Clifford semialgebras

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Abstract

Continuing the theory of systems, we introduce a theory of Clifford semialgebra systems, with application to representation theory via Hasse-Schmidt derivations on exterior semialgebras. Our main result, after the construction of the Clifford semialgebra, is a formula describing the exterior semialgebra as a representation of the Clifford semialgebra, given by the endomorphisms of the first wedge power.

Keywords: Clifford semialgebras, exterior semialgebras, Schubert derivations, exterior semialgebra representation of endomorphisms, bosonic vertex operator representation of Lie semialgebras of endomorphisms.
2010 MSC: Primary 15A75; Secondary 17B69, 14M15, 05E05

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[*]The research of the second author was supported by Finanziamento di Base della Ricerca,(no. 53_RBA17GATLET). The research of the third author was supported by Israel Science Foundation Grant No. 1623/16.

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1. Introduction

The main purpose of this paper is twofold. On one hand we intend to walk our first steps towards a representation theory of Lie semialgebras, in the sense of [22, Section 8.3], by giving a closer look to the case of Clifford semialgebras, introduced in Section 5, where we work out the first theoretical basics. Secondly, we put our abstract picture at work in section 7 and 8, by providing a proper framework to our Theorem 8.1, which computes, in the same spirit of [4] and [5], within the classical algebra framework, the shape of a generating function that describes the exterior $\mathcal{A}$–semialgebra, introduced and studied in [7], as a representation of endomorphisms of a free $\mathcal{A}$–module. The aim is that of excavating the semialgebraic roots of classical mathematical phenomenologies, to investigate to which extent certain classical results in the theory of representation of certain infinite dimensional Lie algebras can be extended to a tropical context, in which one typically works with modules over semialgebras.

Tropical algebra has been used in a variety of applications; one of them, relevant to this paper is the use of a tropical exterior algebra in matroid theory in [12]. The approach here is in the more general framework of "triples," "surpassing relations" and "systems," as explained in [17, 22, 23], which unifies the classical theory and tropical theory and other examples including hyperfields, as explained in [2], and provides a framework for studying linear algebra and module theory [16] in semialgebras, by providing a formal substitution for negation. In brief, one starts with a semimodule spanned by a distinguished set of "tangible" elements. "Triples" are endowed with an abstract negation map, and "systems" are triples endowed with "surpassing relations," which often replace equality in classical theorems. Our main result is Theorem 8.1, in which we propose a more transparent analog for the "Fermionic version" of the generating function that Date, Jimbo, Kashiwara and Miwa (DJKM) [6] provided to describe the representation of Lie algebras of matrices of infinite size having all but finitely many entries zero. (See [18, Section 5.3] for a concise readable account and [9] and [5] for generalizations.)

To be more precise, we explicitly describe the product of elementary matrices by the basis elements of our exterior semialgebra. To do so in a uniform way, we use the actions on the basis elements by the generating functions of the elementary matrices. This procedure which is classical and widely used, requires care in the semialgebra situation, due to many minor verifications needed to show that many equalities that hold true in the classical and well-established case can be replaced by *surpassing relations*. We put the basis element in a generating function to get a uniform (and not a case by case) description. See Section 8 for more explanation.

Additional abundant sources of motivation come from other places in well established literature, showing the potential of our subject to interact with other parts of mathematics. One first remark in this sense is that most scholars working in the theory of symmetric functions are nowadays familiar with the result by Laksov and Thorup [20] showing that the ring $S := \mathbb{Z}[e_1, \ldots, e_r]$ of (commutative) polynomials in r indeterminates is isomorphic, as a $\mathbb{Z}$-module, to the r-th exterior power $\bigwedge^r \mathbb{Z}[X]$. Recall that the former is endowed with its distinguished basis of Schur polynomials parametrized by partitions of length at most r. A deep theorem says that the product of two Schur polynomials is an integral non-negative linear combination of Schur polynomials [21, p. 142], i.e. all the Littlewood-Richardson coefficients are non-negative. Since the Schur polynomials are parametrized by partitions, one may define the module $\bigwedge^r \mathbb{N}[X]$ over the semiring $\mathbb{N}$, as the $\mathbb{N}$–module generated by partitions of length at most r. Therefore the direct sum $\bigwedge \mathbb{N}[X] := \bigoplus_{r\geq 0} \bigwedge^r \mathbb{N}[X]$ can be thought of as a prototype of the exterior semialgebra to which we add a further structure of negation naturally provided by the switch map defined in degree ≥ 2, following the philosophy of [17, 22].

In the present paper the negation map is extended to degrees 1 and 0 via a trick using operators. In conclusion, the construction of the exterior semialgebra alluded to above, besides being interesting in its own right, is natural and motivated by the semialgebra of polynomials. This was performed in detail in [7], giving as an application the extension of the Cayley-Hamilton theorem in the case of endomorphisms of semialgebras.

Looking more closely at the semialgebra of endomorphisms of the exterior semialgebra, it is easy to realize that it naturally satisfies certain properties (namely, the axioms listed in Section 5) which inspired our construction of the Clifford semialgebras as an abstract framework in which to put our concrete observations. On one hand this should not be surprising, because it agrees with the known classical picture concerning usual exterior algebras, but on the other hand we were impressed that the same yoga works, taking care about some delicate technical detail, in a more encompassing context. Accordingly, in this paper some basic representation theory of the Lie semialgebra of the endomorphisms of the free module $\mathcal{A}[x]$ over a semialgebra $\mathcal{A}$ is then proposed. This program could be launched because the apparatus of Hasse-Schmidt derivations on exterior algebras (as in [8]) and of the distinguished ones, called *Schubert*, can be extended almost verbatim to the case of exterior semialgebras. The price we pay is in replacing (but nontrivially, see e.g., Proposition 7.13) equality by the surpassing relation. Hasse-Schmidt derivations, in our context, are formal power series with coefficients in the endomorphisms of the exterior semialgebra. The latter, in a nutshell, can be roughly thought of as the free $\mathcal{A}$ module generated by the set $\mathcal{P}$ of all partitions, additionally endowed with a negation. The reason why the Schubert derivation $\sigma_+(z) = \sum_{i\geq 0} \sigma_i z^i$ is so called is that $\sigma_i \boldsymbol{\lambda} = \sum \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ runs over all the partitions $(\mu_1, \ldots, \mu_r)$ such that $\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_r \geq \lambda_r$ and $|\boldsymbol{\mu}| = |\boldsymbol{\lambda}| + i$; this is the classical rule of Pieri holding in Schubert calculus when multiplying cohomology classes in Grassmannians.

In [10] one deals with finite wedge powers of possibly finite dimensional vector spaces, and the DJKM representation translates to proclaim that the singular cohomology of the Grasmannian is a module over the Lie algebra of $n\times n$ matrices. We extend this result (and hence, in a sense, also DJKM) to the case of the Lie semialgebra of the endomorphisms of $\mathcal{A}[x]$. Most of the equalities are replaced by surpassing relations, which are among the key notions of the theory of systems. Another detail in the context of this paper – we develop a theory of power series over the exterior semialgebra, viewed in terms of negation maps.

Plan of the Paper. The plan of the paper is as follows. Section 2 contains the preliminaries concerning the systemic (and thus tropical) framework developed along the lines of [22, 23] and references therein. Basic motivating examples are given in Section 3.

Section 4 reviews the construction of the exterior semialgebra from [7], and recasts it in terms of the regular representation (Definition 3.2), which utilizes the endomorphism system of (§2.11). We are able thereby to show how the tensor functor takes modules to triples (Theorem 4.12).

Section 5 is devoted to studying Clifford semialgebras in general, to provide the appropriate framework for the second part of the paper. The definition of Clifford semialgebra in Definition 5.6 is rather concise, and a generic construction is proposed in 5.7, but, in contrast to the exterior semialgebra, complications arising when attempting to introduce the negation map, cf. Construction 5.13. Thus we must examine the situation both with and without a negation map. The abstract construction is given in Construction 5.7, whereas the natural model of Clifford semialgebras arising from the exterior semialgebra is given already in Section 4.

(In contrast with other applications of systems, we need a generalization incorporating the "surpassing relation" $\preceq$ in our negation map, to obtain what here we call $\preceq$-systems.) In the case of polynomial semirings, i.e. $\mathcal{A}[x^1, \dots, x^n]$, where $\mathcal{A}$ is a semialgebra, we obtain the Weyl semialgebra, the "symmetric" counterpart of the Clifford semialgebra seen as a semialgebra of endomorphisms of the exterior semialgebra. The "base square-free" symmetric semialgebra (the Giansirancusa tropical exterior semialgebra of [12]), an epimorphic image of $\mathcal{A}[x^1, \dots, x^n]$, is obtained by setting all the squares of basis elements to zero.

We then repeat the constructions of Section 5 in the case of the base square-free symmetric semialgebra, and much of the combinatorial formalism for exterior semialgebras can be mimicked in this case. We describe the semialgebra of endomorphisms of the tropical Grassmannian, and designate it as the tropical Clifford semialgebra.

In Section 6 we study the relevant example of the exterior semialgebra as a regular representation of a canonical Clifford semialgebra, which arises naturally in a concrete context. As shown in the subsequent section, this is also relevant for application to representation theory and mathematical physics.

Vertex operators are also defined in this context in Section 7, and they take a classical familiar form over a semialgebra $\mathcal{A}$ which contains the non-negative rational numbers. Our culminating result is Theorem 8.1, our version of [10, Theorem 6.4] or [1, Theorem 5.9]. It provides a formula expressing the generating function of the action of the Lie semialgebra of endomorphisms on fixed degree elements of the exterior semialgebra by means of the elementary matrices E_{ij}. Clearly this suffices to fully describe the module structure as each endomorphism is a finite linear combination of the elementary ones.

2. Preliminaries and Notation

As usual we denote as $\mathbb{N}$ the monoid of natural numbers (including 0), and $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. Throughout $\mathcal{A}$ denotes a fixed commutative associative base semiring cf. [13] (i.e., satisfies all the axioms of ring except negatives, e.g. $\mathbb{N}$ or $\mathbb{Q}_+$ or the max-plus algebra). Let $n \in \mathbb{N} \cup \{\infty\}$. Modules are defined as usual. $(\mathcal{T}, \cdot, 1)$ denotes an Abelian multiplicative monoid with unit element 1.

Definition 2.1. A (left) $\mathcal{T}$-**(monoid) module** is an additive monoid $(V, +, 0_V)$ together with scalar multiplication $\mathcal{T} \times V \to V$ satisfying distributivity over $\mathcal{T}$ in the sense that

$$a(v_1 + v_2) = av_1 + av_2, \quad \forall a \in \mathcal{T}, \ v_i \in V,$$

also stipulating that $a \cdot 0_V = 0_V$ for all a in $\mathcal{T}$, $1_{\mathcal{T}} v = v$, and $(a_1 a_2)v = a_1(a_2 v)$, for all $a_i \in \mathcal{T}$ and $v \in V$.

$\mathcal{T}$**-semialgebras** are $\mathcal{T}$-modules which are semirings. $\mathcal{A}$ will always be a commutative $\mathcal{T}$**-semialgebra**. $\mathrm{End}_{\mathcal{A}} V$ denotes the semialgebra of module homomorphisms $V \to V$, i.e., group homomorphisms $f : V \to V$ satisfying $f(av) = af(v), \forall a \in \mathcal{A}, v \in V$. For notational convenience, we designate $0_V \in \mathrm{End}_{\mathcal{A}} V$ for the 0 homomorphism, i.e., $0_V \cdot V = 0$. Also there are maps $\mathcal{A} \to \mathrm{End}_{\mathcal{A}} V$ sending $a \mapsto l_a$, the left multiplication map $v \mapsto av$, and when V is a semialgebra, $V \to \mathrm{End}_{\mathcal{A}} V$ given by $v \mapsto l_v$.

Definition 2.2. The free $\mathcal{A}$-module of countable rank is denoted as

$$V = \mathcal{A}[x] := \bigoplus_{i \geq 0} \mathcal{A} x^i.$$

If J is any countable index set, we denote the factor module

$$V_J := \frac{V}{\sum \mathcal{A} x^i : i \notin J}$$

If $J = [n] = \{0 \leq i < n\}$, for $n \in \mathbb{N} \cup \infty$, we write V_n for $\bigoplus_{0 \leq i < n} \mathcal{A} x^i$, and its base will be denoted $\mathbf{x}^n$.

Every element of V is a polynomial with coefficients in $\mathcal{A}$, i.e. a finite linear combination of powers of x. This can be thought of as $\dfrac{\mathcal{A}[x]}{\sum \mathcal{A} x^j : j \geq n.}$ [3]

Definition 2.3. A **negation map** on a $\mathcal{T}$-module V is an injective semigroup homomorphism $(-) : V \to V$ of order ≤ 2, together with a map $(-) : \mathcal{T} \to \mathcal{T}$ also of order ≤ 2, written $v \mapsto (-)v$, satisfying

$$(2.1) \qquad (-)((-)v) = v, \quad ((-)a)v = (-)(av) = a((-)v), \quad \forall a \in \mathcal{T}, \quad v \in V.$$

When V is a semialgebra $\mathcal{A}$, we require (2.1) for all $a, v \in \mathcal{A}$.

We write $v(-)w$ for $v + (-)w$, and v° for $v(-)v$, called a **quasi-zero**. $V^\circ := \{v^\circ : v \in V\}$. A **submodule with negation map** is a submodule W of V satisfying $(-)W = W$.

Lemma 2.4. *If a $\mathcal{A}$-module V has a negation map $(-)$, then:*

(i) V° is a submodule of V.

(ii) $V \times V$ is a module with negation, under the diagonal action.

[3]Technically we are dealing with congruences, so the notation, which we use repeatedly, means that we are modding out the congruence generated by all $(x^j, 0)$, $j \geq n$.

(iii) $\mathrm{End}_{\mathcal{A}} V$ *has a negation map given by*

$$((-)f)(v) = (-)(f(v)).$$

Proof. Easy verifications.

Definition 2.5. A **triple** is a collection $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$, where $\mathcal{A}$ is a $\mathcal{T}$-module and $(-)$ is a negation map on $\mathcal{A}$, and $\mathcal{T}_{\mathcal{A}}$ is a subset of $\mathcal{A}$ closed under $(-)$, satisfying:

(i) $\mathcal{T}_{\mathcal{A}} \cap \mathcal{A}^{\circ} = \emptyset$,

(ii) $\mathcal{T}_{\mathcal{A}} \cup \{0\}$ generates $(\mathcal{A}, +)$ additively.

(We do not require in general that $\mathcal{T}_{\mathcal{A}}$ is a monoid.)

Lemma 2.6. *To verify that a group homomorphism* $f : V \to V$ *is in* $\mathrm{End}_{\mathcal{A}} V$ *it is enough to check that* $(\sum a_i)v = \sum (a_i v)$ *and* $f(av) = af(v), \forall a_i, a \in \mathcal{T}, v \in V$.

Proof. If $b = \sum a_i$ for $a_i \in \mathcal{T}$ then

$$f(bv) = f(\sum a_i v) = \sum f(a_i v) = \sum a_i f(v) = bf(v).$$

■

By **partial order** we always mean a partial order $\preceq$ on $\mathcal{A}$ as a $\mathcal{T}$-module, i.e., satisfying the following conditions for all $b, b' \in \mathcal{A}$:

Conditions 2.7.

(i) If $b \preceq c$ *and* $b' \preceq c'$ *then* $b + b' \preceq c + c'$.

(ii) If $a \in \mathcal{T}$ *and* $b \preceq b'$ *then* $ab \preceq ab'$.

A **surpassing relation** is a partial order, also satisfying

Conditions 2.8.

(iii) $b \preceq b'$ *whenever* $b + c^{\circ} = b'$ *for some* $c \in \mathcal{A}$.

(iv) If $b \preceq b'$ *then* $(-)b \preceq (-)b'$.

(In other words $(-)$ is preserved under $(-)$. If $0 \preceq b'$ then $0 = (-)0 \preceq (-)b'$.) We also write $b \succeq b'$ to denote that $b' \preceq b$.

A **system** $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$ is a triple together with a surpassing relation, also satisfying:

(v) If $a \preceq a'$ for $a, a' \in \mathcal{T}_{\mathcal{A}}$, then $a = a'$.

(vi) **unique negation**: If $a + a' \succeq 0$ for $a, a' \in \mathcal{T}_{\mathcal{A}}$, then $a' = (-)a$.

A consequence of (i) and (iv) is $c(-)c \succeq 0$. We have (v) in order to have equality on $\mathcal{T}$ generalizing classical formulas.

The surpassing relation is of utmost importance, since it replaces equality in many classical formulas.

The most common surpassing relation defined on a triple is $\preceq_\circ$, given by $b \preceq_\circ b'$ whenever $b + c^\circ = b'$ for some $c \in \mathcal{A}$. In this case $b' \succeq 0$ iff $b' = c^\circ$ for some c.

Lemma 2.9. *(i) Conditions 2.7 translate to: A partial order is a subset S of $\mathcal{A} \times \mathcal{A}$ satisfying for all b, b', c:*

(a) $(b, b) \in S$.

(b) If $(b, c) \in S$ and $(b', c') \in S$ then $(b + b', c + c') \in S$.

(c) If $a \in \mathcal{T}$ and $(b, b') \in S$ then $(ab, ab') \in S$.

Conditions 2.8 for surpassing relation translate to:

(d) $(b, b + c^\circ) \in S$.

(e) If $(b, b') \in S$ then $((-)b, (-)b') \in S$.

(ii) Given any subset $S_0 \subseteq \mathcal{A} \times \mathcal{A}$, we can obtain a partial order by taking S to be the smallest $\mathcal{T}$-submodule of $\mathcal{A} \times \mathcal{A}$ containing S_0. We obtain a surpassing relation by taking S to be the smallest $\mathcal{T}$-submodule (with negation) of $\mathcal{A} \times \mathcal{A}$ containing $S_0 + (0 \times \mathcal{A}^\circ)$.

Proof. (i) Conditions 2.7 translate to these conditions of (i), under the familiar interpretation of a relation on $\mathcal{A}$ as a subset S of $\mathcal{A} \times \mathcal{A}$.

(ii) The conditions (a),(b) are the definition of submodule. (a), (d),(e) are those involving the surpassing relation and negation map. ∎

Definition 2.10. A **systemic module** over a semiring with preorder $(\mathcal{A}, \preceq)$ is an $\mathcal{A}$-module M together with a submodule $\mathcal{T}_M$ and negation map $(-)$ and surpassing relation $\preceq$ preserving $\mathcal{T}_M$ and satisfying $(-)(ay) = a((-)y)$ as well as $ay \preceq a'y'$ for $a \preceq a' \in \mathcal{A}$ and $y \preceq y' \in M$.

(When $\mathcal{A}$ has a multiplicative unit 1 and a negation map then the negation map on M could be given by $(-)y := ((-)1)y$ for $y \in M$). In our applications, M often will be an ideal of a semiring system $\mathcal{A}$.

Proposition 2.11. *If M is a systemic module over a semiring $\mathcal{A}$, then taking $\mathcal{T}_{\mathrm{End}M}$ to be the set of endomorphisms sending $\mathcal{T}_M$ to $\mathcal{T}_M$ and $\overline{\mathrm{End}M}$ to be the subalgebra of $\mathrm{End}M$ spanned by $\mathcal{T}_{\mathrm{End}M}$, $(\overline{\mathrm{End}M}, \mathcal{T}_{\mathrm{End}M}, (-), \preceq)$ is a system where we define $(-)f : y \mapsto (-)(f(y))$ and $f \preceq g$ when $f(y) \preceq g(y)$ for all $y \in M$.*

Proof. Easy verification. ∎

$(\overline{\mathrm{End}M}, \mathcal{T}_{\mathrm{End}M}, (-), \preceq)$ is called the **endomorphism system** of M.

Recall from [17, Definition 2.37] that the systemic $\mathcal{A}$-modules comprise a category, where a morphism, which we will call a **$\preceq$-morphism**

$$f : (\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq) \to (\mathcal{M}', \mathcal{T}_{\mathcal{M}'}, (-)', \preceq')$$

is a map $f : \mathcal{M} \to \mathcal{M}'$ satisfying the following properties for $a \in \mathcal{A}$ and c, c_i in $\mathcal{M}$:

(i) $f(0) = 0$.

(ii) $f((-)c_1) = (-)f(c_1)$;

(iii) $f(c_1 + c_2) \preceq' f(c_1) + f(c_2)$;

(iv) $f(ac) = af(c)$.

(v) $f(c_1) \preceq' f(c_2)$ if $c_1 \preceq c_2$.

2.12. The standard dual base. The *standard dual base* $\boldsymbol{\partial} = (\partial^j)_{j\in\mathbb{N}} \subset \mathrm{Hom}(V, \mathcal{A})$ of $(x^i)_{i\in J}$ is defined by $\partial^j x^i = \delta_{ij}$. In the classical case where $\mathcal{A}$ contains the positive rationals, it may be identified with the differential operator

$$\partial^j := \frac{1}{j!}\frac{\partial}{\partial x^j}\bigg|_{x=0}$$

which motivates the notation which we use to emphasize the analogy with the Weyl algebra acting on $\mathcal{A}[x]$, generated by the multiplication by x and the partial derivative ∂^1 subject to the relation $x\partial^1 = \partial^1 x + \mathrm{id}$. $V^* := \bigoplus_{j\geq 0} \mathcal{A}\partial^j$ is called the **restricted dual** of V. Define an action $\lrcorner$ of $\mathcal{V}^*$ on $\mathcal{V}$ as:

$$\partial \lrcorner v = \partial(v), \qquad \forall(\partial, v) \in \mathcal{V}^* \times \mathcal{V}.$$

3. Main motivating examples

Here are the examples to be used throughout this paper.

3.1. The regular representation. We present a way of embedding a semialgebra into a system, using Proposition 2.11.

Definition 3.2. Suppose V is a module over an $\mathcal{A}$-semialgebra $\mathcal{A}$. The **regular representation** $\Psi : \mathcal{A} \to \mathrm{End}_{\mathcal{A}}V$ sends $b \in \mathcal{A}$ to the map $\Psi(b) : v \mapsto bv$.

As in usual algebra, this maps $\mathcal{A}$ into $\mathrm{End}_{\mathcal{A}}V$, and is an injection when V is faithful over $\mathcal{A}$, which will always be the case in this paper.

Lemma 3.3. *When $(V, \mathcal{T}_V, (-), \preceq)$ is a systemic module, the image of $\mathcal{A}$ in the system $(\overline{\mathrm{End}_{\mathcal{A}} V}, \mathcal{T}_{\mathrm{End}_{\mathcal{A}} V}, (-), \preceq)$ is a subsystem.*

Proof. $(-)\Psi(b) : v \to (-)v$. If $b = \sum_{i=1}^t a_i$ then $\Psi(b) = \sum_{i=1}^t \Psi(a_i) \in \overline{\mathrm{End}_{\mathcal{A}} V}$. ∎

Lemma 3.3 will be used to put our basic notions (exterior semialgebra, Clifford semialgebra) in the context of systems. We turn to the notion of Lie semialgebra with a preorder.

Definition 3.4. A **Lie $\preceq$-semialgebra with a preorder** $\preceq$ is an $\mathcal{A}$-module L with a negation map $(-)$, endowed with an $\mathcal{A}$-bilinear **$\preceq$-Lie bracket** $[\quad]$ satisfying, for all $x, y \in L$:

(i) $[xy] + [yx] \succeq 0_L$,

(ii) $[xx] \succeq 0_L$,

(iii) $\mathrm{ad}_{[xy]} + \mathrm{ad}_y \mathrm{ad}_x \preceq \mathrm{ad}_x \mathrm{ad}_y$, where $\mathrm{ad}_x(z) = [xz]$.

It is more intuitive to work with systemic modules.

Definition 3.5. A **systemic Lie $\preceq$-semialgebra** is a systemic $\mathcal{A}$-module L with a negation map $(-)$, endowed with an $\mathcal{A}$-bilinear **$\preceq$-Lie bracket** $[\quad]$ satisfying, for all $x, y \in L$:

(i) $[xy] = (-)[yx]$,

(ii) $[xx] \succeq 0_L$,

(iii) $\mathrm{ad}_{[xy]} \preceq \mathrm{ad}_x \mathrm{ad}_y (-) \mathrm{ad}_y \mathrm{ad}_x$, where $\mathrm{ad}_x(z) = [xz]$.

Lemma 3.6. *Any systemic Lie $\preceq$-semialgebra is a Lie $\preceq$-semialgebra in the sense of Definition 3.4.*

Proof. $[xy] - [yx] = [xy](-)[xy] \succeq 0$;

$$\mathrm{ad}_{[xy]} \preceq \mathrm{ad}_{[xy]} + \mathrm{ad}_y \mathrm{ad}_x (-) \mathrm{ad}_y \mathrm{ad}_x \preceq \mathrm{ad}_x \mathrm{ad}_y (-) \mathrm{ad}_y \mathrm{ad}_x .$$

Lemma 3.7 (Jacobi $\preceq$-identity, [22, Lemma 10.5])**.** *(i)* $[[bb'] v] + [b'[bv]] \preceq [b[b'v]]$ *for all $b, b', v \in L$ for a Lie $\preceq$-semialgebra.*

(ii) $\mathrm{ad}_b(\mathrm{ad}_{b'}(v))(-)[b'[bv]][b'[bv]]$ *for all $b, b', v \in L$ in a systemic Lie semialgebra.*

Proof. (i) $[[bb']v] + [b'[bv]] = \mathrm{ad}_{[bb']}(v) + \mathrm{ad}_{b'}(\mathrm{ad}_b(v)) \preceq \mathrm{ad}_b(\mathrm{ad}_{b'}(v)) = [b[b'v]]$.

(ii) $[[bb']v] \preceq [[bb']v] + [b'[bv]](-) \preceq \mathrm{ad}_b(\mathrm{ad}_{b'}(v))(-)[b'[bv]][b'[bv]]$.

Although (i) in Lemma 3.7 is put in greater generality, (ii) is the $\preceq$-surpassing version of Jacobi's identity.

In any semiring with a negation map $(-)$, we write $[b, b']$ for the **Lie commutator** $bb'(-)b'b$.

Proposition 3.8 ([22, Proposition 10.7])**.** *Any associative semialgebra A with negation map becomes a Lie $\preceq_\circ$-semialgebra, denoted $A^{(-)}$, under the Lie product $[bb'] = [b, b']$.*

Define $\mathrm{ad}\, L = \{\mathrm{ad}_b : b \in L\}$, a Lie sub-semialgebra of $\mathrm{End}_{\mathcal{A}} L$ under the Lie product.

Proposition 3.9 ([22, Proposition 10.6])**.** *If L is a Lie semialgebra, then there is a Lie $\preceq$-morphism $\mathrm{ad} : L \to \mathrm{ad}\, L$, given by $b \mapsto \mathrm{ad}_b$ (In fact ad respects the Lie product and preserves addition).*

$\mathcal{V}_n$ is the free module, as in Definition 2.2, over $\mathcal{A}$ generated by $\mathbf{x}^n := \{x^j : 0 \le j < n\}$. Let

$$S(\mathcal{V}) = \mathcal{A}[x^i : 0 \le i < n] \tag{3.1}$$

be the symmetric (i.e., commutative polynomial) semialgebra of $\mathcal{V}$,

$$\overline{S(V)} := \frac{S(\mathcal{V})}{(x^i x^i \sim 0)}, \tag{3.2}$$

the base square-free symmetric semialgebra (which is the Giansirancusa tropical Grassmannian of [12]). There is an obvious $\mathcal{T}$-semialgebra epimorphism $S(\mathcal{V}) \longrightarrow \overline{S(V)}$ mapping to zero all the words in $\mathbf{x}$ involving a repetition of a letter.

(This suggests that we could define a more general Weyl semialgebra in $\mathrm{End}_{\mathcal{A}} S(\mathcal{V})$ associated to a bilinear form, cf. Definition 5.2.)

3.10. The tensor system and extended tensor system. Tensor products are defined naturally in the semialgebra context, cf. [19]. Putting $V^{\otimes 0} := \mathcal{A}$, $V^{\otimes 1} := V$, and $V^{\otimes n} := V^{\otimes n-1} \otimes_{\mathcal{A}} V$, we let $T^k(V) = V^{\otimes k}$, $\quad T^{\ge 2}(V) = \bigoplus_{n \ge 2} V^{\otimes n}$. The direct sum

$$T(V) = \bigoplus_{n \ge 0} V^{\otimes n}.$$

is called the **tensor semialgebra**, an important structure from which we extract a natural negation map.

Motivated by the wish of getting a working notion of exterior semialgebra, we define a negation map on $T^{\ge 2}(V)$ by

$$(-) v_1 \otimes v_2 \otimes \cdots = v_2 \otimes v_1 \otimes \dots .$$

Define $\mathcal{T}^{\ge 2}$ to be the multiplicative monoid of simple tensors at length at least two of basis elements of V, i.e., $x^{i_1} \otimes x^{i_2} \otimes \dots .$

Lemma 3.11. *As an $\mathcal{A}$-module, $(T^{\geq 2}(V), \mathcal{T}^{\geq 2}, (-), \preceq_\circ)$ is a system.*

Proof. In fact $\mathcal{T}^{\geq 2}$ is a base of $T^{\geq 2}(V)$, where $T^{\geq 2}(V)^\circ$ is spanned by $(\mathcal{T}^{\geq 2})^\circ$, so unique negation is clear. ∎

This has the flavor of exterior algebras, so we call it the **pre-exterior system**, which is a useful tool in representation theory. But this only works as a module, not as a semialgebra, since

$$(-)(v_1 \otimes v_2)(v_3 \otimes v_4) = (v_2 \otimes v_1) \otimes (v_3 \otimes v_4),$$

whereas

$$(v_1 \otimes v_2)((-)(v_3 \otimes v_4)) = (v_1 \otimes v_2) \otimes (v_4 \otimes v_3).$$

To make $(-)$ work for semialgebras we define

$$\overline{T}^k(V) = T^k(V)/\mathfrak{R}, \tag{3.3}$$

where $\mathfrak{R}$ is the congruence equating $v_1 \otimes \cdots \otimes v_k$ with $v_{\pi(1)} \otimes \cdots \otimes v_{\pi(k)}$ for every even permutation $\pi \in \mathrm{Sym}_k$.

Lemma 3.12. *As an $\mathcal{A}$-semialgebra, $(\bar{T}^{\geq 2}(V), \bar{\mathcal{T}}^{\geq 2}, (-), \preceq_\circ)$ is a system.*

Proof. We identify $((-)v_1 \otimes \cdots \otimes v_k) \otimes w_1 \otimes \cdots \otimes w_\ell = v_2 \otimes v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$ with $(v_1 \otimes \cdots \otimes v_k) \otimes (w_2 \otimes w_1 \otimes \cdots \otimes w_\ell) = (v_1 \otimes \cdots \otimes v_k) \otimes ((-)w_1 \otimes w_2 \otimes \cdots \otimes w_\ell)$ since the subscripts differ by a product of two transpositions, which is an even permutation. ∎

As seen in [7], the system of Lemma 3.12 suffices for many applications, but we would like to extend this to a system that also contains the degree 1 part. Thus our pre-exterior system must be enlarged, using the regular representation.

Consider the maps:

$$\mathrm{id}, (-) : \mathcal{A} \to \mathrm{End}_{\mathcal{A}}(\bar{T}^{\geq 2}(V))$$

and

$$l, r : V \to \mathrm{End}_{\mathcal{A}}(\bar{T}^{\geq 2}(V))$$

defined by

$$\mathrm{id}(a)(v_1 \otimes v) = av_1 \otimes v \qquad \text{and} \qquad (-)a(v_1 \otimes v) = (-)av_1 \otimes v,$$

$$l(u)(v_1 \otimes v) = u \otimes v_1 \otimes v \qquad \text{and} \qquad r(u)(v_1 \otimes v) = v_1 \otimes u \otimes v,$$

for all $(u, v_1 \otimes v) \in V \times \bigwedge(V)$ such that $v_1 \in V$.

We shall identify V with $Im(l)$, and also set $(-)V := Im(r)$, which becomes an $\mathcal{A}$–module when one defines $a((-)v) = (-)av \in (-)V$ for all $(a, u) \in \mathcal{A} \times V$. We write $(-)u$ for $r(u)$, and $(-)((-)u) = u$.

Definition 3.13. The **degree zero component** for $\widetilde{\overline{T}}(V)$ is the $\mathcal{A}$-module $\widetilde{\overline{T}}^0(V) := Im(\mathrm{id}) \bigoplus Im((-))$.

The **degree one component** for $\widetilde{\overline{T}}(V)$ is the $\mathcal{A}$-module $\widetilde{\overline{T}}^1(V) := Im(l) \bigoplus Im(r)$.
$\widetilde{\overline{T}}^{\geq 2}(V) := \overline{T}^{\geq 2}(V)$.

There is an obvious $\mathcal{A}$–epimorphism $\rho : \widetilde{\overline{T}}^1(V) \to V$, given by $l(u) \mapsto u$ and $r(u) \mapsto u$, so that $\rho^{-1}(u) = \{l(u), r(u)\}$.

Definition 3.14. Define $\mathrm{gl}(\widetilde{\overline{T}}^1(V))$ as the set of all $f \in \mathrm{End}_{\mathcal{A}}(\bar{T}(V))$ such that

$$\mathrm{gl}(\widetilde{\overline{T}}^1(V)) = \left\{ f \in \mathrm{End}_{\mathcal{A}}(\bar{T}(V)) \;\middle|\; \begin{array}{rcl} ((-)f)(l(u)) & = & r(f(u)) \\ \\ ((-)f)(r(u)) & = & l(f(u)) \end{array} \right\}$$

i.e., the set of all $\mathcal{A}$-linear maps f such that $(-)f$ switches r and l.

Proposition 3.15. *If $f, g \in \mathrm{gl}(\widetilde{\overline{T}}^1(V))$, then*

$$(f \circ (-)g)(u) = (-)f(g(u))$$

i.e.

$$(f \circ (-)g)(u) \wedge v = v \wedge f(g(u)) \qquad v \in V$$

Proof. Let $u := l(u) \in \tilde{T}_1(V)$. Then

$$f(-g)(u) = f((-)g(u)) = (-)f(g(u)).$$

■

The map $(-)v : V \longrightarrow \mathrm{End}_{\mathcal{A}}\overline{T}(V)$ is given by

$$(-)v = \_ \otimes v : \overline{T}(V) \to \overline{T}(V)$$

mapping $u_1 \otimes u$ (for $u_1 \in V$ and $u \in \overline{T}(V)$) to

$$((-)v)(u_1 \otimes u) = (u_1 \otimes v) \otimes u.$$

We extend this to all degrees ≥ 1.

Remark 3.16. The regular representation $\Psi_{\overline{T}^{\geq 1}(V)} : \overline{T}(V) \to \mathrm{End}_{\mathcal{A}}\overline{T}(V)$ (see Definition 3.2) sends $v_1 \otimes \cdots \otimes v_m$ to the map

$$v_1' \otimes \cdots \otimes v_k' \mapsto v_1 \otimes \cdots \otimes v_m \otimes v_1' \otimes \cdots \otimes v_k'.$$

Lemma 3.17. *The regular representation $\Psi_{\overline{T}^{\geq 1}(V)}$ is a homogeneous $\mathcal{A}$-injection of semialgebras, of degree 1, i.e., $v \otimes \_ : \overline{T}^i(V) \to \overline{T}^{i+1}(V)$. Furthermore, the negation map on $\overline{T}^{\geq 2}(V)$ induces a negation map on $\Psi_{\overline{T}(V)}(\overline{T}(V))$ by*

$$(-)\Psi_{\overline{T}(V)}(v_1 \otimes v_2 \otimes \cdots \otimes v_m)(v_1' \otimes \cdots \otimes v_k') = v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes v_2' \otimes v_1' \otimes \cdots \otimes v_k',$$

which can be combined with our previous negation map on $\widetilde{\overline{T}}^1(V)$.

Proof. The analog of the usual regular representation also holds for semialgebras. The map switching the order of the tensors serves as the negation map throughout.

More precisely, we extend the negation map defined on $\overline{T}^{\geq 2}(V)$ to a negation map

$$(-) : \widetilde{\overline{T}}(V) \to \widetilde{\overline{T}}(V),$$

by defining $(-)a$ to be the element of $\widetilde{\overline{T}}^0$ such that

$$((-)a)v_1 \wedge \cdots \wedge v_k = av_2 \wedge v_1 \wedge \cdots \wedge v_k,$$

and defining $(-)v$ to be the element of $\widetilde{\overline{T}}^1$ such that

$$((-)v) \wedge v_1 \wedge \cdots \wedge v_k = v_1 \wedge v \wedge v_2 \wedge \cdots \wedge v_k.$$

Notice that

$$(-)v \wedge w = v \wedge (-)w,$$

so the negation of $\widetilde{\overline{T}}^1(V)$ is compatible with that of $\overline{T}^{\geq 2}(V)$. ∎

The upshot of this is:

Proposition 3.18. *The data* $(\widetilde{\overline{T}}(V), \widetilde{\overline{T}}(V), (-), \preceq_\circ)$ *is a system.*

Remark 3.19. $\widetilde{\overline{T}}^0(V)$ is just the direct sum of two copies of $\mathcal{A}$. In our applications, it is superfluous, since we take tensors of length ≥ 1.

4. Exterior systems

In this section we refine the construction of the exterior semialgebra in [7, Definition 3.1], first to obtain a system, but also because we want later to construct a Clifford semialgebra as its semialgebra of endomorphisms. We start with two constructions from [7].

4.1. Exterior semialgebras of types 1 and 2. We define

$$\bigwedge^k V := \frac{\widetilde{\overline{T}}^k(V)}{(x^i \otimes x^i \sim 0,\ i \in I)}; \qquad \bigwedge^{\geq 2} V = \oplus_{k \geq 2} \bigwedge^k V; \qquad \bigwedge V := \oplus_{k \geq 0} \bigwedge^k V.$$

Definition 4.2. The exterior semialgebra of **type 1** is $\bigwedge(\mathcal{A}[x])$, considered with respect to the juxtaposition product on the $\bigwedge^k V$.

Definition 4.3. The exterior semialgebra of **type 2** is:

$$\overline{\bigwedge V} : \frac{\widetilde{\overline{T}}(V)}{(v^{\otimes 2} \sim 0, v \in \mathcal{A}[x])}.$$

Remark 4.4. The switch map on $\overline{T}^{\geq 2}(V)$ induces a negation map $(-)$ on $\overline{T}^{\geq 2}(V)$ given by the formula

$$(-)v_1 \otimes v_2 \otimes \ldots \otimes v_k = v_2 \otimes v_1 \otimes \ldots \otimes v_k, \tag{4.1}$$

which combined with the negation maps on $\widetilde{\overline{T}}^0(V)$ and $\widetilde{\overline{T}}^1(V)$ provides a negation map on $\widetilde{\overline{T}}(V)$.

Together with (3.3) we have

$$v_{\pi(1)} \otimes \cdots \otimes v_{\pi(k)} = (-)^{\pi} v_1 \otimes v_2 \otimes \ldots \otimes v_k, \tag{4.2}$$

Remark 4.5. In type 1 we are not requiring the elements of V to be square 0; for instance $(x^i + x^j)^{\otimes 2} = x^i \otimes x^j + x^j \otimes x^i$. The "exterior" structure is built in via the switch map.

In both types we denote the product by $\wedge$, abusing notation. In particular

$$(-)(u \wedge v) = ((-)u) \otimes v = u \wedge (-)v = v \wedge u, \qquad \forall (u, v) \in V^2.$$

Remark 4.6. $(\bigwedge^{\geq 2} V, \mathcal{T}^{\geq 2}(V), (-), \preceq_\circ)$ is a system. It differs from the system of Lemma 3.12 only insofar as we have modded out the $v \otimes v$, but the notation remains the same.

4.7. Construction of the extended exterior system. The system of Remark 4.6 suffices for the applications in [7], but we would like to have a system that also includes the degree 1 part of the exterior semialgebra. To attain this, our exterior system will be slightly bigger than the exterior $\mathcal{A}$-algebra. We repeat the argument used in constructing $\tilde{T}$, modding out $(x^i \otimes x^i \sim 0,\ i \in I)$ for type 1, and modding out $(v \otimes v \sim 0,\ v \in V)$ for type 2.

Lemma 4.8. *The regular representation $\Psi_{\bigwedge^{\geq 1} V}$ is a homogeneous $\mathcal{A}$-injection of semialgebras of degree 1, i.e., $v \otimes \_ : \bigwedge^i V \to \bigwedge^{i+1} V$. Furthermore, as in Lemma 3.3, the negation map on $\bigwedge^{\geq 2}(V)$ induces a negation map on $\Psi_{\bigwedge V}(\bigwedge^{\geq 2} V)$ by*

$$(-)\Psi(v_1 \otimes v_2 \otimes \cdots \otimes v_m)(v_1' \otimes \cdots \otimes v_k') = v_2 \otimes v_1 \otimes \cdots \otimes v_m \otimes v_1' \otimes \cdots \otimes v_k',$$

which can be combined with our previous negation map on $\widetilde{\bigwedge}^1 V$.

We write $\widetilde{\bigwedge}^k V$ for $\Psi_{\bigwedge^k V}$. Then we have the negation maps $(-)\Psi_\alpha(v_1 \otimes v_2 \otimes \ldots) = v_2 \otimes v_1 \otimes \ldots$ for $\alpha \in \bigwedge^0 V = \mathcal{A}$ and $(-)\Psi_v(v_1 \otimes v_2 \otimes \ldots) = v \otimes v_2 \otimes v_1 \otimes \ldots$.

Definition 4.9. The **extended exterior** $\mathcal{A}$-semialgebra of type 1 is

$$\left(\widetilde{\bigwedge} V, \wedge\right) := \frac{(\widetilde{\bigwedge}^0 V \oplus (\widetilde{\bigwedge}^1 V \oplus \bigwedge^{\geq 2}(V))}{(x^i \otimes x^i \sim 0)}, \tag{4.3}$$

all viewed inside $\mathrm{End}_{\mathcal{A}}(\bigwedge^{\geq 2} V)$ via the regular representation.

(Note that in the classical algebraic situation, $(-)\widetilde{\bigwedge^1 V} = \widetilde{\bigwedge^1 V}$.) We identify $\bigwedge^k(V)$ with $\widetilde{\bigwedge^k}(V)$ for $k \geq 2$, so $\mathcal{T}_{\geq 2}(V)$ is viewed inside $\widetilde{\bigwedge} V$.

Definition 4.10. Let $\mathcal{T}(\bigwedge V) = \bigwedge^1 V \cup (-)\bigwedge^1 V \cup \mathcal{T}_{\geq 2}(V)$ with respect to the **juxtaposition product** that we denote classically by $\wedge$. The **extended exterior system** (type 1) is

$$\left(\widetilde{\bigwedge} V, \mathcal{T}(\bigwedge V), (-), \preceq_\circ\right)$$

The **extended exterior semialgebra** of type 2, with its system, is defined analogously, with

$$\left(\overline{\widetilde{\bigwedge} V}, \wedge)\right) = \frac{(\widetilde{\bigwedge}^0 V \oplus (\widetilde{\bigwedge}^1 V \oplus \bigwedge^{\geq 2}(V)))}{(v \otimes v \sim 0)}, \qquad \forall v \in V. \tag{4.4}$$

Remark 4.11. Both cases were studied in [7] without the extension of degree 1, and although we use the notation of Case I, the same observations hold for Case II). In fact, there is a $\preceq$-epimorphism from $\widetilde{\bigwedge} V$ to $\overline{\widetilde{\bigwedge} V}$, sending every quasi-zero to 0.

Putting everything together, we have

Theorem 4.12. *For any semiring F, there is a functor from F-modules to exterior systems, sending V to $(\widetilde{\bigwedge} V, \mathcal{T}(\bigwedge V), (-), \preceq_\circ)$,*

which is the main statement enabling us to view the exterior theory in terms of systems.

4.13. Partitions. We can reinterpret the extended exterior semialgebra combinatorially via Young Diagrams. Let

$$\mathcal{P}_{r,n} := \{(\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{N}^r \quad | \quad \lambda_1 \leq n + 1 - r\}$$

be the set of all partitions (typically denoted as $\boldsymbol{\lambda}$) with at most r components, whose Young diagrams are contained in an $r \times (n - r)$ rectangle. We write $\mathcal{P}_r$ for $\mathcal{P}_{r,\infty}$ and $\mathcal{P}$ for $\mathcal{P}_\infty$. The **weight** of a partition is $|\boldsymbol{\lambda}| = \sum \lambda_i$. The partition $\boldsymbol{\lambda}'$ **conjugate** to $\boldsymbol{\lambda}$ is the partition whose Young diagram is the transpose of that of $\boldsymbol{\lambda}$. For example if $\boldsymbol{\lambda} = (3, 3, 2, 1, 1)$ then $\boldsymbol{\lambda}' = (5, 3, 2)$.

Let $\mathcal{A}$ be a semialgebra, $n \in \mathbb{N}^+ \cup \{\infty\}$, and $0 \leq r < n$. Let $n \in \mathbb{N}^+ \cup \{\infty\}$, and V_n be the free $\mathcal{A}$-module of rank n with base $\mathbf{x}^n := \{x^i : \ 0 \leq i < n\}$ (cf. Definition 2.2), i.e. $V_n = \bigoplus_{0 \leq i < n} \mathcal{A}x^i$.

Definition 4.14. Let $0 \leq r < n$. The **extended exterior $\mathcal{A}$-module** $E^r(V_n)$ is defined as follows: $E^0(V_n) := \mathcal{A}$, $E^1(V_n) := \mathcal{A}^{(n)}$, and for $r \geq 2$,

$$E^r(V_n) := \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_{r,n}} \mathcal{A}\boldsymbol{\lambda},$$

the free $\mathcal{A}$-module generated by $\mathcal{P}_{r,n}$.

Following traditional notation we write

$$E^r(V_n) := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathcal{A}\cdot[\mathbf{x}]^r_{\boldsymbol{\lambda}},$$

where $[\mathbf{x}]^1_{\boldsymbol{\lambda}} = x^{\lambda_1}$ and for $r \geq 2$,

$$[\mathbf{x}]^r_{\boldsymbol{\lambda}} = x^{r-1+\lambda_1} \wedge x^{r-2+\lambda_2} \wedge \cdots \wedge x^{\lambda_r}, \tag{4.5}$$

although this notation is redundant since r is understood from the partition. By convention $E^r(V_n) = 0$ if $r \geq n$. (This follows from $x^i \bigwedge x^i = 0$.) The total **extended exterior $\mathcal{A}$–module** is, by definition, the free $\mathcal{A}$-module of rank 2^n:

$$E(V_n) = \bigoplus_{r\geq 0} E^r(V_n)$$

Let $\{e_1, \dots, e_r)$ be commuting indeterminates over $\mathcal{A}$.

Proposition 4.15. *There is an $\mathcal{A}$-module epimorphism*

$$\mathcal{B}_r(\mathcal{A}) := \mathcal{A}[e_1, \dots, e_r] \to E^r(V_n),$$

given in the proof, which is an isomorphism if and only if $n = \infty$.

Proof. The natural base of $\mathcal{B}_r(\mathcal{A})$ is given by all the monomials $e_1^{i_1}\cdots e_r^{i_r}$. Let us denote by $\boldsymbol{\lambda} \in \mathcal{P}_r$ the partition with at most r parts conjugated to $(1^{i_1}\cdots r^{i_r})$. Therefore the claimed epimorphism is given by

$$e_1^{i_1}\cdots e_r^{i_r} = e^{\boldsymbol{\lambda}} \mapsto [\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

and the image is zero if and only if $r \geq n$. ∎

4.16. The action of the symmetric group. Let $\mathbb{X}^r \subset \mathbb{N}^r$ be the set of all r-tuples of distinct natural numbers, acted on by the subgroup Sym_r^+ of even permutations of Sym_r on $\{0, 1, \dots, r-1\}$ (whose action on $\mathbb{X}$ is switching the first two components). Let $\rho_r := (r-1, r-2, \dots, 0) \in \mathbb{X}^r$ (we write ρ when r is understood), and fix the transposition $\tau = (12) \in \mathrm{Sym}_r$. Then $\mathbb{X}^r$ decomposes into the disjoint union

$$\mathbb{X}^r = \mathrm{Sym}_r^+\cdot(\rho + \mathcal{P}_r) \cup \tau(\mathrm{Sym}_r^+\cdot(\rho+\mathcal{P}_r)), \tag{4.6}$$

which also defines a map

$$\Lambda : \mathbb{X}^r \to \mathcal{P}_r$$

given by

$$\Lambda(i_1, \dots, i_r) = \Lambda(\tau(i_1, \dots, i_r)) = \boldsymbol{\lambda}, \quad \forall (i_1, \dots, i_r) \in \mathrm{Sym}_r^+\cdot(\rho + \boldsymbol{\lambda}).$$

We also have a map $\mathbb{X}^r/\mathrm{Sym}_r^+ \to \{\rho, \tau\rho\}$ sending $(\rho + \boldsymbol{\lambda}) \mapsto \rho,\ \tau(\rho+\boldsymbol{\lambda}) \mapsto \tau\rho$.

4.17. The combinatorial construction. We define a negation map on $\mathbb{X}^r$ by putting

$$(-)(i_1,\dots,i_r)=\tau(i_1,\dots,i_r),$$

noting that $\tau(i_1,\dots,i_r)\mathrm{Sym}_r^+=(i_2,i_1,\dots,i_r)\mathrm{Sym}_r^+$, and define $(-)E^r(V_n)$ to be

$$\{v_2\wedge v_1\wedge\cdots\wedge v_r : v_1\wedge v_2\wedge\cdots\wedge v_r\in E^r(\mathcal{V}_n)\}.$$

For all $r\geq 2$ we define

$$\widetilde{\bigwedge}^r V_n := E^r(V_n)(-)E^r(V_n)$$

where $[\mathbf{x}]^r_{(-)\boldsymbol{\lambda}}=(-)[\mathbf{x}]^r_{\boldsymbol{\lambda}}:=x^{r-2+\lambda_2}\wedge x^{r-1+\lambda_1}\wedge\cdots\wedge x^{\lambda_r}$. It is an algebra with respect to juxtaposition

$$[\mathbf{x}]^r_{\boldsymbol{\lambda}}\wedge[\mathbf{x}]^s_{\boldsymbol{\mu}}=[\mathbf{x}]^{r+s}_{\Lambda(\rho_r+\boldsymbol{\lambda},\rho_s+\boldsymbol{\mu})}.$$

The map $(-):\widetilde{\bigwedge}^r V_n\to\widetilde{\bigwedge}^r V_n$ interchanging the sheets over $E^r_n(V_n)$ is the negation map of $\widetilde{\bigwedge}^r V_n$. Define

$$\widetilde{\bigwedge}^{\geq 2} V_n := \bigoplus_{r\geq 2}\widetilde{\bigwedge}^r V_n.$$

Proposition 4.18. *One can define two maps $V_n\times V_n\to\widetilde{\bigwedge}^{\geq 2}V$, which are respective $\mathcal{A}$-linear extensions of the set-theoretic maps $(x^i,x^j)\mapsto x^i\wedge x^j$ and $(x^i,x^j)\mapsto(-)x^i\wedge x^j=x^j\wedge x^i$.*

Proof. Clear. ∎

We view our subsequent examples in this notation.

5. Clifford semialgebras

As in the classical case, Clifford semialgebras are obtained by generalizing exterior semialgebras, using the $\preceq$-version of a quadratic form motivated by [15]. Moreover, the semialgebra of endomorphisms of an exterior semialgebra provides a natural example of Clifford semialgebra in our sense.

5.1. $\preceq$-Bilinear and quadratic forms. We begin by recalling the following:

Definition 5.2 (cf. [14, Definition 6.1])**.** A **$\preceq$-bilinear form** on a systemic module $(V,\mathcal{T}_V,(-),\preceq)$ over a semiring $\mathcal{A}$ is a map

$$\mathcal{B}:V\times V\to\mathcal{A}$$

which satisfies

1. $\mathcal{B}(a_0v_0 + a_1v_1, w) \succeq a_0\mathcal{B}(v_0, w) + a_1\mathcal{B}(v_1, w)$,
2. $\mathcal{B}(v, a_0w_0 + a_1w_1) \succeq a_0\mathcal{B}(v, w_0) + a_1\mathcal{B}(v, w_1)$,

$\forall v, v_i, w, w_i \in V$.

A $\preceq$-bilinear form $\mathcal{B} : V \times V \to \mathcal{A}$ is **symmetric** if $\mathcal{B}(v, w) = \mathcal{B}(w, v)$ for all $v, w \in V$.

Example 5.3. (i) A **bilinear form** is a $\preceq$-bilinear form where $=$ replaces $\preceq$, and as in the classical theory is obtained by an $n \times n$ matrix B; the form is symmetric iff B is a symmetric matrix. In particular, one could take the 0 form.

(ii) Suppose A is zero sum free, in the sense that $a + a' = 0$ implies $a = a' = 0$. Fix a base $\{x^1, \dots, x^n\}$ of V and define $\mathrm{ind}(v)$ for a vector $v = \sum a_i x^i$ to be the number of nonzero coefficients. Given a $\preceq$-bilinear form $\mathcal{B} : V \times V \to \mathcal{A}$, define

$$\mathcal{B}'(v, w) = \mathrm{ind}(v)\mathrm{ind}(w)\mathcal{B}(v, w)^\circ.$$

Since $\mathrm{ind}(v_0 + v_1) \geq \max\{\mathrm{ind}(v_0), \mathrm{ind}(v_1)\}$ we see that

$$\begin{aligned}
&(5.1)\\
\mathcal{B}'(a_0v_0 + a_1v_1, w) &= \mathrm{ind}(v_0 + v_1)\mathrm{ind}(w)\mathcal{B}(a_0v_0 + a_1v_1, w)^\circ \\
&\succeq \mathrm{ind}(v_0 + v_1)\mathrm{ind}(w)(a_0\mathcal{B}(v_0, w)^\circ + a_1\mathcal{B}(v_1, w)^\circ) \\
&\succeq \mathrm{ind}(v_0)\mathrm{ind}(w)a_0\mathcal{B}(v_0, w)^\circ + \mathrm{ind}(v_1)\mathrm{ind}(w)a_1\mathcal{B}(v_1, w)^\circ \\
&\succeq a_0\mathcal{B}'(v_0, w) + a_1\mathcal{B}'(v_1, w).
\end{aligned}$$

(iii) The sum of two $\preceq$-bilinear forms is a $\preceq$-bilinear form, so (ii) gives us a considerable range of examples.

The reason we use $\preceq$ is to prepare for Construction 5.7, Construction 5.10, and Construction 5.13 below; its use is consistent with the next definition.

Definition 5.4. A **$\preceq$-quadratic form** on a systemic $\mathcal{A}_0$-module $(V, \mathcal{T}_V, (-), \preceq)$ is a function $q : V \to \mathcal{A}_0$ with

$$q(av) \succeq a^2 q(v) \tag{5.2}$$

for any $a \in \mathcal{A}$, $v \in V$, together with a symmetric $\preceq$-bilinear form $\mathcal{B} : V \times V \to \mathcal{A}$ (not necessarily uniquely determined by q) such that for any $v, w \in V$,

$$2q(v) = \mathcal{B}(v, v), \qquad q(v + w) \succeq q(v) + q(w) + \mathcal{B}(v, w). \tag{5.3}$$

We call $(q, \mathcal{B})$ a **$\preceq$-quadratic pair** over $\mathcal{A}$.

A **quadratic pair** is a $\preceq$-quadratic pair $(q, \mathcal{B})$ for which $\mathcal{B}$ is a bilinear form and $q(v + w) = q(v) + q(w) + \mathcal{B}(v, w)$ for all $v, w \in V$.

5.5. Abstract definition of Clifford $\preceq$-semialgebra.

Just as Clifford algebras are essential to the theory of quadratic forms, we need a systemic version for this paper and for further research in quadratic forms. As we shall see, this is a more delicate issue than exterior semialgebras. We start with the desired defining properties.

Definition 5.6. A **Clifford $\preceq$-semialgebra** $\mathcal{C}(q, B, V)$ of a $\preceq$-quadratic pair (q, B) over $\mathcal{A}$, is a semialgebra $\mathcal{C}$ generated by $\mathcal{A}$ and V, together with a product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying

$$v^2 \succeq q(v) \qquad \text{for} \qquad v \in V, \tag{5.4}$$

$$v_1 v_2 + v_2 v_1 \succeq \mathcal{B}(v_1, v_2), \quad \forall v_i \in V. \tag{5.5}$$

Lacking negation, it could be unlikely to achieve equality in (5.5). Nonetheless we have:

Construction 5.7. *The* ***standard Clifford semialgebra*** $\widetilde{\mathcal{C}}(q, B, V)$ *is defined by imposing the relation $\preceq$ on $T(V)$ via Lemma 2.9, taking*

$$S_0 = \{(\mathcal{B}(v_1, v_2), v_1 \otimes v_2 + v_2 \otimes v_1),\ (q(v), v \otimes v) :\ v, v_i \in V\},$$

and S the $\mathcal{T}$-submodule of $T(V) \times T(V)$ generated by S_0.

Thus a standard Clifford semialgebra becomes an exterior semialgebra of type 1 when $q = B = 0$. But this does not yet have a negation map when $q \neq 0$. When $\mathcal{A}$ already possesses a negation map, we can define a negation map by defining $(-)v = ((-)1)v$ and $(-)(v_1 \otimes v_2 \otimes \cdots \otimes v_k)$.

Accordingly, we equip $\mathcal{A}$ with a negation map. This can just be the identity, as in tropical algebra, or else we use to the "symmetrization process," which we recall from [22, Definition 3.6]:

Definition 5.8. Define $\widehat{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ with componentwise addition, with multiplication given by the **twist action**

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0), \quad a_i, , b_i \in \mathcal{A}_0.$$

The negation map on $\widehat{\mathcal{A}}$ is the **switch map** $(-)_{\mathrm{sw}}$ given by

$$(-)_{\mathrm{sw}}(b_0, b_1) = (b_1, b_0). \tag{5.6}$$

The surpassing relation $\preceq_\circ$ is given by:

$$(b_0, b_1) \preceq_\circ (b_0', b_1') \quad \text{iff} \quad b_i' = b_i + c \text{ for some } c \in \mathcal{A},\ i = 0, 1. \tag{5.7}$$

(The same c is used for both components.)

So we first symmetrize $\mathcal{A}$ and V to $\widehat{\mathcal{A}}$ and $\widehat{V}$, and then form $\mathcal{C} = \widetilde{\mathcal{C}}(q, B, \widehat{V})$ and the Clifford system $(\mathcal{C}, \mathcal{T}_{\mathcal{C}}, (-), \preceq_\circ)$, with $(-)$ as in (5.6).

Lemma 5.9. *$\mathcal{C}$ as defined in the paragraph above is a Clifford semialgebra, and $(\mathcal{C}, \mathcal{T}_{\mathcal{C}}, (-), \preceq_\circ)$ is a system.*

Proof. It follows from the fact that (5.6) defines a negation map. ∎

We can construct a Clifford semialgebra a little more efficiently using ideas of §4.17.

Construction 5.10. *The **standard Clifford semialgebra with negation map** $\mathcal{C}(q, B, V, (-))$ is defined as follows:*

Define the tensor module $T_{\geq 0}(V)$ of a free $\mathcal{A}$-module V, as the submodule of $T(V)$ spanned by $\{x^{i_1} \otimes \cdots \otimes x^{i_k} : k \in \mathbb{N},\ i_1 \leq i_2 \leq \cdots \leq i_k\}$ and take $T = \mathcal{A} \oplus T_{\geq 0}(V) \oplus (-)T_{\geq 0}(V)$ (where $(-)T_{\geq 0}(V)$ is another copy of $T_{\geq 0}(V)$) under multiplication

$$x^j x^i = \mathcal{B}(x^i, x^j) + (-)x^i \otimes x^j,\ \forall j > i, \tag{5.8}$$

and inductively

$$(x^{i_1} \otimes \cdots \otimes x^{i_k}) \otimes x^j = \begin{cases} x^{i_1} \otimes \cdots \otimes x^{i_k} \otimes x^j, & k \leq j \\ \mathcal{B}(x^k, x^j) x^{i_1} \otimes \cdots \otimes x^{i_{k-1}} + ((-)x^{i_1} \otimes \cdots \otimes x^{i_{k-1}} \otimes x^j \otimes x^{i_k}), & \\ \quad k > j, & \end{cases} \tag{5.9}$$

where $(-)x^i = x^i$, extended distributively.

Theorem 5.11. *$\mathcal{C}(q, B, V, (-))$ of Construction 5.10 is a Clifford semialgebra with negation map. Define*

$$\preceq := \preceq_\circ \cup \bigcup_{k \in \mathbb{N}^+} \{x^{i_1} \otimes \cdots \otimes x^{i_j} \otimes x^{i_j} \otimes \cdots \otimes x^{i_k},\ \ q(x^{i_j}) x^{i_1} \otimes \cdots \otimes x^{i_k}\}, \tag{5.10}$$

i running over $\mathbb{N}^+$, where x^{i_j} is deleted in the tensors on the right side. (In other words we declare that $x^{i_j} \otimes x^{i_j} \preceq q(x^{i_j})$.)

Taking $\mathcal{T}_{\mathcal{C}(q,B,V,(-))}$ to be the simple tensors and their negations,

$$(\mathcal{C}(q, B, V, (-)), \mathcal{T}_{\mathcal{C}(q,B,V,(-))}, (-), \preceq)$$

is a system. There is an injection $\Phi : \widetilde{\mathcal{C}}(q, B, V) \to \mathcal{C}(q, B, V, (-))$ given by

$$x^{i_1} \otimes \cdots \otimes x^{i_k} \mapsto (-)^\pi x^{\pi(i_1)} \otimes \cdots \otimes x^{\pi(i_k)},$$

where π is the permutation rearranging $i_1, \ldots, i_k$ in ascending order.

Proof. We start by verifying the conditions of Definition 5.6.

$x^i \otimes x^j + x^j \otimes x^i = \mathcal{B}(x^i, x^j) + (x^i \otimes x^j)^\circ \succeq \mathcal{B}(x^i, x^j)$ for $i < j$, so we take sums.

To prove associativity of multiplication, we need only check tensors of basis elements, and thus, by induction on length, we appeal to (5.9). ∎

Unfortunately this construction is not finitely spanned over $\mathcal{A}$, since the tensor powers of x^i are independent. We can reduce $x^i \otimes x^i$ to $q(x^i)$, at the cost of losing associativity. Accordingly, we weaken associativity.

Definition 5.12. A nonassociative semialgebra $\mathcal{C}$ with a surpassing relation $(\preceq)$ is **$\preceq$-associative** if $a_1(a_2a_3) + (a_1a_2)a_3 \succeq 0$, and $\mathcal{C}$ is called a **$\preceq$-semialgebra**.

Construction 5.13. *The* ***reduced Clifford $\preceq$-semialgebra with negation map*** $\mathcal{C}_{\preceq}(q, B, V, (-))$ *is defined as follows:*

Define the tensor module $T_>(V)$ of a free $\mathcal{A}$-module V, as the submodule of $T(V)$ spanned by $\{x^{i_1} \otimes \cdots \otimes x^{i_k} : k \in \mathbb{N}, \ : i_1 < i_2 < \cdots < i_k\}$, and take

$$T := \mathcal{A} \oplus T_>(V) \oplus (-)T_>(V)$$

(where $(-)T_>(V)$ is another copy of $T_>(V)$) under multiplication

$$x^i \otimes x^i = q(x^i), \qquad x^j x^i = \mathcal{B}(x^i, x^j) + ((-)x^i \otimes x^j), \ \forall j > i, \tag{5.11}$$

and inductively

$$(x^{i_1} \otimes \cdots \otimes x^{i_k}) \otimes x^j = \begin{cases} x^{i_1} \otimes \cdots \otimes x^{i_k} \otimes x^j, \quad k < j \\ q(x^j) x^{i_1} \otimes \cdots \otimes x^{i_{k-1}}, \quad k = j \\ x^{i_1} \otimes \cdots \otimes x^{i_{k-1}} \mathcal{B}(x^k, x^j) + ((-)x^{i_1} \otimes \cdots \otimes x^{i_{k-1}} \otimes x^j \otimes x^{i_k}), \\ \quad k > j, \end{cases} \tag{5.12}$$

where $(-)(-)x^i = x^i$, extended distributively.

Remark 5.14. Associativity in the reduced Clifford $\preceq$-semialgebra with negation map fails in general since

$$x^2 \otimes (x^2 \otimes x^1) = x^2 \otimes (\mathcal{B}(x^1, x^2)(-)x^1 \otimes x^2) = q(x^2)x^1 + \mathcal{B}(x^1, x^2)(x^2)^\circ$$

whereas $(x^2 \otimes x^2) \otimes x^1 = q(x^2)x^1$. Likewise

$$(x^2 \otimes x^1) \otimes x^1 = \mathcal{B}(x^1, x^2)(-)x^1 \otimes x^2) \otimes x^1 = q(x^1)x^2 + (\mathcal{B}(x^1, x^2)(x^2)^\circ$$

whereas $x^2 \otimes (x^1 \otimes x^1) = q(x^1)x^2$.

Fortunately in our application of representing the exterior algebra, we do have associativity since $\mathcal{B}$ is zero where this difficulty would arise.

Theorem 5.15. *$\mathcal{C}_{\preceq}(q, B, V, (-))$ is a Clifford $\preceq$-semialgebra.*

$$(\mathcal{C}_{\preceq}(q, B, V, (-)), \mathcal{T}_{\mathcal{C}(q,B,V,(-))}, (-), \preceq)$$

is a system, taking $\mathcal{T}_{\mathcal{C}(q,B,V,(-))}$ to be the simple tensors and their negations,

Proof. We start by verifying the conditions of Definition 5.6.

$$x^i \otimes x^j + x^j \otimes x^i = \mathcal{B}(v, w) + (x^i \otimes x^j)^\circ \succeq \mathcal{B}(v, w)$$

for $i < j$, so we take sums.

To prove $\preceq$-associativity of multiplication, we need only check tensors of basis elements, and thus, by induction on length, we appeal to (5.12). ■

Since the indices increase, there must be at most n of them in a product of basis elements. Thus the dimension is 2^{n+1} rather than 2^n in the classical case, because we need to count negatives as well. Note that $\widetilde{\mathcal{C}}(q, B, V) \to \mathcal{C}_{\preceq}(q, B, V, (-))$ is no longer an injection since $x^i \otimes x^i$ and $q(x^i)$ have the same image.

Remark 5.16. In summary, the standard Clifford algebra is associative but lacks the important classical property of finite generation. The reduced standard Clifford algebra seems more in line with the spirit of this paper, replacing equality by $\preceq$ in the key property of associativity.

When $\mathcal{A}$ is taken to be a field, the formal negation map coincides with the negation in the field, and thus both the standard and the reduced Clifford algebras boil down to the classical Clifford algebra of a quadratic form over a field.

Example 5.17 (The standard reduced Clifford $\preceq$-semialgebra with $(-)$ for $n = 2$)**.** We consider the base $\{(\pm)1,\ x := x^1,\ (-)x,\ y := x^2,\ (-)y\}$ over $\mathcal{A}$, and given a quadratic pair $(q, \mathcal{B})$, put $\gamma = \mathcal{B}(x, y)$. Then $x \cdot x = q(x)$, $y \cdot y = q(y)$, and $y \cdot x = (-)x \cdot y + \gamma$.

We turn to involutions.

Definition 5.18. A **$\preceq$-involution** (σ) of a $\preceq$- system $\mathcal{A}$ is a $\preceq$-antihomomorphism of order ≤ 2, in the sense that $(a^\sigma)^\sigma \succeq a$;

$$(a_0 a_1)^\sigma \preceq a_1^\sigma a_0^\sigma, \quad \forall a, a_i \in \mathcal{T}. \tag{5.13}$$

Lemma 5.19. *The standard Clifford $\preceq$-semialgebra with negation map $\mathcal{C}(q, B, V, (-))$ has an involution σ given by the identity on $\mathcal{A}$ and $v^\sigma = (-)v$ and*

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k)^\sigma = (-)^k v_k \otimes \cdots \otimes v_2 \otimes v_1.$$

Proof. It is enough to check (5.13) for tensors of basis elements. For $j > i_k$,

(5.14)
$$\begin{aligned}((x^{i_1}\otimes\cdots\otimes x^{i_k})\otimes x^j)^\sigma &= (-)^k x^{i_k}\otimes\cdots\otimes x^{i_2}\otimes x^{i_1}\\ &= (-)^k x^{i_k}\otimes((-)^{k-1}x^{i_{k-1}}\cdots\otimes x^{i_2}\otimes x^{i_1}) = x^{j^\sigma}\otimes(x^{i_1}\otimes\cdots\otimes x^{i_k})^\sigma.\end{aligned}$$

For $j = k$,

$$((x^{i_1}\otimes\cdots\otimes x^{i_k})\otimes x^{i_k})^\sigma = q(x^{i_k})(x^{i_1}\otimes\cdots\otimes x^{i_{k-1}})^\sigma = x^{k^\sigma}\otimes(x^{i_1}\otimes\cdots\otimes x^{i_k})^\sigma.$$

For $j < k$,

(5.15)
$$\begin{aligned}((x^{i_1}\otimes\cdots\otimes x^{i_k})\otimes x^j)^\sigma &= ((x^{i_1}\otimes\cdots\otimes x^{i_{k-1}})\otimes(\mathcal{B}(x^j,x^{i_k})(-)(x^j\otimes x^{i_k})^\sigma\\ &= (-)^{i_{k-1}}\mathcal{B}(x^j,x^{i_k})x^{i_{k-1}}\otimes\cdots\otimes x^2\otimes x^1 + (-)^{k+1}x^{i_k}\otimes x^j\otimes\cdots\otimes x^2\otimes x^1\\ &= (-)^{k-1}\mathcal{B}(x^j,x^{i_k})^\circ x^{i_{k-1}}\otimes\cdots\otimes x^{i_2}\otimes x^{i_1} + (-)^{k+1}x^{i_k}\otimes x^j\otimes\cdots\otimes x^{i_2}\otimes x^{i_1}\\ &= (x^j)^\sigma(x^{i_1}\otimes\cdots\otimes x^{i_k})^\sigma.\end{aligned}$$

Example 5.20. The involution for Example 5.17 is given by $x^\sigma = (-)x$, $y^\sigma = (-)y$, $(xy)^\sigma = yx = (-)xy+\gamma$, $(yx)^\sigma = xy$. Thus

$$(xy)^\sigma = ((-)xy+\gamma)^\sigma = xy\,(-)\gamma+\gamma \succeq xy$$

and

$$((xy)y)^\sigma = q(y)x^\sigma = x^\sigma q(y) \preceq xy^2 + (\gamma y)^\circ = y(xy) - \gamma y) = y^\sigma(xy)^\sigma$$

$$((xy)x)^\sigma = ((-)x^2y+\gamma x)^\sigma = q(x)y+\gamma x = x(xy)-\gamma x = x^\sigma(xy)^\sigma$$

6. The dual space and the Clifford semialgebra

We bring in our major use of the Clifford semialgebra, removing $\otimes$ from the notation. For simplicity, we continue to deal with the free $\mathcal{A}$-module $V = \mathcal{A}[x]$ of infinite countable rank with base $\mathbf{x} := (x^0, x^1, x^2, \ldots)$, the power of the indeterminate x.

Definition 6.1. Suppose we are given an $(\mathcal{A},\preceq)$-bilinear form B and a module $V = (V, \mathcal{T}_V, (-)))$ over $\mathcal{A}$. Define $\partial^j : V \to \mathcal{A}$ by $\partial^j(v) := B(u, x^j)$ for all $v \in V$. In other words, the restricted dual V^* will be the $\mathcal{A}$-span of the B-dual basis ∂^j such that $\partial^j(x^i) = B^{ij} = B(x^i, x^j)$. Symbolically we could also denote V^* by $\mathcal{A}[\partial]$. If $B(x^i, x^j) = \delta_{ij}$ then ∂^j is the element of the usual dual base. Write $\mathcal{T}_V^*$ for $\{f|_{\mathcal{T}_V} : f \in V^* \text{ with } f(\mathcal{T}_V) \subseteq \mathcal{T}_\mathcal{A}\}$.

Define the set $\mathcal{W}(\mathbf{x},\partial) = \mathcal{W}(\mathbf{x}) \cup \mathcal{W}(\partial)$ of all the words in the union of the two infinite alphabets $\mathcal{W}(\mathbf{x}) = \{x^0, x^1, x^2, \}$ and $\mathcal{W}(\partial) = \{\partial^0, \partial^1, \partial^2, \ldots, \}$. Recall that if $v_1, v_2 \in V$ and $w \in \widetilde{\bigwedge}^n V$, then

$$(-)v_1 \wedge (v_2 \wedge w) = v_2 \wedge v_1 \wedge w,$$

which defines a negation map on $\tilde{V}$ compatible with that on $\widetilde{\bigwedge}^{\geq 2} V$.

Denote by $\mathcal{C} = \mathcal{C}(V)$ the set of formal finite linear combinations of words of $\mathcal{W}(\mathbf{x}, (-)\mathbf{x}, \partial, (-)\partial)$ with $1_{\mathcal{A}}$ adjoined, with respect to the juxtaposition product extended distributively, and with relations given (for $i \geq j$) by

$$x^i x^j = (-)x^j x^i, \qquad \partial^i \partial^j = (-)\partial^j \partial^i, \qquad \partial^j x^i = \mathcal{B}(x^i, x^j)(-)x^i \partial^j. \tag{6.1}$$

Define $\mathcal{C} := \mathcal{C}(V)$ as being the $\mathcal{A}$-algebra generated by $1_{\mathcal{A}} \cup \mathcal{C}_+ \cup \mathcal{C}_-$, where $\mathcal{C}_+$ and $\mathcal{C}_-$ are graded:

$$\mathcal{C}_+ = C_1 \oplus_{h \geq 1} \mathcal{A} \cdot \underbrace{x^{i_1} \cdots x^{i_h}}_{words\ of\ length\ h}$$

where

$$\mathcal{C}_1 = \bigoplus_{j \geq 0} \mathcal{A} x^j \oplus \bigoplus_{j \geq 0} \mathcal{A}((-)x^j)$$

and

$$\mathcal{C}_- = C_{-1} \oplus_{h \geq 1} \mathcal{A} \cdot \underbrace{\partial^{i_1} \cdots \partial^{i_h}}_{words\ of\ length\ h}$$

where

$$\mathcal{C}_{-1} = \bigoplus_{j \geq 0} \mathcal{A} \cdot \partial^j \oplus \bigoplus_{j \geq 0} \mathcal{A}((-)\partial^j),$$

defining

$$(-)\partial^j (x^h x^k) = \partial^j ((-)x^h x^k) = \partial^j (x^k x^h).$$

If $u(x) = \sum_i a_i x^i \in V = \mathcal{A}[x]$ we define its B-dual over V^* as $u(\partial) := \sum_i a_i \partial^i$, i.e. $u(\partial)(v(x)) = B(u(x), v(x))$. We make $\mathcal{C}$ into a semialgebra over $\mathcal{A}$ with respect to the juxtaposition product, by imposing the following commutation rules on the elements of degree ± 1:

$$u(x)v(x) \quad + \quad v(x)u(x) \succeq 0,$$

$$u(\partial)v(\partial) \quad + \quad v(\partial)u(\partial) \succeq 0,$$

$$u(\partial) \cdots v(x) \quad \succeq \quad u(\partial)(v(x))(-)v(x)u(\partial), \tag{6.2}$$

which basically descend from the fact that we are requiring the elements of V^* acting on $\mathcal{C}$ as skew-derivations, namely

$$u(\partial)(v(x)w(x)) = (u(\partial)v(x))w(x)(-)v(x)u(\partial)w(x). \tag{6.3}$$

To be more precise, equality (6.3) should be understood in the sense of operators, i.e. for all $f \in \mathcal{C}_1$:

$$(u(\partial)v(x)w(x)(-)v(x)u(\partial)(w(x)))f = u(\partial)(v(x))f + u(\partial)(w(x))fv(x).$$

Since each element of $\mathcal{C}$ is a finite linear combination of words of elements of degree 1, the above suffices for the definition of the commutation relations on $\mathcal{C}$.

Example 6.2. This example serves to motivate commutations (6.2). Let $\widetilde{\bigwedge}V$ be the exterior semi algebra. Any element of degree ≥ 2 is a sum of elements of the form $w := w_1 \wedge w_2$ with $w_1, w_2 \in V$ (possibly equal to 1). Now $u(\partial)$ acts on $w := w_1 \wedge w_2$ as

$$u(\partial)(w_1 \wedge w_2) \quad = \quad u(\partial)(w_1) \wedge w_2(-)w_1 \wedge u(\partial)(w_2)$$

which means that

$$u(\partial)(w_1 \wedge w_2) \wedge w_3 = u(\partial)(w_1)(w_2 \wedge w_3) + u(\partial)(w_2)w_1 \wedge w_3.$$

This motivates our requirement (6.3). Furthermore a simple computation give:

$$(u(\partial)v(x) + v(x)u(\partial))w \succeq B(u(x), v(x))w$$

i.e.

$$u(\partial)v(x) + v(x)u(\partial) \succeq B(u(x), v(x))$$

or

$$u(\partial)v(x) \succeq B(u(x), v(x))(-)v(x)u(\partial)$$

whence the commutation relation in the Clifford semialgebra

$$u(\partial)v(x) = u(\partial)(-)v(x)u(\partial)$$

6.3. For notational simplicity write $u^* := u(\partial) \in V^*$ and $v := v(x) \in V$. We also denote by "$\cdot$" the product of $\mathcal{C}$ to avoid potential confusions. The product u^*v in $\mathcal{C}$ defines a linear map $\mathcal{C} \to \mathcal{C}$ given by

$$(u^* \cdot v)w = u^*(v)(w) = u^*(vw). \tag{6.4}$$

Then

$$(u^* \cdot v) \cdot w \quad = \quad u^*(v) \cdot w(-)u^*(w)v$$

which means that

$$[(u^* \cdot v) \cdot w]z = u^*(v)w \cdot z + u^*(w)z \cdot v.$$

Notice that by construction the product in $\mathcal{C}$ is associative because we are composing operators, and the composition of operators is associative.

Remark 6.4. (Negation on $\mathcal{A}$) . We endow $\mathcal{A}$ with a negation map, by setting $((-)1_{\mathcal{A}})u = (-u)$ in the sense of operators, namely $((-)u)v = vu$. Write $\widetilde{\mathcal{A}} = \mathcal{A} \oplus ((-)\mathcal{A}$. $\widetilde{\mathcal{A}}$ has a negation map $(-)$ given by the switch. Likewise, take a copy $(-)V$ of V, and view $\widetilde{V} := V \oplus ((-)V)$ as an $\widetilde{\mathcal{A}}$-module via

$$(a_1, a_2)(v_1, v_2) = (a_1v_1 + a_2v_2, a_2v_1 + a_1v_2),$$

and $(-)(v_1, v_2) := (v_2, v_1)$.

We take the dual $\widetilde{V}^*$ of $\widetilde{V}$ which can be thought of as pairs $w^* := (w_1^*, w_2^*)$, where for $v = (v_1, v_2) \in \widetilde{V}$,

$$w^*(v) = (w_1^*(v_1) + w_2^*(v_2), w_1^*(v_2) + w_2^*(v_1)).$$

Then $((-)w)^*(v) = (w_1^*(v_2) + w_2^*(v_1), w_1^*(v_1) + w_2^*(v_2))$.

Notice that

$$(-)u \cdot v = v \cdot u = (-)(-)v \cdot u = u \cdot (-)v$$

so that the relations $(-)1_{\mathcal{A}}(u \cdot v) = (-)u \cdot v = u \cdot (-)v$ hold. We extend the bilinear form B to $V \oplus (-)V$, by putting

$$B((x^i, (-)x^{i'}), (x^j, x^{j'})) = (B(x^i, x^j) + B(x^{i'}, x^{j'})(-)(B(x^i, x^{j'}) + B(x^{i'}, x^j))$$

Now we define $(-)B(x^i, x^j) := B((-)x^i, x^j)$, and our negation map also induces a negation map on V^* in $\mathcal{C}$, given by $(-)(x^i)^*(x^j) = (-)\mathcal{B}(x^i, x^j) = B((-)x^i, x^j)$.

Definition 6.5. An element of $\mathcal{C}$ is in **normal form** if it is a finite linear combination of words w_1w_2 such that $w_1 \in \mathcal{W}(\mathbf{x})$ and $w_2 \in \mathcal{W}(\boldsymbol{\partial})$.

Example 6.6. Here are a couple of examples of the basic yoga to put products in normal form using the appropriate commutation relations.

1. To put $u_1^*u_2^*v$ in normal form we exploit the fact that the elements of V^* act as derivations on $\widetilde{\bigwedge} V$. One has

$$\begin{aligned} u_1^* \cdot u_2^* \cdot v &= u_1^* \cdot (u_2^* \cdot v) = u_1^*\big(B(u_2, v) + ((-)v) \cdot u_2^*\big) \\ &= B(u_2, v)u_1^* + B(u_1, (-)v)u_2^* + v \cdot u_1^* \cdot u_2^* \end{aligned}$$

   where we agree that $B(u, (-v)) = (-)B(u, v)$, acting on the product $w - 1 \cdot w_2$ as $(-)B(u, v)w_1w_2 = B(u, v)w_2 \cdot w_1$

2. As a second example we propose the usual expression one may happen to deal with, such as the monomial $x^i\partial^j x^k$. To put it in normal form:

$$x^i\partial^j x^k = x^i(B_{jk}\,(-)\,x^k\partial^j) = B_{jk}x^i(-)\,x^k\partial^j$$

   where $B_{ij} := B(x^i, x^j)$.

Proposition 6.7. *All elements of $\mathcal{C}$ can be put into normal form.*

Proof. A matter of a routine induction. ∎

We do not claim that the normal form is unique, but we do have uniqueness up to $\circ$: If $\sum a_i w_{1,i} w_{2,i} = a'_i w_{1,i} w_{2,i}$ for $a_i, a'_i \in \mathcal{A}$ then each $a_i(-)a'_i \in \mathcal{A}^\circ$.

Next, we apply some material from [17, §9] to the Clifford semialgebra.

Remark 6.8. $(V^*_{\mathcal{B}}, \mathcal{T}^*_V, \preceq)$ is a systemic module, where we define $v^*_1 \preceq v^*_2$ if $v^*_1(w) \preceq v^*_2(w)$ for every $w \in V$.

6.9. The action on $\widetilde{\bigwedge} V$ (and on $\overline{\bigwedge V}$)

We have the following:

Definition 6.10. Define a left action $\lrcorner$ (possibly depending on the choice of the $\preceq$ bilinear form B) of V^*_n on $\bigwedge V_n$ by:

$$\partial \lrcorner v = \partial(v), \qquad \forall (\partial, v) \in V^*_n \times V_n,$$

and for $(u_1, \dots, u_k) \in V^k_n$,

$$\partial \lrcorner (u_1 \wedge \cdots \wedge u_k) = \begin{vmatrix} \partial(u_1) & \partial(u_2) & \dots & \partial(u_k) \\ u_1 & u_2 & \dots & u_k \end{vmatrix} \in \widetilde{\bigwedge}^{k-1}_{\mathcal{A}} V_n, \tag{6.5}$$

where the expression in the RHS of (6.5) means the linear combination

$$\sum_j (-)^j \partial(u_j) u_1 \wedge \cdots \wedge u_{j-1} \wedge u_{j+1} \wedge \cdots \wedge u_k \in \widetilde{\bigwedge}^{k-1}_{\mathcal{A}} V_n.$$

The cases $k = 2$ and 3 merit explicit descriptions.

Example 6.11. Let $u \wedge v \in \bigwedge^2 V_n$ and $\partial \in V^*_n$. Then

$$\partial \lrcorner (u \wedge v) = \begin{vmatrix} \partial(u) & \partial(v) \\ u & v \end{vmatrix} = \partial(u)v(-)\partial(v)u \in \widetilde{\bigwedge}^1_{\mathcal{A}} V_n,$$

where

$$(\partial(u)v(-)\partial(v)u)(w \wedge w_1) = \partial(u) \cdot v \wedge w \wedge w_1 + \partial(v) \cdot w \wedge u \wedge w_1 \qquad (w, w_1) \in V_n \times \bigwedge V_n.$$

Similarly:

$$\partial \lrcorner (u \wedge v \wedge w) = \begin{vmatrix} \partial(u) & \partial(v) & \partial(w) \\ u & v & w \end{vmatrix} = \partial(u) v \wedge w (-) \partial(v) w \wedge u + \partial(w) u \wedge v.$$

Lemma 6.12. *For all $\partial \in V^*_n$, and $v_1, v_2, \dots, v_k$ $(k \geq 2)$, denote*

$$\begin{vmatrix} \partial(\mathbf{v}) \\ \mathbf{v} \end{vmatrix} := \begin{vmatrix} \partial(v_1) & \cdots & \partial(v_k) \\ v_1 & \dots & v_k \end{vmatrix}$$

Then

$$\partial \lrcorner (u \wedge v_1 \wedge \cdots \wedge v_k) = \begin{vmatrix} \partial(u) & \partial(v_1) & \cdots & \partial(v_k) \\ u & v_1 & \cdots & v_k \end{vmatrix} = \partial(u) v_1 \wedge \cdots \wedge v_k \; (-) \; u \wedge \begin{vmatrix} \partial(\mathbf{v}) \\ \mathbf{v} \end{vmatrix}.$$

Proof. By direct substitution. ∎

The notation of (6.5) is useful in simplifying the combinatorics.

Proposition 6.13. *For $\partial \in V_n^*$, the map $\partial \,\lrcorner : \widetilde{\bigwedge} V \to \widetilde{\bigwedge} V$ is a V-skew derivation, i.e., for $u \in \widetilde{\bigwedge}^1 V$ and $v \in \bigwedge^k V$,*

$$\partial \,\lrcorner (u \wedge v) = \partial(u)v \ (-) \ u \wedge (\partial \,\lrcorner v). \tag{6.6}$$

Proof. The map ∂ is clearly $\mathcal{A}$-linear. On $u \wedge v \in \bigwedge^2 V_n$ one has:

$$\partial \,\lrcorner (u \wedge v) = \begin{vmatrix} \partial(u) & \partial(v) \\ u & v \end{vmatrix} = \partial(u)v \,(-)\, \partial(v)u.$$

Assume that the assertion holds for $k-1 \geq 1$ and let $v \in \bigwedge^j V$ with $j \geq 2$. Then

$$\partial \lrcorner (u \wedge v) = \partial(u)v(-) \ u \wedge \begin{vmatrix} \partial(v) \\ v \end{vmatrix},$$

which proves the assertion for k. ∎

Another way of interpreting (6.6) is $\partial\, u = \partial(u)(-)\, u\partial$, which implies that for all $j \geq 0$,

$$\partial^j \, u + u\partial^j \succeq \partial^j(u) = \mathcal{B}(u, x^j).$$

This is our desired connection with Clifford algebras, which we now formulate.

6.14. The Clifford Algebra action on $\bigwedge V$. The words of the form $x^{i_1} \cdots x^{i_k}$ act on $\bigwedge V$ by wedging:

$$x^{i_1} \cdots x^{i_k}(u) = x^{i_1} \wedge \cdots \wedge x^{i_k} \wedge u, \qquad \forall u \in \bigwedge V.$$

The words involving only the ∂_j act by contraction:

$$\partial^{j_1} \cdots \partial^{j_l} u = \partial^{j_1} \,\lrcorner (\cdots \lrcorner (\partial^{j_k} \,\lrcorner u)) \cdots)$$

Define $<,>: \mathcal{C}_{\pm} \to \mathcal{A}$ by

$$< u_1 \oplus v_1^*, u_2 \oplus v_2^* >= \ B(u_1, v_2) + B(u_2, v_1)$$

This is a nondegenerate inner product on $V \oplus V^*$.

Proposition 6.15. *The inner product possesses an orthogonal base.*

Proof. Let us consider $x^i := (x^i \oplus 0)$, $(x^j)^* := (0 \oplus \partial_j)$, so that $x^i \oplus (x^j)^* = x^i + (x^j)^*$ and $\partial_j(e_i) = \mathcal{B}(e_i, e_j)$. Then $(x^0, \dots, x^{n-1}, x^{0*}, \dots, x^{n-1*})$ is an orthogonal base.

6.16. The Clifford representation of an exterior semialgebra.

Theorem 6.17. *The exterior semialgebra is an irreducible representation of the Clifford semialgebra $\mathcal{C}$.*

Proof. First of all we show that for any choice of a $\preceq$-bilinear form B, the exterior semialgebra represents $\mathcal{C}$. It is obvious that each word of of $\mathcal{C}$ defines an $\mathcal{A}$-endomorphism of $\widetilde{\bigwedge}V$. We have to check the commutation relations. It is obvious that the relation $u \cdot v + v \cdot u$ is mapped to the endomorphism $u \wedge v + v \wedge v$ of $\widetilde{\bigwedge}V$, which is a quasi-zero in $\mathrm{End}_{\mathcal{A}}(\widetilde{\bigwedge})$. The same holds with $u^*v^* + v^*u^*$ acting on w as $u^* \lrcorner (v^* \lrcorner w) + w^* \lrcorner (v^* \lrcorner w) \geq 0$. It remains to check the action of $u^*v + v \cdot u^*$. Let us consider $w := w_1 \wedge w_2$ where $w_1 \in V$ and $w_2 \in \widetilde{\bigwedge}V$. Then

$$u^*(v \wedge w_1 \wedge w_2) + v \wedge (u^*(w_1 \wedge w_2)) = u^*(v)w_1 \wedge w_2 + v \wedge u^*(w_1)w_2(-)$$

To prove the irreducibility, suppose on the contrary that the exterior semialgebra has a proper invariant sub-module W under the action of $\mathcal{C}$. Let U be the submodule of W of elements of minimal degree, say r, whose typical elements are finite $\mathcal{A}$-linear combinations of $u_1 \wedge \cdots \wedge u_r$. But for any $v \in V$, vU is a submodule of elements of degree $r+1$, and thus W cannot be invariant. ∎

Proposition 6.18. *There is an $\mathcal{A}$-semialgebra homomorphism*

$$\mathrm{gl}(\widetilde{\bigwedge}^1 V) := \widetilde{\bigwedge}^1_{\mathcal{A}} V \otimes_{\mathcal{A}} \widetilde{\bigwedge}^1 V^*_{\mathcal{A}} \to \mathrm{End}_{\mathcal{A}}(\bigwedge^1 V).$$

Proof. A general element of $V \otimes_{\mathcal{A}} V^*$ is of the form

$$\sum_{0 \leq i,j \leq n-1} x^i \otimes (a_{ij}\partial_j + c_{ij}((-)\partial_j)) + \sum_{0 \leq i,j \leq n-1} (-)x^i \otimes (a'_{ij}j + c'_{ij}((-)\partial_j)),$$

which gives the endomorphism

$$u \mapsto \sum_{0 \leq i,j \leq n-1} x^i(a_{ij}\partial_j(u) + c_{ij}((-)\partial_j(u))) + \sum_{0 \leq i,j \leq n-1} (-)x^i(a'_{ij}\partial_j(u) + c'_{ij}((-)\partial_j(u))).$$

Conversely each $\phi \in \mathrm{End}_{\mathcal{A}}(V)$ can be uniquely written as a $\mathcal{A}$–linear combination of

$$x^i \otimes \partial_j, \qquad (-)x^i \otimes \partial_j, \qquad x^i \otimes ((-)\partial_j), \qquad (-)x^i \otimes ((-)\partial_j)$$

since, writing $u = \sum_{0 \leq i < n-1} u_i x^i + u'_i((-)x^i)$,

$$\begin{aligned}
\phi(u) &= \phi\left(\sum_{0 \leq i < n-1} u_i x^i + u'_i((-)x^i)\right) = \sum_{0 \leq i < n-1} u_i \phi(x^i) + u'_i \phi((-)x^i) \\
&= \sum_{i=0}^{n-1} u_i a_{ij} x^j + a'_{ij}((-)x^j) + u'_i c_{ij} x^j + u'_i c'_{ij}((-)x^j)
\end{aligned}$$

$$\begin{aligned} &= \sum_{i,j=0}^{n-1} \left[x^i \otimes (a_{ij}\partial_j + c_{ij}((-)\partial_j)) \right. \\ &+ \left. (-)x^i \otimes (a'_{ij}\partial_j + c'_{ij}((-)\partial_j)) \right] (u). \end{aligned}$$ ■

Proposition 6.18 defines a natural action of $\mathrm{gl}(\widetilde{\bigwedge}^1 V)$ on V.

Definition 6.19. The **Lie bracket** $[\phi, \psi]$ of $\phi, \psi \in \mathrm{gl}(\widetilde{\bigwedge}^1 V)$ is defined by:

$$[\phi, \psi] = \phi \circ \psi \ (-) \ \psi \circ \phi.$$

Proposition 6.20. *The $\mathcal{A}$-semialgebra $\mathrm{gl}(\tilde{T}^1(V))$ is a Lie $\preceq$-semialgebra with a negation map in the sense of Definition 3.5.*

Proof. Items (i),(ii) are obvious, by construction. For all $v \in V$

$$[\phi, \psi](u) \wedge v = \big(\phi(\psi(u))(-)\psi(\phi(u))\big) \wedge v = \phi(\psi(u) \wedge v + v \wedge \psi(\phi(u)),$$

and

$$[\psi, \phi](u) \wedge v = \big(\psi(\phi(u))(-)\phi(\psi(u))\big) \wedge v = \psi(\phi(u) \wedge v + v \wedge \phi(\psi(u)).$$

Thus

$$[\psi, \phi](u) \wedge v = \big((-)[\phi, \psi](u)\big) \wedge v.$$

(iii) is a routine verification, done in [22]. ■

Example 6.21. Let $b \in V$ and $\partial \in V^*$. Then $b \otimes \partial$ acts on $v \in \widetilde{\bigwedge} V$ as:

$$(b \otimes \partial)(v) = b \wedge (\partial \lrcorner v).$$

It is a $\mathcal{A}$-derivation, i.e. is trivial on $\mathcal{A}$ and

$$(b \otimes \partial)(u \wedge v) = ((b \otimes \partial)(u)) \wedge v + u \wedge ((b \otimes \partial)(v)).$$

The critical case to verify is for $u = v_1$ and $v = v_2 \in \widetilde{\bigwedge}^1 V$, for which we have

$$(b \otimes \partial)(v_1 \wedge v_2) = \partial(v_1))b \wedge v_2 + \partial(v_2)v_1 \wedge b.$$

The general case follows by induction. In fact, if $u \in \widetilde{\bigwedge}^1 V$:

$$\begin{aligned} (b \otimes \partial)(u \wedge v) &= b \wedge (\partial(u)v \ (-) \ u\partial(v))\partial(u)b \wedge v + \partial(v)u \wedge b \\ &= \partial(u)b \wedge v + u \wedge b\partial(v) = (b \otimes \partial)(u) \wedge v + u \wedge (b \otimes \partial)(v). \end{aligned}$$

6.22. Extending the $\mathrm{gl}(\widetilde{\bigwedge}^1 V)$-action to $\widetilde{\bigwedge}_{\mathcal{A}} V$.

Consider the map

$$\left\{\begin{array}{ccccc} \delta & : & \mathrm{gl}(\widetilde{\bigwedge}^1 V) & \longrightarrow & \mathrm{End}_{\mathcal{A}}(\widetilde{\bigwedge} V) \\ & & \phi & \longmapsto & \delta(\phi) \end{array}\right. \tag{6.7}$$

defined by $\delta(\phi)(u) = \phi(u)$ if $u \in \widetilde{\bigwedge}^1 V$ and inductively

$$\delta(\phi)(u \wedge v) = \delta(\phi)(u) \wedge v + u \wedge \delta(\phi)(v), \qquad \forall u \in \mathrm{gl}(\widetilde{\bigwedge}^1 V),\ v \in \mathrm{gl}(\widetilde{\bigwedge}^{k-1} V).$$

Proposition 6.23. *The exterior $\mathcal{A}$–semialgebra $\widetilde{\bigwedge}_{\mathcal{A}} V$ is a module over the Lie semialgebra $\mathrm{gl}(\bigwedge^1 V)$ (we say in short that it is a $\mathrm{gl}(\bigwedge^1 V)$-module), in the sense that*

$$\phi(\psi(u))(-)\psi(\phi(u)) = [\phi, \psi](u). \tag{6.8}$$

Proof. For all $v \in V$,

$$\big(\phi(\psi(u))(-)\psi(\phi(u))\big) \wedge v = \phi(\psi(u)) \wedge v + v \wedge \psi(\phi(u)) = [\phi, \psi](u) \wedge v.$$

By induction,

$$\delta([\phi, \psi]) = [\delta(\phi), \delta(\psi)]$$

where the Lie bracket of $\mathrm{End}_{\mathcal{A}}(\widetilde{\bigwedge} V)$ is defined analogously. ∎

In the sequel we consider the action of $V \otimes V^*$ on $\widetilde{\bigwedge} V$ given by:

$$(u \otimes \partial)(\mathbf{v}) = u \wedge (\partial \lrcorner \mathbf{v}).$$

which corresponds to the case $B(x^i, x^j) = \delta_{ij}$.

Proposition 6.24. *The following commutation rules hold for any $u \in \widetilde{\bigwedge} V$:*

$$x^i \wedge x^j \wedge u \,+\, x^j \wedge x^i \wedge u \succeq 0, \tag{6.9}$$

$$\partial_i \lrcorner (\partial_j \lrcorner u) +\, \partial_j \lrcorner (\partial_i \lrcorner u) \succeq 0, \tag{6.10}$$

$$\partial_i \lrcorner (x^j \wedge u) + x^i \wedge (\partial_j \lrcorner u)\ \succeq \delta_{ij} \mathcal{B}(x^i, x^j) u. \tag{6.11}$$

Proof. For $u \in \widetilde{\bigwedge} V$, the first rule

$$x^i \wedge x^j \wedge u + x^j \wedge x^i \wedge u \succeq 0$$

is immediate. As for (6.10),

$$\partial_i \lrcorner (\partial_j \lrcorner u) + \partial_j \lrcorner (\partial_i \lrcorner u) = (\partial_i \wedge \partial_j + \partial_j \wedge \partial^i) \lrcorner u \succeq 0.$$

Let us finally check (6.11). For $u \in \widetilde{\bigwedge}^1 V$ we claim that

$$\partial^j \lrcorner (x^i \wedge u) + x^i \wedge (\partial^j \lrcorner u) \succeq \delta_{ij}\mathcal{B}(x^i, x^j)u. \tag{6.12}$$

Since both sides are elements of $\widetilde{\bigwedge}^1 V$, (6.12) means that

$$\left[\partial^j \lrcorner (x^i \wedge u) + x^i \wedge (\partial^j \lrcorner u)\right] \wedge v \succeq \delta_{ij}\mathcal{B}(x^i, x^j)u \wedge v$$

for all $v \in \widetilde{\bigwedge}^{\geq 1} V$. Without loss of generality we may assume that $v \in V$. Then:

$$\begin{aligned} \partial^j \lrcorner (x^i \wedge u) \wedge v + x^i \wedge (\partial^j \lrcorner u) \wedge v =& \delta_{ij}\mathcal{B}(x^i, x^j)u \wedge v + \partial^j(u)v \wedge x^i + \partial^j(u)x^i \wedge v \\ \succeq& \ \delta_{ij}\mathcal{B}(x^i, x^j)u \wedge v, \end{aligned} \tag{6.13}$$

whence (6.11), since v is arbitrary.

For $u \in \bigwedge^{\geq 2} V$, the proof consists in writing $u = u_1 \wedge v$ for $u_1 \in V$, and is analogous to that for $u \in \bigwedge^1 V$:

$$\begin{aligned} \partial^j \lrcorner (x^i \wedge u) + x^i \wedge \partial^j \lrcorner u &= \delta_{ij}\mathcal{B}(x^i, x^j)u \,(-)\, x^i \wedge \partial^j \lrcorner (u_1 \wedge v) + x^i \wedge \partial^j \lrcorner (u_1 \wedge v) \\ &= \delta_{ij}\mathcal{B}(x^i, x^j)u + \partial^j(u_1)v \wedge x^i + \partial^j(u_1)x^i \wedge v \\ &\succeq \delta_{ij}\mathcal{B}(x^i, x^j)\, u. \end{aligned}$$

7. Schubert Derivations on exterior semialgebras of type 1

Definition 7.1. A **Hasse-Schmidt** (HS) derivation on $(\widetilde{\bigwedge}_{\mathcal{A}} V, (-))$ is a map

$$\mathcal{D}(z) : \widetilde{\bigwedge}_{\mathcal{A}} V \to \widetilde{\bigwedge}_{\mathcal{A}} V[[z]]$$

such that

$$\mathcal{D}(z)(u \wedge v) = \mathcal{D}(z)u \wedge \mathcal{D}(z)v.$$

Definition 7.2. Let $f \in \mathrm{End}_{\mathcal{A}}(\bigwedge^1 V)$. We denote by $(-)f$ the map such that

$$((-)f)(u) = (-)f(u).$$

In other words $((-)f)(u) \wedge v = v \wedge f(u)$. Then $\mathcal{D}_f(z)$ and $\overline{\mathcal{D}}_f(z)$ are mutually inverses, where, according to 3.14

$$(-f)(u) \wedge v = v \wedge f(u).$$

Definition 7.3. Let $\mathcal{D}(z)$ be a HS–derivation on $\widetilde{\bigwedge}_{\mathcal{A}} V$. Its transpose $\mathcal{D}(z)^T$ is defined by

$$\mathcal{D}(z)^T(\partial) \lrcorner u = \partial \lrcorner \mathcal{D}(z)u \qquad \forall (u, \partial) \in \widetilde{\bigwedge} V \times \widetilde{\bigwedge}_{\mathcal{A}} V^*.$$

Proposition 7.4. *If $\mathcal{D}(z)$ is a HS–derivation on $\widetilde{\bigwedge}_{\mathcal{A}} V$, then $\mathcal{D}(z)^T$ is a HS–derivation on $\widetilde{\bigwedge}_{\mathcal{A}} V^*$.*

Proof. $\mathcal{D}(z)^T(\partial\wedge\gamma)\lrcorner(u\wedge v)=(\partial\wedge\gamma)\lrcorner\mathcal{D}(z)(u\wedge v)=(\partial\wedge\gamma)\lrcorner(\mathcal{D}(z)u\wedge\mathcal{D}(z)v)=$ $\partial\lrcorner\mathcal{D}(z)u\wedge\gamma\lrcorner\mathcal{D}(z)v=\mathcal{D}(z)^T(\partial)\lrcorner u\wedge\mathcal{D}(z)^T(\gamma)(v)$. ∎

Proposition 7.5 ([7, Theorem 3.12])**.** *If $f(z)\in\mathrm{End}_{\mathcal{A}}(V)[[z]]$, there is a unique HS-derivation $\mathcal{D}_{f(z)}(z)$ on $\widetilde{\bigwedge}V$ such that $\mathcal{D}_f(z)_{|V}=\sum_{i\geq 0}f^iz^i$.*

Proposition 7.6. *The unique HS-derivations $\mathcal{D}_f(z)$ and $\overline{\mathcal{D}}_f(z)$ such that*

$$\mathcal{D}_f(z)=\sum_{n\geq 0}f^nz^n \qquad \text{and} \qquad \overline{\mathcal{D}}_f(z)=1\,(-)\,f\cdot z$$

are quasi–inverses.

Proof. Let $(u,v_1,v_2)\in\widetilde{\bigwedge}^1V\times\widetilde{\bigwedge}^1V\times\widetilde{\bigwedge}V$. Then

$$\begin{aligned}
(\overline{\mathcal{D}}_f(z)\mathcal{D}_f(z)u)\wedge(v_1\wedge v_2) &= (\mathcal{D}_f(z)u\,(-)\,\mathcal{D}_f(z)f(u))\wedge v_1\wedge v_2\\
&= (\mathcal{D}_f(z)u\wedge v_1+v_1\wedge\mathcal{D}_f(z)f(u))\wedge v_2\\
&= \big(u\wedge v_1+z\mathcal{D}_f(z)f(u)\wedge v_1+v_1\wedge\mathcal{D}_f(z)f(u)\big)\wedge v_2\\
&\succeq u\wedge(v_1\wedge v_2).
\end{aligned}$$

Similarly

$$\begin{aligned}
(\mathcal{D}_f(z)\overline{\mathcal{D}}_f(z)u)\wedge(v_1\wedge v_2) &= [\mathcal{D}_f(z)(u(-)f(u)z)]\wedge v_1\wedge v_2\\
&= [\mathcal{D}_f(z)u(-)\mathcal{D}_f(z)f(u)]\wedge v_1\wedge v_2\\
&= (\mathcal{D}_f(z)u\wedge v_1+v_1\wedge\mathcal{D}_f(z)f(u))\wedge v_2\\
&= (\mathcal{D}_f(z)u\wedge v_1+v_1\wedge\mathcal{D}_f(z)f(u))\wedge v_2\\
&= u\wedge v_1+z\mathcal{D}_f(z)f(u)\wedge v_1+v_1\wedge\mathcal{D}_f(z)f(u))\wedge v_2\\
&\succeq u\wedge(v_1\wedge v_2).
\end{aligned}$$

∎

Definition 7.7. The **Schubert derivation** of $\widetilde{\bigwedge}_{\mathcal{A}}V$ is the unique HS–derivation $\sigma_+(z):=\sum_{i\geq 0}\sigma_iz^i\in\mathrm{End}\,(\widetilde{\bigwedge}_{\mathcal{A}}V)[[z]]$ satisfying

$$\sigma_+(z)x^j=\sum_{i\geq 0}x^{j+i}z^i, \qquad (x^k=0 \text{ if } k\geq n),$$

and

$$\sigma_+(z)(u\wedge v)=\sigma_+(z)u\wedge\sigma_+(z)v\in\widetilde{\bigwedge}_{\mathcal{A}}V[[z]].$$

Proposition 7.8. *For all $i \geq 1$,*

$$\sigma_i(u \wedge v) = \sum_{j=0}^{i} \sigma_j u \wedge \sigma_{i-j} v.$$

Proof.

(7.1)

$$u \wedge v + \sigma_1(u \wedge v)z + \sigma_2(u \wedge v)z^2 + \cdots = c$$
$$(u + \sigma_1 u \cdot z + \sigma_2 u \cdot z^2 + \cdots) \wedge (v + \sigma_1 v \cdot z + \sigma_2 v \cdot z^2 + \cdots)$$

The result follows by matching the coefficients of the powers of z. ∎

In particular $\sigma_1(u \wedge v) = \sigma_1 u \wedge v + u \wedge \sigma_1 v$ shows that σ_1 is an $\mathcal{A}$ -derivation.

Example 7.9.

$$\sigma_3(x^3 \wedge x^1) = x^6 \wedge x^1 + x^5 \wedge x^2 + x^4 \wedge x^3 + x^3 \wedge x^4 \succeq x^6 \wedge x^1 + x^5 \wedge x^2.$$

Definition 7.10. Let z, w be two formal variables. The respective generating series of the bases $\mathbf{x}$ and $\boldsymbol{\partial}$ are

$$\mathbf{x}(z) := \sum_{i \geq 0} x^i z^i \in V[[z]] \qquad \text{and} \qquad \boldsymbol{\partial}(w^{-1}) = \sum_{j \geq 0} \partial_j w^{-j} \in V^*[[w^{-1}]].$$

Remark 7.11. Because of definition 7.7 we have

$$\mathbf{x}(z) = \sigma_+(z)x^0. \tag{7.2}$$

Moreover:

$$\boldsymbol{\partial}(w^{-1}) = \sigma_-(w)^T \partial_0. \tag{7.3}$$

In fact $\sigma_-(z) = \sum_{i \geq 0} \sigma_{-i} w^{-i}$, where $\sigma_{-i} x^j = x^{j-i}$ if $j \geq i$ and 0 otherwise. Now, by definition of transpose:

$$\sigma_{-i}^T \partial_j(x^k) = \partial_j(\sigma_{-i} x^k) = \partial_j(x^{k-i}) = \partial_{j+i}(x^k).$$

Therefore:

$$\sigma_-(z)^T \partial_0 = \sum_{j \geq 0} \sigma_{-j}^T \partial^0 w^{-j} = \sum_{j \geq 0} \partial^j w^{-j}.$$

Definition 7.12. We denote by $\sigma_-(z), \overline{\sigma}_-(z) : \widetilde{\bigwedge V} \to \bigwedge \widetilde{V}[z^{-1}]$ the unique HS derivations on $\widetilde{\bigwedge} V$ such that

$$\sigma_-(z)x^j = \sum_{i=0}^{j} \frac{x^{j-i}}{z^i}$$

and

$$\overline{\sigma}_-(z)x^j = x^j(-)\frac{x^{j-1}}{z}$$

with the convention that $x^i = 0$ if $i < 0$.

Notice that $\overline{\sigma}_-(z)x^0 = \sigma_-(z)x^0 = x^0$, i.e. $\overline{\sigma}_-(z)$ and $\sigma_-(z)$ act as the identity on x^0. By general facts proved in [7] we know that $\sigma_-(z)$ is a quasi-inverse of $\overline{\sigma}_-(z)$, and as defined in this broader context we obtain the other side. Let us check it again in this particular case. By Proposition 7.6, the Schubert derivations $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ are quasi-inverses of each other. We offer here an extra check to focus better the proof of Proposition 7.6.

Proposition 7.13. *The Schubert derivations $\overline{\sigma}_-(z)$ and $\overline{\sigma}_-(z)$ are mutual quasi-inverses, i.e.,*

$$\sigma_-(z)\overline{\sigma}_-(z)[\mathbf{x}]^r_{\boldsymbol{\lambda}} \succeq [\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

and

$$\overline{\sigma}_-(z)\sigma_-(z)[\mathbf{x}]^r_{\boldsymbol{\lambda}} \succeq [\mathbf{x}]^r_{\boldsymbol{\lambda}}.$$

Proof. Let v be any test element of the form $v_1 \wedge v_2$ with $v_1 \in V$ and v_2 arbitrary in $\widetilde{\bigwedge} V$. Then

$$\begin{aligned}
\sigma_-(z)(\overline{\sigma}_+(z)x^i) \wedge v &= \overline{\sigma}_-(z)\left(x^i - \frac{x^{i-1}}{z}\right) \wedge v \\
&= \left(\sigma_-(z)x^i \wedge v_1 + v_1 \wedge \frac{1}{z}\sigma_-(z)x^{i-1}\right) \wedge v_2 \\
&= \left(x^i \wedge v_1 + \frac{1}{z}\sigma_-(z)x^{i-1} \wedge v_1 + \ v_1 \wedge \frac{1}{z}\sigma_-(z)x^{i-1}\right) \wedge v_2 \\
&\succeq x^i \wedge v_1 \wedge v_2 = x^i \wedge v,
\end{aligned}$$

which proves the property for $r = 1$. If $r \geq 2$

$$\sigma_-(z)(\overline{\sigma}_+(z)[\mathbf{x}]^r_{\boldsymbol{\lambda}} = \sigma_-(z)\overline{\sigma}_-(z)x^{r-1+\lambda_1} \wedge \sigma_-(z)\overline{\sigma}_-(z)[\mathbf{x}]^{r-1}_{\boldsymbol{\lambda}^{(1)}} \succeq [\mathbf{x}]^r_{\boldsymbol{\lambda}},$$

where $\boldsymbol{\lambda}^{(1)} := (\lambda_2 \geq \ldots \geq \lambda_r)$. Conversely

$$\begin{aligned}
\overline{\sigma}_-(z)(\sigma_+(z)x^i) \wedge v_1 \wedge v_2 &= (\sigma_+(z)x^i(-)z^{-1}\sigma_+(z)x^{i-1}) \wedge v_1 \wedge v_2 \\
&= (x^i \wedge v_1 + z^{-1}\sigma_-(z)x^{i-1} \wedge v_1 + v_1 \wedge z^{-1}\sigma_+(z)x^{i-1}) \wedge v_2 \\
&\succeq x^i \wedge v_1 \wedge v_2 = x^i \wedge v.
\end{aligned}$$

■

Theorem 7.14. *We have*

$$\sigma_+(z)x^0 \wedge [\mathbf{b}]^r_{\boldsymbol{\lambda}} \preceq z^r\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{x}]^{r+1}_{\boldsymbol{\lambda}} \tag{7.4}$$

for all $\boldsymbol{\lambda} \in \mathcal{P}_r$ such that $\lambda_1 \leq n - r$.

Proof. We argue by induction. We first check the claim for $r = 1$, i.e., by analyzing

$$\sigma_+(z)x^0 \wedge x^{\lambda}, \qquad \lambda < n.$$

We have

$$\begin{aligned}
z\sigma_+(z)\overline{\sigma}_-(z)(x^{\lambda+1}\wedge x^0) &= z\sigma_+(z)(\overline{\sigma}_-(z)x^{\lambda+1}\wedge \overline{\sigma}_-(z)x^0) \\
&= z\sigma_+(z)\left[\left(x^{\lambda+1}(-)\frac{x^\lambda}{z}\wedge x^0\right)\right] \\
&= z\sigma_+(z)\left(x^{\lambda+1}\wedge x^0+\frac{1}{z}x^0\wedge x^\lambda\right) \\
&= z\sigma_+(z)x^{\lambda+1}\wedge \sigma_+(z)x^0+\sigma_+(z)x^0\wedge\sigma_+(z)x^\lambda \\
&= z\sigma_+(z)x^{\lambda+1}\wedge \sigma_+(z)x^0+(\sigma_+(z)x^0\wedge\sigma_+(z)x^{\lambda+1}) \\
&= z\sigma_+(z)x^{\lambda+1}\wedge \sigma_+(z)x^0+\sigma_+(z)x^0\wedge x^\lambda + z\sigma_+(z)x^0\wedge\sigma_+(z)x^{\lambda+1} \\
&\succeq \sigma_+(z)x^0\wedge x^\lambda,
\end{aligned}$$

and the property is verified for $r=1$. Assume now that $r-1\geq 1$. Then:

$$\begin{aligned}
& z^r\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{x}]^{r+1}_{\boldsymbol{\lambda}} \\
= \;& z^{r-1}(\sigma_+(z)\overline{\sigma}_-(z)x^{r+\lambda_1}\wedge\cdots\wedge x^{2+\lambda_{r-1}})\wedge z\sigma_+(z)\overline{\sigma}_-(z)(x^{1+\lambda_r}\wedge x^0) \\
\succeq \;& z^{r-1}\sigma_+(z)\overline{\sigma}_-(z)(x^{r+\lambda_1}\wedge\cdots\wedge x^{2+\lambda_{r-1}})\wedge\sigma_+(z)x^0\wedge x^{\lambda_r} \\
= \;& z^{r-1}\sigma_+(z)\overline{\sigma}_-(z)(x^{r+\lambda_1}\wedge\cdots\; x^{2+\lambda_{r-1}}\wedge x^0)\wedge x^{\lambda_r} \\
\succeq \;& [\sigma_+(z)x^0\wedge x^{r-1+\lambda_1}\wedge\cdots\wedge x^{1+\lambda_{r-1}}]\wedge x^{\lambda_r}=\sigma_+(z)x^0\wedge[\mathbf{x}]^r_{\boldsymbol{\lambda}}.
\end{aligned}$$

■

Definition 7.15. The Lie semialgebra $\mathrm{gl}(\widetilde{\bigwedge}^1_{\mathcal{A}}\mathcal{A}[x])\cong\widetilde{\bigwedge}^1 V\otimes\widetilde{\bigwedge}^1 V^*$ acts on $\widetilde{\bigwedge}V$ as follows:

$$(p\otimes\partial)(u)=p\wedge(\partial\lrcorner u).$$

A standard procedure in representation theory is using generating functions.

Definition 7.16. Let $\mathbf{x}:=\{x^i : i\geq 0\}$ and $\boldsymbol{\partial}:=(\{\partial_j : j\geq 0\}$ be the bases of V and V^*. The generating functions of $\mathbf{x}$ and $\boldsymbol{\partial}$ are by definition:

$$\mathbf{x}(z)=\sum_{i\geq 0}x^iz^i\qquad\text{and}\qquad\boldsymbol{\partial}(w^{-1})=\sum_{i\geq 0}\partial_jw^{-j}.$$

In particular $\mathbf{x}(z)\otimes\boldsymbol{\partial}(w^{-1})$ defines a map $\widetilde{\bigwedge}^k_{\mathcal{A}}V \to \widetilde{\bigwedge}^k_{\mathcal{A}}V[[z,w^{-1}]]$. The image of a basis element is prescribed as follows:

$$(\mathbf{x}(z)\otimes\boldsymbol{\partial}(w^{-1}))[\mathbf{x}]^k_{\boldsymbol{\lambda}} \quad := \quad \mathbf{x}(z)\wedge(\boldsymbol{\partial}(w^{-1})\lrcorner[\mathbf{x}]^k_{\boldsymbol{\lambda}}. \tag{7.5}$$

By definition $\boldsymbol{\partial}(w^{-1}) = \partial^0+\partial^1w^{-1}+\partial^2w^{-2}+\cdots$. So $\boldsymbol{\partial}(w^{-1})\lrcorner x^i = w^{-i}$ for all $0\le i<n$. Now we apply Definition 6.10 of contraction, obtaining

$$\mathbf{x}(z)\wedge(\boldsymbol{\partial}(w^{-1})\lrcorner[\mathbf{x}]^k_{\boldsymbol{\lambda}} = \mathbf{x}(z)\wedge\begin{vmatrix} w^{-k+1-\lambda_1} & \dots & w^{-\lambda_k} \\ \\ x^{k-1+\lambda_1} & \dots & x^{\lambda_k}\end{vmatrix} = \sigma_+(z)x^0\wedge\begin{vmatrix} w^{-k+1-\lambda_1} & \dots & w^{-\lambda_k} \\ \\ x^{k-1+\lambda_1} & \dots & x^{\lambda_k}\end{vmatrix},$$

where in the last equality we used 7.7. ∎

Example 7.17. The image of $[\mathbf{x}]^3_{(3,2,1)}$ through the endomorphism $x^2\otimes\partial_3$ of V is easily computable by hand:

$$\begin{aligned}(x^2\otimes\partial_3)\mathbf{x}^3_{(3,2,1)} = x^2\wedge(\partial_3\lrcorner(x^5\wedge x^3\wedge x^1)) \quad &= \quad x^2\wedge\begin{vmatrix}\partial_3(x^5) & \partial_3(x^3) & \partial_3(x^1) \\ \\ x^5 & x^3 & x^1\end{vmatrix} \\ &= \quad x^2\wedge\begin{vmatrix} 0 & 1 & 0 \\ \\ x^5 & x^3 & x^1\end{vmatrix} = x^2\wedge x^1\wedge x^5.\end{aligned}$$

On the other hand $(x^2\otimes\partial_3)[\mathbf{x}]^3_{(3,2,1)}$ should be the coefficient of z^2w^{-3} of the expansion of

$$\sigma_+(z)x^0\wedge\begin{vmatrix} w^{-5} & w^{-3} & w^{-1} \\ x^5 & x^3 & x^1\end{vmatrix}. \tag{7.6}$$

The coefficient of w^{-3} in expression (7.6)

$$\sigma_+(z)x^0\wedge(x^1\wedge x^5)$$

whose coefficient of z^2 is precisely $x^2\wedge x^1\wedge x^5$, as expected.

Remark 7.18. Let us consider the standard bilinear form $B(x^i,x^j)=\delta_{ij}$. Recall by formula (6.11) that

$$x^i\wedge\partial^j\lrcorner[\mathbf{x}]^r_{\boldsymbol{\lambda}} \succeq \delta_{ij}[\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

which are Clifford semialgebra relations. Passing to the generating series:

$$\sum_{i,j\ge 0} x^iz^i\wedge\partial^jw^{-j}\lrcorner[\mathbf{x}]^r_{\boldsymbol{\lambda}} \succeq \delta_{ij}z^iw^{-j}[\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

from which, remembering Definition 7.16 for $\boldsymbol{\partial}(w^{-1})$ and Definition 7.7 for the Schubert derivation:

$$\mathbf{x}(z)\wedge\boldsymbol{\partial}(w^{-1})\lrcorner[\mathbf{x}]^r_{\boldsymbol{\lambda}} \;+\; \boldsymbol{\partial}(w^{-1})\lrcorner(\mathbf{x}(z)\wedge[\mathbf{x}]^r_{\boldsymbol{\lambda}}) \succeq \sum_{i\ge 0}\frac{z^i}{w^i}[\mathbf{x}]^r_{\boldsymbol{\lambda}} = \left(1-\frac{z}{w}\right)^{-1}[\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

a relation providing the connection of the commutation rules of Schubert derivations to Clifford semi Algebras.

Remark 7.19. If $\mathcal{A}$ contains the positive rational numbers then

$$\sigma_+(z) = \exp\left(\sum_{i\geq 1} \frac{1}{i}\delta(x^i)z^i\right),$$

where left multiplication $x\cdot : V \to V$ is defined by $x^i \mapsto x^{i+1}$ if $i < n-1$ and $x^{n-1} \mapsto 0$, and $\delta : \mathrm{End}_{\mathcal{A}}(V) \to \mathrm{End}_{\mathcal{A}}(\bigwedge V)$ is defined as in Example 6.21.

Theorem 7.20. *We use* (7.3)*. Then*

$$\sigma_-(w)^T\partial_0 \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}} = w^{-r+1}\overline{\sigma}_+(w)(\partial_0 \lrcorner \sigma_-(w)[\mathbf{x}]^r_{\boldsymbol{\lambda}})$$

Proof.

$$\begin{aligned}
\sigma_-(w)^T\partial_0 \lrcorner [\mathbf{b}]^r_{\boldsymbol{\lambda}} &= \boldsymbol{\partial}(w^{-1}) \lrcorner [\mathbf{b}]^r_{\boldsymbol{\lambda}} = \begin{vmatrix} \partial^0(\sigma_-(w)x^{r-1+\lambda_1}) & \cdots & \partial^0(\sigma_-(w)x^{\lambda_r}) \\ \\ x^{r-1+\lambda_1} & \cdots & x^{\lambda_r} \end{vmatrix} \\
&= \begin{vmatrix} \partial_0(\sigma_-(w)x^{r-1+\lambda_1}) & \cdots & \partial^0(\sigma_-(w)x^{\lambda_r}) \\ \\ x^{r-1+\lambda_1} & \cdots & x^{\lambda_r} \end{vmatrix} \\
&= \overline{\sigma}_-(w)\begin{vmatrix} \partial^0(\sigma_-(w)x^{r-1+\lambda_1}) & \cdots & \partial^0(\sigma_-(w)x^{\lambda_r}) \\ \sigma_-(w)x^{r-1+\lambda_1} & \cdots & \sigma_-(w)x^{\lambda_r} \end{vmatrix} = \\
&= \frac{1}{w^{r-1}}\overline{\sigma}_+(w)\overline{\sigma}_-(w)(\partial^0 \lrcorner \sigma_-(w)[\mathbf{b}]^r_{\boldsymbol{\lambda}})
\end{aligned}$$

7.21. The explicit description of the $\widetilde{\bigwedge}_{\mathcal{A}}V$ representation of $\mathrm{gl}(\tilde{T}^1(V))$. Theorem 8.1 below gives a first version of the structure of $\widetilde{\bigwedge}V$ as a representation of $\mathrm{gl}(\widetilde{\bigwedge}^1 V)$. To this purpose we need a few preliminaries.

Lemma 7.22. *For all* $\boldsymbol{\lambda} \in \mathcal{P}_r$,

$$x^0 \wedge [\mathbf{b}]^r_{\boldsymbol{\lambda}+(1^r)} \preceq \frac{1}{w^r}\overline{\sigma}_+(w)\sigma_-(w)[\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0 \tag{7.7}$$

Proof. For $r = 1$ one has

$$x^0 \wedge [\mathbf{x}]^1_{(1+\lambda)} = x^0 \wedge x^{1+\lambda} \preceq x^0 \wedge \overline{\sigma}_-(w)\sigma_-(w)x^{1+\lambda} = x^0 \wedge \overline{\sigma}_-(w)\left(\sum_{j=0}^{1+\lambda} \frac{x^{1+\lambda-j}}{w^j}\right)$$

$$\begin{aligned}
&= x^0 \wedge \sum_{j=0}^{1+\lambda} \frac{1}{w^j} \overline{\sigma}_-(w) x^{1+\lambda-j} = x^0 \wedge \sum_{j=0}^{1+\lambda} \frac{1}{w^j} \left(x^{1+\lambda-j} (-) \frac{x^{\lambda-j}}{w} \right) \\
&= \frac{1}{w} \left(\sum_{j=0}^{\lambda} \frac{1}{w^j} \overline{\sigma}_+(w) x^{\lambda-j} + \frac{x^0}{w^{\lambda+1}} \right) \wedge x^0 \\
&= \frac{1}{w} \left(\overline{\sigma}_+(w) \sigma_-(w) x^\lambda + \frac{x^0}{w^{\lambda+1}} \right) \wedge x^0 = \frac{1}{w} (\overline{\sigma}_+(w) \sigma_-(w) x^\lambda) \wedge x^0
\end{aligned}$$

as desired. For $r > 1$ we argue by induction. Suppose the formula holds for $r - 1 \geq 1$. Then

$$\begin{aligned}
x^0 \wedge [\mathbf{x}]^r_{\boldsymbol{\lambda}+(1^r)} &= x^0 \wedge x^{r+\lambda_1} \wedge x^{r-1+\lambda_2} \wedge \cdots \wedge x^{1+\lambda_r} \\
&\preceq \frac{1}{w} \overline{\sigma}_+(w) \sigma_-(w) x^{r-1+\lambda_1} \wedge x^0 \wedge [\mathbf{x}]^{r-1}_{(r-1+\lambda_2,\ldots,1+\lambda_r)} \\
&= \frac{1}{w^r} \overline{\sigma}_+(w) \sigma_-(w) x^{r-1+\lambda_1} \wedge \overline{\sigma}_+(w) \sigma_-(w) (x^{r-2+\lambda_2} \wedge \cdots \wedge x^{\lambda_r}) \wedge x^0 \\
&= \frac{1}{w^r} \overline{\sigma}_+(w) \sigma_-(w) [\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0.
\end{aligned}$$

■

Lemma 7.23. *The following commutation rule holds:*

$$\overline{\sigma}_-(z) \overline{\sigma}_+(w) [[\mathbf{x}]]^r_{\boldsymbol{\lambda}} \wedge x^0 = \overline{\sigma}_+(w) \overline{\sigma}_-(z) [\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0$$

Proof. We first prove the case $r = 1$. If $\lambda > 0$ one has

$$\begin{aligned}
\overline{\sigma}_-(z) \overline{\sigma}_+(w) x^\lambda = \overline{\sigma}_-(z) (x^\lambda (-) x^{\lambda+1} z) &= \sigma_-(w) \overline{\sigma}_+(z) [\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0 \\
&= \left(1 + \frac{z}{w}\right) x^\lambda (-) \left(x^{\lambda+1} + \frac{1}{w} x^{\lambda-1} \right) \\
&= \overline{\sigma}_+(w) \overline{\sigma}_-(z) x^\lambda
\end{aligned}$$

from which it is obvious that $\overline{\sigma}_+(z) \sigma_-(w) x^\lambda \wedge x^0 = \sigma_-(w) \overline{\sigma}_+(z) x^\lambda \wedge x^0$. It remains to check the commutations of the two oprators against x^0. In this case

$$\overline{\sigma}_+(z) \overline{\sigma}_-(w) x^0 \wedge x^0 = x^0 \wedge x^1 z = \overline{\sigma}_-(w) \overline{\sigma}_+(z) x^0 \wedge x^0$$

as a straightforward check shows. It remains to check the general case for which we argue by induction. Suppose the property holds for $r - 1 \geq 0$. Then

$$\begin{aligned}
&\overline{\sigma}_-(z) \overline{\sigma}_+(w) [\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0 \\
&= \overline{\sigma}_-(z) \overline{\sigma}_+(w) x^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_-(z) \overline{\sigma}_+(w) x^{1+\lambda_{r-1}} \wedge (\overline{\sigma}_-(z) \overline{\sigma}_+(w) x^{\lambda_r} \wedge x^0)
\end{aligned}$$

$$\begin{aligned}
&= \overline{\sigma}_-(z)\overline{\sigma}_+(w)x^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_-(z)\overline{\sigma}_+(w)x^{1+\lambda_{r-1}} \wedge (\overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{\lambda_r} \wedge x^0) \\
&= (-)\overline{\sigma}_-(z)\overline{\sigma}_+(w)x^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_-(z)\overline{\sigma}_+(w)x^{1+\lambda_{r-1}} \wedge x^0) \wedge \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{\lambda_r}
\end{aligned}$$

from which, by induction

$$\begin{aligned}
(-) \quad & (\overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{1+\lambda_{r-1}} \wedge x^0) \wedge \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{\lambda_r} \\
= \quad & \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{1+\lambda_{r-1}} \wedge \overline{\sigma}_+(w)\overline{\sigma}_-(z)x^{\lambda_r} \wedge x^0 \\
= \quad & \overline{\sigma}_+(w)\overline{\sigma}_-(z)[\mathbf{x}]^r_{\boldsymbol{\lambda}} \wedge x^0
\end{aligned}$$

■

8. The Main Theorem

In this section we will present and prove our main theorem after relating it to previous literature in the classical context of $\mathbb{Q}$-algebras. To begin with, a rather easy, though non trivial, observation shows that the polynomial ring $B := \mathbb{Q}[x_1, x_2, \ldots]$ in infinitely many indeterminates $\mathbf{x} = (x_1, x_2, \ldots)$ is a module over the Lie algebra $gl_\infty(\mathbb{Q}) = \bigoplus_{i,j\in\mathbb{Z}} \mathbb{Q}\cdot E_{ij}$, where $E_{ij} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Q}$ is the elementary matrix with all entries zero but 1 in position (i, j). This fact comes from the fact that the ring B is isomorphic to a suitable projective limit, $\varprojlim_r \bigwedge^r \mathbb{Q}[X]$, in the category of graded $\mathbb{Q}$-algebras, called *Fermionic Fock space*. The result then basically follows from the fact that a) the polynomial ring in $r < \infty$ variables is a vector space isomorphic to the r-th exterior power of $\mathbb{Q}[X]$ and b) the latter is a module over $gl(\mathbb{Q}[X]) := \bigoplus_{i,j\in\mathbb{N}} \mathbb{Q} \cdot E_{ij}$.

It is more difficult to give an explicit description of B as a representation of $gl_\infty(\mathbb{Q})$. This was achieved in the 1980's by Date, Jimbo, Kashiwara and Miwa in [6], who computed the action of the generating function $E(z, w) = \sum_{i,j\in\mathbb{Z}} E_{i,j} z^i w^{-j}$ on a polynomial in the variables $x_1, x_2, \ldots$, in such a way that to know the product $E_{ij}p$ amounts to looking at the coefficient of $z^i w^{-j}$ in the expansion of $E(z, w)p$, for all $p \in B$. The description relies on some vertex operators which first arose in the Skyrme model [Sk] of the interaction of a meson–like field. The result by Date, Jimbo, Kashiwara and Miwa was put in a more general framework in the paper [10], where a generating function of the action of the elementary matrices on the ring B_r of polynomials in $r < \infty$ indeterminates has been computed. In our case partial derivatives with respect to the variables $(x_1, x_2, \ldots)$ are no longer available anymore for describing the module structure. This is an issue which has compelled us to find suitable substitutes. These are provided by the essential use of the Schubert derivations as defined in e.g. [9] and in Section 7.7 in the context of exterior semialgebras. The elementary matrix $E_{i,j}$ can be identified, using our notation, with the basis element $x^i \otimes \partial^j$ of $V \otimes V^*$, and then the generating function $E(z,w)$ is nothing but $\mathbf{x}(z) \otimes \boldsymbol{\partial}(w^{-1})$, acting on $\bigwedge^r V$ according to the rule

$$\mathbf{x}(z) \otimes \boldsymbol{\partial}(w^{-1})[\mathbf{x}]^r_{\boldsymbol{\lambda}} = \mathbf{x}(z) \wedge \boldsymbol{\partial}(w^{-1}) \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}}.$$

The last member can be computed as $\sigma_+(z)x^0 \wedge \boldsymbol{\partial}(w^{-1}) \wedge [\mathbf{x}]^r_{\boldsymbol{\lambda}}$, and this was done in [10] (with substantial improvement in the preprint [3]), where the generating function of the Clifford algebra basis element $x^i \otimes \partial^j$ against $[\mathbf{x}]^r_{\boldsymbol{\lambda}}$, $x^i \wedge (\partial^j \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}})$ was computed. The resulting expression explicitly involves Schubert derivations which, when r goes to ∞, take the shape of the vertex operators occurring in the DJKM representation of $gl_\infty(\mathbb{Q})$, as shown e.g. in [9]. Our main Theorem 8.1 shows that the same results hold in the semialgebra context and then it is a substantial generalization of [10], which motivates further investigation to check if even the more general pictures as in [4, 5] can be reproduced in the more general systemic framework.

Therefore our main result is the following transparent tropical version of [10, Theorem 6.4] for the Grassmann semialgebra, describing precisely the multiplication of an elementary matrix by a basis element of $\widetilde{\bigwedge}_{\mathcal{A}} V$.

Theorem 8.1. *For all $r \geq 1$ the following formula holds:*

(8.1)

$$\begin{aligned}
\mathbf{x}(z) \wedge \boldsymbol{\partial}(w^{-1}) \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}} &= \sigma_+(z)x^0 \wedge (\sigma_-(w)^T \partial^0 \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}}) \\
&\preceq z^{r-1}w^{r-1} \begin{vmatrix} w^{-r+1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{r+\lambda_1} & \cdots & \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{1+\lambda_r} & \sigma_+(z)x^0 \end{vmatrix} \\
&= \frac{z^{r-1}}{w^{r-1}}\sigma_+(z) \begin{vmatrix} w^{-r+1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{r+\lambda_1} & \cdots & \overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{1+\lambda_r} & x^0 \end{vmatrix}.
\end{aligned}$$

The formula means that the image of the element $[\mathbf{x}]^r_{\boldsymbol{\lambda}}$ through $x^i \otimes \partial^j$ is surpassed by the coefficient of $z^i w^{-j}$ in expression (8.1).

Proof. We have

$$\mathbf{x}(z) \wedge \boldsymbol{\partial}(w^{-1}) \lrcorner [\mathbf{x}]^r_{\boldsymbol{\lambda}}$$

$$
\begin{aligned}
&= \quad \sigma_+(z)x^0 \wedge \begin{vmatrix} w^{-r+1-\lambda_1} & \cdots & w^{-\lambda_r} \\ x^{r-1+\lambda_1} & \cdots & x^{\lambda_r} \end{vmatrix} && \text{(definition of } \sigma_+(z) \text{ and Proposition 7.6)} \\
&\preceq \quad z^{r-1}\sigma_+(z)\overline{\sigma}_-(z) \begin{vmatrix} w^{-r+1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ x^{r+\lambda_1} & \cdots & x^{1+\lambda_r} & x^0 \end{vmatrix} && \text{(Theorem 7.14)} \\
&\preceq \quad z^{r-1}\sigma_+(z)\overline{\sigma}_-(z) \begin{vmatrix} w^{-r-1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \overline{\sigma}_-(w)\sigma_-(w)x^{r+\lambda_1} & \cdots & \overline{\sigma}_-(w)\sigma_-(w)x^{1+\lambda_r} & x^0 \end{vmatrix} && \text{(Proposition 7.13)} \\
&\preceq \quad \frac{z^{r-1}}{w^{-r+1}}\sigma_+(z) \begin{vmatrix} w^{-r-1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w)x^{r+\lambda_1} & \cdots & \overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w)x^{1+\lambda_r} & x^0 \end{vmatrix} && \text{(Theorem 7.20)} \\
&= \quad \frac{z^{r-1}}{w^{-r+1}}\sigma_+(z) \begin{vmatrix} w^{-r-1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{r+\lambda_1} & \cdots & \overline{\sigma}_+(w)\overline{\sigma}_-(w)\sigma_-(w)x^{1+\lambda_r} & x^0 \end{vmatrix} && \text{(Lemma 7.22)}
\end{aligned}
$$

or, using the fact that $\sigma_+(z)$ is a HS derivation

$$
\frac{z^{r-1}}{w^{-r+1}} \begin{vmatrix} w^{-r-1-\lambda_1} & \cdots & w^{-\lambda_r} & 0 \\ \sigma + (z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)x^{r+\lambda_1} & \cdots & \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(w)\sigma_-(w)x^{1+\lambda_r} & \sigma_+(z)x^0 \end{vmatrix}
$$

■

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