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Research article

Solutions to a cubic Schrödinger system with mixed attractive and repulsive forces in a critical regime†

Simone Dovetta and Angela Pistoia*

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via Scarpa 16, 00161 Roma, Italy

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* **Correspondence:** Email: angela.pistoia@uniroma1.it.

Abstract: We study the existence of solutions to the cubic Schrödinger system

$$-\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^2 u_i + \lambda_i u_i \text{ in } \Omega,\ u_i = 0 \text{ on } \partial\Omega,\ i = 1, \dots, m,$$

when Ω is a bounded domain in $\mathbb{R}^4$, λ_i are positive small numbers, β_{ij} are real numbers so that $\beta_{ii} > 0$ and $\beta_{ij} = \beta_{ji}$, $i \neq j$. We assemble the components u_i in groups so that all the interaction forces β_{ij} among components of the same group are attractive, i.e., $\beta_{ij} > 0$, while forces among components of different groups are repulsive or weakly attractive, i.e., $\beta_{ij} < \overline{\beta}$ for some $\overline{\beta}$ small. We find solutions such that each component within a given group blows-up around the same point and the different groups blow-up around different points, as all the parameters λ_i's approach zero.

Keywords: cubic Schrödinger system; attractive and repulsive forces; blow–up phenomenon; Ljapunov–Schmidt reduction

1. Introduction

The study of solitary waves $\Phi_i = \mathtt{exp}(\iota\omega_i t)u_i$ of the nonlinear Schrödinger system

$$-\iota\partial_t\Phi_i = \Delta\Phi_i + \Phi_i \sum_{j=1}^{m} \beta_{ij}|\Phi_j|^2,\ \Phi_i : \Omega \to \mathbb{C},\ i = 1, \dots, m,$$

where Ω is a smooth domain in $\mathbb{R}^N$ naturally leads to study the elliptic system

$$-\Delta u_i + \omega_i u_i = \sum_{j=1}^{m} \beta_{ij} u_j^2 u_i,\ u_i : \Omega \to \mathbb{R},\ i = 1, \ldots, m. \tag{1.1}$$

Here ω_i and $\beta_{ij} = \beta_{ji}$ are real numbers and $\beta_{ii} > 0$. This type of systems arises in many physical models such as incoherent wave packets in Kerr medium in nonlinear optics (see [1]) and in Bose–Einstein condensates for multi–species condensates (see [25]). The coefficient β_{ij} represents the interaction force between components u_i and u_j. The sign of β_{ij} determines whether the interactions between components are *repulsive* (or *competitive*), i.e., $\beta_{ij} < 0$, or *attractive* (or *cooperative*), i.e., $\beta_{ij} > 0$. In particular, one usually assumes $\beta_{ii} > 0$. We observe that system (1.1) has always the trivial solution, namely when all the components vanish. If one or more components are identically zero, then system (1.1) reduces to a system with a smaller number of components. Therefore, we are interested in finding solutions whose all components are not trivial. These are called *fully nontrivial* solutions.

In low dimensions $1 \le N \le 4$, problem (1.1) has a variational structure: solutions to (1.1) are critical points of the energy $J : H \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2} \sum_{i=1}^{m} \int_\Omega \left(|\nabla u_i|^2 + \omega_i u_i^2 \right) - \frac{1}{4} \sum_{i,j=1}^{m} \beta_{ij} \int_\Omega u_i^2 u_j^2,$$

where the space H is either $H^1(\Omega)$ or $H_0^1(\Omega)$, depending on the boundary conditions associated to u_i in (1.1) in the case of not empty $\partial\Omega$. Therefore, the existence and multiplicity of solutions can be obtained using classical methods in critical point theory. However, there is an important difference between the dimensions $1 \le N \le 3$ and the dimension $N = 4$. Actually, in dimension $N = 4$ the nonlinear part of J has a *critical* growth and the lack of compactness of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ makes difficult the search for critical points. On the other hand, in dimensions $1 \le N \le 3$ the problem has a *subcritical* regime and the variational tools can be successfully applied to get a wide number of results. We refer to the introduction of the most recent paper [6] for an overview on the topic and for a complete list of references. Up to our knowledge, the higher dimensional case $N \ge 5$ is completely open, because the problem does not have a variational structure and new ideas are needed.

In this paper, we will focus on problem (1.1) when Ω is a smooth bounded domain in $\mathbb{R}^4$ with Dirichlet boundary condition. We shall rewrite (1.1) in the form

$$-\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^2 u_i + \lambda_i u_i \text{ in } \Omega,\ u_i = 0 \text{ on } \partial\Omega,\ i = 1, \ldots, m, \tag{1.2}$$

where λ_i are real numbers, as this way it can be seen as a generalization of the celebrated *Brezis–Nirenberg* problem [5]

$$-\Delta u = u^3 + \lambda u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

It is worthwhile to remind that the existence of solutions to (1.3) strongly depends on the geometry of Ω. In particular, if Ω is a starshaped domain, then Pohozaev's identity ensures that (1.3) has no solution when $\lambda \le 0$. On the other hand, Brezis and Nirenberg [5] proved that (1.3) has a positive solution if and only if $\lambda \in (0, \Lambda_1(\Omega))$ where Λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet condition on $\partial\Omega$. These solutions are often referred to as *least energy* solutions, as they can be obtained also by minimizing the functional

$$\frac{1}{2}\int_{\Omega}\left(|\nabla u|^2-\lambda|u|^2\right)-\frac{1}{4}\int_{\Omega}|u|^4$$

restricted to the associated Nehari manifold. Later, Han [12] and Rey [18] studied the asymptotic behaviour of this solution as $\lambda\to 0$ and proved that it *blows–up* at a point $\xi_0\in\Omega$ which is a critical point of the Robin's function, whereas far away from ξ_0 his shape resembles the *bubble*

$$U_{\delta,\xi}(x):=\alpha\frac{\delta}{\delta^2+|x-\xi|^2},\ \alpha=2\sqrt{2}. \tag{1.4}$$

Recall that it is well known (see [2,23]) that $\{U_{\delta,\xi}:\ \delta>0,\ \xi\in\mathbb{R}^4\}$ is the set of all the positive solutions to the critical problem

$$-\Delta U=U^3 \text{ in } \mathbb{R}^4. \tag{1.5}$$

Let us also remind that the Robin's function is defined by $\mathtt{r}(x):=H(x,x)$, $x\in\Omega$, where $H(x,y)$ is the regular part of the Green function of $-\Delta$ in Ω with Dirichlet boundary condition.

Successively, relying on the profile of the bubble as a first order approximation, the Ljapunov–Schmidt procedure has been fruitfully used to build both positive and sign–changing solutions to (1.3) blowing–up at different points in Ω as the parameter λ approaches zero (see for example Rey [18] and Musso and Pistoia [14]).

As far as we know, few results are available about existence and multiplicity of solutions to the critical system (1.2). The first result is due to Chen and Zou [9], who considered (1.2) with 2 components only

$$\begin{cases}-\Delta u_1=\mu_1u_1^3+\beta u_1u_2^2+\lambda_1u_1 & \text{in } \Omega\\ -\Delta u_2=\mu_2u_2^3+\beta u_1^2u_2+\lambda_2u_2 & \text{in } \Omega\\ u_1=u_2=0 \quad \text{on } \partial\Omega.\end{cases} \tag{1.6}$$

When $0<\lambda_1,\lambda_2<\Lambda_1(\Omega)$, they proved the existence of a least energy positive solution in the competitive case (i.e., $\beta<0$) and in the cooperative case (i.e., $\beta>0$) if $\beta\in(0,\underline{\beta}]\cup[\overline{\beta},+\infty)$, for some $\overline{\beta}\ge\max\{\mu_1,\mu_2\}>\min\{\mu_1,\mu_2\}\ge\underline{\beta}>0$. In the cooperative case, when $\lambda_1=\lambda_2$ the least energy solution is *synchronized*, i.e., $(u_1,u_2)=(c_1u,c_2u)$ where u is the least energy positive solution of the Eq (1.3) and (c_1,c_2) is a positive solution to the algebraic system

$$\begin{cases}1=\mu_1c_1^2+\beta c_2^2\\ 1=\mu_2c_2^2+\beta c_1^2.\end{cases}$$

In the competitive case, the authors studied also the limit profile of the components of the least energy solution and proved that the following alternative occurs: either one of the components vanishes and the other one converges to a least energy positive solution of the Eq (1.3), or both components survive and their limits separate in different regions of the domain Ω, i.e., a *phase separation* phenomenon takes place. In the subcritical regime such a phenomenon has been studied by Noris, Tavares, Terracini and Verzini [15].

Afterwards, Chen and Lin [8] studied the asymptotic behavior of the least energy solution of (1.6) in the cooperative case as $\max\{\lambda_1,\lambda_2\}\to 0$ and found that both components blow–up at the same critical point of the Robin's function, in the same spirit of the result by Han and Rey for the single equation (1.3).

The existence of blowing–up solutions for system (1.2) with an arbitrary number of components has been studied by Pistoia and Tavares [17]. Using a Ljapunov–Schmidt procedure, they built solutions to (1.2) whose m components blow–up at m different non–degenerate critical points of the Robin's function as $\lambda^* := \max\{\lambda_1, \dots, \lambda_m\} \to 0$, provided the interaction forces are either negative or not too large, namely $\beta^* := \max_{ij} \beta_{ij} \leq \overline{\beta}$ for some $\overline{\beta} > 0$. For example, their result holds in dumbbell shaped domains which are obtained by connecting m mutually disjoint connected domains $D_1, \dots, D_m$ by thin handles. In this case the Robin's function has m distinct local minimum points which are non–degenerate for a generic choice of the domain as proved by Micheletti and Pistoia [13]. Moreover, if, as $\lambda^* \to 0$, we let $\beta^* := \max_{i,j} \beta_{ij}$ approach $-\infty$ with a *sufficiently low velocity* (depending on λ^*), then it is still possible to show that all the components blow–up at different points and a segregation phenomen occurs.

To conclude the state of the art, we would like to mention some recent results obtained by exploiting a variational point of view. Guo, Luo and Zou [11] proved the existence of a least energy solution to (1.2) in the purely cooperative regime (i.e., $\min_{i \neq j} \beta_{ij} \geq 0$) when $\lambda_1 = \cdots = \lambda_m$ and showed that such a solution is synchronized under some additional technical conditions on the coupling coefficients. Tavares and You [24] generalized the previous result to a mixed competitive/weakly cooperative regime (i.e., $\max_{i \neq j} \beta_{ij}$ not too large). Clapp and Szulkin [10] found a least energy solution in the purely competitive regime (i.e., $\max_{i \neq j} \beta_{ij} < 0$), which is not synchronized when the coupling terms β_{ij} diverge to $-\infty$.

Now, let us go back to the result obtained by Pistoia and Tavares [17] concerning the existence of solutions to (1.2) with all the components blowing–up around different points in Ω when all the mixed forces are repulsive or weakly attractive. It is natural to ask what happens for more general mixed repulsive and attractive forces. Our idea is to assemble the components u_i in groups so that all the interaction forces β_{ij} among components of the same group are attractive, while forces among components of different groups are repulsive or weakly attractive. In this setting, we address the following question:

(Q) *is it possible to find solutions such that each component within a given group concentrates around the same point and different groups concentrate around different points?*

Taking the notation introduced in [16], given $1 \leq q \leq m$, let us introduce a q−decomposition of m, namely a vector $(l_0, \dots, l_q) \in \mathbb{N}^{q+1}$ such that

$$0 = l_0 < l_1 < \cdots < l_{q-1} < l_q = m.$$

Given a q–decomposition of m, we set, for $h = 1, \dots, q$,

$$I_h := \{i \in \{1, \dots, m\} : l_{h-1} < i \leq l_h\}.$$

In this way, we have partitioned the set $\{1, \dots, m\}$ into q groups $I_1, \dots, I_q$, and we can consequently split the components of our system into q groups $\{u_i : i \in I_h\}$. Notice that if $l_h - l_{h-1} = 1$, then I_h reduces to the singleton $\{i\}$, for some $i \in \{1, \dots, m\}$. We will assume that for every $h = 1, \dots, q$.

(A1) the algebraic system
$$1 = \sum_{j\in I_h} \beta_{ij} c_j^2, \qquad i \in I_h, \tag{1.7}$$
has a solution $\mathfrak{c}_h = (c_i)_{i\in I_h}$ with $c_i > 0$ for every $i \in I_h$;

(A2) the matrix $\left(\beta_{ij}\right)_{i,j\in I_h}$ is invertible and all the entries are positive.

We observe that (A1) is satisfied for instance if for every $i \neq j$ (see [3])
$$\beta_{ij} =: \beta > \max_{i\in I_h} \beta_{ii} \text{ for every } i \in I_h.$$

Remark 1.1. Assumptions ($A1$) and ($A2$) are necessary to build the solutions to (1.2) using the classical Ljapunov-Schmidt procedure, namely ($A1$) allows to find a good ansatz which is non-degenerate because of ($A2$). Let us be more precise.

From a PDE point of view, assumption ($A1$) is equivalent to require that the nonlinear PDE sub-system
$$-\Delta W_i = W_i \sum_{j\in I_h} \beta_{ij} W_j^2 \quad \text{in } \mathbb{R}^n, \qquad i \in I_h, \tag{1.8}$$
has a *synchronized* solution $W_i = c_i U$, $i \in I_h$, where the positive function
$$U(x) := \alpha \frac{1}{1+|x|^2}, \ \alpha = 2\sqrt{2},$$
solves the critical Eq (1.5). The first key point in the reduction procedure is done: the main order term of the components u_i with $i \in I_h$ is nothing but the syncronized solution of the sub-system (1.8).

Assumption (A2) ensures that such a synchronized solution of (1.8) is *non–degenerate* (see [16, Proposition 1.4]), in the sense that the linear system (obtained by linearizing system (1.8) around the synchronized solution)
$$-\Delta v_i = U^2 \left[\left(3\beta_{ii} c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij} c_j^2\right) v_i + 2 \sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij} c_i c_j v_j\right] \text{ in } \mathbb{R}^n, \quad i \in I_h, \tag{1.9}$$
has a 5–dimensional set of solutions
$$(v_1, \ldots, v_{|I_h|}) \in \mathtt{span}\left\{\mathfrak{e}_h \psi^\ell \mid \ell = 0, 1, \ldots, 4\right\} \subset \left(H_0^1(\Omega)\right)^{|I_h|} \tag{1.10}$$
where $\mathfrak{e}_h \in \mathbb{R}^{|I_h|}$ is a suitable vector (see [16, Lemma 6.1]) and the functions
$$\psi^0(y) = \frac{1-|y|^2}{(1+|y|^2)^2} \quad \text{and} \quad \psi^\ell(y) = \frac{y_\ell}{(1+|y|^2)^2}, \ \ell = 1, \ldots, 4,$$
solve the linear equation
$$-\Delta\psi = 3U^2\psi \quad \text{in } \mathbb{R}^4.$$
The non-degeneracy of the building block is ensured and the second key point in the reduction procedure is also done.

We are now in position to state our main result.

Theorem 1.2. *Assume (A1) and (A2). Assume furthermore that the Robin's function has q distint non–degenerate critical points* $\xi_1^0, \dots, \xi_q^0$. *There exist* $\overline{\beta} > 0$ *and* $\lambda_0 > 0$ *such that, if* $\beta^* := \max_{\substack{(i,j)\in I_h\times I_k\\ h\neq k}} \beta_{ij} < \overline{\beta}$ *then, for every* $(\lambda_i)_{i=1}^m$ *with* $\lambda_i \in (0, \lambda_0)$, $i = 1, \dots, m$, *there exists a solution* $(u_1, \dots, u_m)$ *to* (1.2) *such that, for every* $h = 1, \dots, q$, *each group of components* $\{u_i \ : \ i \in I_h\}$ *blows–up at* ξ_h^0 *as* $\lambda^* := \max_{i=1,\dots,m} \lambda_i \to 0$.

Moreover, if, as $\lambda^* \to 0$, β^* *approaches* $-\infty$ *slowly enough (depending on* λ^**), i.e.,* $|\beta^*| = O\left(e^{\frac{d^*}{\lambda^*}}\right)$ *for some* d^* *sufficiently small, then all the components belonging to different groups blow–up at different points and segregate, while the components belonging to the same group blow–up at the same point and aggregate.*

Remark 1.3. We remind that in dumbbell shaped domains which are obtained by connecting q mutually disjoint connected domains by thin handles, the Robin's function has q distinct critical points and in a domain with holes the Robin's function has at least 2 critical points (see Pistoia and Tavares [17, Examples 1.5 and 1.6]). All these critical points are non–degenerate for a generic choice of the domain as proved by Micheletti and Pistoia [13].

Remark 1.4. Theorem 1.2 deals with systems with mixed aggregating and segregating forces (i.e., some β_{ij}'s are positive, and some others are negative). This is particularly interesting since there are few results about systems with mixed terms. The subcritical regime has been recently investigated by Byeon, Kwon and Seok [6], Byeon, Sato and Wang [7], Sato and Wang [19, 20], Soave and Tavares [22], Soave [21] and Wei and Wu [26]. As far as we know, there are only a couple of results concerning the critical regime. The first one has been obtained by Pistoia and Soave in [16], where the authors studied system (1.2) when all the λ_i's are zero and the domain has some holes whose size approaches zero. The second one is due to Tavares and You, who in [24] found a least energy solution to system (1.2) provided all the parameters λ_i are equal. It would be interesting to compare this least energy solution with the blowing-up solutions found in the present paper.

Remark 1.5. We strongly believe that the solutions found in Theorem 1.2 are positive, because they are constructed as the superposition of positive function and small perturbation term. This is true for sure if the attractive forces β_{ij} are small, as proved in [17]. In the general case, the proof does not work and some refined L^∞−estimates of the small terms are needed. We will not afford this issue in the present paper, because the study of the invertibility of the linear operator naturally associated to the problem (see Proposition 3.1) should be performed in spaces equipped with different norms (i.e., L^∞–weighted norms) that may deserve further investigations.

The proof of Theorem 1.2 relies on the well known Ljapunov–Schmidt reduction. The main steps are described in Section 3, where the details of the proof are omitted whenever it can be obtained, up to minor modifications, by combining the arguments in Pistoia and Tavares [17] and in Pistoia and Soave [16]. Here we limit ourselves to give a detailed proof of the first step of the scheme, as it suggests how to adapt the ideas of [16, 17] to the present setting. The technical details of this part are developed in the Appendix. Before getting to this, in Section 2 we recall some well known results that are needed in the following.

2. Preliminaries

We denote the standard inner product and norm in $H_0^1(\Omega)$ by

$$\langle u, v\rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v, \quad \|u\|_{H_0^1(\Omega)} := \left(\langle u, u\rangle_{H_0^1(\Omega)}\right)^{\frac{1}{2}},$$

and the L^q-norm ($q \geq 1$) by $|\cdot|_{L^q(\Omega)}$. Whenever the domain of integration Ω is out of question, we also make use of the shorthand notation $\|u\|$ for $\|u\|_{H_0^1(\Omega)}$ and $|u|_q$ for $|u|_{L^q(\Omega)}$.

Let $i : H_0^1(\Omega) \to L^4(\Omega)$ be the canonical Sobolev embedding. We consider the adjoint operator $(-\Delta)^{-1} : L^{\frac{4}{3}}(\Omega) \to H_0^1(\Omega)$ characterized by

$$(-\Delta)^{-1}(u) = v \quad \Longleftrightarrow \quad \begin{cases} -\Delta v = u & \text{in } \Omega \\ v \in H_0^1(\Omega) \end{cases}$$

It is well known that $(-\Delta)^{-1}$ is a continuous operator, and relying on it we can rewrite (1.2) as

$$u_i = (-\Delta)^{-1}\left(\sum_{j=1}^m \beta_{ij} u_j^2 u_i + \lambda_i u_i\right), \qquad i = 1, \ldots, m. \tag{2.1}$$

From now on, we will focus on problem (2.1).

We are going to build a solution $\mathbf{u} = (u_1, \ldots, u_m)$ to (2.1), whose main term, as the parameters λ_i approach zero, is defined in terms of the bubbles $U_{\delta,\xi}$ given in (1.4). More precisely, let us consider the projection $PU_{\delta,\xi}$ of $U_{\delta,\xi}$ into $H_0^1(\Omega)$, i.e., the unique solution to

$$-\Delta(PU_{\delta,\xi}) = -\Delta U_{\delta,\xi} = U_{\delta,\xi}^3 \text{ in } \Omega, \; PU_{\delta,\xi} = 0 \text{ on } \partial\Omega.$$

We shall use many times the fact that $0 \leq PU_{\delta,\xi} \leq U_{\delta,\xi}$, which is a simple consequence of the maximum principle. Moreover it is well known that

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \alpha\delta H(x, \xi) + O\left(\delta^3\right).$$

Here $G(x, y)$ is the Green function of $-\Delta$ with Dirichlet boundary condition in Ω and $H(x, y)$ is its regular part.

Now, we search for a solution $\mathbf{u} := (u_1, \ldots, u_m)$ to (2.1) as

$$\mathbf{u} = \mathbf{W} + \boldsymbol{\phi}, \text{ where } \mathbf{W} := \left(\mathfrak{c}_1 PU_{\delta_1,\xi_1}, \ldots, \mathfrak{c}_q PU_{\delta_q,\xi_q}\right) \in (H_0^1(\Omega))^{|I_1|} \times \cdots \times (H_0^1(\Omega))^{|I_q|}, \tag{2.2}$$

where each vector $\mathfrak{c}_h \in \mathbb{R}^{|I_h|}$ is defined in (1.7), the concentration parameters $\delta_h = e^{-\frac{d_h}{\lambda_h^*}}$ with $\lambda_h^* := \max_{i \in I_h} \lambda_i$, and the concentration points $\xi_h \in \Omega$ are such that $(\mathbf{d}, \boldsymbol{\xi}) = (d_1, \ldots, d_q, \xi_1, \ldots, \xi_q) \in X_\eta$, with

$$X_\eta := \left\{(\mathbf{d}, \boldsymbol{\xi}) \in \mathbb{R}^q \times \Omega^q : \; \eta < d_h < \eta^{-1}, \; \text{dist}(\xi_h, \partial\Omega) \geq \eta, \; |\xi_h - \xi_k| \geq \eta \; \text{ if } h \neq k\right\}, \tag{2.3}$$

for some $\eta \in (0, 1)$. Recall that $|I_1| + \cdots + |I_q| = m$. The higher order term $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m) \in \left(H_0^1(\Omega)\right)^m$ belongs to the space $\boldsymbol{K}^\perp$ whose definition involves the solutions of the linear equation

$$-\Delta\psi = 3U_{\delta,\xi}^2 \psi \quad \text{in } \mathbb{R}^4, \quad \psi \in \mathcal{D}^{1,2}(\mathbb{R}^4). \tag{2.4}$$

More precisely, we know that the set of solutions to (2.4) is a 5−dimensional space, which is generated by (see [4])

$$\psi^0_{\delta,\xi} := \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha \frac{|x-\xi|^2-\delta^2}{(\delta^2+|x-\xi|^2)^2}$$

$$\psi^\ell_{\delta,\xi} := \frac{\partial U_{\delta,\xi}}{\partial \xi_\ell} = 2\alpha_N \delta \frac{x_\ell-\xi_\ell}{(\delta^2+|x-\xi|^2)^2}, \quad \ell = 1,\dots,4.$$

It is necessary to introduce the projections $P\psi^\ell_{\delta,\xi}$ of $\psi^\ell_{\delta,\xi}$ ($\ell = 0,\dots,N$) into $H^1_0(\Omega)$, i.e.,

$$-\Delta(P\psi^\ell_{\delta,\xi}) = -\Delta\psi^\ell_{\delta,\xi} = 3U^2_{\delta,\xi}\psi^\ell_{\delta,\xi} \text{ in } \Omega, \quad P\psi^\ell_{\delta,\xi} = 0 \text{ on } \partial\Omega, \tag{2.5}$$

and it is useful to recall that

$$P\psi^0_{\delta,\xi}(x) = \psi^0_{\delta,\xi}(x) - \alpha H(x,\xi) + O\left(\delta^2\right),$$

$$P\psi^\ell_{\delta,\xi}(x) = \psi^\ell_{\delta,\xi}(x) - \alpha\delta\partial_\ell H(x,\xi) + O\left(\delta^3\right), \quad \ell = 1,\dots,4.$$

Now, we define the space $\boldsymbol{K}^\perp$ as

$$\mathbf{K} := K_1 \times \cdots \times K_q \text{ and } \boldsymbol{K}^\perp = K_1^\perp \times \cdots \times K_q^\perp, \tag{2.6}$$

where (see (1.10))

$$K_h := \operatorname{span}\left\{\mathfrak{e}_h P\psi^\ell_{\delta_h,\xi_h} : \ \ell = 0,\dots,4\right\} \subset \left(H^1_0(\Omega)\right)^{|I_h|}, \quad h = 1,\dots,q. \tag{2.7}$$

The unknowns in (2.2) are the rates of the concentration parameters d_h's, the concentration points ξ_h's and the remainder terms ϕ_i's. To identify them, we will use a Ljapunov–Schmidt reduction method. First, we rewrite system (2.1) as a couple of systems. Let us introduce the orthogonal projections

$$\boldsymbol{\Pi} := (\Pi_1,\dots,\Pi_q) : (H^1_0(\Omega))^{|I_1|} \times \cdots \times H^1_0(\Omega))^{|I_q|} \to \mathbf{K}$$

and

$$\boldsymbol{\Pi}^\perp := (\Pi_1^\perp,\dots,\Pi_q^\perp) : (H^1_0(\Omega))^{|I_1|} \times \cdots \times H^1_0(\Omega))^{|I_q|} \to \mathbf{K}^\perp,$$

where $\Pi_h : \left(H^1_0(\Omega)\right)^{|I_h|} \to K_h$ and $\Pi_h^\perp : \left(H^1_0(\Omega)\right)^{|I_h|} \to K_h^\perp$ denote the orthogonal projections, for every $h = 1,\dots,q$.

It is not difficult to check that (2.1) is equivalent to the couple of systems

$$\boldsymbol{\Pi}^\perp \left[\boldsymbol{\mathcal{L}}(\boldsymbol{\phi}) + \boldsymbol{\mathcal{N}}(\boldsymbol{\phi}) + \boldsymbol{\mathcal{E}}\right] = 0 \tag{2.8}$$

and

$$\boldsymbol{\Pi} \left[\boldsymbol{\mathcal{L}}(\boldsymbol{\phi}) + \boldsymbol{\mathcal{N}}(\boldsymbol{\phi}) + \boldsymbol{\mathcal{E}}\right] = 0, \tag{2.9}$$

where the linear operator $\boldsymbol{\mathcal{L}}(\boldsymbol{\phi}) = (\mathcal{L}^1(\boldsymbol{\phi}),\dots,\mathcal{L}^m(\boldsymbol{\phi})) : \left(H^1_0(\Omega)\right)^m \to \left(H^1_0(\Omega)\right)^m$ is defined for every $i \in I_h$ and $h = 1,\dots,q$ as

$$\begin{aligned}\mathcal{L}^i(\boldsymbol{\phi}) :=& \phi_i - (-\Delta)^{-1}\left\{\left[\left(3\beta_{ii}{c_i}^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}{c_j}^2\right)\phi_i + 2\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_jc_i\phi_j\right](PU_{\delta_h,\xi_h})^2\right.\\ &\left.+\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}\left((c_jPU_{\delta_k,\xi_hk})^2\phi_i + 2c_ic_jPU_{\delta_h,\xi_h}PU_{\delta_k,\xi_hk}\phi_j\right) + \lambda_i\phi_i\right\},\end{aligned} \tag{2.10}$$

the nonlinear term $\boldsymbol{\mathcal{N}}(\boldsymbol{\phi}) = (\mathcal{N}^1(\boldsymbol{\phi}), \ldots, \mathcal{N}^m(\boldsymbol{\phi})) \in \left(L^{\frac{4}{3}}(\Omega)\right)^m$ is defined for every $i \in I_h$ and $h = 1, \ldots, q$ as

$$\begin{aligned}\mathcal{N}^i(\boldsymbol{\phi}) := &- (-\Delta)^{-1} \Bigg\{ \sum_{\substack{j\in I_h \\ j\neq i}} \beta_{ij}(c_i PU_{\delta_h,\xi_h}\phi_j^2 + 2c_j PU_{\delta_h,\xi_h}\phi_j\phi_i + \phi_j^2\phi_i) \\ &+ \sum_{k\neq h}\sum_{j\in I_k} \beta_{ij}(c_i PU_{\delta_h,\xi_h}\phi_j^2 + 2c_j PU_{\delta_k,\xi_k}\phi_j\phi_i + \phi_j^2\phi_i) + \beta_{ii}\left(3c_i PU_{\delta_h,\xi_h}\phi_i^2 + \phi_i^3\right)\Bigg\},\end{aligned} \tag{2.11}$$

and the error term $\boldsymbol{\mathcal{E}} = (\mathcal{E}^1, \ldots, \mathcal{E}^m) \in \left(L^{\frac{4}{3}}(\Omega)\right)^m$ is defined for every $i \in I_h$ and $h = 1, \ldots, q$ as

$$\begin{aligned}\mathcal{E}^i := &- (-\Delta)^{-1} \Bigg\{ \left(\sum_{j\in I_h} \beta_{ij} {c_j}^2 c_i\right)\left[(PU_{\delta_h,\xi_h})^3 - (U_{\delta_h,\xi_h})^3\right] \\ &+ \sum_{k\neq h}\sum_{j\in I_k} \beta_{ij}(c_j PU_{\delta_k,\xi_k})^2(c_i PU_{\delta_h,\xi_h}) + \lambda_i c_i PU_{\delta_h,\xi_h}\Bigg\}.\end{aligned} \tag{2.12}$$

In the above computation, we used (1.7) and (1.8), so that for every $i \in I_h$ and $h = 1, \ldots, q$

$$c_i PU_{\delta_h,\xi_h} = (-\Delta)^{-1}\left[\left(\sum_{j\in I_h} \beta_{ij} {c_j}^2 c_i\right)(U_{\delta_h,\xi_h})^3\right].$$

The proof of our main result consists of two main steps. First, for fixed $\mathbf{d} = (d_1, \ldots, d_q)$, and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_q)$ we solve the system (2.8), finding $\boldsymbol{\phi} = \boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi}) \in \mathbf{K}^\perp$. Plugging this choice of $\boldsymbol{\phi}$ into the second system (2.9), we obtain a finite dimensional problem in the unknowns $\mathbf{d}$ and $\boldsymbol{\xi}$, whose solution is identified as a critical point of a suitable function.

3. Proof of Theorem 1.2

We briefly sketch the main steps of the proof.

3.1. The linear theory

As a first step, it is important to understand the solvability of the linear problem naturally associated to (2.8), i.e., given $\boldsymbol{\mathcal{L}}$ as in (2.10)

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\phi}) = \mathbf{h}, \quad \text{with} \quad \mathbf{h} \in \mathbf{K}^\perp.$$

Proposition 3.1. *For every $\eta > 0$ small enough there exist $\bar\beta > 0$, $\lambda_0 > 0$ and $C > 0$, such that if $\lambda_i \in (0, \lambda_0)$, for every $i = 1, \ldots, m$, and*

$$\beta^* := \max_{\substack{(i,j)\in I_h\times I_k \\ h\neq k}} \beta_{ij} \leq \overline{\beta}, \tag{3.1}$$

then

$$\|\boldsymbol{\mathcal{L}}(\boldsymbol{\phi})\|_{(H_0^1(\Omega))^m} \geq C\|\boldsymbol{\phi}\|_{(H_0^1(\Omega))^m} \qquad \forall \boldsymbol{\phi} \in \mathbf{K}^\perp, \tag{3.2}$$

for every $(\mathbf{d}, \boldsymbol{\xi}) \in X_\eta$. Moreover, $\boldsymbol{\mathcal{L}}$ is invertible in $\mathbf{K}^\perp$ with continuous inverse.

Proof. It is postponed to Appendix. □

3.2. The error term

We need to estimate the error term $\boldsymbol{\mathcal{E}}$ defined in (2.12).

Lemma 3.2. *For every $\eta > 0$ small enough there exist $\lambda_0 > 0$ and $C > 0$ such that, if $\lambda_i \in (0, \lambda_0)$ for every $i = 1, \dots, m$, then*

$$\|\boldsymbol{\mathcal{E}}\|_{(H_0^1(\Omega))^m} \leq C \sum_{h=1}^{q} \left(O(\delta_h^2) + O(\lambda_h^* \delta_h) + \sum_{k \neq h} O(|\beta^*| \delta_h \delta_k) \right) \tag{3.3}$$

for every $(\mathbf{d}, \boldsymbol{\xi}) \in X_\eta$, where $\lambda_h^* := \max_{i \in I_h} \lambda_i$ and $\beta^* := \max_{\substack{(i,j) \in I_h \times I_k \\ i \neq k}} \beta_{ij}$.

Proof. We argue as in [17, Lemma A.1–A.3]. Note first that, by the continuity of $(-\Delta)^{-1}$, for every $i \in I_h$

$$\begin{aligned}\|\mathcal{E}^i\| \leq & C \left(\sum_{j \in I_h} |\beta_{ij}| {c_j}^2 c_i \right) \left| (PU_{\delta_h, \xi_h})^3 - (U_{\delta_h, \xi_h})^3 \right|_{\frac{4}{3}} \\ & + C \sum_{k \neq h} \sum_{j \in I_k} |\beta_{ij}| c_i c_j^2 \left| PU_{\delta_h, \xi_h} (PU_{\delta_k, \xi_k})^2 \right|_{\frac{4}{3}} + \lambda_i c_i \left| PU_{\delta_h, \xi_h} \right|_{\frac{4}{3}} .\end{aligned}$$

Moreover,

$$\left| (PU_{\delta_h, \xi_h})^3 - (U_{\delta_h, \xi_h})^3 \right|_{\frac{4}{3}} = O(\delta_h^2),$$

$$\left| PU_{\delta_h, \xi_h} PU^2_{\delta_k, \xi_k} \right|_{\frac{4}{3}} = O(\delta_h \delta_k)$$

and

$$\left| PU_{\delta_h, \xi_h} \right|_{\frac{4}{3}} = O(\delta_h) .$$

Then the claim follows. □

3.3. Solving (2.8)

We combine all the previous results and a standard contraction mapping argument and we prove the solvability of the system (2.8).

Proposition 3.3. *For every $\eta > 0$ small enough there exist $\bar{\beta} > 0$, $\lambda_0 > 0$ and $C > 0$ such that, if $\lambda_i \in (0, \lambda_0)$ for every $i = 1, \dots, m$ and (3.1) holds, then for every $(\mathbf{d}, \boldsymbol{\xi}) \in X_\eta$ there exists a unique function $\boldsymbol{\phi} = \boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi}) \in \boldsymbol{K}^\perp$ solving system (2.8). Moreover,*

$$\|\boldsymbol{\phi}\|_{\left(H_0^1(\Omega)\right)^m} \leq C \sum_{h=1}^{q} \left(O(\delta_h^2) + O(\lambda_h^* \delta_h) + \sum_{k \neq h} O(|\beta^*| \delta_h \delta_k) \right) \tag{3.4}$$

and $(\mathbf{d}, \boldsymbol{\xi}) \mapsto \boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi})$ is a C^1–function.

Proof. The claim follows by Proposition 3.1 and Lemma 3.2 arguing exactly as in [17, Proposition 3.2 and Lemma 3.3], noting that the nonlinear part $\boldsymbol{\mathcal{N}}$ given in (2.11) has a quadratic growth in $\boldsymbol{\phi}$. In particular (3.4) follows by (3.3). □

3.4. The reduced problem

Once the first system (2.8) has been solved, we have to find a solution to the second system (2.9) and so a solution to system (1.2).

Proposition 3.4. *For any $\eta > 0$ small enough there exist $\overline{\beta} > 0$ and $\lambda_0 > 0$ such that, if $\lambda_i \in (0, \lambda_0)$ for every $i = 1, \ldots, m$ and (3.1) holds, then $\mathbf{u} = \mathbf{W}(\mathbf{d}, \boldsymbol{\xi}) + \boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi})$ defined in (2.2) solves system (1.2), i.e., it is a critical point of the energy*

$$J(\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{m}\int_{\Omega}|\nabla u_i|^2 - \frac{1}{4}\sum_{i,j=1}^{m}\beta_{ij}\int_{\Omega}u_i^2u_j^2 - \frac{1}{2}\sum_{i=1}^{m}\int_{\Omega}\lambda_i u_i^2$$

if and only if $(\mathbf{d}, \boldsymbol{\xi}) \in X_\eta$ is a critical point of the reduced energy

$$\tilde{J}(\mathbf{d}, \boldsymbol{\xi}) := J(\mathbf{W} + \boldsymbol{\phi}).$$

Moreover, the following expansion holds true

$$\tilde{J}(\boldsymbol{\delta}, \boldsymbol{\xi}) = \sum_{h=1}^{q}\left(\sum_{i\in I_h}c_i^2\right)\left(A_0 + A_1\delta_h^2\mathtt{r}(\xi_h) + A_2\lambda_h^*\delta_h^2|\ln\delta_h| + o(\delta_h^2)\right) \tag{3.5}$$

C^1−uniformly in X_η. Here the A_i's are positive constants, $\mathtt{r}$ is the Robin's function and $\lambda_h^* = \max_{i\in I_h}\lambda_i$.

Proof. The proof follows by combining the arguments in [17, Section 3 and Section 5] and [16, Section 5]. We remark that in this case the fact that $\boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi})$ solves (2.8) is equivalent to claim that it solves the system

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\phi}) - \boldsymbol{\mathcal{N}}(\boldsymbol{\phi}) - \boldsymbol{\mathcal{E}} = \left(\sum_{\ell=0}^{4}a_1^\ell \mathfrak{e}_1 P\psi^\ell_{\delta_1,\xi_1}, \ldots, \sum_{\ell=0}^{4}a_h^\ell \mathfrak{e}_h P\psi^\ell_{\delta_h,\xi_h}\right),$$

for some real numbers a_i^ℓ. Therefore, $\mathbf{W}(\mathbf{d}, \boldsymbol{\xi}) + \boldsymbol{\phi}(\mathbf{d}, \boldsymbol{\xi})$ solves system (2.9) if and only if all the a_i^ℓ's are zero. We also point out that it is quite standard to prove that $J(\mathbf{W} + \boldsymbol{\phi}) \approx J(\mathbf{W})$ and moreover by (1.7) we deduce

$$\begin{aligned}J(\mathbf{W}) = \sum_{h=1}^{q}&\underbrace{\left(\sum_{i\in I_h}\frac{1}{2}c_i^2\int_{\Omega}|\nabla PU_{\delta_h,\xi_h}|^2 - \frac{1}{4}\sum_{i,j\in I_h}\beta_{ij}(c_ic_j)^2\int_{\Omega}(PU_{\delta_h,\xi_h})^4\right)}_{=\left(\sum_{i\in I_h}c_i^2\right)\left(\frac{1}{2}\int_{\Omega}|\nabla PU_{\delta_h,\xi_h}|^2 - \frac{1}{4}\int_{\Omega}(PU_{\delta_h,\xi_h})^4\right)}\\ &- \frac{1}{2}\sum_{\substack{h,k=1\\ h\neq k}}\beta_{ij}\int_{\Omega}(c_iPU_{\delta_h,\xi_h})^2(c_jPU_{\delta_k,\xi_k})^2 - \sum_{h=1}^{q}\sum_{i\in I_h}\frac{1}{2}\lambda_i\int_{\Omega}(c_iPU_{\delta_h,\xi_h})^2\end{aligned}$$

so that the claim follows just arguing as in [17].

□

3.5. Proof of Theorem 1.2: completed

Arguing exactly as in [17, Proof of Theorem 1.3, p. 437], we prove that the reduced energy (3.5) has a critical point $(\mathbf{d}_\lambda, \boldsymbol{\xi}_\lambda)$ provided $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is small enough and $\boldsymbol{\xi}_\lambda \to (\xi_1^0, \ldots, \xi_q^0)$ as $\lambda^* = \max_i \lambda_i \to 0$. Theorem 1.2 immediately follows by Proposition 3.4. Moreover, if the β_{ij}'s depend on the λ_i's and β^* satisfies $|\beta^*| = O\left(e^{\frac{d^*}{\lambda^*}}\right)$ with $d^* < \min_{h=1,\ldots,q} d_h$, then for every $h = 1, \ldots, m$

$$|\beta^*|\delta_h \lesssim e^{\frac{d^*}{\lambda^*} - \frac{d_h}{\lambda_h^*}} \lesssim e^{\frac{d^* - d_h}{\lambda^*}} = o(1)$$

and by estimate (3.4) we can still conclude the validity of (3.5), and so the last part of Theorem 1.2 follows (see also [17, Section 5.3, p.438].

Conflict of interest

The authors declare no conflict of interest.

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A. Appendix

Proof of Proposition 3.1

We argue combining ideas of [17, Lemma 3.1] and [16, Lemma 5.4]. We first prove (3.2) by contradiction. Assume thus that there exist $\{(\mathbf{d}_n, \boldsymbol{\xi}_n)\}_n \subset X_\eta$ so that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ as $n \to +\infty$, $\boldsymbol{\lambda}_n := (\lambda_{1,n}, \ldots, \lambda_{m,n}) \to 0$ as $n \to +\infty$, and $\boldsymbol{\phi}^n := (\phi_1^n, \ldots, \phi_m^n) \in \mathbf{K}^\perp$ so that $\|\boldsymbol{\phi}^n\| = 1$ for every $n \in \mathbb{N}$ and

$$\|\mathcal{L}(\boldsymbol{\phi}^n)\| \to 0 \qquad \text{as } n \to +\infty\,.$$

We recall that the spaces introduced in (2.6) and (2.7) depend on $\mathbf{d_n}$ and $\boldsymbol{\xi}_n$, so for the sake of clarity, let us introduce the following notation. For every $h = 1, \dots, q$, let

$$K_h^n := K_{d_{h,n},\xi_{h,n}}, \qquad (K_h^n)^\perp := K^\perp_{d_{h,n},\xi_{h,n}},$$
$$U_h^n := U_{\delta_{h,n},\xi_{h,n}}, \qquad PU_h^n := PU_{\delta_{h,n},\xi_{h,n}},$$
$$\psi^l_{h,n} := \psi^l_{\delta_{h,n},\xi_{h,n}}, \qquad P\psi^l_{h,n} := P\psi^l_{\delta_{h,n},\xi_{h,n}}, \qquad l = 0, \dots, 4,$$

where $\delta_{h,n} := e^{-\frac{d_{h,n}}{\lambda^*_{h,n}}}$ and $\lambda^*_{h,n} := \max_{i\in I_h}\lambda_{i,n}$. Moreover, set $\mathbf{h}_n := \mathcal{L}(\boldsymbol{\phi}^n)$.

By definition of $\mathcal{L}$ and the fact that $\boldsymbol{\phi}^n \in \mathbf{K}^\perp$, we have, for every $h = 1, \dots, q$ and $i \in I_h$

$$\begin{aligned}\phi_i^n =&(-\Delta)^{-1}\left\{\left[\left(3\mu_i c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_j^2\right)\phi_i^n + 2\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_ic_j\phi_j^n\right](PU_h^n)^2\right.\\ &\left.+\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}\left[c_j^2(PU_k^n)^2\phi_i^n + 2c_ic_jPU_k^nPU_h^n\phi_j^n\right] + \lambda_{i,n}\phi_i^n\right\} + h_i^n - w_i^n,\end{aligned} \tag{A.1}$$

for a suitable $\mathbf{w}_n := (w_i^n)_{i\in I_h} \in K_h^n$. Here $\mu_i := \beta_{ii}$.

Step 1: $\|\mathbf{w}_n\| \to 0$ *as* $n \to +\infty$. Multiplying (A.1) by $\delta^2_{h,n}w_i^n$ and recalling the definition of $(-\Delta)^{-1}$ yields

$$\begin{aligned}\delta^2_{h,n}\|w_i^n\|^2 =&\delta^2_{h,n}\langle h_i^n - \phi_i^n, w_i^n\rangle + \delta^2_{h,n}\left(3\mu_ic_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_j^2\right)\int(PU_h^n)^2\phi_i^nw_i^n\\ &+2\delta^2_{h,n}\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_ic_j\int(PU_h^n)^2\phi_j^nw_i^n + \delta^2_{h,n}\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j^2\int(PU_k^n)^2\phi_i^nw_i^n\\ &+2\delta^2_{h,n}c_i\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j\int PU_k^nPU_h^n\phi_j^nw_i^n + \delta^2_{h,n}\lambda_{i,n}\int\phi_i^nw_i^n,\end{aligned}$$

so that, summing over $i \in I_h$ and making use of $(\phi_i^n)_{i\in I_h}, (h_i^n)_{i\in I_h} \in (K_h^n)^\perp$,

$$\begin{aligned}&\underbrace{\delta^2_{h,n}\sum_{i\in I_h}\|w_i^n\|^2}_{I}\\ =&\underbrace{\delta^2_{h,n}\sum_{i\in I_h}\left[\left(3\mu_ic_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_j^2\right)\int(PU_h^n)^2\phi_i^nw_i^n + 2\delta^2_{h,n}\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_ic_j\int(PU_h^n)^2\phi_j^nw_i^n\right]}_{II}\\ &+\underbrace{\delta^2_{h,n}\sum_{i\in I_h}\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j^2\int(PU_k^n)^2\phi_i^nw_i^n}_{III}+\underbrace{2\delta^2_{h,n}\sum_{i\in I_h}c_i\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j\int PU_k^nPU_h^n\phi_j^nw_i^n}_{IV}\\ &+\underbrace{\delta^2_{h,n}\sum_{i\in I_h}\lambda_{i,n}\int\phi_i^nw_i^n}_{V}.\end{aligned} \tag{A.2}$$

Note first that, since $(w_i^n)_{i\in I_h} \in K_h^n$, for $l = 0, \dots, 4$ there are $a_{h,n}^l \in \mathbb{R}$ for which it holds (see (2.6))

$$(w_i^n)_{i\in I_h} = \sum_{l=0}^{n} a_{h,n}^l \mathfrak{e}_h P\psi_{h,n}^l \,,$$

so that arguing as in [17, p. 417] and for sufficiently large n we can write

$$I = \delta_{h,n}^2 \sum_{i\in I_h} \sum_{l,p=0}^{n} a_{h,n}^l a_{h,n}^p |\mathfrak{e}_{i,h}|^2 \int \nabla P\psi_{h,n}^l \cdot \nabla P\psi_{h,n}^p = \sum_{l=0}^{n} (a_{h,n}^l)^2 \sigma_{ll} + o(1) \sum_{l=0}^{n} a_{h,n}^l a_{h,n}^p \,, \tag{A.3}$$

for suitable positive constants σ_{ll}, $l = 0, \dots, 4$.

Let us thus estimate terms III and IV in (A.2). On the one hand, for every $h, k = 1, \dots, q$, $k \neq h$, $i \in I_h$ and $l = 0, \dots, 4$,

$$\begin{aligned} \left|\int (PU_k^n)^2 \phi_i^n P\psi_{h,n}^l\right| \leq & \left|\int (PU_k^n)^2 \phi_i^n \psi_{h,n}^l\right| + \left|\int (PU_k^n)^2 \phi_i^n (P\psi_{h,n}^l - \psi_{h,n}^l)\right| \\ \leq & \|\phi_i^n\| \left|(PU_n^k)^2 \psi_{h,n}^l\right|_{\frac{4}{3}} + \|\phi_i^n\| \left|(PU_k^n)^2 (P\psi_{h,n}^l - \psi_{h,n}^l)\right|_{\frac{4}{3}} \,, \end{aligned} \tag{A.4}$$

where we made use of Hölder and Sobolev inequality. Then, by [17, Lemma A.1] we get

$$\left|(PU_k^n)^2 (P\psi_{h,n}^0 - \psi_{h,n}^0)\right|_{\frac{4}{3}} \leq C(\delta_{h,n} + o(\delta_{h,n})) \left(\int (PU_k^n)^{\frac{8}{3}} H(\cdot, \xi_{k,n})^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq C'\delta_{h,n} + o(\delta_{h,n}) \tag{A.5}$$

and

$$\left|(PU_k^n)^2 (P\psi_{h,n}^l - \psi_{h,n}^l)\right|_{\frac{4}{3}} \leq C(\delta_{h,n}^2 + o(\delta_{h,n}^2)) \left(\int (PU_k^n)^{\frac{8}{3}} \frac{\partial H}{\partial \xi}(\cdot, \xi_{k,n})^{\frac{4}{3}}\right) \leq C'\delta_{h,n}^2 + o(\delta_{h,n}^2) \tag{A.6}$$

for every $l = 1, \dots, 4$. Moreover, since direct calculations show

$$\begin{aligned} |\psi_{\delta,\xi}^0| &\leq \frac{C}{\delta} U_{\delta,\xi} \\ |\psi_{\delta,\xi}^l| &\leq \frac{C}{\delta} U_{\delta,\xi}^2 |x_l - \xi_l|, \quad l = 1, \dots, 4 \,, \end{aligned}$$

recalling that $0 \leq PU_k^n \leq U_k^n$ by the maximum principle and making use of [17, Lemma A.2–A.4], we also have

$$\begin{aligned} \left|(PU_k^n)\psi_{h,n}^0\right|_{\frac{4}{3}} \leq & \frac{C}{\delta_{h,n}} \left|(PU_k^n)^2 U_h^n\right|_{\frac{4}{3}} \leq \frac{C}{\delta_{h,n}} \left|(U_k^n)^2 U_h^n\right|_{\frac{4}{3}} \\ \leq & \frac{C'}{\delta_{h,n}} \left(O(\delta_{h,n}\delta_{k,n}) + O(\delta_{k,n}^2 \delta_{h,n})\right) = O(\delta_{k,n}) \end{aligned}$$

and, for $l = 1, \dots, 4$,

$$\left|(PU_k^n)\psi_{h,n}^l\right|_{\frac{4}{3}} \leq \frac{C}{\delta_{h,n}} \left|(U_k^n)^2 (U_h^n)^2\right|_{\frac{4}{3}} \leq \frac{C'}{\delta_{h,n}} \left(O(\delta_{k,n}^2 \delta_{h,n}) + O(\delta_{k,n}\delta_{h,n}^2)\right) = o(\delta_{k,n}) \,.$$

Combining the previous estimates with (A.4) and the fact that $\|\boldsymbol{\phi}_n\| = 1$ thus leads to

$$\delta_{h,n}^2 |III| \leq o(\delta_{h,n}^2) \sum_{l=0}^{n} |a_{h,n}^l| \,. \tag{A.7}$$

Similarly, by Hölder and Sobolev inequality, $\|\boldsymbol{\phi}_n\| = 1$ and [17, Lemma A.5],

$$\left| \int PU_k^n PU_h^n \phi_j^n P\psi_{h,n}^l \right| \leq C \left| PU_k^n PU_h^n P\psi_{h,n}^l \right|_{\frac{4}{3}} = O(\delta_{k,n}\delta_{h,n}) \,,$$

for every $k, h = 1, \dots, q$, $k \neq h$, $j \in I_k$ and $l = 0, \dots, 4$, so that

$$\delta_{h,n}^2 |IV| \leq o(\delta_{h,n}^2) \sum_{l=0}^{n} |a_{h,n}^l| \,. \tag{A.8}$$

Furthermore, by Hölder inequality and recalling that $\lambda_n \to 0$ as $n \to +\infty$ and $\|\boldsymbol{\phi}_n\| = 1$, we also have

$$\delta_{h,n}^2 |V| \leq o(\delta_{h,n}^2) \|(w_i^n)_{i \in I_h}\| \,. \tag{A.9}$$

We are thus left to estimate the term II in (A.2). To this purpose, we set, for every $i, j \in I_h$,

$$\alpha_{ii} := 3\mu_i c_i^2 + \sum_{\substack{j \in I_h \\ j \neq i}} \beta_{ij} c_j^2, \qquad \alpha_{ij} := 2\beta_{ij} c_i c_j \,,$$

so that

$$\begin{aligned} II &= \sum_{i \in I_h} \int \left(\alpha_{ii}\phi_i^n + \sum_{\substack{j \in I_h \\ j \neq i}} \alpha_{ij}\phi_j^n \right) (PU_h^n)^2 w_i^n \\ &= \underbrace{\sum_{i \in I_h} \int \left(\alpha_{ii}\phi_i^n + \sum_{\substack{j \in I_h \\ j \neq i}} \alpha_{ij}\phi_j^n \right) \left((PU_h^n)^2 - (U_h^n)^2 \right) w_i^n}_{II.1} \\ &\quad + \underbrace{\sum_{i \in I_h} \int \left(\alpha_{ii}\phi_i^n + \sum_{\substack{j \in I_h \\ j \neq i}} \alpha_{ij}\phi_j^n \right) (U_h^n)^2 w_i^n}_{II.2} \,. \end{aligned}$$

As for $II.1$, we have, for every $i, j \in I_h$

$$\left| \int \left((PU_h^n)^2 - (U_h^n)^2 \right) \phi_j^n w_i^n \right| \leq \left| (PU_h^n)^2 - (U_h^n)^2 \right|_2 |\phi_j^n|_4 |w_i^n|_4 \leq \left| (PU_h^n)^2 - (U_h^n)^2 \right|_2 \|w_i^n\|$$

by Hölder and Sobolev inequality and $\|\boldsymbol{\phi}_n\| = 1$. Furthermore, by [17, Lemma A.3] and $0 \leq PU_h^n \leq U_h^n$

$$\left| (PU_h^n)^2 - (U_h^n)^2 \right|_2 \leq C \left(\left| U_h^n (PU_h^n - U_h^n) \right|_2 + \left| (PU_h^n - U_h^n)^2 \right|_2 \right) \leq C' \left| U_h^n (PU_h^n - U_h^n) \right|_2 \,,$$

and by [17, Lemma A.1–A.2]

$$\left|U_h^n(PU_h^n - U_h^n)\right|_2 = \left(\int (U_h^n)^2(PU_h^n - U_h^n)^2\right)^{\frac{1}{2}} = \left(\int (U_h^n)^2(\delta_{h,n}AH(\cdot,\xi_{h,n}) + o(\delta_{h,n}))^2\right)^{\frac{1}{2}}$$
$$\leq (C\delta_{h,n} + o(\delta_{h,n}))\left(\int (U_h^n)^2\right)^{\frac{1}{2}} \leq C'\delta_{h,n}^2|\ln\delta_{h,n}|^{\frac{1}{2}} + o(\delta_{h,n}^2),$$

in turn yielding

$$|II.1| \leq C\sum_{i,j\in I_h}\left|\int\left((PU_h^n)^2 - (U_h^n)^2\right)\phi_j^n w_i^n\right| \leq C'(\delta_{h,n}^2|\ln\delta_{h,n}| + o(\delta_{h,n}^2))\|(w_i^n)_{i\in I_h}\|. \tag{A.10}$$

To estimate $II.2$, note first that

$$\begin{aligned} II.2 =& \sum_{i\in I_h}\int\left(\alpha_{ii}\phi_i^n + \sum_{\substack{j\in I_h\\ j\neq i}}\alpha_{ij}\phi_j^n\right)(U_h^n)^2\sum_{l=0}^n a_{h,n}^l \mathfrak{e}_{i,h}P\psi_{h,n}^l \\ =& \sum_{l=0}^n a_{h,n}^l\int (U_h^n)^2 P\psi_{h,n}^l\sum_{i\in I_h}\left(\alpha_{ii}\mathfrak{e}_{i,h}\phi_i^n + \mathfrak{e}_{i,h}\sum_{\substack{j\in I_h\\ j\neq i}}\alpha_{ij}\phi_j^n\right) \\ =& \sum_{l=0}^n a_{h,n}^l\int (U_h^n)^2 P\psi_{h,n}^l\sum_{i\in I_h}\phi_i^n\left(\alpha_{ii}\mathfrak{e}_{i,h} + \sum_{\substack{j\in I_h\\ j\neq i}}\alpha_{ij}\mathfrak{e}_{j,h}\right) \\ =& 3\sum_{l=0}^n a_{h,n}^l\int (U_h^n)^2 P\psi_{h,n}^l\sum_{i\in I_h}\phi_i^n\mathfrak{e}_{i,h}, \end{aligned} \tag{A.11}$$

since by construction $\mathfrak{e}_h$ is an eigenvector of the matrix $(\alpha_{ij})_{i,j\in I_h}$ corresponding to the eigenvalue $\Lambda_1 = 3$ (see [16, Lemma 6.1]). Recalling that $(\phi_i^n)_{i\in I_h} \in (K_h^n)^{\perp}$, so that by (2.5) and (2.6)

$$0 = \sum_{i\in I_h}\int \mathfrak{e}_{i,h}\nabla P\psi_{h,n}^l\cdot\nabla\phi_i^n = 3\sum_{i\in I_h}\int (U_h^n)^2\mathfrak{e}_{i,h}\psi_{h,n}^l\phi_i^n$$

for every $l = 0, \ldots, 4$, we can then rewrite (A.11) as

$$II.2 = 3\sum_{l=0}^n\sum_{i\in I_h}a_{h,n}^l\mathfrak{e}_{i,h}\int (U_h^n)^2(P\psi_{h,n}^l - \psi_{h,n}^l)\phi_i^n.$$

Arguing as in (A.4)–(A.5)–(A.6) above we get

$$\left|\int (U_h^n)^2(P\psi_{h,n}^l - \psi_{h,n}^l)\phi_i^n\right| \leq C\delta_{h,n} + o(\delta_{h,n})$$

for every $l = 0, \ldots, 4$, $h = 1, \ldots, q$ and $i \in I_h$, thus implying

$$|II.2| \leq (C\delta_{h,n} + o(\delta_{h,n}))\sum_{l=0}^n |a_{h,n}^l|. \tag{A.12}$$

Coupling (A.10)–(A.12) then gives

$$|II| \le C'(\delta_{h,n}^2|\ln\delta_{h,n}| + o(\delta_{h,n}^2))\|\mathbf{w}_n\| + (C\delta_{h,n} + o(\delta_{h,n}))\sum_{l=0}^{n}|a_{h,n}^l|\,,$$

and combining with (A.2),(A.7),(A.8),(A.9) we finally obtain

$$\delta_{h,n}^2\|(w_i^n)\|_{i\in I_h}^2 \le o(\delta_{h,n}^2)\|(w_i^n)\|_{i\in I_h} + o(\delta_{h,n}^2)\sum_{l=0}^{n}|a_{h,n}^l|\,.$$

Together with (A.3), this ensures that $\|(w_i^n)\|_{i\in I_h}\| \to 0$ as $n\to+\infty$, and repeating the argument for every $h=1,\dots,q$, gives $\|\mathbf{w}_n\|\to 0$ as desired.

Step 2. For every $h=1,\dots,q$ and $i\in I_h$, we set

$$\widetilde{\phi}_i^n(y) := \begin{cases} \delta_{h,n}\phi_i^n(\xi_{h,n}+\delta_{h,n}y) & \text{if } y\in\widetilde{\Omega}_{h,n} := \frac{\Omega-\xi_{h,n}}{\delta_{h,n}} \\ 0 & \text{if } y\in\mathbb{R}^n\setminus\widetilde{\Omega}_{h,n}\,. \end{cases}$$

By definition, $\|\widetilde{\phi}_i^n\|_{H_0^1(\mathbb{R}^n)} = \|\phi_i^n\|_{H_0^1(\Omega)}$, so that $\widetilde{\phi}_i^n \rightharpoonup \widetilde{\phi}_i$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $n\to+\infty$, for some $\widetilde{\phi}_i$. Let us thus show that $\widetilde{\phi}_i\equiv 0$ for every $i=1,\dots,m$. To this aim, note first that (A.1) can be rewritten as

$$\begin{aligned}
&\int_{\widetilde{\Omega}_{h,n}}\nabla\widetilde{\phi}_i^n\cdot\nabla\varphi \\
&= \underbrace{\delta_{h,n}^2\int_{\widetilde{\Omega}_{h,n}}\left[\left(3\mu_i c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_j^2\right)\widetilde{\phi}_i^n + 2\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_ic_j\widetilde{\phi}_j^n\right](PU_h^n)^2(\xi_{h,n}+\delta_{h,n}y)\varphi}_{A_n} \\
&+ \underbrace{\delta_{h,n}^2\int_{\widetilde{\Omega}_{h,n}}\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j^2(PU_k^n)^2(\xi_{h,n}+\delta_{h,n}y)\widetilde{\phi}_i^n\varphi}_{B_n} \\
&+ \underbrace{2\delta_{h,n}^3\int_{\widetilde{\Omega}_{h,n}}\sum_{k\neq h}\sum_{j\in I_k}c_ic_jPU_k^n(\xi_{h,n}+\delta_{h,n}y)PU_h^n(\xi_{h,n}+\delta_{h,n}y)\phi_j^n(\xi_{h,n}+\delta_{h,n}y)\varphi}_{C_n} \\
&+\delta_{h,n}^2\int_{\widetilde{\Omega}_{h,n}}\lambda_{i,n}\widetilde{\phi}_i^n\varphi + \int_{\widetilde{\Omega}_{h,n}}\nabla(\widetilde{h}_i^n - \widetilde{w}_i^n)\cdot\nabla\varphi,
\end{aligned}$$

for every $\varphi\in C_C^\infty(\mathbb{R}^n)$, where

$$\widetilde{h}_i^n(y) := \begin{cases} \delta_{h,n}h_i^n(\xi_{h,n}+\delta_{h,n}y) & \text{if } y\in\widetilde{\Omega}_{h,n} \\ 0 & \text{if } y\in\mathbb{R}^n\setminus\widetilde{\Omega}_{h,n} \end{cases}$$

$$\widetilde{w}_i^n(y) := \begin{cases} \delta_{h,n}w_i^n(\xi_{h,n}+\delta_{h,n}y) & \text{if } y\in\widetilde{\Omega}_{h,n} \\ 0 & \text{if } y\in\mathbb{R}^n\setminus\widetilde{\Omega}_{h,n}\,. \end{cases}$$

Let now φ be so that $K_\varphi := \text{supp}\varphi \subset \widetilde{\Omega}_{h,n}$, which is always true for any given $\varphi \in C_c^\infty(\mathbb{R}^n)$ and n large enough. On the one hand, it is readily seen that

$$\delta_{h,n}^2 \int_{\widetilde{\Omega}_{h,n}} \lambda_{i,n} \widetilde{\phi}_i^n \varphi \to 0$$

$$\int_{\widetilde{\Omega}_{h,n}} \nabla(\widetilde{h}_i^n - \widetilde{w}_i^n) \cdot \nabla\varphi \to 0$$

as $n \to +\infty$, since $\boldsymbol{\lambda}_n \to 0$, $\|\mathbf{h}_n\| \to 0$ and $\|\mathbf{w}_n\| \to 0$.

On the other hand, for every $i \in I_h$, $j \in I_k$, $h \neq k$

$$\begin{aligned}\int_{\widetilde{\Omega}_{h,n}} (PU_k^n)^2(\xi_{h,n}+\delta_{h,n}y)\widetilde{\phi}_i^n(y)\varphi(y) &= \int_{\widetilde{\Omega}_{h,n}} (U_k^n)^2(\xi_{h,n}+\delta_{h,n}y)\widetilde{\phi}_i^n(y)\varphi(y) + o(1)\\ &= \delta_{k,n}^2 \int_{\widetilde{\Omega}_{h,n}\cap K_\varphi} \frac{\alpha_4^2}{(\delta_{k,n}^2 + |\delta_{h,n}y + \xi_{h,n} - \xi_{k,n}|^2)^2} \widetilde{\phi}_i^n(y)\varphi(y) + o(1) \to 0\end{aligned}$$

and

$$\begin{aligned}&\delta_{h,n}^3 \int_{\widetilde{\Omega}_{h,n}} PU_k^n(\xi_{h,n}+\delta_{h,n}y)PU_h^n(\xi_{h,n}+\delta_{h,n}y)\phi_j^n(\xi_{h,n}+\delta_{h,n}y)\varphi(y)\\ &\quad= \delta_{h,n}^2 \int_{\widetilde{\Omega}_{h,n}\cap K_\varphi} \frac{\varphi(y)}{1+|y|^2} \frac{\alpha_4\delta_{k,n}}{\delta_{k,n}^2 + |\delta_{h,n}y + \xi_{h,n} - \xi_{k,n}|^2} \phi_j^n(\xi_{h,n}+\delta_{h,n}y) + o(1) \to 0\end{aligned}$$

as $y \in K_\varphi$, which is fixed and bounded, and $|\xi_{h,n} - \xi_{k,n}| \geq \eta$ by assumption. Hence,

$$B_n \to 0, \quad C_n \to 0 \quad \text{as } n \to +\infty\,.$$

Furthermore, for every $j \in I_h$,

$$\begin{aligned}&\delta_{h,n}^2 \int_{\widetilde{\Omega}_{h,n}} (PU_h^n)^2\,(\xi_{h,n}+\delta_{h,n}y)\widetilde{\phi}_j^n(y)\varphi(y)\\ &\qquad= \delta_{h,n}^2 \int_{\widetilde{\Omega}_{h,n}} (U_h^n)^2\,(\xi_{h,n}+\delta_{h,n}y)\widetilde{\phi}_j^n(y)\varphi(y) + o(1)\\ &\qquad= \int_{\widetilde{\Omega}_{h,n}\cap K_\varphi} \frac{\alpha_4^2}{(1+|y|^2)^2}\widetilde{\phi}_j^n(y)\varphi(y) + o(1) \to \int_{\mathbb{R}^n} \frac{\alpha_4^2}{(1+|y|^2)^2}\widetilde{\phi}_j(y)\varphi(y)\\ &\qquad= \int_{\mathbb{R}^n} U_{1,0}^2(y)\widetilde{\phi}_j(y)\varphi(y)\end{aligned}$$

since $\widetilde{\phi}_j^n \rightharpoonup \widetilde{\phi}_j$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $U_{1,0} \in L^n(\mathbb{R}^n)$. Therefore,

$$A_n \to \int_{\mathbb{R}^n} \left[\left(3\mu_i c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij}c_j^2\right)\widetilde{\phi}_i + 2\sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij}c_ic_j\widetilde{\phi}_j\right](U_{1,0})^2\,\varphi \qquad \text{as } n \to +\infty$$

that is, for every $i \in I_h$

$$-\Delta\widetilde{\phi}_i = \left[\left(3\mu_i c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij}c_j^2\right)\widetilde{\phi}_i + 2\sum_{\substack{j\in I_h\\ j\neq i}} \beta_{ij}c_ic_j\widetilde{\phi}_j\right](U_{1,0})^2 \quad \text{in } \mathbb{R}^n, \quad \widetilde{\phi}_i \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

Therefore, the weak limit $(\widetilde{\phi}_i)_{i\in I_h}$ solves the linearized system (1.9) for every $h = 1, \dots, q$. Thus, $(\widetilde{\phi}_i)_{i\in I_h} \in \operatorname{span}\{\mathfrak{e}_h\psi^l_{1,0} : l = 0, \dots, 4\}$. However, since $(\phi^n_i)_{i\in I_h} \in (K^n_h)^\perp$ for every n, then it follows

$$\begin{aligned}0 =&\delta_{h,n}\langle(\phi^n_i)_{i\in I_h}, \mathfrak{e}_h P\psi^0_{h,n}\rangle = 3\sum_{i\in I_h}\delta_{h,n}\int_\Omega (U^n_h)^2\mathfrak{e}_{i,h}\psi^0_{h,n}\phi^n_i\\ =&3\sum_{i\in I_h}\int_{\widetilde{\Omega}_{h,n}}\mathfrak{e}_{i,h}\alpha_4^3\frac{|y|^2-1}{(1+|y|^2)^n}\widetilde{\phi}^n_i = 3\sum_{i\in I_h}\int_{\widetilde{\Omega}_{h,n}} U^2_{1,0}\psi^0_{1,0}\mathfrak{e}_{i,h}\widetilde{\phi}^n_i\end{aligned}$$

and, for every $l = 1, \dots, 4$,

$$\begin{aligned}0 =&\delta_{h,n}\langle(\phi^n_i)_{i\in I_h}, \mathfrak{e}_h P\psi^l_{h,n}\rangle = 3\sum_{i\in I_h}\delta_{h,n}\int_\Omega (U^n_h)^2\mathfrak{e}_{i,h}\psi^l_{h,n}\phi^n_i\\ =&3\sum_{i\in I_h}\int_{\widetilde{\Omega}_{h,n}}\mathfrak{e}_{i,h}2\alpha_4^3\frac{y_l}{(1+|y|^2)^n}\widetilde{\phi}^n_i = 3\sum_{i\in I_h}\int_{\widetilde{\Omega}_{h,n}} U^2_{1,0}\psi^l_{1,0}\mathfrak{e}_{i,h}\widetilde{\phi}^n_i.\end{aligned}$$

Passing to the limit as $n \to +\infty$ and making use of $\widetilde{\phi}^n_i \rightharpoonup \widetilde{\phi}_i$, we obtain

$$3\sum_{i\in I_h}\int_{\mathbb{R}^n} U^2_{1,0}\psi^l_{1,0}\mathfrak{e}_{i,h}\widetilde{\phi}_i = 0, \qquad l = 0, \dots, 4\,.$$

This shows that $(\widetilde{\phi}_i)_{i\in I_h} \in (\operatorname{span}\{\mathfrak{e}_h\psi^l_{1,0} : l = 0, \dots, 4\})^\perp$, thus implying $\widetilde{\phi}_i \equiv 0$ for every $i \in I_h$ and concluding Step 2.

Step 3. We now prove that $\phi^n_i \to 0$ strongly in $H^1_0(\Omega)$ for every $i = 1, \dots, m$, which in turn concludes the proof of (3.2) as it contradicts the assumption $\|\boldsymbol{\phi}_n\| = 1$ for every n.

To this aim, let us test (A.1) with ϕ^n_i, so to have

$$\begin{aligned}\|\phi^n_i\|^2 =&\underbrace{\left(3\mu_i c_i^2 + \sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_j^2\right)\int_\Omega (PU^n_h)^2\,(\phi^n_i)^2}_{I} \underbrace{+2\sum_{\substack{j\in I_h\\ j\neq i}}\beta_{ij}c_ic_j\int_\Omega (PU^n_h)^2\,\phi^n_j\phi^n_i}_{II}\\ &\underbrace{+\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j^2\int_\Omega (PU^n_k)^2(\phi^n_i)^2}_{III} \underbrace{+\sum_{k\neq h}\sum_{j\in I_k}2\beta_{ij}c_ic_j\int_\Omega PU^n_kPU^n_h\phi^n_j\phi^n_i}_{IV}\\ &+\lambda_{i,n}\|\phi^n_i\|^2 + \langle h^n_i - w^n_i, \phi^n_i\rangle\,.\end{aligned} \tag{A.13}$$

Since $\boldsymbol{\lambda}_n \to 0$, $\|\mathbf{h}_n\| \to 0$, $\|\mathbf{w}_n\| \to 0$ and $\boldsymbol{\phi}_n$ is bounded in $H^1_0(\Omega)$ uniformly on n,

$$\lambda_{i,n}\|\phi^n_i\|^2 + \langle h^n_i - w^n_i, \phi^n_i\rangle \to 0 \qquad \text{as } n \to +\infty\,. \tag{A.14}$$

Moreover, recalling that $0 \le PU^n_h \le U^n_h$ for every $h = 1, \dots, q$, we have

$$\begin{aligned}&\int_\Omega (PU^n_h)^2\,(\phi^n_i)^2 \le \int_\Omega (U^n_h)^2\,(\phi^n_i)^2 = \int_{\widetilde{\Omega}_{h,n}} U^2_{1,0}(\widetilde{\phi}^n_i)^2 \to 0\\ &\int_\Omega (PU^n_h)^2\,\phi^n_j\phi^n_i = \int_\Omega (U^n_h)^2\,\phi^n_j\phi^n_i + o(1) = \int_{\widetilde{\Omega}_{h,n}} (U_{1,0})^2\widetilde{\phi}^n_j\widetilde{\phi}^n_i + o(1) \to 0\end{aligned}$$

as $n \to +\infty$ and for every $i, j \in I_h$, since $\widetilde{\phi}_i^n, \widetilde{\phi}_j^n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $U_{1,0}^2 \in L^2(\mathbb{R}^n)$, so that

$$|I| \to 0 \quad \text{and} \quad |II| \to 0 \quad \text{as } n \to +\infty\,. \tag{A.15}$$

As for term IV, for every $i \in I_h$, $j \in I_k$, $h \neq k$, by Hölder and Sobolev inequalities and by [17, Lemma A.2–A.4]

$$\begin{aligned}\left|\int_\Omega PU_k^n PU_h^n \phi_j^n \phi_i^n\right| \leq &\left(\int_\Omega (PU_k^n)^2 (PU_h^n)^2\right)^{\frac{1}{2}} \left(\int_\Omega (\phi_j^n)^2 (\phi_i^n)^2\right)^{\frac{1}{2}} \leq C\left(\int_\Omega (U_k^n)^2 (U_h^n)^2\right)^{\frac{1}{2}} \\ \leq &C'\left(O(\delta_{h,n}^2)\int_\Omega (U_k^n)^2 + O(\delta_{k,n}^2)\int_\Omega (U_k^n)^2 + O(\delta_{h,n}^2\delta_{k,n}^2)\right)^{\frac{1}{2}} \\ \leq &C''\left(O(\delta_{h,n})\delta_{k,n}\sqrt{|\ln\delta_{k,n}|} + O(\delta_{k,n})\delta_{h,n}\sqrt{|\ln\delta_{h,n}|} + O(\delta_{h,n}\delta_{k,n})\right)\,,\end{aligned}$$

thus ensuring

$$|IV| \to 0 \qquad \text{as } n \to +\infty\,. \tag{A.16}$$

We are left to discuss term III. On the one hand, if for every $h = 1, \dots, q$ it holds

$$\max_{\substack{(i,j)\in I_h\times I_k \\ h\neq k}} \beta_{ij} \leq 0\,,$$

then we simply have

$$III \leq 0.$$

On the other hand, if there exist $i \in I_h$, $j \in I_k$ with $\beta_{ij} > 0$, then

$$\beta_{ij}\int_\Omega (PU_k^n)^2(\phi_i^n)^2 \leq \beta_{ij}\left|\phi_i^n\right|_4^2 \left|U_k^n\right|_4^2 \leq C\beta_{ij}\|\phi_i^n\|^2\,.$$

Let then $\overline{\beta} > 0$ be a positive constant so that, whenever

$$\max_{\substack{(i,j)\in I_h\times I_k \\ h\neq k}} \beta_{ij} \leq \overline{\beta}\,,$$

we have

$$|III| \leq C\sum_{k\neq h}\sum_{j\in I_k}\beta_{ij}c_j^2\|\phi_i^n\|^2 \leq \frac{1}{2}\|\phi_i^n\|^2\,. \tag{A.17}$$

Summing up, coupling (A.14), (A.15), (A.16) and (A.17) with (A.13), we conclude that $\|\phi_i^n\| \to 0$ as $n \to +\infty$, for every $i = 1, \dots, m$.

Step 4: invertibility. Note first that $(-\Delta)^{-1} : L^{\frac{4}{3}}(\Omega) \to H_0^1(\Omega)$ is a compact operator, so that $\mathcal{L}$ restricted to $\mathbf{K}^\perp$ is a compact perturbation of the identity. Furthermore, (3.2) implies that $\mathcal{L}$ is injective, and thus surjective by Fredholm alternative. Henceforth, it is invertible, and the continuity of the inverse operator is guaranteed by (3.2).