is still possible to find the explicit expression in a very special case.

Theorem 4.3. *Assume that* $\kappa = \kappa_1 = \cdots = \kappa_d \geq 0$ *and* $\delta = \delta_1 = \cdots = \delta_d = (d/(d-1))\kappa$ *and* $\lambda = \lambda_1 = \cdots = \lambda_d > 0$. *Then, model* (2.1) *has a unique stationary distribution* $\Pi(\mathbf{a})$ *whose expression is* (4.3) *with* $\nu(n)$ *given by* (3.5) *and* (4.2) *and with* $\pi(\cdot \mid n)$ *given by a uniform distribution*

$$\pi(\mathbf{a}|n) = \frac{n!(d-1)!}{(n+d-1)!} \tag{4.7}$$

on the simplex $\{\mathbf{a} \in \{0, \ldots n\}^d : \|\mathbf{a}\|_1 = n\}$.

The proof is by direct verification, substituting expression (4.7) into equation (4.3) with κ_{ij} set to zero for all $j \neq (i+1)_d$ and to the same value κ otherwise. Calculations are displayed in appendix C. If the rate constants are scaled in the volume as in equation (4.5) and further set to (4.6), the critical value of the volume that makes the distribution flat is $V = d/((d-1)D)$, in agreement with the result for $d = 2$. In 4-dim, in [1], it was noticed from simulation that the order of the magnitude of this critical value should be around $V \sim 1/D$. However, determining the exact value was pursued. Our result allows us to ensure that the exact value is $V = 4/3D$.

5. Conclusion

We provide a theoretical analysis of some autocatalytic reaction networks motivated by the original TK model. These models exhibit the phenomenon of DITs. For specific classes of models with symmetry in the rate constants, it is possible to find the stationary distribution exactly. This allows us to further analyse the models and determine the exact tipping point of DITs, namely for $V = d/((d-1)D)$ for the TK model (in every dimension d, both odd and even; see [30]) and $V = d/D$ for model (4.1). In general, this is too much to hope for. With asymmetric rate constants the networks are still positive recurrent, therefore a unique stationary distribution exists, but their analytical expression remains unknown and it might be less clear when the phase transition happens. However, it would be interesting to investigate whether the effect of volume scaling we observe for small and large V pertains to the autocatalytic reaction networks in general.

Data accessibility. This article has no additional data.

Authors' contributions. E.B. proposed the research topic, and stated the analytical solutions of the stationary distributions. J.K. proved theorem 2. E.B., J.K. and C.W. contributed to the research in general and jointly wrote the manuscript.

Competing interests. The authors declare that they have no competing interests.

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Appendix

In this appendix, we report the detailed proofs of the three main theorems (theorems 4.1–4.3) that are stated in §4. Theorem 3.1 is not proved separately since it is a special case of theorem 4.2. The most general high-dimensional model that we consider is that in (4.1), which we repeat here

$$A_i + A_j \xrightarrow{\kappa_{ij}} 2A_j \quad A_i \underset{\lambda_i}{\overset{\delta_i}{\rightleftharpoons}} \emptyset,$$

Appendix A

As we stated in the main text, the associated CTMC for the general dimensional model (4.1) is positive recurrent and admits a unique stationary distribution. We prove this in the following theorem. We further show that the CTMC is *exponentially ergodic*, meaning that the associated distribution P^t at time t converges to the unique stationary distribution exponentially fast. The proof relies on the Foster–Lyapunov criterion [27]. We begin with a formal statement and necessary concepts for the Foster–Lyapunov criterion.

Definition A.1. For a CTMC $X(t)$, $t \geq 0$, defined on a countable state space χ, the infinitesimal generator $\mathcal{L}$ is the operator

$$\mathcal{L}V(x) = \sum_{\eta} \lambda_\eta(x)(V(x+\eta) - V(x)),$$

where η is a transition of $X(t)$, λ_η is the transition rate associated with η and V is any real function defined on the state space.

Let $\tau_M = \inf\{t > 0 : |X(t)| \geq M\}$. For a CTMC $X(t)$, $t \geq 0$, we define a truncated process X_M such that $X_M(t) = X(t)$ if $t < \tau_M$ and $X_M(t) = x_M$ otherwise for some fixed state x_M with $|x_M| \geq M$. We denote by $\mathcal{L}_M$ the infinitesimal generator of X_M.

We further call $V(x)$ a norm-like function if $V(x)$ is a positive function such that $|V(x)| \to \infty$, as $|x| \to \infty$.

The following theorem is theorem 6.1 in [27], in the case of a countable state space. It is one version of the Foster–Lyapunov criterion for exponential ergodicity.

Theorem A (Foster–Lyapunov criterion [27]). Let $X(t)$, $t \geq 0$, be a CTMC defined on a countable state space χ. Then $X(t)$, $t \geq 0$, is non-explosive and positive recurrent if there exist a norm-like function V on χ and positive constants C and D such that for any $M > 0$

$$\mathcal{L}_M V(x) \leq -CV(x) + D \quad \text{for all } |x| < M.$$

Furthermore, $X(t)$, $t \geq 0$, admits a unique stationary distribution π on each irreducible component, and there exist $B > 0$ and $\beta \in (0, 1)$ such that

$$\sup_A |P^t(x, A) - \pi(A)| \leq BV(x)\beta^t \quad \text{for all } x \in \chi.$$

To show positive recurrence and exponential ergodicity of the CTMC associated with the general model (4.1), it is therefore sufficient to prove that there exist a norm-like function V and positive constants C, D such that

$$\mathcal{L}V(x) \leq -CV(x) + D \quad \text{for all } x. \tag{A 1}$$

In the proof of the following theorem, we prove (A 1) for an exponential function V. Using this specific function, we also show that all moments of the unique stationary distribution of $X(t)$, $t \geq 0$, are finite.

Proof of theorem 4.1. Let $X(t)$, $t \geq 0$, be the CTMC associated with the system (4.1). Let $V(x) = e^{\|x\|_1}$, where $\|x\|_1 = \sum_{i=1}^d |x_i|$. Then we show that (A 1) holds for some positive constants C and D.

Let $e_i \in \mathbb{N}^d$ be the vector with ith component 1 and zero otherwise. We have

$$\begin{aligned}\mathcal{L}V(x) &= \sum_{i,j} \kappa_{ij} x_i x_j (V(x - e_i + e_j) - V(x)) \\ &\quad + \sum_{i=1}^d \delta_i x_i (V(x - e_i) - V(x)) + \sum_{i=1}^d \lambda_i (V(x + e_i) - V(x)) \\ &= \sum_{i=1}^d \delta_i x_i (V(x - e_i) - V(x)) + \sum_{i=1}^d \lambda_i (V(x + e_i) - V(x)).\end{aligned}$$

Let $K_n = \{x \in \mathbb{N}^d : x_i \geq n \text{ for each } i\}$. Then note that, for $x \in K_n$,

$$\begin{aligned}\mathcal{L}V(x) &= V(x)\left(\sum_{i=1}^d \delta_i x_i (e^{-1} - 1) + \sum_{i=1}^d \lambda_i (e - 1)\right) \\ &\leq \left((\min_i \delta_i)(e^{-1} - 1)dn + \sum_{i=1}^d \lambda_i (e - 1)\right) V(x).\end{aligned}$$

Hence, by choosing sufficiently large N such that

$$C = -\left((\min_i \delta_i)(e^{-1} - 1)\, \mathrm{d}N + \sum_{i=1}^d \lambda_i (e - 1)\right) > 0,$$

we conclude that (A 1) holds with $D = 2C \max_{x \in K_N^c} V(x)$. This implies that $X(t)$, $t \geq 0$, is non-explosive, positive recurrent and exponentially ergodic by theorem A. This implies the existence of a unique stationary distribution π.

To show that π has finite mth moment for any $m \in \mathbb{N}^d$, we use (A 2) below combined with the ergodic theorem [31]. Then by using Dynkin's formula [32,33] and (A 1), we have

$$\begin{aligned}\mathbb{E}_x(V(X(t \wedge \tau_M))) &= V(x) + \mathbb{E}_x\left(\int_0^{t \wedge \tau_M} \mathcal{L}V(X(s))\, \mathrm{d}s\right) \\ &\leq V(x) - C\mathbb{E}_x\left(\int_0^{t \wedge \tau_M} V(X(s))\, \mathrm{d}s\right) + Dt,\end{aligned} \tag{A 2}$$

where $\mathbb{E}_x$ denotes the expectation of $X(t)$ with $X(0) = x$ and $t \wedge \tau_M = \min\{t, \tau_M\}$. By rearranging terms in (A 2) and dividing by t, C, it follows that

$$\mathbb{E}_x\left(\frac{1}{t}\int_0^{t \wedge \tau_M} V(X(s))\mathrm{d}s\right) \leq \frac{V(x)}{Ct} + \frac{D}{C}. \tag{A 3}$$

Then by the monotone convergence theorem, taking lim for $M \to \infty$ on both sides in (A 3) gives that

$$\begin{aligned}&\lim_{M \to \infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^{t \wedge \tau_M} V(X(s))\mathrm{d}s\right) \\ &= \mathbb{E}_x\left(\frac{1}{t}\int_0^t V(X(s))\mathrm{d}s\right) \leq \frac{V(x)}{Ct} + \frac{D}{C}.\end{aligned}$$

Then the ergodic theorem and Fatou's lemma apply for $t \to \infty$ to conclude that $\sum_{x \in \mathbb{N}^d} V(x)\pi(x) \leq D/C$. Since $V(x) = e^{\|x\|_1}$, any moment of π is finite. ∎

Appendix B. Stationary distribution

Proof of theorem 4.2. Under the assumption of equal outflow rates, the process $X(t)$ that counts the molecules of each species is lumpable on the partition $\{E_n\}_{n \in \mathbb{N}}$, where $E_n = \{\mathbf{a} \in \mathbb{N}^d : \sum_{i=1}^d a_i = n\}$.

The lumped process $\overline{X}(t) = \sum_{i=1}^d X_i(t)$ has Poisson stationary distribution $\nu(n)$ with intensity (4.2). As stated earlier, the stationary distribution $\Pi(\mathbf{a})$ factorizes as $\Pi(\mathbf{a}) = \pi(\mathbf{a}|n)\nu(n)$. Under the given assumptions on the parameters, $\pi(\mathbf{a}|n)$ solves the equation, similar to (3.7),

$$R_n = L_{n-1} + L_n + L_{n+1}, \tag{B 1}$$

where

$$R_n = \pi(\mathbf{a}|n)\left[\sum_{i=1}^d \lambda_i + \delta n + \sum_{i=1}^d \sum_{j \neq i} \kappa a_i a_j\right],$$

$$L_{n-1} = \frac{\delta n}{\sum_{i=1}^d \lambda_i} \sum_{i=1}^d \pi(\mathbf{a} - \mathbf{e}_i|n-1)\lambda_i,$$

$$L_n = \sum_{i,j=1}^d \pi(\mathbf{a} - \mathbf{e}_i + \mathbf{e}_j|n)\kappa(a_i - 1)(a_j + 1)$$

and

$$L_{n+1} = \frac{\sum_{i=1}^d \lambda_i}{(n+1)} \sum_{i=1}^d \pi(\mathbf{a} + \mathbf{e}_i|n+1)(a_i + 1).$$

The proof now proceeds by showing that the ansatz $\pi(\cdot\,|\,n)$ specified by equation (4.4) solves equation (B 1). First, we note that if the ansatz is true, then the following

recurrence relations hold:

$$\left.\begin{aligned}\pi(\mathbf{a}|n) &= \frac{1}{n+1}\sum_{i=1}^{d}(a_i+1)\,\pi(\mathbf{a}+\mathbf{e}_i|n+1),\\ \pi(\mathbf{a}-\mathbf{e}_i|n-1) &= \frac{a_i(n-1+\sum_{i=1}^{d}\alpha_i)}{n(a_i-1+\alpha_i)}\,\pi(\mathbf{a}|n)\\ \text{and}\quad \pi(\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j|n) &= \frac{a_i(a_j+\alpha_j)}{(a_j+1)(a_i-1+\alpha_i)}\,\pi(\mathbf{a}|n).\end{aligned}\right\} \tag{B 2}$$

Applying (B 2) and dividing by $\pi(\mathbf{a}|n)$ in (B 1) we get

$$\delta n+\sum_{i=1}^{d}\sum_{j\neq i}\kappa a_i a_j=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j\neq i}\frac{\kappa a_i(a_i-1)(a_j+\alpha_j)}{a_i-1+\alpha_i}. \tag{B 3}$$

By fixing $a_i=n$, the following condition is necessary:

$$\delta(n-1+\alpha_i)=\delta\left(n-1+\sum_{i=1}^{d}\alpha_i\right)\frac{\lambda_i}{\sum_{i=1}^{d}\lambda_i}+\kappa(n-1)\sum_{j\neq i}\alpha_j. \tag{B 4}$$

If we further set $n=1$, we get

$$\frac{\lambda_i}{\sum_{i=1}^{d}\lambda_i}=\frac{\alpha_i}{\sum_{i=1}^{d}\alpha_i}. \tag{B 5}$$

Moreover, if we take equation (B 4) and sum over all $i=1,\ldots,d$, we get

$$(d-1)\delta(n-1)+\delta\sum_{i=1}^{d}\alpha_i=\delta\sum_{i=1}^{d}\alpha_i+\kappa(n-1)(d-1)\sum_{i=1}^{d}\alpha_i,$$

which further implies

$$\sum_{i=1}^{d}\alpha_i=\frac{\delta}{\kappa}. \tag{B 6}$$

Together with equation (B 5), this implies

$$\alpha_i=\frac{\delta\lambda_i}{\kappa\sum_{i=1}^{d}\lambda_i}. \tag{B 7}$$

Taking again equation (B 3), we can manipulate the third summand on the right-hand side (r.h.s.) in order to recast the equation into the following form:

$$\delta n=-\sum_{i=1}^{d}\sum_{j\neq i}\kappa a_i a_j+\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j\neq i}\frac{\kappa a_i(a_i-1+\alpha_i)(a_j+\alpha_j)}{a_i-1+\alpha_i}-\sum_{i=1}^{d}\sum_{j\neq i}\frac{\kappa a_i\alpha_i(a_j+\alpha_j)}{a_i-1+\alpha_i}.$$

In this way, the fraction in the third summand on the r.h.s. can be simplified, and part of what remains cancels out with the first summand of the same side, getting

$$\begin{aligned}\delta n&=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j\neq i}\kappa a_i\alpha_j-\sum_{i=1}^{d}\sum_{j\neq i}\frac{\kappa a_i\alpha_i(a_j+\alpha_j)}{a_i-1+\alpha_i}\\&=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j=1}^{d}\kappa a_i\alpha_j-\sum_{i=1}^{d}\kappa a_i\alpha_i\\&\quad-\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\kappa a_i\alpha_i(a_j+\alpha_j)}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i(a_i+\alpha_i)}{a_i-1+\alpha_i}.\end{aligned}$$

In the first summand on the last line, we can collect all terms that do not depend on the index j, getting

$$\begin{aligned}\delta n&=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j=1}^{d}\kappa a_i\alpha_j\\&\quad-\sum_{i=1}^{d}\kappa a_i\alpha_i-\left(n+\sum_{i=1}^{d}\alpha_i\right)\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i}{a_i-1+\alpha_i}\\&\quad+\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i(a_i+\alpha_i)}{a_i-1+\alpha_i},\end{aligned}$$

and manipulating further the last summand, we find

$$\begin{aligned}\delta n&=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j=1}^{d}\kappa a_i\alpha_j\\&\quad-\sum_{i=1}^{d}\kappa a_i\alpha_i-\left(n+\sum_{i=1}^{d}\alpha_i\right)\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i}{a_i-1+\alpha_i}\\&\quad+\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i(a_i+\alpha_i-1)}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i}{a_i-1+\alpha_i}.\end{aligned}$$

Finally, cancelling the third summand on the r.h.s. with the sixth and by combining the fourth with the last, we get

$$\delta n=\frac{\delta(n-1+\sum_{i=1}^{d}\alpha_i)}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}+\sum_{i=1}^{d}\sum_{j=1}^{d}\kappa a_i\alpha_j-\left(n+\sum_{i=1}^{d}\alpha_i-1\right)\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i}{a_i-1+\alpha_i}. \tag{B 8}$$

Now, using (B 6) and (B 7), we have

$$\sum_{i=1}^{d}\kappa a_i\sum_{j=1}^{d}\alpha_j=n\delta$$

and

$$\frac{\delta}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\frac{\lambda_i a_i}{a_i-1+\alpha_i}=\sum_{i=1}^{d}\frac{\kappa a_i\alpha_i}{a_i-1+\alpha_i},$$

making equation (B 8) identically satisfied. ■

Appendix C

Proof of theorem 4.3. We first rewrite equation (B 1), where the κ_{ij} are set to zero for all $j \neq (i+1)_d$ and to a constant value κ otherwise. We get the condition

$$R_n = L_{n-1} + L_n + L_{n+1}, \tag{C 1}$$

where

$$R_n = \pi(\mathbf{a}|n)\left[\sum_{i=1}^{d}\lambda_i + \delta n + \sum_{i=1}^{d}\kappa a_i a_{(i+1)_d}\right],$$

$$L_{n-1} = \frac{\delta n}{\sum_{i=1}^{d}\lambda_i}\sum_{i=1}^{d}\pi(\mathbf{a}-\mathbf{e}_i|n-1)\lambda_i,$$

$$L_n = \sum_{i=1}^{d}\pi(\mathbf{a}+\mathbf{e}_i-\mathbf{e}_{(i+1)_d}|n)\kappa(a_i+1)(a_{(i+1)_d}-1)$$

and $$L_{n+1} = \frac{\sum_{i=1}^{d}\lambda_i}{(n+1)}\sum_{i=1}^{d}\pi(\mathbf{a}+\mathbf{e}_i|n+1)(a_i+1).$$

We now note that, if the uniform ansatz is true, the following recurrence relations also hold:

$$\pi(\mathbf{a}+\mathbf{e}_i|n+1) = \frac{n+1}{n+d}\,\pi(\mathbf{a}|n)$$

and $$\pi(\mathbf{a}-\mathbf{e}_i|n-1) = \frac{n+d-1}{n}\,\pi(\mathbf{a}|n).$$

Plugging the ansatz (4.7) and these recurrence relations into (C 1), we get that equation (C 1) holds if and only if

$$\delta n + \sum_{i=1}^{d}\kappa a_i a_{(i+1)_d} = \delta(n+d-1) + \sum_{i=1}^{d}\kappa(a_i+1)(a_{(i+1)_d}-1).$$

This simplifies to $0 = \delta(d-1) - \kappa d$. Such a condition is identically satisfied under the hypothesis of the theorem which guarantees

$$\kappa = \frac{d-1}{d}\,\delta. \qquad \blacksquare$$

References

1. Togashi Y, Kaneko K. 2001 Transitions induced by the discreteness of molecules in a small autocatalytic system. *Phys. Rev. Lett.* **86**, 2459–2462. (doi:10.1103/PhysRevLett.86.2459)
2. Awazu A, Kaneko K. 2007 Discreteness-induced transition in catalytic reaction networks. *Phys. Rev. E* **76**, 041915. (doi:10.1103/PhysRevE.76.041915)
3. Berti S, López C, Vergni D, Vulpiani A. 2007 Discreteness effects in a reacting system of particles with finite interaction radius. *Phys. Rev. E* **76**, 031139. (doi:10.1103/PhysRevE.76.031139)
4. Togashi Y, Kaneko K. 2004 Molecular discreteness in reaction-diffusion systems yields steady states not seen in the continuum limit. *Phys. Rev. E* **70**, 020901. (doi:10.1103/PhysRevE.70.020901)
5. Alarcón T, Page KM. 2006 Stochastic models of receptor oligomerization by bivalent ligand. *J. R. Soc. Interface* **3**, 545–559. (doi:10.1098/rsif.2006.0116)
6. Biancalani T, Dyson L, McKane AJ. 2014 Noise-induced bistable states and their mean switching time in foraging colonies. *Phys. Rev. Lett.* **112**, 038101. (doi:10.1103/PhysRevLett.112.038101)
7. Lente G. 2005 Stochastic kinetic models of chiral autocatalysis: a general tool for the quantitative interpretation of total asymmetric synthesis. *J. Phys. Chem. A* **109**, 11 058–11 063. (doi:10.1021/jp054613f)
8. Sardanyés J, Alarcón T. 2018 Noise-induced bistability in the fate of cancer phenotypic quasispecies: a bit-strings approach. *Sci. Rep.* **8**, 1027. (doi:10.1038/s41598-018-19552-2)
9. de Anna P, Di Patti F, Fanelli D, McKane AJ, Dauxois T. 2010 Spatial model of autocatalytic reactions. *Phys. Rev. E* **81**, 056110. (doi:10.1103/PhysRevE.81.056110)
10. Sardanyés J, Arderiu A, Elena SF, Alarcón T. 2018 Noise-induced bistability in the quasi-neutral coexistence of viral RNAs under different replication modes. *J. R. Soc. Interface* **15**, 20180129. (doi:10.1098/rsif.2018.0129)
11. Plesa T, Zygalakis KC, Anderson DF, Erban R. 2018 Noise control for molecular computing. *J. R. Soc. Interface* **15**, 20180199. (doi:10.1098/rsif.2018.0199)
12. Saito N, Kaneko K. 2015 Theoretical analysis of discreteness-induced transition in autocatalytic reaction dynamics. *Phys. Rev. E* **91**, 022707. (doi:10.1103/PhysRevE.91.022707)
13. Nakagawa M, Togashi Y. 2016 An analytical framework for studying small-number effects in catalytic reaction networks: a probability generating function approach to chemical master equations. *Front. Physiol.* **7**, 89. (doi:10.3389/fphys.2016.00089)
14. Biancalani T, Rogers T, McKane AJ. 2012 Noise-induced metastability in biochemical networks. *Phys. Rev. E* **86**, 010106. (doi:10.1103/PhysRevE.86.010106)
15. Houchmandzadeh B, Vallade M. 2015 Exact results for a noise-induced bistable system. *Phys. Rev. E* **91**, 022115. (doi:10.1103/PhysRevE.91.022115)
16. Plesa T, Erban R, Othmer HG. 2019 Noise-induced mixing and multimodality in reaction networks. *Eur. J. Appl. Math.* **30**, 887–911. (doi:10.1017/S0956792518000517)
17. Al-Radhawi MA, Del Vecchio D, Sontag ED. 2019 Multi-modality in gene regulatory networks with slow promoter kinetics. *PLoS Comput. Biol.* **15**, e1006784. (doi:10.1371/journal.pcbi.1006784)
18. Hoessly L, Mazza C. 2019 Stationary distributions and condensation in autocatalytic reaction networks. *SIAM J. Appl. Math.* **79**, 1173–1196. (doi:10.1137/18M1220340)
19. Anderson DF, Kurtz TG. 2015 *Stochastic analysis of biochemical systems*. Mathematical Biosciences Institute Lecture Series, vol. 1. Stochastics in Biological Systems. Cham, Switzerland: Springer; Columbus, OH: MBI Mathematical Biosciences Institute, Ohio State University.
20. Anderson DF, Cotter SL. 2016 Product-form stationary distributions for deficiency zero networks with non-mass action kinetics. *Bull. Math. Biol.* **78**, 2390–2407. (doi:10.1007/s11538-016-0220-y)
21. Ethier SN, Kurtz TG. 1986 *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. New York, NY: John Wiley & Sons.
22. Mozgunov P, Beccuti M, Horvath A, Jaki T, Sirovich R, Bibbona E. 2018 A review of the deterministic and diffusion approximations for stochastic chemical reaction networks. *React. Kinet. Mech. Cat.* **123**, 289–312. (doi:10.1007/s11144-018-1351-y)
23. Kurtz TG. 1970 Solutions of ordinary differential equations as limits of pure jump Markov processes. *J. Appl. Probab.* **7**, 49–58. (doi:10.2307/3212147)
24. Ball F, Yeo G. 1993 Lumpability and marginalisability for continuous-time Markov-chains. *J. Appl. Probab.* **30**, 518–528. (doi:10.2307/3214762)
25. Anderson DF, Cappelletti D. 2019 Discrepancies between extinction events and boundary equilibria in reaction networks. *J. Math. Biol.* **79**, 1253–1277. (doi:10.1007/s00285-019-01394-9)
26. Agazzi A, Mattingly JC. 2018 Seemingly stable chemical kinetics can be stable, marginally stable, or unstable. (https://arxiv.org/abs/1810.06547)
27. Meyn SP, Tweedie RL. 1993 Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. Appl. Probab.* **25**, 518–548. (doi:10.2307/1427522)
28. Mosimann JE. 1962 On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions. *Biometrika* **49**, 65–82. (doi:10.1093/biomet/49.1-2.65)
29. Hoadley B. 1969 The compound multinomial distribution and Bayesian analysis of categorical

data from finite populations. *J. Am. Stat. Assoc.* **64**, 216–229. (doi:10.1080/01621459.1969.10500965)

30. Togashi Y, Kaneko K. 2005 Discreteness-induced transitions in autocatalytic systems. (http://arxiv.org/abs/physics/0501017).
31. Norris J. 1997 *Markov chains*. Cambridge, UK: Cambridge University Press.
32. Dynkin EB. 1965 Markov processes. In: *Markov processes. Die Grundlehren der Mathematischen Wissenschaften (In Einzeldarstellungen mit Besonderer Berücksichtigung der Anwendungsgebiete)*, vol. 121/122. Berlin, Germany: Springer.
33. Oksendal B. 2013 *Stochastic differential equations: an introduction with applications*. Berlin, Germany: Springer Science & Business Media.