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A New Preconditioner for the EFIE Based on Primal and Dual Graph Laplacian Spectral Filters

Lyes Rahmouni* and Francesco P. Andriulli
Department of Electronics and Telecommunications, Politecnico di Torino, Turin, Italy
lyes.rahmouni@polito.it, francesco.andriulli@polito.it

***Abstract*—The Electric Field Integral Equation (EFIE) is notorious for its ill-conditioning both in frequency and h-refinement. Several techniques exist for fixing the equation conditioning problems based on hierarchical strategies, Calderon techniques, and related technologies. This work leverages on a new approach, based on the construction of tailored spectral filters for the EFIE components which allow the block renormalization of the EFIE spectrum resulting in a provably constant condition number for the equation. This is achieved without the need for a barycentric refinement and with low computational overhead compared with other schemes. In particular, only sparse matrices are required in addition to the EFIE original matrix. Numerical results will show the robustness of our scheme and its application to the solution of realistic problems.**

***Index Terms*—EFIE, preconditioning, projectors, loop-star decomposition**

I. Introduction

Numerous problems in electromagnetics can be formulated as boundary integral equations. Surface integral equations are especially efficient because they require the discretization of scatterers boundaries only but, however, they give rise to dense linear systems that are generally solved with iterative methods. The quality of the solution as well as the number of iterations required to achieve convergence intrinsically depends on the spectral properties of the discretized operators. Unfortunately, a large subset of integral formulations produce matrices whose condition number grows unbounded with respect to the inverse of the mesh parameter h (the average edge length) [1]. This ill-conditioning is mainly due to the fact that some of the underlying continuous operators are Fredholm integral operators of the first-kind, which are known to have eigenvalues accumulating at zero and/or at infinity [2], [3]. This is a critical limitation as it becomes prohibitively expensive to solve problems of practical interest. At the same time, electromagnetic integral equations are often unstable when the frequency decreases. This is mainly due to an unfavorable scaling of the operator components and to a numerical cancellation phenomenon of the electric current solutions.

Several efforts have been profused in developing strategies for addressing the above problems. On the one hand algebraic preconditioners have been proposed (see for example [4] and [5]) which, although improving the conditioning properties, they are still showing growing condition numbers for finer meshes. A second class of preconditioners are tuned to the spectral properties of the involved operators and encompass hierarchical, Calderon strategies, and related methods ([1], [6] and references therein). These schemes, however, may often results in computational overheads either because a barycentric refinement is required or because extra dense operators must be computed.

A different approach is adopted here: we leverage on the design suitably conceived spectral filters for the EFIE components which allow for the block renormalization of the EFIE spectrum. This results in a constant condition number without the need for a barycentric refinement and with low computational overhead compared with other schemes. In particular, only sparse matrices are required in addition to the EFIE original matrix. Numerical results will show the robustness of our scheme and its application to the solution of realistic problems.

II. Background and Notation

Let Γ be a Lipschitz boundary representing the surface of a Perfect Electrically Conducting (PEC) object and $\hat{\boldsymbol{n}}$ its outward pointing unit normal. A time harmonic incident wave $\boldsymbol{E}^i$ induces a surface electric current density $\boldsymbol{J}$, which in turn generates a scattered field $\boldsymbol{E}^s$. The latter can be computed by solving the EFIE which reads

$$-\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{E^i} = \mathcal{T}\boldsymbol{J} = \mathcal{T}_A\boldsymbol{J} + \mathcal{T}_\phi\boldsymbol{J} \quad (1)$$

where the $\mathcal{T}_A\boldsymbol{J}$ is the vector potential

$$\mathcal{T}_A\boldsymbol{J} = \hat{\boldsymbol{n}}(\boldsymbol{r}) \times \mathrm{j}k \int_\Gamma \frac{\mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r'}\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r'}\|}\boldsymbol{J}(\boldsymbol{r'})\mathrm{d}S(\boldsymbol{r'}) \quad (2)$$

and $\mathcal{T}_\phi\boldsymbol{J}$ is the scalar potential

$$\mathcal{T}_\phi\boldsymbol{J} = -\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \frac{1}{\mathrm{j}k}\nabla_{\boldsymbol{r}} \int_\Gamma \frac{\mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r'}\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r'}\|}\nabla_{\boldsymbol{r'}}\cdot\boldsymbol{J}(\boldsymbol{r'})\mathrm{d}S(\boldsymbol{r'}) \quad (3)$$

in which $k = \omega\sqrt{\epsilon\mu}$ is the wavenumber. Following standard discretization strategy, we approximate Γ with a triangular mesh of average edge length h. On this mesh, the current density $\boldsymbol{J}$ is expanded as $\boldsymbol{J} = \sum_{n=1}^{N} I_n \boldsymbol{f}_n(\boldsymbol{r})$, where $\boldsymbol{f}_n(\boldsymbol{r})$ are N Rao-Wilton-Glisson (RWG) basis functions [7]. Equation (**??**) is then tested with $\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r})$ to obtain a linear system $\boldsymbol{T}\boldsymbol{j} = \boldsymbol{e}$ where $\boldsymbol{T} = \mathrm{j}k\boldsymbol{T}_A + (\mathrm{j}k)^{-1}\boldsymbol{T}_\phi$, with $\boldsymbol{T}_A = \langle\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r}), \mathcal{T}_A(\boldsymbol{f}_n(\boldsymbol{r}))\rangle$, $\boldsymbol{T}_\phi = \langle\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r}), \mathcal{T}_\phi(\boldsymbol{f}_n(\boldsymbol{r}))\rangle$ and $\boldsymbol{e} = \langle \boldsymbol{f}_n(\boldsymbol{r}), -\boldsymbol{E}^i\rangle$. Unfortunately, this system is ill-conditioned both in low frequencies and with dense discretization, as $\mathrm{cond}(\boldsymbol{T}) \lesssim 1/(hk)^2$.

Analyzing equation (**??**), it is clear that the two components of the operator $\mathcal{T}$ scale inversely in frequency, which is the source of the low frequency breakdown of the EFIE. Using a loop-star decomposition [8], it is possible to decouple the contributions of the solenoidal and irrotational currents, which will allow for a diagonal preconditioner to effectively cure the low frequency breakdown. In particular, multiplying left and right with $\boldsymbol{\Upsilon} = [\boldsymbol{\Lambda}/\sqrt{k} \quad \boldsymbol{\Sigma}\sqrt{k}]$, where $\boldsymbol{\Lambda}$ and $\boldsymbol{\Sigma}$ are the RWG to loop and RWG to star transformation matrices, eliminates the unfavorable frequency ill-scaling. The resulting matrix $\boldsymbol{T}_{LS} = \boldsymbol{\Upsilon}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Upsilon}$ has a condition number stable with decreasing frequency. This, however, comes at the expense of a degraded conditioning in dense discretization. Indeed, the condition number of the new EFIE system grows cubically with the edge parameter, that is, $\mathrm{cond}(\boldsymbol{T}_{LS}) \lesssim 1/h^3$. In the following, we present a new strategy to cure the dense discretization breakdown.

III. A New Preconditioner Based on Primal and Dual Graph Laplacian Spectral Filters

Following the above-mentioned considerations, it is necessary to further regularize the operator $\boldsymbol{T_{LS}}$ h-dependency. The strategy adopted in this work consists in partitioning the spectrum of the operator $\boldsymbol{T_{LS}}$ into N sub-intervals by following the natural ordering of the primal and dual graph Laplacian. Subsequently, we normalize each sub-interval eigenvalues by the largest eigenvalue of the i^{th} interval $\sigma^i_{\max}$. A naive eigen-decomposition of the different operators, however, is not practical given its $\mathcal{O}(N^3)$ and $\mathcal{O}(N^2)$ complexity for dense and sparse matrices, respectively. A computationally efficient approach, proposed in this work, leverages on the sparsity of the graph Laplacian to build spectral filters $\boldsymbol{Q}_i$. We first introduce a family of low-pass spectral filters of length 2^l

$$\boldsymbol{P}_i^{\Sigma} = \frac{\mathbf{I}}{\mathbf{I} + \boldsymbol{\Delta}_i^{\Sigma}} \quad \text{for } i = 1 \dots N_{\Sigma}\,, \tag{4}$$

$$\boldsymbol{P}_i^{\Lambda} = \frac{\mathbf{I}}{\mathbf{I} + \boldsymbol{\Delta}_i^{\Lambda}} \quad \text{for } i = 1 \dots N_{\Lambda}\,, \tag{5}$$

where $N_{\Sigma} = \log_2(\sigma^{\Sigma}_{\max})$, $N_{\Lambda} = \log_2(\sigma^{\Lambda}_{\max})$ and

$$\boldsymbol{\Delta}_i^{\Sigma,\Lambda} = \left(\frac{\boldsymbol{\Delta}^{\Sigma,\Lambda}}{2^i}\right)^n \tag{6}$$

in which the parameter n controls the sharpness of the filters. The primal and dual Laplacian used in the filters are given by

$$\boldsymbol{\Delta}^{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma} \tag{7}$$

$$\boldsymbol{\Delta}^{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda} \tag{8}$$

The filters $\boldsymbol{Q}_i$ are then defined as

$$\boldsymbol{Q}_i = (\boldsymbol{P}_i - \boldsymbol{P}_{i-1})\frac{1}{\sqrt{\sigma^i_{\max}}}. \tag{9}$$

The well-conditioned EFIE we propose is finally given by

$$\begin{bmatrix} \boldsymbol{Q}_i^{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_i^{\Lambda} \end{bmatrix} \begin{bmatrix} k\widetilde{\boldsymbol{\Sigma}}^{\mathrm{T}}(\boldsymbol{T_A} + \boldsymbol{T_\phi})\widetilde{\boldsymbol{\Sigma}} & \widetilde{\boldsymbol{\Sigma}}^{\mathrm{T}}\boldsymbol{T_A}\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{T_A}\widetilde{\boldsymbol{\Sigma}} & \frac{1}{k}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{T_A}\boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}_i^{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_i^{\Lambda} \end{bmatrix} \tag{10}$$

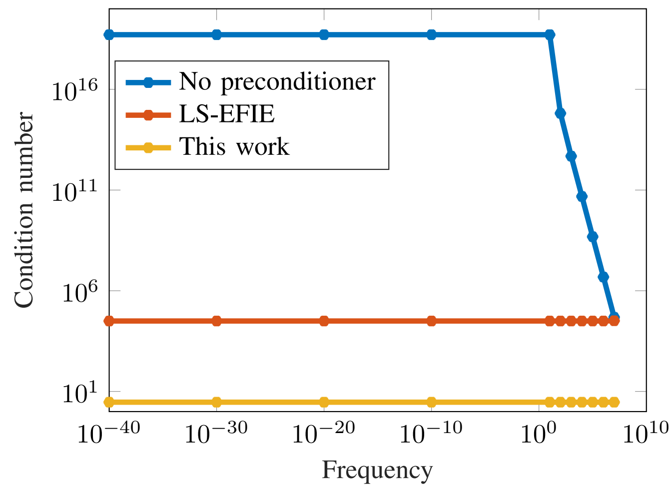

Fig. 1: Condition number as a function of the frequency

where

$$\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma}\right)^{+} \tag{11}$$

The introduction of further Gram matrices can be used to further reduce the condition number especially in case of non uniform meshes, but we omitted this analysis here for the sake of brevity.

IV. Numerical Results

Our first numerical result is intended to demonstrate that the presented technique delivers an EFIE immune from the low frequency breakdown. To that end, we simulated a spherical PEC geometry of radius 1m, discretized with 3270 triangles. Plane waves of different frequencies were used as excitations. The condition numbers of $\boldsymbol{T}$, $\boldsymbol{T_{LS}}$ and our technique are reported in **??**. We can see that even though the Loop-Star decomposition solves the low frequency breakdown, our scheme further improves the condition number of the EFIE.

As second example, we considered a PEC sphere illuminated by a plane wave oscillating at 1 Hz. **??** shows the condition number of the EFIE matrix of the standard loop-tree preconditioner and our new scheme. We can see that the condition number of $\boldsymbol{T}_{LS}$ grows unbounded as the discretization increases. Our formulation offers a regularized and stable solver.

The last example shows the study of a real case scenario. Accordingly, we simulated the aircraft shown in **??**, discretized into 16400 triangles. As an excitation, we considered a plane wave oscillating at 10^5 Hz. Using loop-star decomposition, the conjugate gradient converged in 2400 iterations, while with our techniques, it took 330 iterations.

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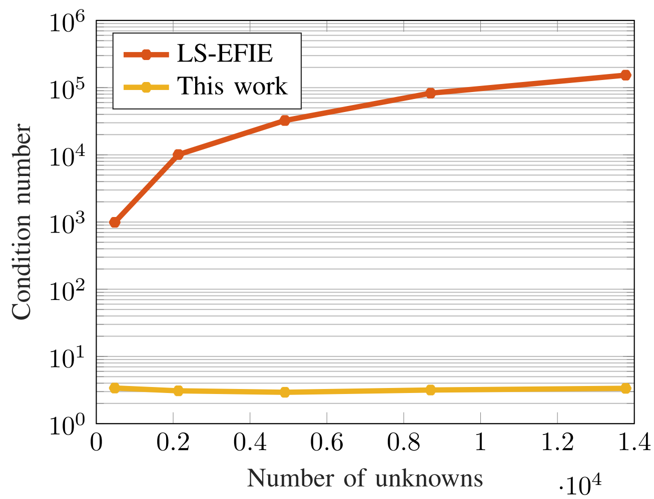

Fig. 2: Condition number as a function of number of elements

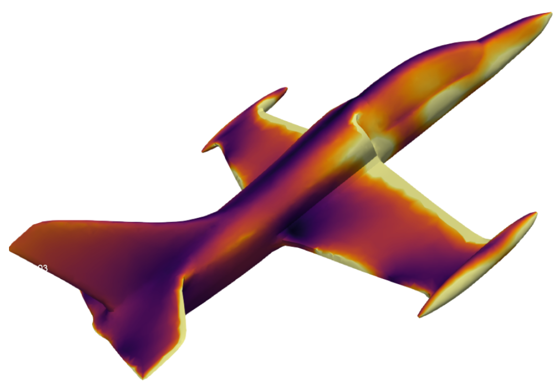

Fig. 3: The induced surface current

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