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(Article begins on next page)

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The Weyl product on quasi-Banach modulation spaces

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We study the bilinear Weyl product acting on quasi-Banach modulation spaces. We find sufficient conditions for continuity of the Weyl product and we derive necessary conditions. The results extend known results for Banach modulation spaces.

Keywords: Pseudodifferential calculus; Weyl product; modulation spaces; quasi-Banach spaces; Gabor frames.

Mathematics Subject Classification: 35S05, 47G30, 42B35, 46A16, 46E35, 46F05

1. Introduction

In this paper, we study the Weyl product acting on weighted modulation spaces with Lebesgue parameters in $(0, \infty]$. We work out conditions on the weights and the Lebesgue parameters that are sufficient for continuity of the Weyl product, and we also prove necessary conditions.

The Weyl product or twisted product is the product of symbols in the Weyl calculus of pseudodifferential operators corresponding to operator composition. This means that the Weyl product

$$(a_1, a_2) \mapsto a_1 \# a_2,$$

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of two distributions a_1 and a_2 defined on the phase space $T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$ is defined by

$$\mathrm{Op}^w(a_1 \# a_2) = \mathrm{Op}^w(a_1) \circ \mathrm{Op}^w(a_2),$$

provided the composition is well-defined.

Our result on sufficient conditions is as follows. Suppose ω_j, $j = 0, 1, 2$, are moderate weights on $\mathbf{R}^{4d}$ that satisfy

$$\omega_0(Z + X, Z - X) \lesssim \omega_1(Y + X, Y - X)\,\omega_2(Z + Y, Z - Y), \quad X, Y, Z \in \mathbf{R}^{2d}.$$

Suppose $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$, satisfy

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2},$$

and either

$$q_1, q_2 \leq q_0 \leq \min(1, p_0),$$

or

$$\min(1, p_0) \leq q_1, q_2 \leq q_0 \quad \text{and} \quad \frac{1}{\min(1, p_0)} + \frac{1}{q_0} \leq \frac{1}{q_1} + \frac{1}{q_2}.$$

Denote the Gelfand–Shilov space of order $\frac{1}{2}$ by $\mathcal{S}_{1/2}$, and the weighted modulation space with Lebesgue parameters $p, q > 0$ and with weight ω by $\mathcal{M}^{p,q}_{(\omega)}$. Then the map $(a_1, a_2) \mapsto a_1 \# a_2$ from $\mathcal{S}_{1/2}(\mathbf{R}^{2d}) \times \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and

$$\|a_1 \# a_2\|_{\mathcal{M}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{\mathcal{M}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{M}^{p_2,q_2}_{(\omega_2)}}. \tag{1.1}$$

As a consequence for unweighted modulation spaces, we obtain new conditions on Lebesgue parameters that are sufficient for $\mathcal{M}^{p,q}(\mathbf{R}^{2d})$ to be an algebra: $p, q \in (0, \infty]$ and $q \leq \min(1, p)$.

The necessary conditions we deduce are as follows. Suppose (1.1) holds for all $a_1, a_2 \in \mathscr{S}(\mathbf{R}^{2d})$, for a triple of polynomial type weights ω_j, $j = 0, 1, 2$ interrelated in a certain way, see (3.7). Then

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p_0} \leq \frac{1}{q_1} + \frac{1}{q_2} \quad \text{and} \quad q_1, q_2 \leq q_0,$$

which are strictly weaker than the sufficient conditions.

Our results for the Weyl product are special cases of results formulated and proved for a family of pseudodifferential calculi parametrized by real matrices $A \in \mathbf{R}^{d \times d}$. In fact, we work with a symbol product indexed by $A \in \mathbf{R}^{d \times d}$, denoted and defined by

$$\mathrm{Op}_A(a \#_A b) = \mathrm{Op}_A(a) \circ \mathrm{Op}_A(b),$$

where $\mathrm{Op}_A(a)$ is the A-indexed pseudodifferential operator with symbol a. This family of calculi contains the Weyl quantization as the special case $A = \frac{1}{2}I$.

The sufficient conditions and the necessary conditions that we find extend results [7, 23] where the same problem was studied for the narrower range of Lebesgue

parameters $[1, \infty]$. In the latter case, modulation spaces are Banach spaces, whereas they are merely quasi-Banach spaces if a Lebesgue parameter is smaller than one.

The Weyl product on Banach modulation spaces has been studied in e.g. [7, 18, 20, 23, 26, 29, 30]. In [7] conditions on the Lebesgue parameters were found that are both necessary and sufficient for continuity of the Weyl product, thus characterizing the Weyl product acting on Banach modulation spaces.

One possible reason that we do not obtain characterizations in the full range of Lebesgue parameters $(0, \infty]$ is that new difficulties arise as soon as a Lebesgue parameter is smaller than one. The available techniques are quite different, and many tools that are useful in the Banach space case, e.g. duality and complex interpolation, are not applicable or fraught with subtle difficulties.

Our technique to prove the sufficient conditions consists of a discretization of the Weyl product by means of a Gabor frame. This reduces the continuity of the Weyl product to the continuity of certain infinite-dimensional matrix operators. A similar idea has been developed in [38].

The paper is organized as follows. Section 2 fixes notation and gives the background on Gelfand–Shilov function and distribution spaces, pseudo-differential calculi, modulation spaces, Gabor frames, and symbol product results for Banach modulation spaces.

Section 3 contains the result on sufficient conditions for continuity on quasi-Banach modulation spaces (Theorem 3.1). Section 4 contains the result on necessary conditions for continuity on quasi-Banach modulation spaces (Theorem 4.3). Finally in Appendix, we show a Fubini type result for Gelfand–Shilov distributions that is needed in the definition of the short-time Fourier transform (STFT) of a Gelfand–Shilov distribution.

2. Preliminaries

2.1. *Weight functions*

A *weight* on $\mathbf{R}^d$ is a positive function $\omega \in L_{\mathrm{loc}}^\infty(\mathbf{R}^d)$ such that $1/\omega \in L_{\mathrm{loc}}^\infty(\mathbf{R}^d)$. We usually assume that ω is (v-)*moderate*, for some positive function $v \in L_{\mathrm{loc}}^\infty(\mathbf{R}^d)$. This means

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \tag{2.1}$$

Here, $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ holds uniformly for all θ in the intersection of the domains of f and g for some constant $c > 0$, and we write $f \asymp g$ when $f \lesssim g \lesssim f$. Note that (2.1) implies the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbf{R}^d. \tag{2.2}$$

If v in (2.1) can be chosen as a polynomial, then ω is called polynomially moderate or a weight of polynomial type. We let $\mathscr{P}(\mathbf{R}^d)$ and $\mathscr{P}_E(\mathbf{R}^d)$ be the sets of all weights of polynomial type and moderate weights on $\mathbf{R}^d$, respectively.

If $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, then there exists $r > 0$ such that ω is v-moderate for $v(x) = e^{r|x|}$ [19]. Hence by (2.2), for any $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, there is $r > 0$ such that

$$e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbf{R}^d. \tag{2.3}$$

A weight v is called submultiplicative if v is even and (2.1) holds with $\omega = v$. In the paper, v and v_j for $j \geq 0$ will denote submultiplicative weights if not otherwise stated.

2.2. *Gelfand–Shilov spaces*

Let $h, s \in \mathbf{R}_+$ be fixed. Then $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}(\alpha!\,\beta!)^s}$$

is finite, where the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}_{s,h}$ is a Banach space which increases with h and s, and it is contained in the Schwartz space $\mathscr{S}$. (Inclusions of function and distribution spaces understand embeddings.) The topological dual $\mathcal{S}'_{s,h}(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbf{R}^d)$ (the tempered distributions). If $s > \frac{1}{2}$, then $\mathcal{S}_{s,h}$ and $\bigcup_{h>0} \mathcal{S}_{1/2,h}$ contain all finite linear combinations of Hermite functions.

The (Fourier invariant) *Gelfand–Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ with respect to h. This implies

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d). \tag{2.4}$$

The topology for $\mathcal{S}_s(\mathbf{R}^d)$ is the strongest topology such that each inclusion $\mathcal{S}_{s,h}(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d)$ is continuous. The projective limit $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with seminorms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. It holds $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$ if and only if $s \geq \frac{1}{2}$, and $\Sigma_s(\mathbf{R}^d) \neq \{0\}$ if and only if $s > \frac{1}{2}$.

For every $\varepsilon > 0$ and $s > 0$,

$$\Sigma_s(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbf{R}^d).$$

The *Gelfand–Shilov distribution spaces* $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$ are the projective and inductive limits respectively of $\mathcal{S}'_{s,h}(\mathbf{R}^d)$. Hence if $s \geq \frac{1}{2}$ and $t > \frac{1}{2}$ then

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma'_t(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{t,h}(\mathbf{R}^d). \tag{2.4a}$$

The space $\mathcal{S}'_s(\mathbf{R}^d)$ is the topological dual of $\mathcal{S}_s(\mathbf{R}^d)$, and if $s > \frac{1}{2}$ then $\Sigma'_s(\mathbf{R}^d)$ is the topological dual of $\Sigma_s(\mathbf{R}^d)$ [12].

The action of a distribution f on a test function ϕ is written $\langle f, \phi \rangle$, and the conjugate linear action is written $(u, \phi) = \langle u, \overline{\phi} \rangle$, consistent with the L^2 inner product $(\cdot, \cdot) = (\cdot, \cdot)_{L^2}$ which is conjugate linear in the second argument.

The Gelfand–Shilov (distribution) spaces enjoy many invariance properties, for instance under translation, dilation, tensorization, coordinate transformations and (partial) Fourier transformation.

We use the normalization

$$\mathscr{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi \rangle} dx, \quad \xi \in \mathbf{R}^d,$$

of the Fourier transform of $f \in L^1(\mathbf{R}^d)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^d$. The Fourier transform $\mathscr{F}$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

The *symplectic Fourier transform* of $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ where $s \geq \frac{1}{2}$ is defined by

$$\mathscr{F}_\sigma a(X) = \pi^{-d} \int_{\mathbf{R}^{2d}} a(Y)e^{2i\sigma(X,Y)} dY,$$

where σ is the symplectic form

$$\sigma(X,Y) = \langle y,\xi \rangle - \langle x,\eta \rangle, \quad X = (x,\xi) \in \mathbf{R}^{2d}, \quad Y = (y,\eta) \in \mathbf{R}^{2d}.$$

Since $\mathscr{F}_\sigma a(x,\xi) = 2^d \mathscr{F}a(-2\xi, 2x)$, the definition of $\mathscr{F}_\sigma$ extends in the same way as $\mathscr{F}$.

Let $\phi \in \mathcal{S}_s(\mathbf{R}^d)\backslash\{0\}$. The *short-time Fourier transform* (STFT) $V_\phi f$ of $f \in \mathcal{S}'_s(\mathbf{R}^d)$ is the distribution on $\mathbf{R}^{2d}$ defined by

$$V_\phi f(x,\xi) = \mathscr{F}(f\overline{\phi(\cdot -x)})(\xi) = (2\pi)^{-\frac{d}{2}}(f, \phi(\cdot -x)e^{i\langle \cdot,\xi \rangle}). \tag{2.5}$$

Note that $f\overline{\phi(\cdot -x)} \in \mathcal{S}'_s(\mathbf{R}^d)$ for fixed $x \in \mathbf{R}^d$, and therefore its Fourier transform is an element in $\mathcal{S}'_s(\mathbf{R}^d)$. The fact that the Fourier transform is actually a smooth function given by the formula (2.5) is proved in Appendix.

If $T(f,\phi) \equiv V_\phi f$ for $f, \phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, then T extends uniquely to sequentially continuous mappings

$$T : \mathcal{S}'_s(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d) \to \mathcal{S}'_s(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d}),$$

$$T : \mathcal{S}'_s(\mathbf{R}^d) \times \mathcal{S}'_s(\mathbf{R}^d) \to \mathcal{S}'_s(\mathbf{R}^{2d}),$$

and similarly when $\mathcal{S}_s$ and $\mathcal{S}'_s$ are replaced by Σ_s and Σ'_s, respectively, or by $\mathscr{S}$ and $\mathscr{S}'$, respectively [6, 33].

Similar properties hold true if instead $T(f,\phi) = W_{f,\phi}$, where $W_{f,\phi}$ is the cross-Wigner distribution of $f \in \mathcal{S}'_s(\mathbf{R}^d)$ and $\phi \in \mathcal{S}_s(\mathbf{R}^d)$, given by

$$W_{f,\phi}(x,\xi) \equiv \mathscr{F}(f(x + \cdot/2)\overline{\phi(x - \cdot/2)})(\xi).$$

If $q \in [1,\infty]$, $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, $f \in L^q_{(\omega)}(\mathbf{R}^d)$ and $\phi \in \Sigma_1(\mathbf{R}^d)$ then $V_\phi f$ and $W_{f,\phi}$ take the forms

$$V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi \rangle} dy, \tag{2.5a}$$

and

$$W_{f,\phi}(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x + y/2)\overline{\phi(x - y/2)}e^{-i\langle y,\xi\rangle} dy.$$

Here, $L^p_{(\omega)}(\mathbf{R}^d)$ for $p \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ denotes the space of all $f \in L^p_{\mathrm{loc}}(\mathbf{R}^d)$ such that $f\omega \in L^p(\mathbf{R}^d)$, and $\|f\|_{L^p_{(\omega)}} = \|f\omega\|_{L^p}$.

For $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ and $\Phi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})\backslash 0$ the *symplectic STFT* $\mathcal{V}_\Phi a$ of a with respect to Φ is defined similarly as the STFT by

$$\mathcal{V}_\Phi a(X,Y) = \mathscr{F}_\sigma\big(a\,\overline{\Phi(\,\cdot\,-X)}\big)(Y), \quad X,Y \in \mathbf{R}^{2d}.$$

There are several ways to characterize Gelfand–Shilov function and distribution spaces, for example in terms of expansions with respect to Hermite functions [13, 24], or in terms of the Fourier transform and the STFT [5, 21, 33, 37].

2.3. *An extended family of pseudodifferential calculi*

We consider a family of pseudodifferential calculi parameterized by the real $d \times d$ matrices, denoted $\mathbf{M}(d,\mathbf{R})$ [3, 36]. Let $s \geq \frac{1}{2}$, let $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ and let $A \in \mathbf{M}(d,\mathbf{R})$ be fixed. The pseudodifferential operator $\mathrm{Op}_A(a)$ is the linear and continuous operator

$$\mathrm{Op}_A(a)f(x) = (2\pi)^{-d} \iint_{\mathbf{R}^{2d}} a(x - A(x - y), \xi)f(y)e^{i\langle x-y,\xi\rangle} dy d\xi, \qquad (2.6)$$

when $f \in \mathcal{S}_s(\mathbf{R}^d)$. For $a \in \mathcal{S}'_s(\mathbf{R}^{2d})$ the operator $\mathrm{Op}_A(a)$ is defined as the linear and continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}'_s(\mathbf{R}^d)$ with distribution kernel

$$K_{a,A}(x,y) = (2\pi)^{-\frac{d}{2}}\mathscr{F}_2^{-1}a(x - A(x - y), x - y). \qquad (2.7)$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}'_s(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense since

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x - A(x - y), x - y), \qquad (2.8)$$

are homeomorphisms on $\mathcal{S}'_s(\mathbf{R}^{2d})$.

An important special case is $A = tI$, with $t \in \mathbf{R}$ and $I \in \mathbf{M}(d,\mathbf{R})$ denoting the identity matrix. In this case, we write $\mathrm{Op}_t(a) = \mathrm{Op}_{tI}(a)$. The normal or Kohn–Nirenberg representation $a(x,D)$ corresponds to $t = 0$, and the Weyl quantization $\mathrm{Op}^w(a)$ corresponds to $t = \frac{1}{2}$. Thus

$$a(x,D) = \mathrm{Op}_0(a) = \mathrm{Op}(a) \quad \text{and} \quad \mathrm{Op}^w(a) = \mathrm{Op}_{1/2}(a).$$

The Weyl calculus is connected to the Wigner distribution with the formula

$$(\mathrm{Op}^w(a)f,g)_{L^2(\mathbf{R}^d)} = (2\pi)^{-\frac{d}{2}}(a, W_{g,f})_{L^2(\mathbf{R}^{2d})},$$

$$a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d}), \quad f,g \in \mathcal{S}_{1/2}(\mathbf{R}^d).$$

For every $a_1 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d,\mathbf{R})$, there is a unique $a_2 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ such that $\mathrm{Op}_{A_1}(a_1) = \mathrm{Op}_{A_2}(a_2)$. The following restatement of [36, Proposition 2.1] explains the relations between a_1 and a_2.

Proposition 2.1. *Let $a_1, a_2 \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$. Then*

$$\mathrm{Op}_{A_1}(a_1) = \mathrm{Op}_{A_2}(a_2) \quad \Leftrightarrow \quad e^{i\langle A_2 D_\xi, D_x \rangle} a_2(x, \xi) = e^{i\langle A_1 D_\xi, D_x \rangle} a_1(x, \xi). \quad (2.9)$$

2.4. *Modulation spaces*

Let $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)\backslash 0$, $p, q \in (0, \infty]$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$. The *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the set of all $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$ such that $V_\phi f \in L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$, and $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is equipped with the quasi-norm

$$f \mapsto \|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}. \quad (2.10)$$

Here $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ is the space of all complex-valued measurable functions F on $\mathbf{R}^{2d}$ such that $\|F\|_{L^{p,q}_{(\omega)}} < \infty$, where

$$\|F\|_{L^{p,q}_{(\omega)}} \equiv \|F_{p,\omega}\|_{L^q} \quad \text{with} \quad F_{p,\omega}(\xi) \equiv \|F(\,\cdot\,, \xi)\omega(\,\cdot\,, \xi)\|_{L^p}, \quad \xi \in \mathbf{R}^d.$$

On the even-dimensional phase space $\mathbf{R}^{2d}$, one may define modulation spaces based on the symplectic STFT. Thus if $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$, $p, q \in (0, \infty]$ and $\Phi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})\backslash 0$ are fixed, the *symplectic modulation space* $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ is obtained by replacing the STFT $a \mapsto V_\Phi a$ by the symplectic STFT $a \mapsto \mathcal{V}_\Phi a$ in (2.10). It holds (cf. [7])

$$\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d}) = M^{p,q}_{(\omega_0)}(\mathbf{R}^{2d}), \quad \omega(x, \xi, y, \eta) = \omega_0(x, \xi, -2\eta, 2y),$$

so all properties that are valid for $M^{p,q}_{(\omega)}$ carry over to $\mathcal{M}^{p,q}_{(\omega)}$.

In the following propositions, we list some properties of modulation spaces and refer to [8–11, 17, 32] for proofs.

Proposition 2.2. *Let $p, q \in (0, \infty]$.*

(1) *If $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ then $\Sigma_1(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq \Sigma'_1(\mathbf{R}^d)$.*

(2) *If $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ satisfies (2.3) for every $r > 0$, then $\mathcal{S}_1(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq \mathcal{S}'_1(\mathbf{R}^d)$.*

(3) *If $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ then $\mathscr{S}(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d)$.*

Proposition 2.3. *Let $r \in (0, 1]$, $p, q, p_j, q_j \in (0, \infty]$ and $\omega, \omega_j, v \in \mathscr{P}_E(\mathbf{R}^{2d})$, $j = 1, 2$, satisfy $r \leq \min(p, q)$, $p_1 \leq p_2$, $q_1 \leq q_2$, $\omega_2 \lesssim \omega_1$, and let ω be v-moderate.*

(1) *If $\phi \in M^r_{(v)}(\mathbf{R}^d)\backslash 0$ then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (2.10) is finite. In particular $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is independent of the choice of $\phi \in M^r_{(v)}(\mathbf{R}^d)\backslash 0$. The space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a quasi-Banach space under the quasi-norm (2.10), and different choices of ϕ give rise to equivalent quasi-norms. If $p, q \geq 1$ then $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a Banach space with norm (2.10).*

(2) *$M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d) \subseteq M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d)$.*

We will rely heavily on Gabor expansions so we need the following concepts. The operators in Definition 2.4 are well defined and continuous by the analysis in [17, Chaps. 11–14].

Definition 2.4. Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, let $\Lambda^2 = \Lambda \times \Lambda \subseteq \mathbf{R}^{2d}$, let $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that ω is v-moderate, and let $\phi, \psi \in M^1_{(v)}(\mathbf{R}^d)$.

(1) The *Gabor analysis operator* $C_\phi = C_\phi^\Lambda$ is the operator from $M^\infty_{(\omega)}(\mathbf{R}^d)$ to $\ell^\infty_{(\omega)}(\Lambda^2)$ given by

$$C_\phi^\Lambda f \equiv \{V_\phi f(j, \iota)\}_{j, \iota \in \Lambda}.$$

(2) The *Gabor synthesis operator* $D_\psi = D_\psi^\Lambda$ is the operator from $\ell^\infty_{(\omega)}(\Lambda^2)$ to $M^\infty_{(\omega)}(\mathbf{R}^d)$ given by

$$D_\psi^\Lambda c \equiv \sum_{j, \iota \in \Lambda} c(j, \iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j).$$

(3) The *Gabor frame operator* $S_{\phi, \psi} = S_{\phi, \psi}^\Lambda$ is the operator on $M^\infty_{(\omega)}(\mathbf{R}^d)$ given by $D_\psi^\Lambda \circ C_\phi^\Lambda$, i.e.

$$S_{\phi, \psi}^\Lambda f \equiv \sum_{j, \iota \in \Lambda} V_\phi f(j, \iota) \, e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j).$$

The following result is a consequence of [17, Theorem 13.1.1] (see also [16, Theorem S]).

Proposition 2.5. *Suppose* $v \in \mathscr{P}_E(\mathbf{R}^{2d})$ *is submultiplicative, and let* $\phi \in M^1_{(v)}(\mathbf{R}^d) \backslash 0$. *There is a constant* $\theta_0 > 0$ *such that the Gabor frame operator* $S_{\phi, \phi}^\Lambda$ *is a homeomorphism on* $M^1_{(v)}(\mathbf{R}^d)$ *when* $\Lambda = \theta \mathbf{Z}^d$ *and* $\theta \in (0, \theta_0]$. *The Gabor systems*

$$\{e^{i\langle \cdot, \iota \rangle} \phi(\cdot - j)\}_{(j, \iota) \in \Lambda} \quad and \quad \{e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j)\}_{(j, \iota) \in \Lambda}, \tag{2.11}$$

are dual frames for $L^2(\mathbf{R}^d)$ *when* $\psi = (S_{\phi, \phi}^\Lambda)^{-1} \phi \in M^1_{(v)}(\mathbf{R}^d)$ *and* $\theta \in (0, \theta_0]$.

Let v, ϕ and Λ be as in Proposition 2.5. Then $(S_{\phi, \phi}^\Lambda)^{-1} \phi$ is called the *canonical dual window* of ϕ, with respect to Λ. We have

$$S_{\phi, \phi}^\Lambda (e^{i\langle \cdot, \iota \rangle} f(\cdot - j)) = e^{i\langle \cdot, \iota \rangle} (S_{\phi, \phi}^\Lambda f)(\cdot - j),$$

when $f \in M^\infty_{(1/v)}(\mathbf{R}^d)$ and $(j, \iota) \in \Lambda$.

The next result concerns Gabor expansion of modulation spaces. It is a special case of [34, Theorem 3.7] (see also [17, Corollaries 12.2.5 and 12.2.6] and [11, Theorem 3.7]).

Proposition 2.6. *Let* $\theta > 0$, $\Lambda = \theta \mathbf{Z}^d$,

$$\Lambda^2 = \Lambda \times \Lambda = \{(j, \iota)\}_{j, \iota \in \Lambda} \subseteq \mathbf{R}^{2d},$$

let $p, q, r \in (0, \infty]$ *satisfy* $r \leq \min(1, p, q)$, *and let* $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ *be such that* ω *is* v-moderate. *Suppose* $\phi, \psi \in M^r_{(v)}(\mathbf{R}^d)$ *are such that* (2.11) *are dual frames for*

$L^2(\mathbf{R}^d)$. *Then the following is true*:

(1) *The operators*

$$C_\phi^\Lambda : M_{(\omega)}^{p,q}(\mathbf{R}^d) \mapsto \ell_{(\omega)}^{p,q}(\Lambda^2) \quad \text{and} \quad D_\psi^\Lambda : \ell_{(\omega)}^{p,q}(\Lambda^2) \mapsto M_{(\omega)}^{p,q}(\mathbf{R}^d)$$

are continuous.

(2) *The operators* $S_{\phi,\psi} \equiv D_\psi \circ C_\phi$ *and* $S_{\psi,\phi} \equiv D_\phi \circ C_\psi$ *are both the identity map on* $M_{(\omega)}^{p,q}(\mathbf{R}^d)$, *and if* $f \in M_{(\omega)}^{p,q}(\mathbf{R}^d)$, *then*

$$
\begin{aligned}
f &= \sum_{j,\iota \in \Lambda} V_\phi f(j,\iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j) \\
&= \sum_{j,\iota \in \Lambda} V_\psi f(j,\iota) e^{i\langle \cdot, \iota \rangle} \phi(\cdot - j),
\end{aligned}
\tag{2.12}
$$

with unconditional quasi-norm convergence in $M_{(\omega)}^{p,q}$ *when* $p, q < \infty$, *and with convergence in* $M_{(\omega)}^\infty$ *with respect to the weak* topology otherwise.*

(3) *If* $f \in M_{(1/v)}^\infty(\mathbf{R}^d)$, *then*

$$\|f\|_{M_{(\omega)}^{p,q}} \asymp \|V_\phi f\|_{\ell_{(\omega)}^{p,q}(\Lambda^2)} \asymp \|V_\psi f\|_{\ell_{(\omega)}^{p,q}(\Lambda^2)}.$$

The series (2.12) are called *Gabor expansions of* f *with respect to* ϕ, ψ *and* Λ.

Remark 2.7. There are many ways to achieve dual frames (2.11) satisfying the required properties in Proposition 2.6. In fact, let $v, v_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$ be submultiplicative such that ω is v-moderate and

$$L_{(v_0)}^1(\mathbf{R}^{2d}) \subseteq \bigcap_{0 < r \leq 1} L^r(\mathbf{R}^{2d}).$$

This inclusion is satisfied e.g. for $v_0(x) = e^{\varepsilon|x|}$ with $\varepsilon > 0$. Proposition 2.5 guarantees that for some choice of $\phi, \psi \in M_{(v_0 v)}^1(\mathbf{R}^d) \subseteq \bigcap_{0 < r \leq 1} M_{(v)}^r(\mathbf{R}^d)$ and lattice $\Lambda \subseteq \mathbf{R}^d$, the sets in (2.11) where $\psi = (S_{\phi,\phi}^\Lambda)^{-1}\phi$, are dual frames.

We usually assume that $\Lambda = \theta \mathbf{Z}^d$, with $\theta > 0$ small enough to guarantee the hypotheses in Propositions 2.5 and 2.6 be fulfilled, and that the window function and its dual belong to $M_{(v)}^r$ for every $r > 0$. This is always possible, in view of Remark 2.7.

We need the following version of Proposition 2.5, which is a consequence of [3, Corollary 3.2] and the Fourier invariance of $\Sigma_1(\mathbf{R}^{2d})$.

Lemma 2.8. *Suppose* $v \in \mathscr{P}_E(\mathbf{R}^{4d})$ *is submultiplicative, let* $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d) \backslash 0$, *and let*

$$\Phi(x,\xi) = \phi_1(x)\overline{\widehat{\phi_2}(\xi)}e^{-i\langle x,\xi \rangle}. \tag{2.13}$$

Then there is a lattice $\Lambda^2 \subseteq \mathbf{R}^{2d}$ such that

$$\{\Phi(x-j,\xi-\iota)e^{i(\langle x,\kappa\rangle+\langle k,\xi\rangle)}\}_{(j,\iota),(k,\kappa)\in\Lambda^2} \tag{2.14}$$

is a Gabor frame for $L^2(\mathbf{R}^{2d})$ with canonical dual frame

$$\{\Psi(x-j,\xi-\iota)e^{i(\langle x,\kappa\rangle+\langle k,\xi\rangle)}\}_{(j,\iota),(k,\kappa)\in\Lambda^2},$$

and

$$\Psi = (S_{\Phi,\Phi}^{\Lambda^2})^{-1}\Phi \in \bigcap_{r>0} M_{(v)}^r(\mathbf{R}^{2d}).$$

The right-hand side of (2.13) is called the cross-Rihaczek distribution of ϕ_1 and ϕ_2 [17].

Remark 2.9. The last conclusion in Lemma 2.8 is a consequence of the sharper result [25, Lemma 2].

2.5. *Pseudodifferential operators and Gabor analysis*

In order to discuss a reformulation of pseudodifferential operators by means of Gabor analysis, we need the following matrix concepts [36].

Definition 2.10. Let $p,q \in (0,\infty]$, $\theta > 0$, let J be an index set and let $\Lambda = \theta\mathbf{Z}^d$ be a lattice, and let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$.

(1) $\mathbb{U}_0'(J)$ is the set of all matrices $A = (\boldsymbol{a}(j,k))_{j,k\in J}$ with entries in $\mathbf{C}$;
(2) $\mathbb{U}_0(J)$ is the set of all $A = (\boldsymbol{a}(j,k))_{j,k\in J} \in \mathbb{U}_0'(J)$ such that $\boldsymbol{a}(j,k) \neq 0$ for at most finitely many $(j,k) \in J \times J$;
(3) for $A = (\boldsymbol{a}(j,k))_{j,k\in\Lambda} \in \mathbb{U}_0'(\Lambda)$ let

$$H_{A,\omega}(j,k) = \boldsymbol{a}(j,j-k)\omega(j,j-k) \quad \text{and} \quad h_{A,p,\omega}(k) = \|H_{A,\omega}(\,\cdot\,,k)\|_{\ell^p}. \tag{2.15}$$

The set $\mathbb{U}^{p,q}(\omega,\Lambda)$ consists of all matrices $A = (\boldsymbol{a}(j,k))_{j,k\in\Lambda}$ such that

$$\|(\boldsymbol{a}(j,k))_{j,k\in\Lambda}\|_{\mathbb{U}^{p,q}(\omega,\Lambda)} \equiv \|h_{A,p,\omega}\|_{\ell^q} \tag{2.16}$$

is finite.

$\mathbb{U}^{p,q}(\omega,\Lambda)$ is a quasi-Banach space, and if $p,q \geq 1$ it is a Banach space.

If J is an index set then $A = (\boldsymbol{a}(j,k))_{j,k\in J} \in \mathbb{U}_0'(J)$ is called properly supported if the sets

$$\{j \in J \,;\, \boldsymbol{a}(j,k_0) \neq 0\} \quad \text{and} \quad \{k \in J \,;\, \boldsymbol{a}(j_0,k) \neq 0\},$$

are finite for every $j_0, k_0 \in J$. The set of properly supported matrices is denoted $\mathbb{U}_{\mathrm{p}}(J)$, and evidently $\mathbb{U}_0(J) \subseteq \mathbb{U}_{\mathrm{p}}(J)$. The sets $\mathbb{U}_0(J)$ and $\mathbb{U}_{\mathrm{p}}(J)$ are rings under matrix multiplication, and $\mathbb{U}_0'(J)$ is a $\mathbb{U}_{\mathrm{p}}(J)$-module with respect to matrix multiplication.

Let $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d)\backslash 0$, let Φ be defined by (2.13), let $\Lambda \subseteq \mathbf{R}^d$ be a lattice such that $\Lambda^2 = \Lambda \times \Lambda \subseteq \mathbf{R}^{2d}$ makes (2.14) a Gabor frame in accordance with Lemma 2.8, and let $\Psi = (S_{\Phi\Phi}^{\Lambda^2})^{-1}\Phi$ be the canonical dual window of Φ. Suppose $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{4d})$ and set

$$\omega(x, \xi y, \eta) = \omega_0(x, \eta, \xi - \eta, y - x). \tag{2.17}$$

Let $a \in M^{p,q}_{(\omega_0)}(\mathbf{R}^{2d})$, define

$$\boldsymbol{a}(\boldsymbol{j}, \boldsymbol{k}) = V_\Psi a(j, \kappa, \iota - \kappa, k - j)e^{i\langle k-j, \kappa\rangle},$$

where

$$\boldsymbol{j} = (j, \iota) \in \Lambda^2 \quad \text{and} \quad \boldsymbol{k} = (k, \kappa) \in \Lambda^2, \tag{2.18}$$

and define the matrix

$$A = (\boldsymbol{a}(\boldsymbol{j}, \boldsymbol{k}))_{\boldsymbol{j}, \boldsymbol{k} \in \Lambda^2}.$$

Then it follows from Propositions 2.5 and 2.6 that

$$\|a\|_{M^{p,q}_{(\omega_0)}} \asymp \|A\|_{\mathbb{U}^{p,q}(\omega, \Lambda^2)}, \tag{2.19}$$

provided θ is sufficiently small.

By identifying matrices with corresponding linear operators, [35, Lemma 3.3] gives

$$\text{Op}(a) = D_{\phi_1} \circ A \circ C_{\phi_2}. \tag{2.20}$$

Hence, if $b \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$,

$$\boldsymbol{b}(\boldsymbol{j}, \boldsymbol{k}) = V_\Psi b(j, \kappa, \iota - \kappa, k - j)e^{i\langle k-j, \kappa\rangle}, \quad \boldsymbol{j}, \boldsymbol{k} \in \Lambda^2,$$

$$B = (\boldsymbol{b}(\boldsymbol{j}, \boldsymbol{k}))_{\boldsymbol{j}, \boldsymbol{k} \in \Lambda^2},$$

and the matrix C is defined as

$$C = C_{\phi_2} \circ D_{\phi_1}, \tag{2.21}$$

then

$$\text{Op}(a\#_0 b) = \text{Op}(a) \circ \text{Op}(b) = D_{\phi_1} \circ (A \circ C \circ B) \circ C_{\phi_2}, \tag{2.22}$$

and

$$\|a\#_0 b\|_{M^{p,q}_{(\omega_0)}} \asymp \|A \circ C \circ B\|_{\mathbb{U}^{p,q}(\omega, \Lambda^2)}. \tag{2.23}$$

2.6. *Composition of pseudodifferential operators with symbols in Banach modulation spaces*

We recall algebraic results for pseudodifferential operators with symbols in modulation spaces with Lebesgue exponents not smaller than one [7, 23, 36].

If $A \in \mathbf{M}(d, \mathbf{R})$, then the product $\#_A$ with N factors

$$(a_1, \ldots, a_N) \mapsto a_1 \#_A \cdots \#_A a_N, \tag{2.24}$$

from $\mathcal{S}_{1/2}(\mathbf{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is defined by the formula

$$\mathrm{Op}_A(a_1 \#_A \cdots \#_A a_N) = \mathrm{Op}_A(a_1) \circ \cdots \circ \mathrm{Op}_A(a_N).$$

The map (2.24) can be extended in different ways, e.g. as in [7, Theorem 2.11] which is stated in a generalized form in Theorem 2.11 below. Assume that the weight functions satisfy

$$\omega_0(T_A(X_N, X_0)) \lesssim \prod_{j=1}^{N} \omega_j(T_A(X_j, X_{j-1})), \quad X_0, \ldots, X_N \in \mathbf{R}^{2d}, \tag{2.25}$$

where

$$T_A(X, Y) = (y + A(x - y), \xi + A^*(\eta - \xi), \eta - \xi, x - y),$$
$$X = (x, \xi) \in \mathbf{R}^{2d}, \quad Y = (y, \eta) \in \mathbf{R}^{2d}. \tag{2.26}$$

Here A^* denotes A transposed. Assume that the Lebesgue parameters satisfy

$$\max(\mathsf{R}_N(\boldsymbol{q}'), 0) \leq \min_{j=1,\ldots,N}\left(\frac{1}{p_0'}, \frac{1}{q_0}, \frac{1}{p_j}, \frac{1}{q_j'}, \mathsf{R}_N(\boldsymbol{p})\right) \tag{2.27}$$

or

$$\mathsf{R}_N(\boldsymbol{p}) \geq 0, \quad \frac{1}{q_0} \leq \frac{1}{p_0'} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{q_j'} \leq \frac{1}{p_j} \leq \frac{1}{2}, \quad j = 1, \ldots, N, \tag{2.28}$$

where

$$\mathsf{R}_N(\boldsymbol{p}) = (N-1)^{-1}\left(\sum_{j=1}^{N}\frac{1}{p_j} - \frac{1}{p_0}\right),$$

$$\boldsymbol{p} = (p_0, p_1, \ldots, p_N) \in [1, \infty]^{N+1}.$$

Theorem 2.11. *Suppose $p_j, q_j \in [1, \infty]$, $j = 0, 1, \ldots, N$ satisfy (2.27) or (2.28), and suppose $\omega_j \in \mathscr{P}_E(\mathbf{R}^{4d})$, $j = 0, 1, \ldots, N$, satisfy (2.25) and (2.26). Then the map (2.24) from $\mathcal{S}_{1/2}(\mathbf{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a continuous and associative map from $M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^{2d}) \times \cdots \times M_{(\omega_N)}^{p_N, q_N}(\mathbf{R}^{2d})$ to $M_{(\omega_0)}^{p_0, q_0}(\mathbf{R}^{2d})$, and*

$$\|a_1 \#_A \cdots \#_A a_N\|_{M_{(\omega_0)}^{p_0, q_0}} \lesssim \prod_{j=1}^{N} \|a_j\|_{M_{(\omega_j)}^{p_j, q_j}},$$

for $a_j \in M_{(\omega_j)}^{p_j, q_j}(\mathbf{R}^{2d})$, $j = 1, \ldots, N$.

Theorem 2.11 follows by similar arguments as in the proof of [7, Theorem 2.11]. The details are left for the reader.

Remark 2.12. We note that the definition of T_A in [7, Eq. (2.30)] is incorrect and should be replaced by (2.26) with $A = tI$, in order for [7, Theorem 2.11] to hold. A corrected version of [7] has been posted on arXiv.

3. Composition of Pseudodifferential Operators with Symbols in Quasi-Banach Modulation Spaces

In this section, we deduce a composition result for pseudodifferential operators with symbols in modulation spaces with Lebesgue parameters in $(0, \infty]$.

If $A \in \mathbf{M}(d, \mathbf{R})$, then the map

$$(a_1, a_2) \mapsto a_1 \#_A a_2 \tag{3.1}$$

from $\mathcal{S}_{1/2}(\mathbf{R}^{2d}) \times \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is defined by

$$\mathrm{Op}_A(a_1 \#_A a_2) = \mathrm{Op}_A(a_1) \circ \mathrm{Op}_A(a_2).$$

The following result is the principal result of this paper. It concerns sufficient conditions for the unique extension of (3.1) to symbols in quasi-Banach modulation spaces.

The weight functions are assumed to obey the estimates

$$\omega_0(T_A(Z, X)) \lesssim \omega_1(T_A(Y, X))\omega_2(T_A(Z, Y)), \quad X, Y, Z \in \mathbf{R}^{2d}, \tag{3.2}$$

where

$$T_A(X, Y) = (y + A(x - y), \xi + A^*(\eta - \xi), \eta - \xi, x - y),$$
$$X = (x, \xi) \in \mathbf{R}^{2d}, \quad Y = (y, \eta) \in \mathbf{R}^{2d}. \tag{3.3}$$

(cf. (2.25) and (2.26)).

Theorem 3.1. *Let $A \in \mathbf{M}(d, \mathbf{R})$ and suppose $\omega_j \in \mathscr{P}_E(\mathbf{R}^{4d})$, $j = 0, 1, 2$, satisfy (3.2) and (3.3). Suppose $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$, satisfy*

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2}, \tag{3.4}$$

and either

$$q_1, q_2 \leq q_0 \leq \min(1, p_0) \tag{3.5}$$

or

$$\min(1, p_0) \leq q_1, q_2 \leq q_0 \quad and \quad \frac{1}{\min(1, p_0)} + \frac{1}{q_0} \leq \frac{1}{q_1} + \frac{1}{q_2}. \tag{3.6}$$

Then the map $(a_1, a_2) \mapsto a_1 \#_A a_2$ *from* $\mathcal{S}_{1/2}(\mathbf{R}^{2d}) \times \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ *to* $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^{2d}) \times M_{(\omega_2)}^{p_2, q_2}(\mathbf{R}^{2d})$ *to* $M_{(\omega_0)}^{p_0, q_0}(\mathbf{R}^{2d})$, *and*

$$\|a_1 \#_A a_2\|_{M_{(\omega_0)}^{p_0, q_0}} \lesssim \|a_1\|_{M_{(\omega_1)}^{p_1, q_1}} \|a_2\|_{M_{(\omega_2)}^{p_2, q_2}},$$

for all $a_1 \in M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^{2d})$ *and* $a_2 \in M_{(\omega_2)}^{p_2, q_2}(\mathbf{R}^{2d})$.

We need some preparations for the proof. The following result contains the needed continuity properties for matrix operators.

Proposition 3.2. *Let* $\Lambda \subseteq \mathbf{R}^d$ *be a lattice, let* $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$, *be such that* (3.4)–(3.6) *hold, and suppose* $\omega_0, \omega_1, \omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ *satisfy*

$$\omega_0(x, z) \lesssim \omega_1(x, y)\, \omega_2(y, z), \quad x, y, z \in \mathbf{R}^d.$$

Then the map $(A_1, A_2) \mapsto A_1 \circ A_2$ *from* $\mathbb{U}_0(\Lambda) \times \mathbb{U}_0(\Lambda)$ *to* $\mathbb{U}_0(\Lambda)$ *extends uniquely to a continuous map from* $\mathbb{U}^{p_1, q_1}(\omega_1, \Lambda) \times \mathbb{U}^{p_2, q_2}(\omega_2, \Lambda)$ *to* $\mathbb{U}^{p_0, q_0}(\omega_0, \Lambda)$, *and*

$$\|A_1 \circ A_2\|_{\mathbb{U}^{p_0, q_0}(\omega_0, \Lambda)} \lesssim \|A_1\|_{\mathbb{U}^{p_1, q_1}(\omega_1, \Lambda)} \|A_2\|_{\mathbb{U}^{p_2, q_2}(\omega_2, \Lambda)}. \tag{3.7}$$

Proof. Let $\mathbb{U}'_{0,+}(\Lambda)$ be the set of all $A \in \mathbb{U}'_0(\Lambda)$ with non-negative entries, let $A_m = (\boldsymbol{a}_m(j, k))_{j, k \in \Lambda} \in \mathbb{U}_0(\Lambda) \cap \mathbb{U}'_{0,+}(\Lambda)$, $m = 1, 2$, denote the matrix elements of $B = A_1 \circ A_2$ by $\boldsymbol{b}(j, k)$, $j, k \in \Lambda$, and set $p = p_0$, $q = q_0$, $\omega = \omega_0$,

$$\mathfrak{a}_m(j, k) \equiv |\boldsymbol{a}_m(j, j - k)| \omega_m(j, j - k) \quad \text{and} \quad \mathfrak{b}(j, k) \equiv |\boldsymbol{b}(j, j - k)| \omega(j, j - k),$$

$m = 1, 2$. Then

$$\|A_m\|_{\mathbb{U}^{p_m, q_m}(\omega_m, \Lambda)} = \|\mathfrak{a}_m\|_{\ell^{p_m, q_m}}, \quad m = 1, 2,$$

$$\|A_1 \circ A_2\|_{\mathbb{U}^{p, q}(\omega, \Lambda)} = \|\mathfrak{b}\|_{\ell^{p, q}},$$

and we first prove

$$\|\mathfrak{b}\|_{\ell^{p, q}} \le \|\mathfrak{a}_1\|_{\ell^{p_1, q_1}} \|\mathfrak{a}_2\|_{\ell^{p_2, q_2}}.$$

We have

$$\mathfrak{b}(j, k) \le \sum_{l \in \Lambda} \mathfrak{a}_1(j, l) \mathfrak{a}_2(j - l, k - l). \tag{3.8}$$

In order to estimate $\|\mathfrak{b}(\,\cdot\,, k)\|_{\ell^p}$ we consider the cases $p < 1$ and $p \ge 1$ separately.

First assume that $p < 1$, and set $r_j = \frac{p_j}{p}$. Then $\frac{1}{r_1} + \frac{1}{r_2} \ge 1$ by assumption (3.4), and therefore Hölder's inequality yields for $k \in \Lambda$

$$\|\mathfrak{b}(\,\cdot\,, k)\|_{\ell^p}^p \le \sum_{j \in \Lambda} \left(\sum_{l \in \Lambda} \mathfrak{a}_1(j, l) \mathfrak{a}_2(j - l, k - l) \right)^p$$

$$\le \sum_{l \in \Lambda} \sum_{j \in \Lambda} (\mathfrak{a}_1(j, l) \mathfrak{a}_2(j - l, k - l))^p$$

$$\leq \sum_{l \in \Lambda} \|\mathfrak{a}_1(\,\cdot\,,l)^p\|_{\ell^{r_1}} \|\mathfrak{a}_2(\,\cdot\,,k-l)^p\|_{\ell^{r_2}}$$

$$= \sum_{l \in \Lambda} \|\mathfrak{a}_1(\,\cdot\,,l)\|_{\ell^{p_1}}^p \|\mathfrak{a}_2(\,\cdot\,,k-l)\|_{\ell^{p_2}}^p,$$

that is

$$\|\mathfrak{b}(\,\cdot\,,k)\|_{\ell^p} \leq (c_1 * c_2(k))^{\frac{1}{p}},$$

with $c_m(k) = \|\mathfrak{a}_m(\,\cdot\,,k)\|_{\ell^{p_m}}^p$, $m = 1,2$.

In order to estimate $(c_1 * c_2)^{\frac{1}{p}}$ we first assume (3.5). Then

$$\|\mathfrak{b}\|_{\ell^{p,q}} \leq \|(c_1 * c_2)^{\frac{1}{p}}\|_{\ell^q} = \|c_1 * c_2\|_{\ell^{q/p}}^{\frac{1}{p}}$$

$$\leq (\|c_1\|_{\ell^{q/p}} \|c_2\|_{\ell^{q/p}})^{\frac{1}{p}} = \|\mathfrak{a}_1\|_{\ell^{p_1,q}} \|\mathfrak{a}_2\|_{\ell^{p_2,q}} \leq \|\mathfrak{a}_1\|_{\ell^{p_1,q_1}} \|\mathfrak{a}_2\|_{\ell^{p_2,q_2}},$$

and the result follows in this case.

If instead (3.6) holds then $q \geq q_1, q_2 \geq p$, and $r_j = q_j/p$, $j = 1,2$, and $r = q/p$ satisfy

$$r_1, r_2, r \geq 1 \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} \geq 1 + \frac{1}{r}.$$

Hence Young's inequality may be applied and gives

$$\|\mathfrak{b}\|_{\ell^{p,q}} \leq \|c_1 * c_2\|_{\ell^r}^{\frac{1}{p}} \leq (\|c_1\|_{\ell^{r_1}} \|c_2\|_{\ell^{r_2}})^{\frac{1}{p}} = \|\mathfrak{a}_1\|_{\ell^{p_1,q_1}} \|\mathfrak{a}_2\|_{\ell^{p_2,q_2}},$$

and the result follows in this case as well.

Next we consider the case $p \geq 1$. By Minkowski's and Hölder's inequalities and the assumption (3.4) we get from (3.8)

$$\|\mathfrak{b}(\,\cdot\,,k)\|_{\ell^p} \leq \sum_{l \in \Lambda} \|\mathfrak{a}_1(\cdot,l)\mathfrak{a}_2(\cdot -l,k-l)\|_{\ell^p}$$

$$\leq \sum_{l \in \Lambda} \|\mathfrak{a}_1(\,\cdot\,,l)\|_{\ell^{p_1}} \|\mathfrak{a}_2(\,\cdot\,,k-l)\|_{\ell^{p_2}} = c_1 * c_2(k), \tag{3.9}$$

where $c_m(k) = \|\mathfrak{a}_m(\,\cdot\,,k)\|_{\ell^{p_m}}$, $m = 1,2$.

If (3.6) holds then $q \geq q_1, q_2 \geq 1$ and Young's inequality gives

$$\|\mathfrak{b}\|_{\ell^{p,q}} \leq \|c_1 * c_2\|_{\ell^q} \leq \|c_1\|_{\ell^{q_1}} \|c_2\|_{\ell^{q_2}} = \|\mathfrak{a}_1\|_{\ell^{p_1,q_1}} \|\mathfrak{a}_2\|_{\ell^{p_2,q_2}}$$

and the result follows. If instead (3.5) holds then $q \leq 1$ and (3.9) gives

$$\|\mathfrak{b}\|_{\ell^{p,q}} \leq \|c_1 * c_2\|_{\ell^q} \leq \|c_1\|_{\ell^q} \|c_2\|_{\ell^q} = \|\mathfrak{a}_1\|_{\ell^{p_1,q}} \|\mathfrak{a}_2\|_{\ell^{p_2,q}}$$

$$\leq \|\mathfrak{a}_1\|_{\ell^{p_1,q_1}} \|\mathfrak{a}_2\|_{\ell^{p_2,q_2}}.$$

Thus we have proved (3.7) when $A_1, A_2 \in \mathbb{U}_0(\Lambda) \cap \mathbb{U}'_{0,+}(\Lambda)$.

By Beppo–Levi's theorem or Fatou's lemma applied to the previous situation, we obtain that $A_1 \circ A_2$ is uniquely defined as an element in $\mathbb{U}^{p,q}(\omega,\Lambda)$ and (3.7) holds, provided $A_m \in \mathbb{U}^{p_m,q_m}(\omega_m,\Lambda) \cap \mathbb{U}'_{0,+}(\Lambda)$ for $m = 1,2$.

For $A_m \in \mathbb{U}^{p_m, q_m}(\omega_m, \Lambda)$, $m = 1, 2$, there are unique

$$A_{m,k} \in \mathbb{U}^{p_m, q_m}(\omega_m, \Lambda) \cap \mathbb{U}'_{0,+}(\Lambda), \quad m = 1, 2, \quad k = 1, \ldots, 4,$$

such that

$$A_m = \sum_{k=1}^{4} i^k A_{m,k},$$

and we have

$$\|A_{m,k}\|_{\mathbb{U}^{p_m, q_m}(\omega_m, \Lambda)} \leq \|A_m\|_{\mathbb{U}^{p_m, q_m}(\omega_m, \Lambda)}, \quad m = 1, 2, \quad k = 1, \ldots, 4.$$

Since the assertion holds true for $A_{1,k}$ and $A_{2,l}$ in place of A_1 and A_2, it follows from the latter estimate that

$$A_1 \circ A_2 = \sum_{k,l=1}^{4} i^{k+l} A_{1,k} \circ A_{2,l} \in \mathbb{U}^{p,q}(\omega, \Lambda),$$

is uniquely defined and that (3.7) holds for $A_m \in \mathbb{U}^{p_m, q_m}(\omega_m, \Lambda)$, $m = 1, 2$. $\qquad\square$

We also need the following result on the composition of the analysis operator and the synthesis operator defined by two Gabor systems.

Lemma 3.3. *Suppose $\Lambda \subseteq \mathbf{R}^d$ is a lattice, $\Lambda^2 = \Lambda \times \Lambda$ and $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d) \backslash 0$. Let $C_{\phi_2} = C_{\phi_2}^\Lambda$ be the Gabor analysis operator and let $D_{\phi_1} = D_{\phi_1}^\Lambda$ be the Gabor synthesis operator defined by ϕ_2 and ϕ_1, respectively, and Λ. Then $C_{\phi_2} \circ D_{\phi_1}$ is the matrix $(\mathbf{c}(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \Lambda^2}$ where*

$$\mathbf{c}(\mathbf{j}, \mathbf{k}) = e^{i\langle k, \kappa - \iota \rangle} V_{\phi_2} \phi_1(\mathbf{j} - \mathbf{k}), \quad \mathbf{j} = (j, \iota), \quad \mathbf{k} = (k, \kappa). \tag{3.10}$$

If $\omega_0(X, Y) = \omega(X - Y)$, $X, Y \in \mathbf{R}^{2d}$ for $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, then

$$(\mathbf{c}(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \Lambda^2} \in \bigcap_{\substack{\omega \in \mathscr{P}_E(\mathbf{R}^{2d}) \\ q > 0}} \mathbb{U}^{\infty, q}(\omega_0, \Lambda^2). \tag{3.11}$$

Proof. Let f be a sequence on Λ^2 such that $f(\mathbf{k}) \neq 0$ for at most a finite number of $\mathbf{k} \in \Lambda^2$. Then

$$D_{\phi_1} f = \sum_{\mathbf{k} \in \Lambda^2} f(\mathbf{k}) \phi_{1,\mathbf{k}}, \quad \phi_{1,\mathbf{k}} \equiv \phi_1(\cdot - k) e^{i\langle \cdot, \kappa \rangle}, \quad \mathbf{k} = (k, \kappa),$$

and

$$C_{\phi_2}(D_{\phi_1} f)(\mathbf{j}) = V_{\phi_2}(D_{\phi_1} f)(\mathbf{j}) = \sum_{\mathbf{k} \in \Lambda^2} V_{\phi_2} \phi_{1,\mathbf{k}}(\mathbf{j}) f(\mathbf{k}).$$

If $\mathbf{j} = (j, \iota)$ then $C_{\phi_2} \circ D_{\phi_1}$ is hence given by the matrix $C = (\mathbf{c}(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \Lambda^2}$ where

$$\mathbf{c}(\mathbf{j}, \mathbf{k}) = V_{\phi_2} \phi_{1,\mathbf{k}}(\mathbf{j}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \phi_{1,\mathbf{k}}(y) \overline{\phi_2(y - j)} e^{-i\langle y, \iota \rangle} dy$$

$$= (2\pi)^{-\frac{d}{2}} e^{i\langle k,\kappa-\iota\rangle} \int_{\mathbf{R}^d} \phi_1(y)\overline{\phi_2(y-(j-k))} e^{-i\langle y,\iota-\kappa\rangle} dy$$

$$= e^{i\langle k,\kappa-\iota\rangle} V_{\phi_2}\phi_1(\boldsymbol{j}-\boldsymbol{k}),$$

which proves (3.10).

It remains to prove (3.11). Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ and $q > 0$. Since $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d)$, we have by [33, Theorem 2.4]

$$|V_{\phi_2}\phi_1(x,\xi)| \lesssim e^{-r(|x|+|\xi|)},$$

for every $r > 0$. From (3.10) and (2.15) we obtain

$$h_{C,\infty,\omega_0}(\boldsymbol{k}) = \sup_{\boldsymbol{j}\in\Lambda^2} |H_{C,\omega_0}(\boldsymbol{j},\boldsymbol{k})| = |V_{\phi_2}\phi_1(\boldsymbol{k})\omega(\boldsymbol{k})|.$$

A combination of these relations and (2.16) now give

$$\|C\|_{\mathbb{U}^{\infty,q}(\omega_0,\Lambda^2)} = \|h_{C,\infty,\omega_0}\|_{\ell^q} = \|V_{\phi_2}\phi_1 \cdot \omega\|_{\ell^q(\Lambda^2)} < \infty.$$

Hence $C \in \mathbb{U}^{\infty,q}(\omega_0,\Lambda^2)$ for any $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ and any $q > 0$. $\qquad\square$

Proof of Theorem 3.1. By [36, Proposition 2.8] and Proposition 2.1 we may assume that $A = 0$. Pick $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d)\backslash 0$ and a lattice $\Lambda \subseteq \mathbf{R}^d$ such that Φ, Ψ and $\Lambda^2 = \Lambda \times \Lambda \subseteq \mathbf{R}^{2d}$ are as in Lemma 2.8. Let finally $a_m \in M^{p_m,q_m}_{(\omega_m)}(\mathbf{R}^{2d})$, $m = 1, 2$.

By (2.17)–(2.20), we have for $m = 1, 2$

$$\|a_m\|_{M^{p_m,q_m}_{(\omega_m)}} \asymp \|A_m\|_{\mathbb{U}^{p_m,q_m}(\vartheta_m,\Lambda^2)}, \tag{3.12}$$

and

$$\mathrm{Op}(a_m) = D_{\phi_1} \circ A_m \circ C_{\phi_2}, \tag{3.13}$$

where

$$A_m = (\boldsymbol{a}_m(\boldsymbol{j},\boldsymbol{k}))_{\boldsymbol{j},\boldsymbol{k}\in\Lambda^2},$$

$$\boldsymbol{a}_m(\boldsymbol{j},\boldsymbol{k}) \equiv e^{i\langle k-j,\kappa\rangle} V_\Psi a_m(j,\kappa,\iota-\kappa,k-j), \quad \boldsymbol{j}=(j,\iota), \quad \boldsymbol{k}=(k,\kappa) \in \Lambda^2,$$

and

$$\vartheta_m(x,\xi,y,\eta) = \omega_m(x,\eta,\xi-\eta,y-x).$$

Condition (3.2) means for the weights ϑ_m, $m = 0, 1, 2$,

$$\vartheta_0(X,Y) \lesssim \vartheta_1(X,Z)\vartheta_2(Z,Y), \quad X,Y,Z \in \mathbf{R}^{2d}. \tag{3.14}$$

Pick $v_1 \in \mathscr{P}_E(\mathbf{R}^d)$ even so that ω_2 is v_2-moderate with

$$v_2 = v_1 \otimes v_1 \otimes v_1 \otimes v_1 \in \mathscr{P}_E(\mathbf{R}^{4d}),$$

set $v = v_1^2 \otimes v_1 \in \mathscr{P}_E(\mathbf{R}^{2d})$ and

$$v_0(X,Y) = v(X-Y) \in \mathscr{P}_E(\mathbf{R}^{4d}), \quad X,Y \in \mathbf{R}^{2d}.$$

Then v_0 is designed to guarantee

$$\vartheta_2(X, Y) \lesssim v_0(X, Z)\vartheta_2(Z, Y), \quad X, Y, Z \in \mathbf{R}^{2d}. \tag{3.15}$$

We have by (2.21) and (2.22)

$$\mathrm{Op}(a_1) \circ \mathrm{Op}(a_2) = D_{\phi_1} \circ A \circ C_{\phi_2},$$

where

$$A = A_1 \circ C \circ A_2$$

and $C = C_{\phi_2} \circ D_{\phi_1}$. By Lemma 3.3

$$C \in \bigcap_{r>0} \mathbb{U}^{\infty,r}(v_0, \Lambda^2).$$

Set $r = \min(1, p_2, q_2)$. Then we obtain from (2.23), (3.14), (3.15) and Proposition 3.2 applied twice

$$
\begin{aligned}
\|a_1 \#_0 a_2\|_{M^{p_0,q_0}_{(\omega_0)}} &\asymp \|A_1 \circ C \circ A_2\|_{\mathbb{U}^{p_0,q_0}(\vartheta_0, \Lambda^2)} \\
&\lesssim \|A_1\|_{\mathbb{U}^{p_1,q_1}(\vartheta_1, \Lambda^2)} \|C \circ A_2\|_{\mathbb{U}^{p_2,q_2}(\vartheta_2, \Lambda^2)} \\
&\lesssim \|A_1\|_{\mathbb{U}^{p_1,q_1}(\vartheta_1, \Lambda^2)} \|C\|_{\mathbb{U}^{\infty,r}(v_0, \Lambda^2)} \|A_2\|_{\mathbb{U}^{p_2,q_2}(\vartheta_2, \Lambda^2)} \\
&\asymp \|A_1\|_{\mathbb{U}^{p_1,q_1}(\vartheta_1, \Lambda^2)} \|A_2\|_{\mathbb{U}^{p_2,q_2}(\vartheta_2, \Lambda^2)} \\
&\asymp \|a_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{M^{p_2,q_2}_{(\omega_2)}}.
\end{aligned}
$$

It remains to prove the claimed uniqueness of the extension. If (3.5) holds then $M^{p_j,q_j}_{(\omega_j)} \subseteq M^{\infty,1}_{(\omega_j)}$, $j = 1, 2$, and $M^{p_0,q_0}_{(\omega_0)} \subseteq M^{\infty,1}_{(\omega_0)}$. Then the claim follows from the uniqueness of the extension

$$M^{\infty,1}_{(\omega_1)} \#_A M^{\infty,1}_{(\omega_2)} \subseteq M^{\infty,1}_{(\omega_0)}, \tag{3.16}$$

which is proved in [7, Theorem 2.11].

Suppose (3.6) holds. Then the same argument applies if $q \leq 1$, and if $p \geq 1$, then the claim is a consequence of the uniqueness of the extension

$$M^{\infty,q_1}_{(\omega_1)} \#_A M^{\infty,q_2}_{(\omega_2)} \subseteq M^{\infty,q}_{(\omega_0)}, \tag{3.17}$$

which is again proved in [7, Theorem 2.11]. Suppose $p < 1 < q$. If $q_1, q_2 \geq 1$, then the uniqueness follows again from the uniqueness of (3.17). If $q_1 \geq 1 > q_2$, then it follows from the uniqueness of (3.17) with q_2 replaced by 1, and analogously for $q_2 \geq 1 > q_1$. Finally if $q_1, q_2 < 1$, then the uniqueness follows from the uniqueness of (3.16). $\qquad\square$

Let $p, q \in (0, \infty]$ and set $r = \min(1, p, q)$. A particular case of Theorem 3.1 is the inclusion

$$M^{p,q}_{(\omega_0)} \#_A M^{\infty,r}_{(\omega_2)} \subseteq M^{p,q}_{(\omega_0)},$$

where the weights $\omega_0, \omega_2 \in \mathscr{P}_E(\mathbf{R}^{4d})$ satisfy

$$\omega_0(T_A(Z, X)) \lesssim \omega_0(T_A(Y, X))\, \omega_2(T_A(Z, Y)), \quad X, Y, Z \in \mathbf{R}^{2d},$$

and T_A is defined by (3.3).

We also note that $M^{p,q}_{(\omega)}$ is an algebra under the product $\#_A$ provided $p, q \in (0, \infty]$, $q \leq \min(1, p)$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$ satisfies

$$\omega(T_A(Z, X)) \lesssim \omega(T_A(Y, X))\, \omega(T_A(Z, Y)), \quad X, Y, Z \in \mathbf{R}^{2d}.$$

4. Necessary Conditions

In this final section, we show that some of the sufficient conditions in Theorem 3.1 are necessary. We need the following lemma that concerns Wigner distributions.

Lemma 4.1. *Let $q_0, q \in (0, \infty]$ satisfy $q_0 < q$, let*

$$\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2}|x|^2} \quad \text{for } x \in \mathbf{R}^d,$$

let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, let $\mathbf{c} = \{c(\kappa)\}_{\kappa \in \Lambda} \in \ell^q(\Lambda) \setminus \ell^{q_0}(\Lambda)$, where $c(\kappa) \geq 0$ for all $\kappa \in \Lambda$, and finally let

$$f(x) = \sum_{\kappa \in \Lambda} c(\kappa) e^{i\langle x, \kappa \rangle} \phi(x) \in \mathscr{S}'(\mathbf{R}^d).$$

Then

$$f \in \bigcap_{p > 0} M^{p,q}(\mathbf{R}^d) \setminus M^{\infty, q_0}(\mathbf{R}^d), \tag{4.1}$$

and

$$W_{f,\phi} \in \bigcap_{p > 0} \mathcal{M}^{p,q}(\mathbf{R}^{2d}). \tag{4.2}$$

Proof. By replacing Λ by a sufficiently dense lattice Λ_0, containing Λ and letting $c(\kappa) = 0$ when $\kappa \in \Lambda_0 \setminus \Lambda$, we reduce ourselves to a situation where the hypothesis in Proposition 2.6 is fulfilled. Hence we may assume that (2.11) are dual frames for $L^2(\mathbf{R}^d)$.

First we show (4.1). (cf. [28, Proposition 2.6]). On one hand, we have $\|f\|_{M^{p,q}} \lesssim \|\mathbf{c}\|_{\ell^q}$ for any $p > 0$ due to Proposition 2.6(1). Thus $f \in \bigcap_{p>0} M^{p,q}(\mathbf{R}^d)$. On the other hand $f \notin M^{\infty, q_0}(\mathbf{R}^d)$.

In fact, set $\phi_1(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4}|x|^2}$ for $x \in \mathbf{R}^d$. Since

$$V_\phi f(0, \iota) = (2\pi)^{-\frac{d}{2}} \sum_{\kappa \in \Lambda} c(\iota - \kappa) e^{-\frac{1}{4}|\kappa|^2} = \mathbf{c} * \phi_1(\iota)$$

we obtain

$$\|\mathbf{c}\|_{\ell^{q_0}}^{q_0} = \sum_{\iota \in \Lambda} c(\iota)^{q_0} \leq (2\pi)^{\frac{dq_0}{2}} \sum_{\iota \in \Lambda} (\mathbf{c} * \phi_1(\iota))^{q_0}$$

$$= (2\pi)^{\frac{dq_0}{2}} \sum_{\iota \in \Lambda} |V_\phi f(0, \iota)|^{q_0} \leq (2\pi)^{\frac{dq_0}{2}} \sum_{\iota \in \Lambda} \left(\sup_{j \in \Lambda} |V_\phi f(j, \iota)| \right)^{q_0}$$

$$= (2\pi)^{\frac{dq_0}{2}} \|V_\phi f\|_{\ell^{\infty, q_0}(\Lambda^2)}^{q_0} \asymp \|f\|_{M^{\infty, q_0}}^{q_0}$$

again by Proposition 2.6. Thus it must hold $f \notin M^{\infty, q_0}(\mathbf{R}^d)$, since otherwise we get the contradiction $\mathbf{c} \in \ell^{q_0}(\Lambda)$. We have now showed (4.1).

In order to prove (4.2), set $a = W_{f,\phi} \in \mathscr{S}'(\mathbf{R}^{2d})$. Since $\mathcal{M}^{p,q}$ is increasing with respect to p and q, it suffices to intersect in (4.2) over $0 < p \leq \min(1, q)$. We have

$$\|a\|_{\mathcal{M}^{p,q}} \asymp \|V_\Phi W_{f,\phi}\|_{\ell^{p,q}(\Lambda^4)},$$

where $\Phi(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{-(|x|^2 + |\xi|^2)}$, and

$$\Lambda^4 = \Lambda \times \Lambda \times \Lambda \times \Lambda \subseteq \mathbf{R}^{4d}.$$

By straightforward computations, we get

$$a(x, \xi) = W_{f,\phi}(x, \xi)$$

$$= (2\pi)^{-\frac{d}{2}} \sum_{\kappa \in \Lambda} c(\kappa) \pi^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{-\frac{1}{2}(|x - \frac{y}{2}|^2 + |x + \frac{y}{2}|^2)} e^{i(\langle x, \kappa \rangle - \langle y, \xi - \frac{\kappa}{2} \rangle)} dy$$

$$= 2^{\frac{d}{2}} \pi^{-\frac{d}{2}} \sum_{\kappa \in \Lambda} c(\kappa) e^{-|x|^2 - |\xi - \frac{\kappa}{2}|^2} e^{i\langle x, \kappa \rangle}.$$

This gives

$$V_\Phi a(x, \xi, \eta, y) = 2^{\frac{d}{2}} \pi^{-\frac{d}{2}} \sum_{\kappa \in \Lambda} c(\kappa) F_\kappa(x, \xi, \eta, y),$$

where

$$F_\kappa(x, \xi, \eta, y) = (2\pi)^{-\frac{3d}{2}} \iint_{\mathbf{R}^{2d}} e^{-(|z|^2 + |\zeta - \frac{\kappa}{2}|^2 + |z - x|^2 - |\zeta - \xi|^2)} e^{-i(\langle z, \eta - \kappa \rangle + \langle y, \zeta \rangle)} dz\, d\zeta$$

$$= 2^{-\frac{5d}{2}} \pi^{-\frac{d}{2}} e^{-\frac{1}{2}(|x|^2 + |\xi - \frac{\kappa}{2}|^2) - \frac{1}{8}(|\eta - \kappa|^2 + |y|^2)} e^{-\frac{i}{2}(\langle x, \eta - \kappa \rangle + \langle y, \xi + \frac{\kappa}{2} \rangle)}.$$

Hence

$$V_\Phi a(x, \xi, \eta, y) = 2^{-2d} \pi^{-d} \sum_{\kappa \in \Lambda} c(\kappa) e^{-\frac{1}{2}(|x|^2 + |\xi - \frac{\kappa}{2}|^2) - \frac{1}{8}(|\eta - \kappa|^2 + |y|^2)}$$

$$\times e^{-\frac{i}{2}(\langle x, \eta - \kappa \rangle + \langle y, \xi + \frac{\kappa}{2} \rangle)}. \tag{4.3}$$

If $q < \infty$ we get, in the third inequality using $p \leq 1$,

$$\|W_{f,\phi}\|_{\mathcal{M}^{p,q}} = \|a\|_{\mathcal{M}^{p,q}} \asymp \|\{V_\Phi a(k_1, \kappa_1, \kappa_2, k_2)\}_{k_j, \kappa_j \in \Lambda}\|_{\ell^{p,q}(\Lambda^4)}$$

$$\lesssim \left(\sum_{k_2, \kappa_2} \left(\sum_{k_1, \kappa_1} \left(\sum_\kappa c(\kappa) e^{-\frac{1}{2}(|k_1|^2 + |\kappa_1 - \frac{\kappa}{2}|^2) - \frac{1}{8}(|\kappa_2 - \kappa|^2 + |k_2|^2)} \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$\asymp \left(\sum_{\kappa_2}\left(\sum_{\kappa_1}\left(\sum_{\kappa}c(\kappa)e^{-(\frac{1}{2}|\kappa_1-\frac{\kappa}{2}|^2+\frac{1}{8}|\kappa_2-\kappa|^2)}\right)\right)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{\kappa_2}\left(\sum_{\kappa,\kappa_1}c(\kappa)^p\,e^{-(\frac{p}{2}|\kappa_1-\frac{\kappa}{2}|^2+\frac{p}{8}|\kappa_2-\kappa|^2)}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\asymp \left(\sum_{\kappa_2}\left(\sum_{\kappa}c(\kappa)^p e^{-\frac{p}{8}|\kappa_2-\kappa|^2}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = (\|\{c^p\}*e^{-\frac{p}{8}|\cdot|^2}\|_{\ell^{\frac{q}{p}}})^{\frac{1}{p}}$$

$$\leq (\|\{c^p\}\|_{\ell^{\frac{q}{p}}}\|e^{-\frac{p}{8}|\cdot|^2}\|_{\ell^1})^{\frac{1}{p}} \asymp \|\mathbf{c}\|_{\ell^q} < \infty,$$

using Young's inequality. The result follows if $q < \infty$. If $q = \infty$, a similar argument proves the result. $\qquad\square$

The preceding lemma is needed in the proof of Theorem 4.3 below on necessary conditions for continuity. We aim at conditions on the exponents p_j, q_j, $j = 0, 1, 2$, that are necessary for

$$\|a\#_A b\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|a\|_{M^{p_1,q_1}_{(\omega_1)}}\|b\|_{M^{p_2,q_2}_{(\omega_2)}}, \tag{4.4}$$

to hold for all $a, b \in \mathscr{S}(\mathbf{R}^{2d})$, for certain weight functions ω_j, $j = 0, 1, 2$. We restrict to weights of polynomial type.

By [36, Proposition 2.8], it suffices to prove the result in the Weyl case $A = 1/2$, and then (4.4) in terms of symplectic modulation spaces is

$$\|a\#b\|_{\mathcal{M}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a\|_{\mathcal{M}^{p_1,q_1}_{(\omega_1)}}\|b\|_{\mathcal{M}^{p_2,q_2}_{(\omega_2)}}, \quad a, b \in \mathscr{S}(\mathbf{R}^{2d}). \tag{4.5}$$

The conditions on the weights (3.2) and (3.3) are then transformed into

$$\omega_0(Z+X, Z-X) \lesssim \omega_1(Y+X, Y-X)\omega_2(Z+Y, Z-Y), \quad X, Y, Z \in \mathbf{R}^{2d}. \tag{4.6}$$

(cf. [7, 23]).

We will consider weights with the particular structure

$$\omega_0(X,Y) = \frac{\vartheta_2(X-Y)}{\vartheta_0(X+Y)}, \quad \omega_1(X,Y) = \frac{\vartheta_2(X-Y)}{\vartheta_1(X+Y)},$$

$$\omega_2(X,Y) = \frac{\vartheta_1(X-Y)}{\vartheta_0(X+Y)}, \tag{4.7}$$

for $\vartheta_j \in \mathscr{P}(\mathbf{R}^{2d})$, $j = 0, 1, 2$. Then (4.6) is automatically satisfied. Without loss, we may assume $\vartheta_j \in C^\infty$ [23, Remark 2.18].

For $\vartheta \in \mathscr{P}(\mathbf{R}^{2d})$, let $S^{(\vartheta)}(\mathbf{R}^{2d})$ denote the space of smooth symbols on $\mathbf{R}^{2d}$ such that $(\partial^\alpha a)/\vartheta \in L^\infty$ for any $\alpha \in \mathbf{N}^{2d}$.

Lemma 4.2. *Let $p, q \in (0, \infty]$, let $\vartheta_j \in \mathscr{P}(\mathbf{R}^{2d})$, $j = 1, 2$ and suppose $\omega(X, Y) = \vartheta_2(X - Y)/\vartheta_1(X + Y)$. Then there exist $a_j \in S^{(\vartheta_j)}(\mathbf{R}^{2d})$ and $b_j \in S^{(1/\vartheta_j)}(\mathbf{R}^{2d})$, $j = 1, 2$ such that*

$$a_j \# b_j = b_j \# a_j = 1, \quad j = 1, 2, \tag{4.8}$$

and the map $a \mapsto a_2 \# a \# b_1$ is continuous on $\mathscr{S}(\mathbf{R}^{2d})$ and extends uniquely to a homeomorphism from $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p,q}(\mathbf{R}^{2d})$.

Proof. According to [2, Corollary 6.6] there exist $a_j \in S^{(\vartheta_j)}(\mathbf{R}^{2d})$ and $b_j \in S^{(1/\vartheta_j)}(\mathbf{R}^{2d})$, $j = 1, 2$, such that (4.8) is satisfied.

By [23, Remark 2.18], we have

$$S^{(\vartheta)}(\mathbf{R}^{2d}) = \bigcap_{N \geq 0} \mathcal{M}^{\infty,r}_{(1/\vartheta_N)}(\mathbf{R}^{2d}), \quad \vartheta_N(X, Y) = \vartheta(X)\langle Y\rangle^{-N},$$

for any $\vartheta \in \mathscr{P}(\mathbf{R}^{2d})$ and any $r > 0$. More precisely the remark gives the equality for $r = 1$, and for general $r > 0$, the equality follows from the embeddings

$$M^{\infty,r_2}_{(1/v_{N+N_0})} \subseteq M^{\infty,r_1}_{(1/v_N)} \subseteq M^{\infty,r_2}_{(1/v_N)}, \quad \text{when} \quad r_1 < r_2, \ N_0 > 2d\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If we set $r = \min(1, p, q)$, then $p_1 = \infty$, $q_1 = r$, $p_2 = p$, $q_2 = q$, as well as $p_2 = \infty$, $q_2 = r$, $p_1 = p$, $q_1 = q$, satisfy the conditions (3.4), and (3.5) or (3.6) of Theorem 3.1.

From these observations, the result follows from Theorem 3.1 and a repetition of the arguments in the proof of [7, Lemma 3.3]. ∎

Theorem 4.3. *Let $p_j, q_j \in (0, \infty]$, suppose $\omega_j \in \mathscr{P}(\mathbf{R}^{4d})$, $j = 0, 1, 2$, are given by (4.7) where $\vartheta_j \in \mathscr{P}(\mathbf{R}^{2d})$, $j = 0, 1, 2$. If (4.5) holds then*

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p_0} \leq \frac{1}{q_1} + \frac{1}{q_2} \quad and \quad q_1, q_2 \leq q_0. \tag{4.9}$$

Proof. By Lemma 4.2, the estimate (4.5) with weights (4.7) implies

$$\|a \# b\|_{\mathcal{M}^{p_0,q_0}} \lesssim \|a\|_{\mathcal{M}^{p_1,q_1}} \|b\|_{\mathcal{M}^{p_2,q_2}}, \quad a \in \mathcal{M}^{p_1,q_1}(\mathbf{R}^{2d}), \quad b \in \mathscr{S}(\mathbf{R}^{2d}). \tag{4.10}$$

It thus suffices to prove the result for $\omega_j \equiv 1$, $j = 0, 1, 2$.

Let $a_{\lambda,\mu}(x, \xi) = e^{-\lambda|x|^2 - \mu|\xi|^2}$ and $a_\lambda = a_{\lambda,\lambda}$, for $\mu, \lambda > 0$. Then by the proof of [23, Proposition 3.1] (cf. [7, Sec. 3])

$$\|a_\lambda\|^{1/d}_{\mathcal{M}^{p,q}} = \pi^{\frac{1}{p} + \frac{1}{q} - 1} p^{-\frac{1}{p}} q^{-\frac{1}{q}} \lambda^{-\frac{1}{p}} (1 + \lambda)^{\frac{1}{p} + \frac{1}{q} - 1},$$

and

$$a_\lambda \# a_\mu(X) = (1 + \lambda\mu)^{-d} \exp\left(-|X|^2 \frac{\lambda + \mu}{1 + \lambda\mu}\right).$$

Hence

$$\|a_\lambda \# a_\lambda\|_{\mathcal{M}^{p,q}}^{1/d} = \pi^{1/p+1/q-1} p^{-1/p} q^{-1/q} (1+\lambda^2)^{-1/q} (2\lambda)^{-1/p} (1+\lambda)^{2(1/p+1/q-1)}.$$

Thus

$$\left(\frac{\|a_\lambda \# a_\lambda\|_{\mathcal{M}^{p_0,q_0}}}{\|a_\lambda\|_{\mathcal{M}^{p_1,q_1}} \|a_\lambda\|_{\mathcal{M}^{p_2,q_2}}}\right)^{1/d} = C \lambda^{\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}}$$

$$\times (1+\lambda^2)^{-\frac{1}{q_0}} (1+\lambda)^{\frac{2}{p_0}-\frac{1}{p_1}-\frac{1}{p_2}+\frac{2}{q_0}-\frac{1}{q_1}-\frac{1}{q_2}},$$

for some constant $C > 0$ which does not depend on λ. The right hand side behaves like $\lambda^{\frac{1}{p_0}-\frac{1}{q_1}-\frac{1}{q_2}}$ when λ is large, and like $\lambda^{\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}}$ when λ is small. The continuity (4.10) hence implies the necessary conditions

$$\frac{1}{p_0} \le \frac{1}{q_1}+\frac{1}{q_2}, \quad \frac{1}{p_0} \le \frac{1}{p_1}+\frac{1}{p_2}.$$

It remains to show $q_1, q_2 \le q_0$. Since $\overline{a_1 \# a_2} = \overline{a}_2 \# \overline{a}_1$ (cf. [23]), it suffices to show $q_1 \le q_0$. We give a proof by contradiction. Suppose (4.10) holds and $q_0 < q_1$. Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice,

$$\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2}|x|^2}, \quad \mathbf{c} = \{c(\kappa)\}_{\kappa \in \Lambda} \in \ell^{q_1}(\Lambda) \backslash \ell^{q_0}(\Lambda),$$

and let

$$f(x) = \sum_{\kappa \in \Lambda} c(\kappa) e^{i\langle x, \kappa \rangle} \phi(x).$$

Then

$$f \in \bigcap_{p_1 > 0} M^{p_1,q_1}(\mathbf{R}^d) \backslash M^{\infty,q_0}(\mathbf{R}^d),$$

$$a = W_{f,\phi} \in \bigcap_{p_1 > 0} \mathcal{M}^{p_1,q_1}(\mathbf{R}^{2d}) \quad \text{and} \quad b = W_{\phi,\phi} \in \mathscr{S}(\mathbf{R}^{2d}),$$

by Lemma 4.1. Since

$$\mathrm{Op}^w(a)g = (2\pi)^{-\frac{d}{2}}(g,\phi)f \quad \text{and} \quad \mathrm{Op}^w(b)g = (2\pi)^{-\frac{d}{2}}(g,\phi)\phi,$$

it follows that

$$\mathrm{Op}^w(a\#b)\phi = (2\pi)^{-d}\|\phi\|_{L^2}^4 f \in \bigcap_{p_1 > 0} M^{p_1,q_1}(\mathbf{R}^d) \backslash M^{\infty,q_0}(\mathbf{R}^d).$$

Therefore $\mathrm{Op}^w(a\#b)$ is not continuous from $\mathscr{S}(\mathbf{R}^d)$ to $M^{\infty,q_0}(\mathbf{R}^d)$.

On the other hand, we have by assumption

$$a\#b \in \mathcal{M}^{p_0,q_0} \subseteq \mathcal{M}^{\infty,q_0}.$$

If $q_0 \in (0,1]$, then $\mathrm{Op}^w(a\#b)$ is continuous from M^{p_0,q_0} to M^{p_0,q_0} when $p_0 \in [q_0,\infty]$, by [35, Theorem 3.1]. This contradicts the fact that $\mathrm{Op}^w(a\#b)$ is not continuous from $\mathscr{S}$ to M^{∞,q_0}. Hence the assumption $q_0 < q_1$ must be false.

If instead $q_0 \in [1, \infty]$, then by [31, Theorem 4.3] $\mathrm{Op}^w(a\#b)$ is continuous from $M^{1,1}$ to M^{q_0,q_0}, which again contradicts the fact that $\mathrm{Op}^w(a\#b)$ is not continuous from $\mathscr{S}$ to M^{∞,q_0}. Hence the assumption $q_0 < q_1$ is again false.

Thus we must have $q_1 \leq q_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 4.4. Let $\mathscr{P}_E^0(\mathbf{R}^d)$ denote all $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ such that ω is v-moderate for a sub-multiplicative weight v satisfying

$$v(x) \lesssim e^{r|x|}, \quad x \in \mathbf{R}^d,$$

for all $r > 0$. Then $\mathscr{P}(\mathbf{R}^d) \subsetneq \mathscr{P}_E^0(\mathbf{R}^d)$. By using the new [1, Theorem 4.1] instead of Lemma 4.2, it follows that Theorem 4.3 holds for $\vartheta_j \in \mathscr{P}_E^0(\mathbf{R}^{2d})$ and ω_j defined by (4.7). The space $\mathscr{S}$ in (4.5) is then replaced by $\mathcal{S}_1$.

Remark 4.5. For Banach modulation spaces with exponents p_j, q_j restricted to $[1, \infty]$, we have found that the following conditions are necessary and sufficient for continuity of the Weyl product [7, Theorems 0.1 and 3.1].

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2}, \tag{4.11}$$

$$q_1, q_2 \leq q_0, \quad 1 \leq \frac{1}{q_1} + \frac{1}{q_2}, \tag{4.12}$$

$$\frac{1}{p_0} + \frac{1}{q_0} \leq \frac{1}{q_1} + \frac{1}{q_2}, \quad 1 + \frac{1}{q_0} \leq \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_j}, \quad j = 1, 2, \tag{4.13}$$

$$1 + \frac{1}{p_0} + \frac{1}{q_0} \leq \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_1} + \frac{1}{p_2}. \tag{4.14}$$

In this paper, we have worked with exponents p_j, q_j in the full range $(0, \infty]$. The sufficient conditions in Theorem 3.1 and the necessary conditions in Theorem 4.3 are not equal, as conditions (4.11)–(4.14) are for exponents in $[1, \infty]$.

In fact, consider the inclusion

$$M^{1,2}\#M^{1,2} \subseteq M^{\infty,2},$$

which holds since the exponents satisfy (4.11)–(4.14). They do however not satisfy (3.4), and (3.5) or (3.6). Hence the sufficient conditions in Theorem 3.1 are not at all necessary.

Appendix

In this appendix, we prove the formula

$$\mathscr{F}(f\overline{\phi})(\xi) = (2\pi)^{-\frac{d}{2}}(f, \phi e^{i\langle \cdot, \xi\rangle}), \quad f \in \mathcal{S}_s'(\mathbf{R}^d), \ \phi \in \mathcal{S}_s(\mathbf{R}^d), \ \xi \in \mathbf{R}^d, \tag{A.1}$$

for $s \geq \frac{1}{2}$, which we claimed to be true in the definition of the STFT (2.5). There is a parallel formula for $f \in \Sigma_s'(\mathbf{R}^d)$, $\phi \in \Sigma_s(\mathbf{R}^d)$ and $s > \frac{1}{2}$, that we also prove.

Let $f \in \mathcal{S}'_s(\mathbf{R}^d)$, let $\phi \in \mathcal{S}_s(\mathbf{R}^d)$ and denote

$$u(\xi) = (2\pi)^{-\frac{d}{2}}(f, \phi\, e^{i\langle\, \cdot\, ,\xi\rangle}) = (2\pi)^{-\frac{d}{2}}\langle f, \overline{\phi}\, e^{-i\langle\, \cdot\, ,\xi\rangle}\rangle, \quad \xi \in \mathbf{R}^d. \qquad (A.2)$$

Then $u \in C^\infty(\mathbf{R}^d)$. We need the following estimate (cf. [5]).

Lemma A.1. *The function* (A.2) *satisfies the estimate*

$$|u(\xi)| \lesssim e^{c|\xi|^{\frac{1}{s}}}, \quad \xi \in \mathbf{R}^d,$$

for any $c > 0$.

Proof. By (2.4) $\phi \in \mathcal{S}_{s,h}(\mathbf{R}^d)$ for all $h \geq h_0$ where $h_0 > 0$. Let $c > 0$ and set

$$h = \max\left(h_0, \left(\frac{ds}{c}\right)^s\right).$$

Let $\alpha, \beta \in \mathbf{N}^d$. Using $|\gamma|! \leq d^{|\gamma|}\gamma!$ (cf. [27, Eq. (0.3.3)]) and $\sum_{\gamma \leq \beta}\binom{\beta}{\gamma} = 2^{|\beta|}$ we estimate for $x, \xi \in \mathbf{R}^d$

$$\frac{\left|x^\alpha \partial_x^\beta(e^{i\langle x,\xi\rangle}\phi(x))\right|}{(\alpha!\beta!)^s (2h)^{|\alpha+\beta|}} \leq 2^{-|\alpha+\beta|}\sum_{\gamma \leq \beta}\binom{\beta}{\gamma}\frac{|\xi|^{|\gamma|}\, h^{-|\gamma|}}{\gamma!^s}\frac{\left|x^\alpha\partial^{\beta-\gamma}\phi(x)\right|}{(\alpha!(\beta-\gamma)!)^s h^{|\alpha+\beta-\gamma|}}\binom{\beta}{\gamma}^{-s}$$

$$\leq \|\phi\|_{\mathcal{S}_{s,h}} 2^{-|\alpha+\beta|}\sum_{\gamma \leq \beta}\binom{\beta}{\gamma}\left(\frac{\left(\frac{c}{s}|\xi|^{\frac{1}{s}}\right)^{|\gamma|}}{|\gamma|!}\right)^s\left(\frac{(ds)^s}{hc^s}\right)^{|\gamma|}$$

$$\lesssim 2^{-|\alpha+\beta|}\sum_{\gamma \leq \beta}\binom{\beta}{\gamma}\left(\frac{\left(\frac{c}{s}|\xi|^{\frac{1}{s}}\right)^{|\gamma|}}{|\gamma|!}\right)^s$$

$$\leq e^{c|\xi|^{\frac{1}{s}}} 2^{-|\alpha+\beta|}\sum_{\gamma \leq \beta}\binom{\beta}{\gamma}$$

$$\leq e^{c|\xi|^{\frac{1}{s}}}.$$

This implies

$$\|\phi\, e^{i\langle\, \cdot\, ,\xi\rangle}\|_{\mathcal{S}_{s,2h}} \lesssim e^{c|\xi|^{\frac{1}{s}}}, \quad \xi \in \mathbf{R}^d,$$

which via (2.4a) finally gives the estimate

$$|u(\xi)| \lesssim \|\phi\, e^{i\langle\, \cdot\, ,\xi\rangle}\|_{\mathcal{S}_{s,2h}} \lesssim e^{c|\xi|^{\frac{1}{s}}}, \quad \xi \in \mathbf{R}^d. \qquad \square$$

The formula (A.1) amounts to the claim $\mathscr{F}(f\,\overline{\phi}) = u$.

A priori $\mathscr{F}(f\,\overline{\phi}) \in \mathcal{S}'_s(\mathbf{R}^d)$ is the distribution

$$\langle\mathscr{F}(f\overline{\phi}), g\rangle = \langle f, \overline{\phi}\hat{g}\rangle, \quad g \in \mathcal{S}_s(\mathbf{R}^d).$$

To prove our claim $\mathscr{F}(f\overline{\phi}) = u$ we must therefore show

$$\langle f, \overline{\phi}\hat{g}\rangle = \int_{\mathbf{R}^d} u(x)g(x)dx = (2\pi)^{-d/2}\int_{\mathbf{R}^d}\langle f, \overline{\phi}g(x)e^{-i\langle\, \cdot\, ,x\rangle}\rangle dx, \quad g \in \mathcal{S}_s(\mathbf{R}^d).$$

$$(A.3)$$

Note that the integral is well defined due to Lemma A.1 and the estimate for
$g \in \mathcal{S}_s(\mathbf{R}^d)$ [33, Lemma 1.6]

$$|g(x)| \lesssim e^{-\varepsilon|x|^{1/s}}, \quad x \in \mathbf{R}^d,$$

which is valid for some $\varepsilon > 0$.

In view of the definition of the Fourier transform $\widehat{g}$, formula (A.3) is true
provided we can switch order in the action of the distribution $1 \otimes f \in \mathcal{S}'_s(\mathbf{R}^{2d})$
with respect to the first and second $\mathbf{R}^d$ variable, when it acts on the test func-
tion $\Phi(x, y) = (2\pi)^{-d/2} \overline{\phi}(y) \, g(x) e^{-i\langle y, x \rangle}$. Note that $\Phi \in \mathcal{S}_s(\mathbf{R}^{2d})$ if $\phi, g \in \mathcal{S}_s(\mathbf{R}^d)$
and $s \geq \frac{1}{2}$, and $\Phi \in \Sigma_s(\mathbf{R}^{2d})$ if $\phi, g \in \Sigma_s(\mathbf{R}^d)$ and $s > \frac{1}{2}$, cf. [3, Theorem 3.1]
and [4, Proposition 3.4].

Thus the claim (A.1) is a consequence of the following Fubini-type result for
Gelfand–Shilov distributions. It corresponds to [22, Theorem 5.1.1] in the Schwartz
distribution theory.

Theorem A.2. *Suppose $s \geq \frac{1}{2}$, and $f_j \in \mathcal{S}'_s(\mathbf{R}^{d_j})$, $j = 1, 2$. Then there exists a
unique tensor product distribution $f = f_1 \otimes f_2 \in \mathcal{S}'_s(\mathbf{R}^{d_1+d_2})$ such that*

$$\langle f_1 \otimes f_2, \phi_1 \otimes \phi_2 \rangle = \langle f_1, \phi_1 \rangle \langle f_2, \phi_2 \rangle, \quad \phi_j \in \mathcal{S}_s(\mathbf{R}^{d_j}), \quad j = 1, 2.$$

It holds

$$\langle f, \phi \rangle = \langle f_1, \langle f_2, \phi(x_1, x_2) \rangle \rangle = \langle f_2, \langle f_1, \phi(x_1, x_2) \rangle \rangle, \quad \phi \in \mathcal{S}_s(\mathbf{R}^{d_1+d_2}),$$

where f_j acts on x_j only, $j = 1, 2$.

*The same conclusion holds for $s > 1/2$ and $f_j \in \Sigma'_s(\mathbf{R}^{d_j})$, $j = 1, 2$, with test
functions in Σ_s.*

Proof. We use the Hermite functions

$$h_\alpha(x) = \pi^{-\frac{d}{4}} (-1)^{|\alpha|} (2^{|\alpha|} \alpha!)^{-\frac{1}{2}} e^{\frac{|x|^2}{2}} (\partial^\alpha e^{-|x|^2}), \quad x \in \mathbf{R}^d, \ \alpha \in \mathbf{N}^d,$$

and formal series expansions with respect to Hermite functions:

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha,$$

where $\{c_\alpha\}$ is a sequence of complex coefficients defined by $c_\alpha = c_\alpha(f) = (f, h_\alpha)$.

It is known that Gelfand–Shilov spaces and their distribution duals can be
identified by means of such series expansions, with characterizations in terms of the
corresponding sequence spaces (see [14, 15, 37] and the references therein).

In fact, let

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha,$$

and

$$\phi = \sum_{\alpha \in \mathbf{N}^d} d_\alpha h_\alpha,$$

with sequences $\{c_\alpha\}$ and $\{d_\alpha\}$ of finite support. Then the sesquilinear form

$$(f, \phi) = \sum_{\alpha \in \mathbf{N}^d} c_\alpha \overline{d_\alpha}, \tag{A.4}$$

agrees with the inner product on $L^2(\mathbf{R}^d)$ due to the fact that $\{h_\alpha\}_{\alpha \in \mathbf{N}^d} \subseteq L^2(\mathbf{R}^d)$ is an orthonormal basis. The form (A.4) extends uniquely to the duality on $\mathcal{S}'_s(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d)$ for $s \geq \frac{1}{2}$, and to the duality on $\Sigma'_s(\mathbf{R}^d) \times \Sigma_s(\mathbf{R}^d)$ for $s > \frac{1}{2}$. All spaces are then expressed in terms of the Hilbert sequence spaces

$$\ell_r^2 = \ell_r^2(\mathbf{N}^d) = \left\{ \{c_\alpha\}; \ \sum_{\alpha \in \mathbf{N}^d} |c_\alpha|^2 e^{r|\alpha|^{\frac{1}{2s}}} < \infty \right\},$$

where $r \in \mathbf{R}$. For $s \geq \frac{1}{2}$ the space $\mathcal{S}_s(\mathbf{R}^d)$ is identified topologically as the inductive limit

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \, ; \, \{c_\alpha\} \in \ell_r^2 \right\},$$

and $\mathcal{S}'_s(\mathbf{R}^d)$ is identified topologically as the projective limit

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \, ; \, \{c_\alpha\} \in \ell_{-r}^2 \right\}.$$

For $s > \frac{1}{2}$, the space $\Sigma_s(\mathbf{R}^d)$ is identified topologically as the projective limit

$$\Sigma_s(\mathbf{R}^d) = \bigcap_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha : \{c_\alpha\} \in \ell_r^2 \right\},$$

and $\Sigma'_s(\mathbf{R}^d)$ is identified topologically as the inductive limit

$$\Sigma'_s(\mathbf{R}^d) = \bigcup_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \, ; \, \{c_\alpha\} \in \ell_{-r}^2 \right\}.$$

We have for $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^{d_1 + d_2}$ with $\alpha_j \in \mathbf{N}^{d_j}$, $j = 1, 2$, $h_\alpha = h_{\alpha_1} \otimes h_{\alpha_2}$. This gives for $f_j \in \mathcal{S}'_s(\mathbf{R}^{d_j})$, $j = 1, 2$,

$$c_\alpha = c_\alpha(f_1 \otimes f_2) = (f_1, h_{\alpha_1})(f_2, h_{\alpha_2}), \quad \alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^{d_1 + d_2},$$

so $c_\alpha = c_{1,\alpha_1} c_{2,\alpha_2}$ if we denote $c_{j,\alpha_j} = (f_j, h_{\alpha_j})$ where $\alpha_j \in \mathbf{N}^{d_j}$ for $j = 1, 2$.

Let $\phi \in \mathcal{S}_s(\mathbf{R}^{d_1+d_2})$ and denote $d_\alpha(\phi) = (\phi, h_\alpha)$ for $\alpha \in \mathbf{N}^{d_1+d_2}$. This gives for any $r > 0$

$$\langle f_1 \otimes f_2, \phi \rangle = \sum_{(\alpha_1, \alpha_2) \in \mathbf{N}^{d_1+d_2}} c_{1,\alpha_1} c_{2,\alpha_2} \, e^{-r|(\alpha_1, \alpha_2)|^{\frac{1}{2s}}} \, \overline{d_{\alpha_1, \alpha_2}} \, e^{r|(\alpha_1, \alpha_2)|^{\frac{1}{2s}}}. \qquad (A.5)$$

From

$$e^{-r|(\alpha_1, \alpha_2)|^{\frac{1}{2s}}} \leq e^{-\frac{r}{2}|\alpha_1|^{\frac{1}{2s}}} e^{-\frac{r}{2}|\alpha_2|^{\frac{1}{2s}}},$$

$\{c_{1,\alpha_1}\} \in \ell^2_{-r}(\mathbf{N}^{d_1})$, $\{c_{2,\alpha_2}\} \in \ell^2_{-r}(\mathbf{N}^{d_2})$ for any $r > 0$, $\{d_{\alpha_1,\alpha_2}\} \in \ell^2_r(\mathbf{N}^{d_1+d_2})$ for some $r > 0$, and the Cauchy–Schwarz inequality, we may now conclude that the sum (A.5) converges absolutely.

The conclusion of the theorem is thus a consequence of the well-known Fubini theorem with respect to the counting measure. □

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