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MAXIMIZING THE RATIO OF EIGENVALUES OF NON-HOMOGENEOUS PARTIALLY HINGED PLATES

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Abstract. We study the spectrum of non-homogeneous partially hinged plates having structural engineering applications. A possible way to prevent instability phenomena is to maximize the ratio between the frequencies of certain oscillating modes with respect to the density function of the plate; we prove existence of optimal densities and we investigate their analytic expression. This analysis suggests where to locate reinforcing material within the plate; some numerical experiments give further information and support the theoretical results.

1. Introduction

In recent years the trend in bridge design is to replace expensive experiments in wind tunnels with numerical tests; hopefully, these tests should be preceded by a suitable mathematical modelling and, possibly, by analytic arguments. In particular, since it is by now well-established that reliable models for suspension bridges should have enough degrees of freedom to display torsional oscillations, it is convenient to model the deck of the bridge by means of a long narrow rectangular thin plate $\Omega \subset \mathbb{R}^2$, hinged at short edges and free on the remaining two, see [23] and problem (1.1) below.

When the wind comes up against the deck of the bridge, a form of dynamic instability arises, which appears as uncontrolled vortices and it is usually named *flutter*. The origin of asymmetric vortices generates a forcing lift which launches vertical oscillations of the deck; this phenomenon finds confirmation in wind tunnel tests, see e.g. [30]. In particular, a transition between these vertical oscillations to torsional ones may happen which, in some cases, leads to the collapse of the bridge; we refer to [25, Chapter 1] for a survey of historical events where this phenomenon occurred, among which the infamous Tacoma Narrows Bridge collapse. Therefore, it becomes extremely important preventing flutter instability to provide a structure strong and safe. Rocard [33] suggested that for common bridge there exists a threshold of wind velocity V_c at which flutter arises. The computation of V_c is not an easy task, since it depends on the wind and on the geometric features of the deck; a possible way is to determine it experimentally. On the other hand, in engineering literature there exist some closed formulas for V_c; even if the debate on these formulas is still open, it seems to be accordance in thinking that the critical velocity depends on the frequencies or, equivalently, on the eigenvalues of the normal modes of the deck, see [19, 27, 33]. More precisely, since V_c represents the critical threshold at which an energy transfer occurs between the j-th and the i-th mode of oscillation, most of the authors propose V_c directly proportional to the difference between the square of the corresponding eigenvalues $\lambda_i > \lambda_j$, i.e.

$$V_c \propto (\lambda_i^2 - \lambda_j^2)\,.$$

It follows that a way to increase the critical velocity V_c, and in turn to prevent instability, is by increasing the distance between λ_i^2 and λ_j^2; this purpose is achievable moving the ratio $(\lambda_i/\lambda_j)^2$ away as much as possible from 1. A theoretical explanation of this fact was given in [10], within the classical stability theory of Mathieu equations, by relating large ratios of eigenvalues to the situation in which the instability resonant tongues of the Mathieu diagram become very thin.

Coming back to the plate model of the bridge, in order to prevent dynamical instability, different strategies to optimize the design of the plate have been proposed in literature; for instance, one may modify its shape, see [6], or rearrange the materials composing it, see [7, 8]. Within the present research,

2010 *Mathematics Subject Classification.* 35J40; 35P05; 74K20.

Key words and phrases. eigenvalues; plates; mass density.

we exploit the latter approach to maximize the ratio of selected eigenvalues of a partially hinged non-homogeneous plate. More precisely, by rescaling, we assume that the plate has length π and width 2ℓ with $2\ell \ll \pi$ so that

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 ;$$

then we characterize the non-homogeneity of the plate by a density function $p = p(x, y)$ and we consider the weighted eigenvalues problem:

$$\begin{cases} \Delta^2 u = \lambda\, p(x, y)u & \text{in } \Omega \\ u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0 & \text{for } y \in (-\ell, \ell) \\ u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = u_{yyy}(x, \pm\ell) + (2 - \sigma)u_{xxy}(x, \pm\ell) = 0 & \text{for } x \in (0, \pi) . \end{cases} \tag{1.1}$$

The boundary conditions on short edges are of Navier type, see [31], and model the situation in which the deck of the bridge is hinged on $\{0, \pi\} \times (-\ell, \ell)$. Instead, the boundary conditions on large edges are of Neumann type, see [18, 32], they model the fact that the deck is free to move vertically and involve the Poisson ratio σ which, for most of materials, satisfies $\sigma \in (0, 1/2)$. Finally, we focus on densities p satisfying some natural constraints, i.e. for $\alpha, \beta \in (0, +\infty)$ with $\alpha < \beta$ fixed, we assume that p belongs to the following class of weights

$$P_{\alpha,\beta} := \left\{ p \in L^\infty(\Omega) : \alpha \leqslant p \leqslant \beta\,, \quad p(x, y) = p(x, -y) \text{ a.e. in } \Omega \text{ and } \int_\Omega p\, dxdy = |\Omega| \right\} . \tag{1.2}$$

The integral condition in (1.2) represents the preservation of the total mass of the plate, while the symmetry requirement on p means that we focus on designs which are symmetric with respect to the mid-line of the roadway. From a mathematical point of view, the symmetry of p produces two classes of eigenfunctions of (1.1), respectively, even or odd in the y-variable, that we named *longitudinal* and *torsional* modes. In order to prevent the energy transfer from longitudinal to torsional modes, one may study the effect of the weight p on the ratio $\nu(p)/\mu(p)$, where ν and μ are two selected eigenvalues corresponding, respectively, to a torsional and a longitudinal mode. Since the final goal is to find the best rearrangement of materials in Ω which maximizes this ratio, we study, either from a theoretical and a numerical point of view, the optimization problem:

$$\mathcal{R} = \sup_{p \in P_{\alpha,\beta}} \frac{\nu(p)}{\mu(p)}. \tag{1.3}$$

We refer to [4] for optimization results on the ratio of eigenvalues of second order operators subject to domain perturbations and to [28] for optimization results, with respect to the weight, in 1-dimensional domains; see also [26, Chapter 9] and references therein. In particular, in [28] the author proved that the weight maximizing the considered ratio is of *bang-bang* type, namely a piecewise constant function, symmetric with respect to the middle of the string and getting the minimum value there. Unfortunately, the techniques exploited in [28] are closely related to the 1-dimensional nature of the problem and seem not applicable to our situation. Furthermore, here, things are complicated by dealing with a fourth order operator with non standard boundary conditions, for which no general positivity results are known. We refer the interested reader to [8] where a partial positivity property result was proved for the operator in (1.1).

As a consequence of what remarked, at the current state of art, a complete theoretical solution to problem (1.3) is difficult to reach and we proceed by steps. More precisely, we concentrate our efforts in looking for weights increasing $\nu(p)$ or reducing $\mu(p)$, separately. The numerical results we collect in Section 4 reveal that this apparently not rigorous approach turns out to be effective in increasing the ratio (1.3); indeed, as a matter of fact, weights having strong effect on torsional eigenvalues $\nu(p)$ produce very confined effects on longitudinal eigenvalues $\mu(p)$, and viceversa. In this regard, preliminary results were obtained in [8], where the goal was minimizing the *first* eigenvalue of (1.1), see Proposition 3.3 below. The focus of the present paper is on *higher* eigenvalues, furthermore we deal with a supremum problem and different methods are required; hence, the optimization issue (1.3) deserves to be studied independently. About the optimization of the *first* weighted eigenvalue of Δ^2 under Dirichlet or Navier boundary conditions, we mention the papers [2],[3],[16]-[22]. Concerning *higher* eigenvalues we refer to [29] where the authors provide a detailed spectral optimization analysis, upon density variations, of general elliptic operators of arbitrary order subject to several kinds of boundary conditions. In [15] numerical results were given for the Dirichlet and Navier version of of (1.1); while in [18] sharp upper bounds for weighted eigenvalues in the Neumann case were provided.

In order to increase the numerator of (1.3), i.e. the first torsional eigenvalue, we adapt to our situation the approach developed by [21], in the second order case, and which was partially extended to the fourth order by [22], in order to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions. The main novelty of the present paper is the exploitation of the precise information we have from [23] on the spectrum of problem (1.1) with $p \equiv 1$; this fact allows us to partially overcome the loss of positivity results for (1.1). Moreover, since we work with a domain $\Omega \subset \mathbb{R}^2$ rectangular, we perform some computations explicitly; in particular, we obtain upper bounds on longitudinal eigenvalues that, suitable combined with some rearrangements arguments inspired by [13] and [16], give the analytic expression of weights reducing the denominator in (1.3), see Theorem 3.4. Finally, in Section 4 we complete our theoretical results with numerical experiments; they provide weights increasing the ratio (1.3) and suggest a maximizer to (1.3).

The paper is organized as follows. In Section 2 we introduce some preliminaries and notations and we recall the known results in the case $p \equiv 1$. Section 3 is devoted to the main results of the paper, which we prove in Section 5. The theoretical results are complemented with numerical experiments collected in Section 4, where we give some practical suggestions about the location of the reinforcements in the plate. Finally, in the Appendix we complete our analysis of problem (1.1) by providing a Weyl-type asymptotic law for the eigenvalues.

2. Preliminaries and notations

From now onward we fix $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$ with $\ell > 0$ and $\sigma \in (0, 1/2)$. We denote by $\| \cdot \|_q$ the norm related to the Lebesgue spaces $L^q(\Omega)$ with $1 \leqslant q \leqslant \infty$ and we shall omit the set Ω in the notation of the functional spaces, e.g. $V := V(\Omega)$. The natural functional space where to set problem (1.1) is

$$H^2_* = \left\{ u \in H^2 : u = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell) \right\}.$$

Note that the condition $u = 0$ has to be meant in a classical sense because Ω is a planar domain and the energy space H^2_* embeds into continuous functions. Furthermore, H^2_* is a Hilbert space when endowed with the scalar product

$$(u, v)_{H^2_*} := \int_\Omega [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] \, dx \, dy$$

and associated norm

$$\|u\|^2_{H^2_*} = (u, u)_{H^2_*},$$

which is equivalent to the usual norm in H^2, see [23, Lemma 4.1]. Then, we reformulate problem (1.1) in the following weak sense

$$(u, v)_{H^2_*} = \lambda \int_\Omega p(x, y) uv \, dx \, dy \qquad \forall v \in H^2_*, \tag{2.1}$$

where p belongs to the family of weights $P_{\alpha,\beta}$ defined in (1.2) with $\alpha, \beta \in (0, +\infty)$ and $\alpha < \beta$ fixed. We underline that condition $p \in P_{\alpha,\beta}$ implies $\alpha \leqslant 1 \leqslant \beta$ since $\int_\Omega p \, dx \, dy = |\Omega|$. Moreover, it is not restrictive to assume $\alpha < 1 < \beta$ when we consider weights that do not coincide a.e. with the constant function $p \equiv 1$. In fact, if we assume $\beta = 1$, it must be $p = 1$ a.e. in Ω, since otherwise we would have $\int_\Omega p \, dx \, dy < |\Omega|$; similarly, if we put $\alpha = 1$. For these reasons, since the aim of our research is to study the effect of a non-constant weight on the eigenvalues of (1.1), in what follows we will always assume

$$0 < \alpha < 1 < \beta.$$

The bilinear form $(u, v)_{H^2_*}$ is continuous and coercive and $p \in L^\infty$ is positive a.e. in Ω, by standard spectral theory of self-adjoint operators we infer

Proposition 2.1. *Let $p \in P_{\alpha,\beta}$. Then all eigenvalues of* (2.1) *have finite multiplicity and can be represented by means of an increasing and divergent sequence $\lambda_h(p)$ ($h \in \mathbb{N}_+$), where each eigenvalue is repeated according to its multiplicity. Furthermore, the corresponding eigenfunctions form a compete system in H^2_*.*

We refer to [29, Lemma 2.1] for a detailed proof of Proposition 2.1 in a more general setting. On the other hand, it is well-known, see [20, 26], that the following variational representation of eigenvalues holds for every $h \in \mathbb{N}_+$:

$$\lambda_h(p) = \inf_{\substack{W_h \subset H^2_* \\ \dim W_h = h}} \sup_{u \in W_h \setminus \{0\}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}u\|^2_2}. \tag{2.2}$$

When $h = 1$, (2.2) includes the well known characterization for the first eigenvalue

$$\lambda_1(p) = \inf_{u \in H^2_* \setminus \{0\}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}u\|^2_2}. \tag{2.3}$$

If $h \geqslant 2$, the minimum in (2.2) is achieved by the space W_h spanned by the h-th first eigenfunctions. Assuming that the first $h-1$ eigenfunctions, $u_1, u_2, \dots, u_{h-1}$ are known, one also obtains

$$\lambda_h(p) = \inf_{\substack{u \in H^2_* \setminus \{0\} \\ (u,u_i)_{H^2_*} = 0 \ \forall i = 1, \dots, h-1}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}u\|^2_2}. \tag{2.4}$$

When $p \equiv 1$, we recall that the whole spectrum of (1.1) was determined in [23] (see also [6]); we collect what known in the following proposition.

Proposition 2.2. [23] *Consider problem* (1.1) *with $p \equiv 1$. Then:*

(i) for any $m \geqslant 1$ integer there exists a unique eigenvalue $\lambda = \Lambda_{m,1} \in ((1-\sigma^2)m^4, m^4)$ with corresponding eigenfunctions $\phi_{m,1}(y)\sin(mx)$ with $\phi_{m,1}(y)$ given in (2.6);

(ii) for any $m \geqslant 1$ and any $k \geqslant 2$ integers there exists a unique eigenvalue $\lambda = \Lambda_{m,k} > m^4$ satisfying $\left(m^2 + \frac{\pi^2}{\ell^2}\left(k - \frac{3}{2}\right)^2\right)^2 < \mu_{m,k} < \left(m^2 + \frac{\pi^2}{\ell^2}(k-1)^2\right)^2$, with corresponding eigenfunctions $\phi_{m,k}(y)\sin(mx)$ with $\phi_{m,k}(y)$ given in (2.6);

(iii) for any $m \geqslant 1$ and any $k \geqslant 2$ integers there exists a unique eigenvalue $\lambda = \Lambda^{m,k} > m^4$ with corresponding eigenfunctions $\psi_{m,k}(y)\sin(mx)$ with $\psi_{m,k}(y)$ given in (2.6);

(iv) for any $m \geqslant 1$ integer, satisfying $\ell m\sqrt{2}\coth(\ell m\sqrt{2}) > \left(\frac{2-\sigma}{\sigma}\right)^2$, there exists a unique eigenvalue $\lambda = \Lambda^{m,1} \in (\Lambda_{m,1}, m^4)$ with corresponding eigenfunctions $\psi_{m,1}(y)\sin(mx)$ with $\psi_{m,1}(y)$ given in (2.6).

Finally, if

$$\text{the unique positive solution } s > 0 \text{ of: } \tanh(\sqrt{2}s\ell) = \left(\frac{\sigma}{2-\sigma}\right)^2 \sqrt{2}s\ell \quad \text{is not an integer,} \tag{2.5}$$

then the only eigenvalues are the ones given in $(i) - (iv)$.

In the following, we will always assume that (2.5) holds.

At last, we recall the analytic expression of the functions $\phi_{m,k}(y)$ and $\psi_{m,k}(y)$ of Proposition 2.2. For $m,k \in \mathbb{N}_+$, we define:

$$\begin{aligned}
\phi_{m,1}(y) &:= \frac{1}{N_{m,1}}\left\{ \frac{\sigma m^2 - c_{m,1}^2}{\cosh(\ell\,\overline{c}_{m,1})}\cosh(y\,\overline{c}_{m,1}) + \frac{\overline{c}_{m,1}^2 - \sigma m^2}{\cosh(\ell\,c_{m,1})}\cosh(y\,c_{m,1})\right\}\\
\phi_{m,k}(y) &:= \frac{1}{N_{m,k}}\left\{ \frac{\sigma m^2 + c_{m,k}^2}{\cosh(\ell\,\overline{c}_{m,k})}\cosh(y\,\overline{c}_{m,k}) + \frac{\overline{c}_{m,k}^2 - \sigma m^2}{\cos(\ell\,c_{m,k})}\cos(y\,c_{m,k})\right\}\\
\psi_{m,k}(y) &:= \frac{1}{N^{m,k}}\left\{ \frac{\sigma m^2 + d_{m,k}^2}{\sinh(\ell\,\overline{d}_{m,k})}\sinh(y\,\overline{d}_{m,k}) + \frac{\overline{d}_{m,k}^2 - \sigma m^2}{\sin(\ell\,d_{m,k})}\sin(y\,d_{m,k})\right\}\\
\psi_{m,1}(y) &:= \frac{1}{N^{m,1}}\left\{ \frac{\sigma m^2 - d_{m,1}^2}{\sinh(\ell\,\overline{d}_{m,1})}\sinh(y\,\overline{d}_{m,1}) + \frac{\overline{d}_{m,1}^2 - \sigma m^2}{\sinh(\ell\,d_{m,1})}\sinh(y\,d_{m,1})\right\},
\end{aligned} \tag{2.6}$$

where

$$c_{m,k} := \sqrt{|(\Lambda_{m,k})^{1/2} - m^2|} \qquad \overline{c}_{m,k} := \sqrt{(\Lambda_{m,k})^{1/2} + m^2}$$
$$d_{m,k} := \sqrt{|(\Lambda^{m,k})^{1/2} - m^2|} \qquad \overline{d}_{m,k} := \sqrt{(\Lambda^{m,k})^{1/2} + m^2}\,,$$

with $\Lambda_{m,k}$ and $\Lambda^{m,k}$ defined in Proposition 2.2, and $N_{m,k}, N^{m,k} \in \mathbb{R}_+$ are fixed in such a way that $\|\phi_{m,k}(y)\,\sin(mx)\|_2 = \|\psi_{m,k}(y)\,\sin(mx)\|_2 = 1$.

Remark 2.3. *Denote by $\lambda_h(1)$ ($h \in \mathbb{N}_+$) the sequence of eigenvalues of* (1.1) *with $p \equiv 1$; this sequence can be written explicitly by ordering the eigenvalues given by Proposition 2.2. Then, for all $p \in P_{\alpha,\beta}$, the characterization* (2.2) *readily gives the stability inequality*

$$\frac{\lambda_h(1)}{\beta} \leqslant \lambda_h(p) \leqslant \frac{\lambda_h(1)}{\alpha}\,,$$

for every $h \in \mathbb{N}_+$. In applicative terms, if we choose materials having similar densities, we obtain eigenvalues close to those of the homogeneous plate.

By Proposition 2.2 we distinguish two classes of eigenfunctions of problem (1.1) with $p \equiv 1$:
- y-even eigenfunctions $\phi_{m,k}(y)\,\sin(mx)$ corresponding to the eigenvalues $\Lambda_{m,k}$;
- y-odd eigenfunctions $\psi_{m,k}(y)\,\sin(mx)$ corresponding to the eigenvalues $\Lambda^{m,k}$.

As in [11], this suggests to introduce the subspaces of H^2_*:

$$H^2_{\mathcal{E}} := \{u \in H^2_* : u(x,-y) = u(x,y) \quad \forall (x,y) \in \Omega\},$$
$$H^2_{\mathcal{O}} := \{u \in H^2_* : u(x,-y) = -u(x,y) \quad \forall (x,y) \in \Omega\},$$

where

$$H^2_{\mathcal{E}} \perp H^2_{\mathcal{O}}, \qquad H^2_* = H^2_{\mathcal{E}} \oplus H^2_{\mathcal{O}}\,.$$

By the symmetry assumption on $p \in P_{\alpha,\beta}$ it is readily verified that all linearly independent eigenfunctions of (1.1) may be thought in the class $H^2_{\mathcal{E}}$ or in the class $H^2_{\mathcal{O}}$. We call the eigenfunctions belonging to $H^2_{\mathcal{E}}$ *longitudinal* modes and those belonging to $H^2_{\mathcal{O}}$ *torsional* modes. In what follows we order all eigenvalues of (1.1), repeated according to multiplicity, into two increasing and divergent sequences: the sequence of the eigenvalues $\mu_j(p)$ ($j \in \mathbb{N}_+$) corresponding to longitudinal eigenfunctions and the sequence of the eigenvalues $\nu_j(p)$ ($j \in \mathbb{N}_+$) corresponding to torsional eigenfunctions. From Proposition 2.2 we infer that the sequences $\mu_j(1)$ and $\nu_j(1)$ can be written explicitly by ordering, respectively, the numbers $\Lambda_{m,k}$ and $\Lambda^{m,k}$. In particular, we have

$$\lambda_1(1) = \mu_1(1) = \Lambda_{1,1} < \nu_1(1) = \min\{\Lambda^{1,1}, \Lambda^{1,2}\}\,. \tag{2.7}$$

For actual bridges, one usually has $\nu_1(1) = \Lambda^{1,2}$, indeed the inequality required in Proposition 2.2-*iv*) is not satisfied for ℓ small, see Table 1 in Section 4. We note that, even in the case $p \equiv 1$, simplicity of eigenvalues is not know, hence, in principle, the same eigenvalue may correspond either to longitudinal and torsional eigenfunctions. However, our numerical results show that "low" eigenvalues are simple for $\ell \ll \pi$ and $\sigma \in (0, 1/2)$, furthermore "high" modes are activated by bending energy so large that they not appear in realistic situations; it follows that eigenvalues are expected to be simple in the applications.

For future purposes it is convenient to characterize in a variational way longitudinal and torsional eigenvalues. First, for $j \in \mathbb{N}_+$ fixed, we introduce, respectively, the spaces $U_j^{\mathcal{E}} \subset H^2_{\mathcal{E}}$ of the first $(j-1)$ longitudinal eigenfunctions and $U_j^{\mathcal{O}} \subset H^2_{\mathcal{O}}$ of the first $(j-1)$ torsional eigenfunctions of (1.1). Then we define

$$V_j^{\mathcal{E}} := \{u \in H^2_{\mathcal{E}} \,:\, (u,v)_{H^2_*} = 0 \quad \forall v \in U_j^{\mathcal{E}}\}, \qquad V_j^{\mathcal{O}} := \{u \in H^2_{\mathcal{O}} \,:\, (u,v)_{H^2_*} = 0 \quad \forall v \in U_j^{\mathcal{O}}\},$$

where if $j = 1$ we mean $V_1^{\mathcal{E}} = H^2_{\mathcal{E}}$ and $V_1^{\mathcal{O}} = H^2_{\mathcal{O}}$. Finally, using (2.4), we set

$$\mu_j(p) = \inf_{u \in V_j^{\mathcal{E}} \setminus \{0\}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}\, u\|_2^2} \quad \text{and} \quad \nu_j(p) = \inf_{u \in V_j^{\mathcal{O}} \setminus \{0\}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}\, u\|_2^2}. \tag{2.8}$$

3. Main results

As in Section 2 we always assume

$$0 < \sigma < \frac{1}{2} \quad \text{and} \quad \alpha < 1 < \beta \quad (\alpha, \beta \in (0, +\infty)).$$

The final goal of our analysis is to maximize the ratio (1.3) with the family $P_{\alpha,\beta}$ defined in (1.2). To this aim we need first to clarify which eigenvalues we shall consider in the ratio; the model situation we have in mind is a motion concentrated on a longitudinal mode, with corresponding eigenvalue μ_j and we want to prevent the transfer of energy from this mode to the nearest torsional one ν_i, for suitable $i, j \in \mathbb{N}_+$. Rocard [33, p.169] claims that, for the usual design of bridges, the eigenvalues of the observed longitudinal oscillating modes are larger than those of torsional modes, i.e. $\mu_j < \nu_i$. For the homogeneous plate this inequality readily follows from (2.7) if $j = i = 1$. More in general, we set

$$j_0 := \max\{j \in \mathbb{N}_+ : \nu_1(1) - \mu_j(1) > 0\}. \tag{3.1}$$

Clearly, $j_0 \geqslant 1$ and $j_0 = j_0(\ell, \sigma)$; in our numerical experiments, for several values of ℓ and σ chosen, taking into account real bridges, we always obtain $j_0 = 10$. As explained in [9, Section 1] this number is in accordance with what reported in the Federal Report [1], since a moment before the collapse of the Tacoma Narrows Bridge the motion was involving nine or ten longitudinal waves. Coming back to the choice of the eigenvalues in the ratio (1.3), for what observed, we finally focus on the problem

$$\mathcal{R} = \sup_{p \in P_{\alpha,\beta}} \frac{\nu_1(p)}{\mu_{j_0}(p)}. \tag{3.2}$$

Note that if $j_0 > 1$, then $\nu_1(p)/\mu_{j_0}(p) \leqslant \nu_1(p)/\mu_j(p)$ for all $1 \leqslant j < j_0$; therefore weights p increasing the value of $\nu_1(p)/\mu_{j_0}(p)$ also increase the value of $\nu_1(p)/\mu_j(p)$ for all $1 \leqslant j < j_0$.

First we prove

Theorem 3.1. *Let $j_0 \in \mathbb{N}_+$ be as defined in* (3.1)*. Then, problem* (3.2) *admits a solution.*

As already explained in the introduction, a precise theoretical characterization of maximizers to problem (3.2) seems hard to reach at the current state of studies. For this reason, we concentrate our efforts in looking for weights increasing $\nu_1(p)$ or reducing $\mu_{j_0}(p)$, separately. We start by facing the problem

$$\nu_1^{\alpha,\beta} := \sup_{p \in P_{\alpha,\beta}} \nu_1(p), \tag{3.3}$$

where $\nu_1(p)$ is defined in (2.8) taking $j = 1$. We call *optimal pair* for (3.3) a couple $(\widehat{p}, \widehat{u})$ such that $\widehat{p}$ achieves the supremum in (3.3) and $\widehat{u}$ is an eigenfunction of $\nu_1(\widehat{p})$. In the following we will always indicate with χ_D the characteristic function of a set $D \subset \mathbb{R}^2$. In Section 5 we prove

Theorem 3.2. *Problem* (3.3) *admits an optimal pair* $(\widehat{p},\widehat{u}) \in P_{\alpha,\beta} \times H^2_{\mathcal{O}}$. *Furthermore,* $\widehat{u}$ *and* $\widehat{p}$ *are related as follows*

$$\widehat{p}(x,y) = \beta\chi_{\widehat{S}}(x,y) + \alpha\chi_{\Omega\setminus\widehat{S}}(x,y) \quad \text{for a.e. } (x,y) \in \Omega\,,$$

where $\widehat{S} = \{(x,y) \in \Omega\,:\, \widehat{u}^2(x,y) \leqslant \widehat{t}\}$ *for some* $\widehat{t} > 0$ *such that* $|\widehat{S}| = \frac{1-\alpha}{\beta-\alpha}\,|\Omega|$.

Next we focus on longitudinal eigenvalues. For $j \in \mathbb{N}_+$, we set the minimum problem

$$\mu_j^{\alpha,\beta} := \inf_{p\in P_{\alpha,\beta}} \mu_j(p)\,, \tag{3.4}$$

where $\mu_j(p)$ is as defined in (2.8). We call *optimal pair* for (3.4) a couple $(\overline{p}_j,\overline{u}_j)$ such that $\overline{p}_j$ achieves the infimum in (3.4) and $\overline{u}_j$ is an eigenfunction of $\mu_j(\overline{p}_j)$. When $j = 1$ the counterpart of Theorem 3.2 for problem (3.4) is basically known from [8] where the minimization issue for $\lambda_1(p)$, as defined in (2.3), was dealt with. More precisely, the same proof of [8, Theorem 3.2] with minor changes yields the following statement:

Proposition 3.3. [8] *Set* $j = 1$, *then problem* (3.4) *admits an optimal pair* $(\overline{p}_1,\overline{u}_1) \in P_{\alpha,\beta} \times H^2_{\mathcal{E}}$. *Furthermore,* $\overline{u}_1$ *and* $\overline{p}_1$ *are related as follows*

$$\overline{p}_1(x,y) = \alpha\chi_{S_1}(x,y) + \beta\chi_{\Omega\setminus S_1}(x,y) \quad \text{for a.e. } (x,y) \in \Omega\,,$$

where $S_1 = \{(x,y) \in \Omega\,:\, \overline{u}_1^2(x,y) \leqslant t_1\}$ *for some* $t_1 > 0$ *such that* $|S_1| = \frac{\beta-1}{\beta-\alpha}\,|\Omega|$.

Things become more involved for higher longitudinal eigenvalues. Indeed, the proofs of Theorem 3.2 and Proposition 3.3 are based on suitable rearrangement inequalities, see Lemma 5.4 below, involving $\widehat{p}$ and $\overline{p}_1$, respectively; this approach does not carry over to the case $j \geqslant 2$ since, in general, the orthogonality condition in the sets $V_j^{\mathcal{E}}$ of (2.8) is not preserved when changing weights. For this reason, we proceed differently and we lower $\mu_j(p)$ "indirectly". More precisely, we first derive upper bounds for $\mu_j(p)$, where the eigenfunctions $\overline{u}_j$ are, in some sense, replaced by functions suitably chosen in $H^2_{\mathcal{E}}$; then we look for weights effective in lowering the upper bounds and, in turn, $\mu_j(p)$. We do not claim that this indirect approach will give the optimal density, however it suggests explicit weights effective in lowering higher eigenvalues and furthermore it provides a theoretical validation of the numerical results we collect in Section 4.2.

For $j \geqslant 2$ fixed and $m = 1,\dots,j$, we introduce the following functions having disjoint supports

$$w_m(x,y) := \begin{cases} \sin^2(jx) & \text{if } (x,y) \in \Omega_m^j \\ 0 & \text{if } (x,y) \in \Omega\setminus\Omega_m^j, \end{cases} \tag{3.5}$$

where $\Omega_m^j := \left(\dfrac{(m-1)\pi}{j}, \dfrac{m\pi}{j}\right) \times (-\ell,\ell) \subset \Omega$; it is readily checked that $w_m \in C^1(\overline{\Omega}) \cap H^2_{\mathcal{E}}$ for all $m = 1,\dots,j$. Then, we prove:

Theorem 3.4. *Let* $j \geqslant 2$ *integer, then problem* (3.4) *admits an optimal pair* $(\overline{p}_j,\overline{u}_j) \in P_{\alpha,\beta} \times H^2_{\mathcal{E}}$ *and there holds*

$$\mu_j(p) \leqslant \inf_{p\in P_{\alpha,\beta}} \left\{ \max_{m=1,\dots,j} \left\{ \frac{1}{\|\sqrt{p}\,w_m\|_2^2} \right\} \right\} j^3|\Omega|\,. \tag{3.6}$$

In particular, denoting by $P^{per}_{\alpha,\beta} := \{p \in P_{\alpha,\beta} : p(x,y) = p\big(x+\frac{\pi}{j},y\big)$, *for a.e.* $(x,y) \in \Omega\}$, *we have*

$$\mu_j(p) \leqslant \inf_{p\in P^{per}_{\alpha,\beta}} \left\{ \frac{1}{\|\sqrt{p}\,\sin^2(jx)\|_2^2} \right\} j^4|\Omega| \tag{3.7}$$

and the latter infimum is achieved by the functions

$$p_j(x,y) = \alpha\chi_{S_j}(x,y) + \beta\chi_{\Omega\setminus S_j}(x,y) \quad \text{for a.e. } (x,y) \in \Omega\,,$$

where $S_j = \{(x,y) \in \Omega\,:\, \sin^4(jx) \leqslant t_j\}$ *for* $t_j > 0$ *such that* $|S_j| = \frac{\beta-1}{\beta-\alpha}\,|\Omega|$.

Remark 3.5. *A comment on the choice of the functions w_m is in order. The idea of taking functions π/j-periodic in the x-variable comes from the explicit form of the longitudinal eigenfunctions of Proposition 2.2; slightly changes in the analytic expression of functions w_m will qualitatively produce the same weights p_j, e.g. replacing $\sin^2(jx)$ with $\sin^{2n}(jx)$ ($n \geqslant 2$ integer) or $\exp\left[-1/(1-|x|^2)\right]$ properly rescaled and shifted in each Ω_m^j. We underline that there is accordance between the optimal weights found numerically in Section 4.2 and the weights p_j of Theorem 3.4.*

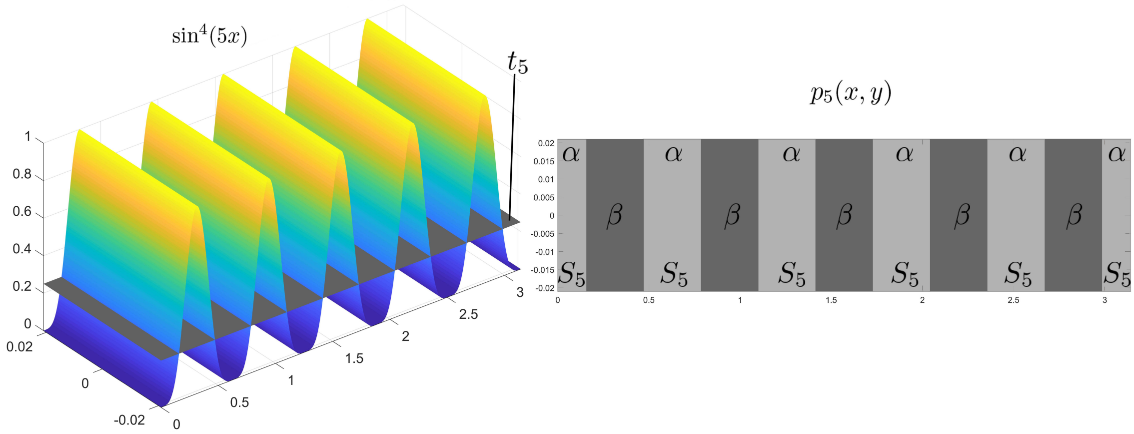


FIGURE 1. Plots of $z = \sin^4(5x)$ intersected with the plane $z = t_5$ and the correspondent set S_5, related to the weight $p_5(x, y)$, for a plate with $\ell = \pi/150$ ($\alpha = 0.5$, $\beta = 1.5$).

We observe that, while the sets $\widehat{S}$ and S_1 of Theorem 3.2 and Proposition 3.3 depend on the unknown functions $\widehat{u}$ and $\overline{u}_1$, the set S_j of Theorem 3.4 is explicitly given once determined $t_j > 0$. As a matter of example, in Figure 1 we plot the function $z = \sin^4(5x)$, the corresponding set S_5 and the related weight $p_5(x, y)$. It is worth noting that the statement of Theorem 3.2 combines nicely with those of Proposition 3.3 and Theorem 3.4 in increasing the ratio in (3.2). This is highlighted by the numerical experiments we collect in Section 4.

4. Numerical results

In the previous section we proved that an optimal weight maximizing the ratio $\nu_1(p)/\mu_{j_0}(p)$, with j_0 defined in (3.1), exists. Then, in order to find information on its analytic expression, we decided to minimize $\mu_{j_0}(p)$ or maximize $\nu_1(p)$, separately. All the theoretical results obtained tell that the optimal weights $p \in P_{\alpha,\beta}$, either for problem (3.3) and (3.4), must be of *bang-bang* type, i.e.

$$p(x, y) = \alpha\chi_S(x, y) + \beta\chi_{\Omega\setminus S}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

for a suitable set $S \subset \Omega$. In other words, the plate must be composed by two different materials properly located in Ω; this is useful in engineering terms, since the assemblage of two materials with constant density is simpler than the manufacturing of a material having variable density. Unfortunately, Theorem 3.2 and Proposition 3.3 give no precise information on the location of the set S; nevertheless, through suitable numerical experiments, we are able to suggest what could be the optimal design of the set S, in problems (3.3) and (3.4), and to guess a possible maximizer to problem (3.2). For the applicative purpose we may strengthen the plate with steel and we may consider the other material composed by a mixture of steel and concrete; following this approach, the denser material has approximately triple density with respect to the weaken, i.e. $\beta = 3\alpha$.

4.1. **Eigenvalues computation.** We propose a numerical method to find approximate solutions of (1.1) which relies on the explicit information we have from Proposition 2.2 ($p \equiv 1$). We expand the solutions u of (1.1) in Fourier series, adopting the orthonormal basis of L^2 given by eigenfunctions of the homogeneous plate. More precisely, denoting by $z_m(x,y) \in H^2_{\mathcal{E}}$ and $\theta_m(x,y) \in H^2_{\mathcal{O}}$, respectively, the (ordered) longitudinal and torsional eigenfunctions of problem (1.1) with $p \equiv 1$, u writes

$$u(x,y) = \sum_{m=1}^{\infty} \left[a_m z_m(x,y) + b_m \theta_m(x,y)\right] , \tag{4.1}$$

for suitable $a_m, b_m \in \mathbb{R}$. In order to get a numerical approximation, we trunk the series in (4.1) at $N \in \mathbb{N}_+$ and we plug the Fourier sum into (2.1). We recall that, for all $m \in \mathbb{N}_+$, z_m and θ_m solve:

$$\begin{aligned}(z_m, v)_{H^2_*} &= \mu_m(1)\,(z_m, v)_{L^2} \quad \forall v \in H^2_*\\ (\theta_m, v)_{H^2_*} &= \nu_m(1)\,(\theta_m, v)_{L^2} \quad \forall v \in H^2_*\,,\end{aligned}$$

where $\mu_m(1)$ and $\nu_m(1)$ are defined in (2.8) with $p \equiv 1$. Therefore, we obtain the following finite dimensional linear system in the unknowns a_n and b_n:

$$\begin{cases} a_n \mu_n(1) = \mu(p) \sum\limits_{m=1}^{N} a_m\, C^p_{n,m} \\ b_n \nu_n(1) = \nu(p) \sum\limits_{m=1}^{N} b_m\, \overline{C}^p_{n,m}, \end{cases} \qquad \text{for } n = 1, \dots, N \tag{4.2}$$

where

$$\begin{aligned} C^p_{n,m} &:= \int_\Omega p(x,y)\, z_n(x,y)\, z_m(x,y)\, dxdy \\ \overline{C}^p_{n,m} &:= \int_\Omega p(x,y)\, \theta_n(x,y)\, \theta_m(x,y)\, dxdy. \end{aligned}$$

In particular, by solving (4.2), it is possible to determine N approximated longitudinal eigenvalues $\mu_n(p)$ and N torsional eigenvalues $\nu_n(p)$. We observe that the decoupling between the unknowns a_n and b_n, which produces eigenfunctions even or odd in y, is due to the assumption on $p \in P_{\alpha,\beta}$, being y-even.

In order to compute numerically the eigenvalues $\mu_n(p)$ and $\nu_n(p)$ for suitable choices of the weight p, we fix from now onward:

$$\sigma = 0.2 \qquad \text{and} \qquad \ell = \frac{\pi}{150}, \tag{4.3}$$

which is a choice consistent with common bridge design. The explicit values of $\mu_n(1)$ and $\nu_n(1)$ are computed by exploiting Proposition 2.2, see Table 1. When condition (4.3) holds, we numerically find that the eigenvalues $\Lambda^{m,1}$ do not exist for $1 \leqslant m \leqslant 2734$ and that

$$\begin{aligned} \mu_m(1) &= \Lambda_{m,1} \quad \text{for} \quad m = 1, \dots, 113 \\ \nu_m(1) &= \Lambda^{m,2} \quad \text{for} \quad m = 1, \dots, 174\,. \end{aligned}$$

Hence, by Proposition 2.2 we know that the basis of eigenfunctions exploited in (4.1) writes $z_m(x,y) = \phi_{m,1}(y)\sin(mx)$ and $\theta_m(x,y) = \psi_{m,2}(y)\sin(mx)$ for $1 \leqslant m \leqslant 113$.

4.2. **Numerical solution of (3.4).** Let us begin by minimizing $\mu_1(p)$, i.e. the first longitudinal eigenvalue of problem (1.1), as characterized in (2.8) with $j = 1$. In order to find the optimal weight given by Proposition 3.3, we adopt a numerical algorithm proposed in [15] that we shortly illustrate in the following. First we solve numerically (4.2) with a given weight $p^{(i)}$ and we determine the corresponding eigenfunction $u_1^{(i)}$. Then, we choose a weight at the next iteration $p^{(i+1)}$ such that

$$\|\sqrt{p^{(i+1)}}u_1^{(i)}\|_2^2 \geqslant \|\sqrt{p^{(i)}}u_1^{(i)}\|_2^2,$$

in order to have

$$\mu_1^{(i+1)} = \min_{u \in H^2_{\mathcal{E}} \setminus \{0\}} \frac{\|u\|^2_{H^2_*}}{\|\sqrt{p^{(i+1)}}\,u\|_2^2} = \frac{\|u_1^{(i+1)}\|^2_{H^2_*}}{\|\sqrt{p^{(i+1)}}\,u_1^{(i+1)}\|_2^2} \leqslant \frac{\|u_1^{(i)}\|^2_{H^2_*}}{\|\sqrt{p^{(i+1)}}\,u_1^{(i)}\|_2^2} \leqslant \frac{\|u_1^{(i)}\|^2_{H^2_*}}{\|\sqrt{p^{(i)}}\,u_1^{(i)}\|_2^2} = \mu_1^{(i)}.$$

	$\mu_m(1) = \Lambda_{m,1}$
$m = 1$	$9.60{\cdot}10^{-1}$
$m = 2$	$1.54{\cdot}10^{1}$
$m = 3$	$7.78{\cdot}10^{1}$
$m = 4$	$2.46{\cdot}10^{2}$
$m = 5$	$6.00{\cdot}10^{2}$
$m = 6$	$1.24{\cdot}10^{3}$
$m = 7$	$2.31{\cdot}10^{3}$
$m = 8$	$3.93{\cdot}10^{3}$
$m = 9$	$6.30{\cdot}10^{3}$
$m = 10$	$9.61{\cdot}10^{3}$
$m = 11$	$1.41{\cdot}10^{4}$
$m = 12$	$1.99{\cdot}10^{4}$

	$\nu_m(1) = \Lambda^{m,2}$
$m = 1$	$1.09{\cdot}10^{4}$
$m = 2$	$4.38{\cdot}10^{4}$
$m = 3$	$9.86{\cdot}10^{4}$
$m = 4$	$1.75{\cdot}10^{5}$
$m = 5$	$2.74{\cdot}10^{5}$
$m = 6$	$3.95{\cdot}10^{5}$
$m = 7$	$5.38{\cdot}10^{5}$
$m = 8$	$7.04{\cdot}10^{5}$
$m = 9$	$8.93{\cdot}10^{5}$
$m = 10$	$1.10{\cdot}10^{6}$
$m = 11$	$1.34{\cdot}10^{6}$
$m = 12$	$1.60{\cdot}10^{6}$

TABLE 1. On the left the lowest longitudinal eigenvalues $\mu_m(1)$ and on the right the lowest torsional eigenvalues $\nu_m(1)$ of (1.1) with $p \equiv 1$.

Note that to select $p^{(i+1)}$ we exploited the rearrangement Lemma 5.4 below. Iterating, we obtain a decreasing sequence of eigenvalues; since the infimum in (3.4) is achieved, the sequence is bounded from below by $\mu_1^{\alpha,\beta}$ so that it is convergent. We stop the algorithm when $|\mu_1^{(i+1)} - \mu_1^{(i)}| < \epsilon$, with $\epsilon = 10^{-4} \div 10^{-3}$. As pointed out in [13] it is not clear a priori if the sequence converges to $\mu_1^{\alpha,\beta}$ or not; to avoid the latter case we repeated the procedure considering different weights at the first iteration and we always obtain the convergence to the same values.

In Figure 2 we plot the set S_1 defined in Proposition 3.3 for the eigenfunction $\overline{u}_1$ of the obtained numerical optimal pair; clearly, the direction is to concentrate the denser material near the maximum of $\overline{u}_1^2$. Since the set $\Omega \setminus S_1$ is similar to a rectangle, we propose the following analytic expression of the **approximated optimal weight** for $\mu_1^{\alpha,\beta}$:

$$\overline{p}_1(x,y) = \overline{p}_1(x) := \beta\chi_{I_1}(x) + \alpha\chi_{(0,\pi)\setminus I_1}(x) \quad \text{for a.e. } (x,y) \in \Omega\,,$$

where $I_1 := \left(\frac{\pi}{2} - \frac{\pi}{2}\frac{(1-\alpha)}{(\beta-\alpha)}, \frac{\pi}{2} + \frac{\pi}{2}\frac{(1-\alpha)}{(\beta-\alpha)}\right)$.

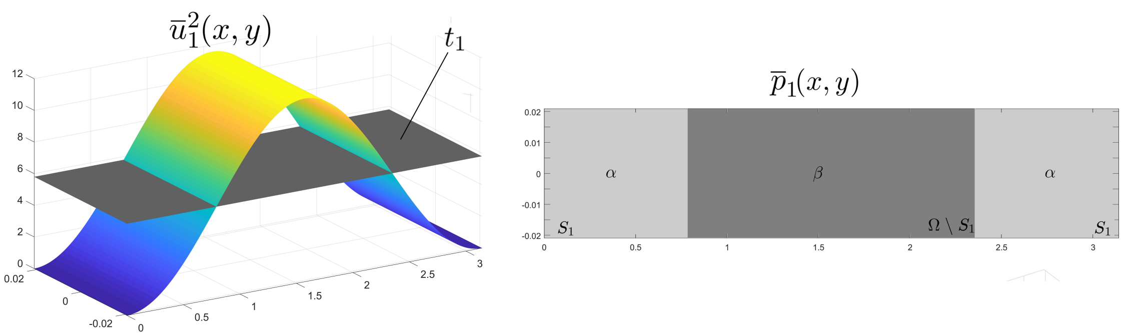


FIGURE 2. Plot of $z = \overline{u}_1^2(x,y)$ intersected with the plane $z = t_1$ and plot of the related set S_1 ($\alpha = 0.5$, $\beta = 1.5$, $N = 20$).

The previous algorithm can be adapted to determine $\mu_j(\overline{p}_j)$ for generic $j \in \mathbb{N}_+$; in particular, if $j > 1$ we apply the characterization (2.4) of eigenvalues, i.e. we consider the minimum onto the space $V_j^{\mathcal{E}}$ instead of $H^2_{\mathcal{E}}$. The further difficulty is that, now, at the end of every iteration, we have to check that $u_j^{(i)}(x,y)$ does not belong to the subspace spanned by $\{u_1^{(i+1)}, \dots, u_{j-1}^{(i+1)}\}$, for more details see [15]. For each $j \in \mathbb{N}_+$, the obtained optimal weight has the denser material concentrated near to the peaks of the associated eigenfunction which are, approximatively, located at $\frac{\pi}{2j}(2h-1)$ with $h = 1, ..., j$; this is aligned with the statement of Theorem 3.4. Therefore, we propose the following **approximated optimal weight** for $\mu_j^{\alpha,\beta}$:

$$\overline{p}_j(x,y) = \overline{p}_j(x) := \beta\chi_{I_j}(x) + \alpha\chi_{(0,\pi)\setminus I_j}(x), \quad \text{for a.e. } (x,y) \in \Omega\,, \tag{4.4}$$

where $I_j := \bigcup_{h=1}^{j} \left(\frac{\pi}{2j}(2h-1) - \frac{\pi}{j}\frac{(1-\alpha)}{2(\beta-\alpha)}, \frac{\pi}{2j}(2h-1) + \frac{\pi}{j}\frac{(1-\alpha)}{2(\beta-\alpha)} \right)$.

The above results show that the optimal weight changes if we change j. Nevertheless, numerically, we observe that the weight $\overline{p}_j(x)$ in (4.4) reduces not only $\mu_j(\overline{p}_j)$, but also all the previous longitudinal eigenvalues $\mu_i(\overline{p}_j)$ with $1 \leqslant i < j$; while it increases $\mu_i(\overline{p}_j)$ with $i > j$. This means that, if it were possible to predict the highest mode of vibration for a plate during its design, then there would be an optimal reinforce for it, reducing at the same time all the previous ones.

4.3. **Numerical solution to (3.3).** About the maximization of the first torsional eigenvalue, we cannot adapt the algorithm in [15] that only works for infimum problems. Nevertheless, Theorem 3.2 suggests to put the denser material in the region $\widehat{S} = \{(x,y) \in \Omega : \widehat{u}^2(x,y) \leqslant \widehat{t}\}$ for some $\widehat{t} > 0$, where $\widehat{u}$ is the eigenfunction corresponding to $\nu_1(\widehat{p})$. Since we do not know explicitly $\widehat{u}$, we proceed by trial and error; we start by replacing $\widehat{u}$ with the first torsional eigenfunction $\theta_1(x,y) = \psi_{1,2}(y)\sin(x)$ of problem (1.1) with $p \equiv 1$ and we define the weight $p^*(x,y) := \beta\chi_{S^*}(x,y) + \alpha\chi_{\Omega\setminus S^*}(x,y)$ where $S^* := \{(x,y) \in \Omega : \theta_1^2(x,y) \leqslant t^*\}$, for $t^* > 0$ such that $|S^*| = \frac{1-\alpha}{\beta-\alpha}|\Omega|$. Then, we proceed by solving (1.1) with p^*, obtaining a new first torsional eigenfunction u^* to which we associate, as done for θ_1, a new weight p^{**} of bang-bang type; iterating the procedure, we observe that the obtained weights are always very close to p^*, so that we conjecture that the theoretical optimal weight $\widehat{p}$ of Theorem 3.2 is qualitatively very similar to p^*. In Figure 3 we plot S^* and in Table 2 we give the corresponding eigenvalues.

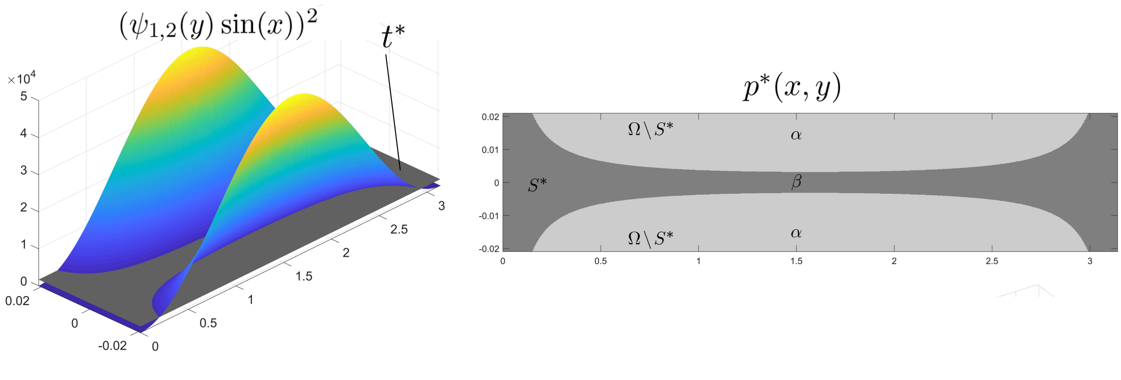


FIGURE 3. Plot of $z = \theta_1^2(x,y) = (\psi_{1,2}(y)\sin(x))^2$ intersected with the plane $z = t^*$ and plot of the related set S^* ($\alpha = 0.5$, $\beta = 1.5$).

In order to find a reinforce more suitable for practical reproduction, inspired by Figure 3, we consider in our experiments a second weight depending only on y and concentrated around the mid-line $y = 0$, i.e.

$$\breve{p}(x,y) = \breve{p}(y) := \beta\chi_{\breve{I}}(y) + \alpha\chi_{(-\ell,\ell)\setminus\breve{I}}(y) \quad \text{for a.e. } (x,y) \in \Omega\,,$$

where $\breve{I} := \big(-\frac{\ell(\beta-1)}{\beta-\alpha}, \frac{\ell(\beta-1)}{\beta-\alpha}\big)$. Clearly, this choice produces some simplifications in the problem and the coefficients in (4.2) become simpler, see also [8, Section 4]. The obtained eigenvalues are again collected in Table 2.

Since the weight $\breve{p}(x,y)$ increases $\nu_1(p)$ less than $p^*(x,y)$, we keep as **approximated optimal weight** for $\nu_1^{\alpha,\beta}$:

$$p^*(x,y) = \beta\chi_{S^*}(x,y) + \alpha\chi_{\Omega\setminus S^*}(x,y) \quad \text{for a.e. } (x,y)\in\Omega\,,$$

where $S^* = \{(x,y)\in\Omega \,:\, (\theta_1)^2(x,y) \leqslant t^*\}$ for $t^*>0$ such that $|S^*| = \frac{1-\alpha}{\beta-\alpha}|\Omega|$.

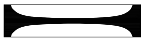
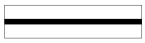
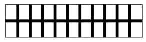
	$p\equiv 1$ 	$\overline{p}_{10}(x)$ 	$p^*(x,y)$ 	$\breve{p}(y)$ 	$\overline{\overline{p}}(x)$ 	$\widetilde{p}(x,y)$ 
$\mu_1(p)$	$9.60{\cdot}10^{-1}$	$9.60{\cdot}10^{-1}$	1.16	$9.60{\cdot}10^{-1}$	1.40	$9.86{\cdot}10^{-1}$
$\mu_2(p)$	$1.54{\cdot}10^{1}$	$1.54{\cdot}10^{1}$	$1.66{\cdot}10^{1}$	$1.54{\cdot}10^{1}$	$1.52{\cdot}10^{1}$	$1.58{\cdot}10^{1}$
$\mu_3(p)$	$7.78{\cdot}10^{1}$	$7.77{\cdot}10^{1}$	$8.06{\cdot}10^{1}$	$7.78{\cdot}10^{1}$	$8.05{\cdot}10^{1}$	$7.98{\cdot}10^{1}$
$\mu_4(p)$	$2.46{\cdot}10^{2}$	$2.46{\cdot}10^{2}$	$2.51{\cdot}10^{2}$	$2.46{\cdot}10^{2}$	$2.96{\cdot}10^{2}$	$2.52{\cdot}10^{2}$
$\mu_5(p)$	$6.00{\cdot}10^{2}$	$5.99{\cdot}10^{2}$	$6.10{\cdot}10^{2}$	$6.01{\cdot}10^{2}$	$6.78{\cdot}10^{2}$	$6.16{\cdot}10^{2}$
$\mu_6(p)$	$1.24{\cdot}10^{3}$	$1.24{\cdot}10^{3}$	$1.27{\cdot}10^{3}$	$1.25{\cdot}10^{3}$	$1.31{\cdot}10^{3}$	$1.28{\cdot}10^{3}$
$\mu_7(p)$	$2.31{\cdot}10^{3}$	$2.28{\cdot}10^{3}$	$2.36{\cdot}10^{3}$	$2.31{\cdot}10^{3}$	$2.60{\cdot}10^{3}$	$2.37{\cdot}10^{3}$
$\mu_8(p)$	$3.93{\cdot}10^{3}$	$3.84{\cdot}10^{3}$	$4.04{\cdot}10^{3}$	$3.94{\cdot}10^{3}$	$4.55{\cdot}10^{3}$	$4.04{\cdot}10^{3}$
$\mu_9(p)$	$6.30{\cdot}10^{3}$	$5.87{\cdot}10^{3}$	$6.48{\cdot}10^{3}$	$6.31{\cdot}10^{3}$	$6.85{\cdot}10^{3}$	$6.47{\cdot}10^{3}$
$\mu_{10}(p)$	$9.61{\cdot}10^{3}$	$7.28{\cdot}10^{3}$	$9.90{\cdot}10^{3}$	$9.62{\cdot}10^{3}$	$1.04{\cdot}10^{4}$	$9.55{\cdot}10^{3}$
$\mu_{11}(p)$	$1.41{\cdot}10^{4}$	$1.68{\cdot}10^{4}$	$1.45{\cdot}10^{4}$	$1.41{\cdot}10^{4}$	$1.61{\cdot}10^{4}$	$1.45{\cdot}10^{4}$
$\mu_{12}(p)$	$1.99{\cdot}10^{4}$	$2.27{\cdot}10^{4}$	$2.05{\cdot}10^{4}$	$2.00{\cdot}10^{4}$	$2.24{\cdot}10^{4}$	$2.05{\cdot}10^{4}$
$\nu_1(p)$	$1.09{\cdot}10^{4}$	$1.09{\cdot}10^{4}$	$1.98{\cdot}10^{4}$	$1.75{\cdot}10^{4}$	$1.56{\cdot}10^{4}$	$1.71{\cdot}10^{4}$
$\nu_2(p)$	$4.38{\cdot}10^{4}$	$4.37{\cdot}10^{4}$	$6.88{\cdot}10^{4}$	$7.01{\cdot}10^{4}$	$4.14{\cdot}10^{4}$	$6.84{\cdot}10^{4}$
$\mathcal{R}$	**1.14**	**1.50**	**2.00**	**1.82**	**1.49**	**1.79**

TABLE 2. The lowest longitudinal eigenvalues $\mu_j(p)$ with $j=1,\dots,12$, the first two torsional $\nu_i(p)$ with $i=1,2$ and the ratio $\mathcal{R}=\frac{\nu_1(p)}{\mu_{10}(p)}$ of (4.2) with different weights, assuming (4.3), $\alpha=0.5$, $\beta=1.5$ and $N=30$.

4.4. **Numerical solution of (3.2).** From the eigenvalues in Table 1 we infer

$$\mu_1(1) < \dots < \mu_{10}(1) < \nu_1(1) < \mu_{11}(1) < \dots$$

This is the reason why, we fix $j_0=10$ in (3.1) and we focus on the ratio $\mathcal{R}$ between $\nu_1(p)$ and $\mu_{10}(p)$. In order to increase $\mathcal{R}$ we test weights raising $\nu_1(p)$ and lowering $\mu_{10}(p)$.

First we consider the optimal weight $\overline{p}_{10}(x)$ as defined in (4.4). As we can see from Table 2, it has a limited effect on the variation of $\nu_1(p)$, so that it makes sense to minimize $\mu_{10}(p)$ in order to increase the ratio (3.2).

Next we consider weights having strong effects on $\nu_1(p)$, such as the weights $p^*(x,y)$ and $\breve{p}(y)$ defined in Section 4.3. Table 2 highlights that they have a confined effect on longitudinal eigenvalues. Moreover, they increase the ratio $\mathcal{R}$ much more than the weights optimal for the longitudinal modes.

We complete the numerical experiments by testing other weights which seem to be reasonable in order to increase $\mathcal{R}$; more precisely, we consider a weight concentrated near the short edges of the plate:

$$\overline{\overline{p}}(x,y) = \overline{\overline{p}}(x) := \alpha\chi_I(x) + \beta\chi_{(0,\pi)\setminus I}(x) \quad \text{for a.e. } (x,y)\in\Omega\,,$$

where $I := \left(\frac{\pi}{2} - \frac{\pi(\beta-1)}{2(\beta-\alpha)}, \frac{\pi}{2} + \frac{\pi(\beta-1)}{2(\beta-\alpha)}\right)$, and the cross-type weight $\widetilde{p}(x, y)$, given in the last column of Table 2, which is obtained by combining $\overline{p}_{10}(x)$ and $\breve{p}(y)$. From Table 2 we observe that these weights have effects both on torsional and on longitudinal eigenvalues, so that they do not seem optimal for the ratio $\mathcal{R}$.

Summing up, since $p^*(x, y)$ increases the ratio $\mathcal{R}$ more than all the other considered weights, we propose as **approximated optimal weight** for $\mathcal{R}$:

$$p^*(x, y) = \beta\chi_{S^*}(x, y) + \alpha\chi_{\Omega\setminus S^*}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $S^* = \{(x, y) \in \Omega : \theta_1^2(x, y) \leqslant t^*\}$ for $t^* > 0$ such that $|S^*| = \frac{1-\alpha}{\beta-\alpha}|\Omega|$, cfr. Figure 3.

Although the present work is focused on the first torsional eigenvalue, in Table 2 we also collect the results obtained for the second torsional eigenvalue $\nu_2(p)$. We observe that the variation of $\nu_2(p)$ follows the same trend of that of $\nu_1(p)$ with respect to the considered weights, hence we may conjecture that the same reinforcement could be adopted to optimize ratios involving subsequent (low) torsional eigenvalues.

5. Proofs

In what follows we will always assume that

$$0 < \sigma < \frac{1}{2} \quad \text{and} \quad \alpha < 1 < \beta \quad (\alpha, \beta \in (0, +\infty))$$

and the family $P_{\alpha,\beta}$ is as defined in (1.2).

5.1. **Proof of Theorem 3.1.** The proof follows by combining three lemmas that we state here below. In the first part of this section we will not need to distinguish between longitudinal and torsional eigenvalues.

Given $p \in P_{\alpha,\beta}$, it is convenient to endow the space L^2 of the weighted scalar product: $(p\,u, v)_{L^2}$, for all $u, v \in L^2$, which defines an equivalent norm in L^2. Then, for $h \in N_+$, we introduce the orthogonal projection of $u \in H^2_*$, with respect to the above weighted scalar product, onto the space generated by the first $(h-1)$ eigenfunctions $u_1, \ldots, u_{h-1}$ of problem (1.1):

$$P_{h-1}(p)u := \sum_{i=1}^{h-1}(p\,u, u_i)_{L^2}\, u_i\,;$$

when $h = 1$ we adopt the convection $P_0(p)u = 0$. Finally, we recall the Auchmuty's principle [5] stated in our framework:

Lemma 5.1. *Let $p \in P_{\alpha,\beta}$ and $\lambda_h(p)$ the $h-$th eigenvalue of (1.1) with $h \in N_+$, then*

$$-\frac{1}{2\lambda_h(p)} = \inf_{u \in H^2_*} \mathcal{A}_h(p, u) \quad \textit{where} \quad \mathcal{A}_h(p, u) := \frac{1}{2}\|u\|^2_{H^2_*} - \|\sqrt{p}\,\left[u - P_{h-1}(p)u\right]\|_2\,.$$

Furthermore, the minimum is achieved at a h-th eigenfunction normalized according to

$$\|u_h\|^2_{H^2_*} = \|\sqrt{p}\,u_h\|_2 = \frac{1}{\lambda_h(p)}$$

Proof. The proof follows arguing as in [22, Lemma 3.3] by simply replacing $H^2 \cap H^1_0$ with H^2_*. In alternative, in [5] one can find the original proof in a general setting. □

Lemma 5.2. *The set $P_{\alpha,\beta}$ is compact for the weak* topology of L^∞.*

Proof. First we prove that $P_{\alpha,\beta}$ is a strongly closed set in L^2.

Let $\{p_m\}_m \subset P_{\alpha,\beta}$ be a sequence such that $p_m \to q$ in L^2 (as $m \to +\infty$) for some $q \in L^2$; then $p_m \to q$ in L^1 (as $m \to +\infty$) and up to a subsequence (still denoted by p_m) we infer that $p_m \to q$ a.e. in Ω. Therefore, $\alpha \leqslant q \leqslant \beta$ and q is y-even a.e. in Ω; moreover, $\int_\Omega p_m\, v\, dx\, dy \to \int_\Omega q\, v\, dx\, dy$ for all $v \in L^2$, so that, choosing $v \equiv 1 \in L^2$, we obtain $|\Omega| = \int_\Omega q\, dx\, dy$. This implies that $q \in P_{\alpha,\beta}$ and $P_{\alpha,\beta}$ is strongly closed in L^2.

Next, we show that any sequence $\{p_m\}_m \subset P_{\alpha,\beta}$ admits a subsequence converging in the weak* topology of L^∞ to an element of $P_{\alpha,\beta}$. By the definition of $P_{\alpha,\beta}$ we have $\|p_m\|_\infty \leqslant \beta$, so that, up to a subsequence, we obtain

$$p_{m_k} \overset{*}{\rightharpoonup} \overline{p} \quad \text{in } L^\infty \quad \text{as } k \to \infty\,.$$

Moreover, we have $\|p_{m_k}\|_2^2 \leqslant \beta|\Omega|$, so that, up to a subsequence, we infer that $p_{m_{k_j}} \rightharpoonup \overline{q}$ in L^2 as $j \to \infty$. It is easy to check that $P_{\alpha,\beta}$ is a convex set and, since convex strongly closed space are weakly closed, we readily infer that $\overline{q} \in P_{\alpha,\beta}$.

Therefore,

$$\int_\Omega p_{m_{k_j}}\, v\, dx\, dy \to \int_\Omega \overline{q}\, v\, dx\, dy \quad \forall v \in L^2 \subset L^1 \quad \text{as } j \to \infty \quad \text{with } \overline{q} \in P_{\alpha,\beta}$$

and, since $p_{m_k} \overset{*}{\rightharpoonup} \overline{p}$ in L^∞ yields $\int_\Omega p_{m_{k_j}}\, v\, dx\, dy \to \int_\Omega \overline{p}\, v\, dx\, dy$ $\forall v \in L^1$, we conclude that $\overline{p} = \overline{q}$ a.e. in Ω. Whence, $\overline{p} \in P_{\alpha,\beta}$ and the proof is complete. □

Lemma 5.3. *Let $\lambda_h(p)$ the $h-$th eigenvalue of* (1.1) *with $h \in N_+$. The map $p \mapsto \lambda_h(p)$ is continuous on $P_{\alpha,\beta}$ for the weak* convergence.*

Proof. Let $\{p_m\}_m \subset P_{\alpha,\beta}$ be a sequence converging in the weak* topology of L^∞ to $\overline{p}$, i.e.

$$p_m \overset{*}{\rightharpoonup} \overline{p} \quad \text{in } L^\infty \quad \text{as } m \to \infty;$$

then $\overline{p} \in P_{\alpha,\beta}$ by Lemma 5.2.

To p_m we associate the h-th eigenvalue $\lambda_h(p_m)$ of (1.1) and an eigenfunction $u_h(p_m)$ normalized with respect to the weighted scalar product, i.e. $\int_\Omega p_m\, u_h(p_m)\, u_r(p_m)\, dx\, dy = \delta_{hr}$, where δ_{hr} is the Kronecker delta for all $h, r \in \mathbb{N}_+$ and $\lambda_h(p_m) = \|u_h(p_m)\|^2_{H^2_*}$.

By (1.2) and (2.2) we have

$$\lambda_h(p) \leqslant \frac{\lambda_h(1)}{\alpha} \qquad \forall p \in P_{\alpha,\beta},$$

where $\lambda_h(1)$ is the h-th eigenvalue of (1.1) with $p \equiv 1$, implying that $\lambda_h(p_m) = \|u_h(p_m)\|_{H^2_*} \leqslant \lambda_h(1)/\alpha$. Therefore, we can extract a subsequence, still denoted by $u_h(p_m)$, such that

$$\begin{aligned} &\lambda_h(p_m) \to \overline{\lambda}_h \quad \text{in } \mathbb{R}, \\ &u_h(p_m) \rightharpoonup \overline{u}_h \quad \text{in } H^2_* \quad \text{as } m \to \infty. \end{aligned}$$

Moreover, due to the compact embedding $H^2_* \hookrightarrow L^2$, we obtain that $u_h(p_m)$ strongly converges to $\overline{u}_h$ in L^2 as $m \to \infty$; this implies, for all $v \in H^2_*$, that

$$\int_\Omega p_m\, u_h(p_m)\, v\, dx\, dy \to \int_\Omega \overline{p}\, \overline{u}_h\, v\, dx\, dy \quad \text{as } m \to \infty$$

indeed

$$\left| \int_\Omega (p_m\, u_h(p_m) - \overline{p}\, \overline{u}_h)\, v\, dx\, dy \right| \leqslant \|p_m v\|_2 \|u_h(p_m) - \overline{u}_h\|_2 + \left| \int_\Omega p_m\, \overline{u}_h v\, dx\, dy - \int_\Omega \overline{p}\, \overline{u}_h v\, dx\, dy \right| \to 0,$$

since $\overline{u}_h v \in H^2_* \subset L^1$. Therefore, we obtain

$$\big(u_h(p_m), v\big)_{H^2_*} - \lambda_h(p_m)\big(p_m\, u_h(p_m), v\big)_{L^2} \to \big(\overline{u}_h, v\big)_{H^2_*} - \overline{\lambda}_h\big(\overline{p}\, \overline{u}_h, v\big)_{L^2} \qquad \forall v \in H^2_* \quad \text{as } m \to \infty\,,$$

inferring that $\overline{\lambda}_h$ is an eigenvalue of (1.1) and $\overline{u}_h$ is a corresponding eigenfunction.

Arguing as before we also obtain $\int_\Omega p_m\, u_h(p_m)\, u_r(p_m)\, dx\, dy \to \int_\Omega \overline{p}\, \overline{u}_h\, \overline{u}_r\, dx\, dy = \delta_{hr}$ for all $h, r \in \mathbb{N}_+$, so that $\overline{\lambda}_h$ is a diverging sequence for $h \to \infty$. To prove that $\overline{\lambda}_h = \lambda_h(\overline{p})$ for every $h \in \mathbb{N}_+$, we assume by contradiction that, for $p = \overline{p}$, there exists an eigenfunction $\overline{u}$ associated with the eigenvalue $\overline{\lambda}$ such that $\big(\overline{p}\, \overline{u}, \overline{u}_h\big)_{L^2} = 0$ for all $h \in \mathbb{N}_+$. We suppose that $\overline{u}$ is normalized in H^2_* so that $\|\sqrt{\overline{p}}\, \overline{u}\|_2 = 1/\overline{\lambda}$; applying Lemma 5.1 we have

$$-\frac{1}{2\lambda_h(p_m)} \leqslant \mathcal{A}_h(p_m, \overline{u}) = \frac{1}{2}\|\overline{u}\|^2_{H^2_*} - \|\sqrt{p_m}\,\left[\overline{u} - P_{h-1}(p_m)\overline{u}\right]\|_2 \to \frac{1}{2}\|\overline{u}\|^2_{H^2_*} - \|\sqrt{\overline{p}}\,\overline{u}\|_2 = -\frac{1}{2\overline{\lambda}}, \tag{5.1}$$

where the convergence comes from

$$P_{h-1}(p_m)\overline{u} = \sum_{i=1}^{h-1} \big(p_m\overline{u}, u_i(p_m)\big)_{L^2} u_i(p_m) \to \sum_{i=1}^{h-1} \big(\overline{p}\,\overline{u}, \overline{u}_i\big)_{L^2} \overline{u}_i = 0 \quad \text{in } L^2.$$

Therefore, by (5.1), letting $m \to \infty$, we obtain

$$\overline{\lambda} \geqslant \lambda_h(p_m) \to \overline{\lambda}_h \qquad \forall h \in \mathbb{N}_+,$$

giving a contradiction since $\overline{\lambda}_h$ is an unbounded sequence for $h \to \infty$. Thus $\overline{\lambda}_h = \lambda_h(\overline{p})$, implying the continuity of $p \mapsto \lambda_h(p)$ for every $h \in \mathbb{N}_+$ fixed. □

Proof of Theorem 3.1 completed. Let us consider the function $F : (0, +\infty) \times (0, +\infty) \mapsto \mathbb{R}$ given by $F(t, s) := \frac{t}{s}$, continuous on its domain. By Lemma 5.3, the maps $p \mapsto \nu_1(p)$ and $p \mapsto \mu_{j_0}(p)$ are continuous on $P_{\alpha,\beta}$ for the weak* convergence; since $\nu_1(p) > \mu_{j_0}(p) > 0$, we infer that $F(\mu_{j_0}(p), \nu_1(p)) = \frac{\nu_1(p)}{\mu_{j_0}(p)}$ is also continuous on $P_{\alpha,\beta}$ for the weak* convergence. Finally, the existence of a maximum (or minimum) of $F(\mu_{j_0}(p), \nu_1(p)) = \frac{\nu_1(p)}{\mu_{j_0}(p)}$ on $P_{\alpha,\beta}$ follows thanks to the compactness proved in Lemma 5.2 of the set $P_{\alpha,\beta}$ for the weak* topology of L^∞.

5.2. **Proof of Theorem 3.2.** The existence of an optimal pair for (3.3) follows as in the proof of Theorem 3.1 by considering the continuous function $F(\mu_{j_0}(p), \nu_1(p)) = \nu_1(p)$. In the sequel we will denote by $(\widehat{p}, \widehat{u}) \in P_{\alpha,\beta} \times H^2_{\mathcal{O}}$ an optimal pair for (3.3) suitable normalized as follows

$$\nu_1^{\alpha,\beta} = \nu_1(\widehat{p}) = \frac{\|\widehat{u}\|^2_{H^2_*}}{\|\sqrt{\widehat{p}}\,\widehat{u}\|_2^2} \quad \text{and} \quad \|\widehat{u}\|^2_{H^2_*} = \frac{1}{\nu_1(\widehat{p})}\,. \tag{5.2}$$

Next we state a couple of lemmas useful to complete the proof.

Lemma 5.4. *Let $u \in H^2_* \setminus \{0\}$ and let $J : P_{\alpha,\beta} \to \mathbb{R}$ be defined as follows*

$$J(p) = \int_\Omega p(x, y)u^2\,dx\,dy\,.$$

Then, the problem

$$I_{\alpha,\beta} := \inf_{p\in P_{\alpha,\beta}} J(p)$$

admits the solution

$$p_u(x, y) = \beta\chi_{\widetilde{S}}(x, y) + \alpha\chi_{\Omega\setminus\widetilde{S}}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $\widetilde{S} = \widetilde{S}(u) \subset \Omega$ is such that $|\widetilde{S}| = \frac{1-\alpha}{\beta-\alpha}\,|\Omega| := \widetilde{C}_{\alpha,\beta}$. Furthermore, define

$$t := \sup\Big\{s \geqslant 0 : |\{(x, y) \in \Omega\, : u^2(x, y) \leqslant s\}| < \widetilde{C}_{\alpha,\beta}\Big\}\,, \tag{5.3}$$

if $t = 0$ we have that

$$\widetilde{S} \subseteq \{(x, y) \in \Omega\, : u^2(x, y) = 0\}$$

while if $t > 0$ we have that

$$\{(x, y) \in \Omega\, : u^2(x, y) < t\} \subseteq \widetilde{S} \subseteq \{(x, y) \in \Omega\, : u^2(x, y) \leqslant t\}\,. \tag{5.4}$$

Similarly, the problem

$$M_{\alpha,\beta} := \sup_{p\in P_{\alpha,\beta}} J(p)$$

admits the solution

$$p_u(x, y) = \alpha\chi_{\check{S}}(x, y) + \beta\chi_{\Omega\setminus\check{S}}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $\check{S} = \check{S}(u) \subset \Omega$ *is such that* $|\check{S}| = \frac{\beta-1}{\beta-\alpha}\,|\Omega| =: \check{C}_{\alpha,\beta}$ *and satisfies* (5.4)-(5.3) *with* $\widetilde{C}_{\alpha,\beta}$ *replaced by* $\check{C}_{\alpha,\beta}$.

Proof. We only prove the statement for $I_{\alpha,\beta}$ since the statement for $M_{\alpha,\beta}$ follows basically by reversing all the inequalities below. Since $p_u \in P_{\alpha,\beta}$ we have

$$I_{\alpha,\beta} \leqslant J(p_u)\,.$$

If we prove that

$$J(p) \geqslant J(p_u) \qquad \forall p \in P_{\alpha,\beta},$$

the thesis is obtained. To this aim, we first consider the case $t > 0$, we have

$$\begin{aligned}
&\int_\Omega u^2(p_u - p)\,dx\,dy \\
&= \int_{\{u^2<t\}} u^2\,(\beta - p)\,dx\,dy + \int_{\{u^2>t\}} u^2\,(\alpha - p)\,dx\,dy + \int_{\{u^2=t\}} u^2\,(p_u - p)\,dx\,dy \\
&\leqslant t\int_{\{u^2<t\}} (\beta - p)\,dx\,dy + t\int_{\{u^2>t\}} (\alpha - p)\,dx\,dy + t\int_{\{u^2=t\}} (p_u - p)\,dx\,dy \\
&= t\int_\Omega (p_u - p)\,dx\,dy = 0\,,
\end{aligned}$$

where the last equality comes from the preservation of the total mass condition.

Similarly, for $t = 0$ we have

$$\int_\Omega u^2(p_u - p)\,dx\,dy = \int_{\{u^2>0\}} u^2\,(p_u - p)\,dx\,dy \leqslant 0.$$

In both the cases we conclude that $J(p) \geqslant J(p_u)$, and in turn that $J(p_u) = I_{\alpha,\beta}$. □

We will also invoke the Auchmuty's principle recalled in Lemma 5.1 that, in terms of $\nu_1(p)$, rewrites

$$-\frac{1}{2\nu_1(p)} = \inf_{u\in H^2_{\mathcal{O}}} \mathcal{A}(p,u) \qquad \mathcal{A}(p,u) := \frac{1}{2}\|u\|^2_{H^2_*} - \|\sqrt{p}\,u\|_2\,. \tag{5.5}$$

By this, if $(\widehat{p},\widehat{u}) \in P_{\alpha,\beta} \times H^2_{\mathcal{O}}$ and (5.2) is satisfied, it is readily deduced that

$$\sup_{p\in P_{\alpha,\beta}} \inf_{u\in H^2_{\mathcal{O}}} \mathcal{A}(p,u) = -\frac{1}{2\nu_1^{\alpha,\beta}} = -\frac{1}{2\nu_1(\widehat{p})} = \inf_{u\in H^2_{\mathcal{O}}} \mathcal{A}(\widehat{p},u) = \mathcal{A}(\widehat{p},\widehat{u})\,. \tag{5.6}$$

Furthermore, we have

Lemma 5.5. *Let* $\mathcal{A}(p,u)$ *be as defined in* (5.5)*, the following equality holds*

$$\sup_{p\in P_{\alpha,\beta}} \inf_{u\in H^2_{\mathcal{O}}} \mathcal{A}(p,u) = \inf_{u\in H^2_{\mathcal{O}}} \sup_{p\in P_{\alpha,\beta}} \mathcal{A}(p,u)\,. \tag{5.7}$$

The proof of Lemma 5.5 is the same of [22, Lemma 3.7] once replaced the set $H^2 \cap H^1_0$ there with our set $H^2_{\mathcal{O}}$ (strongly and weakly closed subspace of H^2). Hence, we omit it and we refer the interested readers to [22] or [21], where the proof was originally given in the second order case.

Finally, we prove

Lemma 5.6. *There exists an optimal pair* $(\widehat{p},\widehat{u}) \in P_{\alpha,\beta} \times H^2_{\mathcal{O}}$ *as in* (5.2) *such that there holds*

$$\int_\Omega \widehat{p}(x,y)\,\widehat{u}^2\,dxdy \leqslant \int_\Omega p(x,y)\,\widehat{u}^2\,dxdy \qquad \forall p \in P_{\alpha,\beta}\,. \tag{5.8}$$

Proof. The idea of the proof is taken from [22, Proposition 3.8], the main difference here is the use of Lemma 5.4.

First we consider the functional $B : H^2_{\mathcal{O}} \to \mathbb{R}$ defined as follows

$$B(u) = \sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,u) = \frac{1}{2}\|u\|^2_{H^2_*} - \inf_{p \in P_{\alpha,\beta}} \|\sqrt{p}\, u\|_2\,,$$

and we claim that

$$\text{there exists } \bar{u} \in H^2_{\mathcal{O}} \text{ such that } B(\bar{u}) = \inf_{u \in H^2_{\mathcal{O}}} B(u) = \inf_{u \in H^2_{\mathcal{O}}} \sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,u) =: I\,. \tag{5.9}$$

To this aim, let $\{u_k\}$ be a minimizing sequence for I, namely

$$\frac{1}{2}\|u_k\|^2_{H^2_*} - \inf_{p \in P_{\alpha,\beta}} \|\sqrt{p}\, u_k\|_2 = I + o(1) \quad \text{as } k \to +\infty\,.$$

By the boundedness of p and the continuity of the embedding $H^2_* \subset L^2$ it is readily deduced that

$$\|u_k\|^2_{H^2_*} \leqslant C\|u_k\|_{H^2_*} + 2I + o(1) \quad \text{as } k \to +\infty$$

and, in turn, that

$$\|u_k\|_{H^2_*} \leqslant \overline{C} \quad \text{for } k \text{ sufficiently large,}$$

with $C, \overline{C} > 0$. Then, up to a subsequence, we have

$$u_k \rightharpoonup \overline{u} \text{ in } H^2_* \quad \text{and} \quad u_k \to \overline{u} \text{ in } L^2 \quad \text{as } k \to +\infty\,.$$

Therefore,

$$\|\overline{u}\|^2_{H^2_*} \leqslant \liminf_{k \to +\infty} \|u_k\|^2_{H^2_*}\,.$$

Next we take $p_{\overline{u}}$ as given in Lemma 5.4, namely such that

$$\inf_{p \in P_{\alpha,\beta}} \int_\Omega p(x,y)\, \overline{u}^2\, dx\, dy = \int_\Omega p_{\overline{u}}(x,y)\, \overline{u}^2\, dx\, dy\,.$$

Clearly,

$$\lim_{k \to +\infty} \int_\Omega p_{\overline{u}}(x,y)\, u_k^2\, dx\, dy = \int_\Omega p_{\overline{u}}(x,y)\, \overline{u}^2\, dx\, dy$$

and

$$\inf_{p \in P_{\alpha,\beta}} \int_\Omega p(x,y)\, u_k^2\, dx\, dy \leqslant \int_\Omega p_{\overline{u}}(x,y)\, u_k^2\, dx\, dy\,.$$

In particular, we conclude that

$$\limsup_{k \to +\infty} \inf_{p \in P_{\alpha,\beta}} \int_\Omega p(x,y)\, u_k^2\, dx\, dy \leqslant \limsup_{k \to +\infty} \int_\Omega p_{\overline{u}}(x,y)\, u_k^2\, dx\, dy = \inf_{p \in P_{\alpha,\beta}} \int_\Omega p(x,y)\, \overline{u}^2\, dx\, dy\,.$$

The above inequalities yield

$$B(\overline{u}) \leqslant \liminf_{k \to +\infty} B(u_k) = I$$

which is the claim (5.9).

By combining (5.6), (5.7) and (5.9), it follows that

$$\sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,\overline{u}) = \inf_{u \in H^2_{\mathcal{O}}} \sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,u) = \sup_{p \in P_{\alpha,\beta}} \inf_{u \in H^2_{\mathcal{O}}} \mathcal{A}(p,u) = \mathcal{A}(\widehat{p},\widehat{u}) = \inf_{u \in H^2_{\mathcal{O}}} \mathcal{A}(\widehat{p},u)\,.$$

In particular, this implies that

$$\mathcal{A}(p,\overline{u}) \leqslant \sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,\overline{u}) = \inf_{u \in H^2_{\mathcal{O}}} \mathcal{A}(\widehat{p},u) \leqslant \mathcal{A}(\widehat{p},\overline{u}) \qquad \forall p \in P_{\alpha,\beta} \tag{5.10}$$

and that

$$\mathcal{A}(\widehat{p},\overline{u}) \leqslant \sup_{p \in P_{\alpha,\beta}} \mathcal{A}(p,\overline{u}) = \inf_{u \in H^2_{\mathcal{O}}} \mathcal{A}(\widehat{p},u) \leqslant \mathcal{A}(\widehat{p},u) \qquad \forall u \in H^2_{\mathcal{O}}\,.$$

From the above inequality we infer that $\overline{u}$ is a minimizer of $\mathcal{A}(\widehat{p},u)$, hence an eigenfunction of $\nu_1(\widehat{p})$, see (5.6). Then, $(\widehat{p},\overline{u})$ is an optimal pair as defined in (5.2) and we may take $\widehat{u}=\overline{u}$ in the statement. Furthermore, by (5.10) with $\widehat{u}=\overline{u}$ we get

$$\mathcal{A}(p,\widehat{u})\leqslant\mathcal{A}(\widehat{p},\widehat{u})\qquad\forall p\in P_{\alpha,\beta}\,,$$

which, recalling the definition of $\mathcal{A}(p,u)$, yields (5.8).

□

Proof of Theorem 3.2 completed.

Step 1. Let $(\widehat{p},\widehat{u})\in P_{\alpha,\beta}\times H^2_{\mathcal{O}}$ be the optimal pair given by Lemma 5.6 and set $p_{\widehat{u}}(x,y):=\beta\chi_{S_{\widehat{u}}}(x,y)+\alpha\chi_{\Omega\setminus S_{\widehat{u}}}(x,y)$ with $S_{\widehat{u}}=\widetilde{S}(\widehat{u})$ as defined in Lemma 5.4; we prove that

$$\|\sqrt{\widehat{p}}\,\widehat{u}\|_2^2=\|\sqrt{p_{\widehat{u}}}\,\widehat{u}\|_2^2. \tag{5.11}$$

By Lemma 5.4 on problem $I_{\alpha,\beta}$ we know that

$$\int_\Omega\widehat{p}(x,y)\,\widehat{u}^2\,dx\,dy\geqslant\int_\Omega p_{\widehat{u}}(x,y)\,\widehat{u}^2\,dx\,dy\,.$$

On the other hand, by (5.8) with $p=p_{\widehat{u}}$ we infer

$$\int_\Omega\widehat{p}(x,y)\,\widehat{u}^2\,dxdy\leqslant\int_\Omega p_{\widehat{u}}(x,y)\,\widehat{u}^2\,dxdy\,.$$

Comparing the above inequalities, the proof of Step 1 follows.

Step 2. Let $(p_{\widehat{u}},\widehat{u})\in P_{\alpha,\beta}\times H^2_{\mathcal{O}}$ be as in Step 1 and let $\widehat{t}\geqslant 0$ be the number corresponding to $\widehat{u}$ in $S_{\widehat{u}}$. We prove that $\widehat{p}=p_{\widehat{u}}$ a.e. in Ω.

By (5.11), if $\widehat{t}=0$ we have

$$0=\int_\Omega(\widehat{p}-p_{\widehat{u}})\widehat{u}^2\,dx\,dy=\int_{\{\widehat{u}^2=0\}}(\widehat{p}-p_{\widehat{u}})\widehat{u}^2\,dx\,dy+\int_{\{\widehat{u}^2>0\}}(\widehat{p}-\alpha)\widehat{u}^2\,dx\,dy,$$

implying $\widehat{p}\,\widehat{u}=\alpha\,\widehat{u}$ a.e. in Ω. On the other hand, since $\widehat{u}\in H^4(\Omega)$ we can write almost everywhere the Euler-Lagrange equation related to the Rayleigh quotient of $\nu_1^{\alpha,\beta}=\nu_1(\widehat{p})$ and, for what observed above, we infer that

$$\Delta^2\widehat{u}=\nu_1^{\alpha,\beta}\,\alpha\,\widehat{u}\quad\text{a.e. in }\Omega\,.$$

Recalling that $\widehat{u}$ satisfies the partially hinged boundary conditions, this means that it must be one of the eigenfunctions listed in Proposition 2.2; since the set of zeroes of any of the eigenfunctions of Proposition 2.2 has zero measure, this contradicts the definition of $S_{\widehat{u}}$ and forces $S_{\widehat{u}}$ to be a set of positive measure. Whence, the above arguments proves that $\widehat{t}>0$.

For $\widehat{t}>0$ we set

$$A_{\widehat{t}}=\{(x,y)\in\Omega\,:\,\widehat{u}^2(x,y)=\widehat{t}\,\}\,.$$

By (5.11) we obtain

$$0=\int_\Omega(\widehat{p}-p_{\widehat{u}})\widehat{u}^2\,dx\,dy=\int_{\{\widehat{u}^2<\widehat{t}\}}(\widehat{p}-\beta)\widehat{u}^2\,dx\,dy+\int_{\{\widehat{u}^2>\widehat{t}\}}(\widehat{p}-\alpha)\widehat{u}^2\,dx\,dy+\int_{A_{\widehat{t}}}(\widehat{p}-p_{\widehat{u}})\widehat{u}^2\,dx\,dy.$$

Assume by contradiction that $\widehat{p}>\alpha$ in a set $A\subseteq\{\widehat{u}^2>\widehat{t}\}$ with $|A|>0$, then we get

$$\int_{\{\widehat{u}^2>\widehat{t}\}}(\widehat{p}-\alpha)(\widehat{u}^2-\widehat{t})\,dx\,dy\geqslant\int_A(\widehat{p}-\alpha)(\widehat{u}^2-\widehat{t})\,dx\,dy>0$$

and, in turn, that $\int_{\{\widehat{u}^2>\widehat{t}\}}(\widehat{p}-\alpha)\widehat{u}^2\,dx\,dy>\widehat{t}\int_{\{\widehat{u}^2>\widehat{t}\}}(\widehat{p}-\alpha)\,dx\,dy$. Whence,

$$0>\int_{\{\widehat{u}^2<\widehat{t}\}}(\widehat{p}-\beta)\widehat{u}^2\,dx\,dy+\widehat{t}\int_{\{\widehat{u}^2>\widehat{t}\}}(\widehat{p}-\alpha)\,dx\,dy+\widehat{t}\int_{A_{\widehat{t}}}(\widehat{p}-p_{\widehat{u}})\,dx\,dy\geqslant\widehat{t}\int_\Omega(\widehat{p}-p_{\widehat{u}})\,dx\,dy=0\,,$$

where the last equality follows from the preservation of the total mass condition. This contradicts the definition of the set A and implies $\widehat{p}=\alpha$ a.e. in $\{\widehat{u}^2>\widehat{t}\}$. Proceeding as before, we suppose that $\widehat{p}<\beta$ in a subset of positive measure of $\{\widehat{u}^2<\widehat{t}\}$ and we obtain a further contradiction

$$0>\widehat{t}\int_{\{\widehat{u}^2<\widehat{t}\}}(\widehat{p}-\beta)\,dx\,dy+\widehat{t}\int_{A_{\widehat{t}}}(\widehat{p}-p_{\widehat{u}})\,dx\,dy=\widehat{t}\int_{\Omega}(\widehat{p}-p_{\widehat{u}})\,dx\,dy=0.$$

It remains to study $\widehat{p}$ in $A_{\widehat{t}}$. When $|A_{\widehat{t}}|>0$, we write the Euler-Lagrange equation related to $\nu_1^{\alpha,\beta}=\nu_1(\widehat{p})$ obtaining

$$\Delta^2\widehat{u}=\nu_1^{\alpha,\beta}\,\widehat{p}\,\widehat{u}\quad \text{a.e. in } A_{\widehat{t}}.$$

Since $\widehat{u}^2=\widehat{t}$ we get $0=\nu_1^{\alpha,\beta}\,\widehat{p}$, that is absurd since $\widehat{p}\geqslant\alpha>0$. This implies that $A_{\widehat{t}}$ must have zero measure, so that $\widehat{p}=p_{\widehat{u}}$ a.e. in Ω.

Step 3. Since $|A_{\widehat{t}}|=0$, also $|A_{\widehat{t}}\setminus S_{\widehat{u}}|=0$, therefore it is not restrictive, up to a set of zero measure, to assume that $A_{\widehat{t}}\setminus S_{\widehat{u}}=\emptyset$ in such way that $A_{\widehat{t}}\subseteq S_{\widehat{u}}$ and, in turn, that

$$S_{\widehat{u}}=\left\{(x,y)\in\Omega:\widehat{u}^2(x,y)\leqslant\widehat{t}\right\}.$$

5.3. **Proof of Theorem 3.4.** The existence issue follows as in the proof of Theorem 3.1 by considering the continuous function $F(\mu_j(p),\nu_1(p))=\mu_j(p)$.

Fixed $j\geqslant 2$, by the characterization (2.2) of $\mu_j(p)$, we may choose $\overline{W}_j=\{w_1,\dots,w_j\}\subset H^2_{\mathcal{E}}$, where the functions w_m for $m=1,\dots,j$ are defined in (3.5). Therefore, we obtain

$$\mu_j(p)\leqslant\sup_{u\in\overline{W}_j\setminus\{0\}}\frac{\|u\|^2_{H^2_*}}{\|\sqrt{p}u\|_2^2}=\max_{m=1,\dots,j}\left\{\frac{\|w_m\|^2_{H^2_*}}{\|\sqrt{p}\,w_m\|_2^2}\right\},$$

where we have exploited the fact that the space $\overline{W}_j$ is generated by disjointly supported functions. Since $\|w_m\|^2_{H^2_*}=|\Omega|j^3$, we finally obtain the upper bound (3.6).

In order to reduce this bound and, in turn $\mu_j^{\alpha,\beta}$, it is convenient to choose a weight p having the same effect on every $\|\sqrt{p}\,w_m\|_2^2$; this suggests to take a weight π/j-periodic in x, i.e. $p\in P^{per}_{\alpha,\beta}$. In this way we obtain

$$\|\sqrt{p}w_m\|_2^2=\frac{1}{j}\|\sqrt{p}\sin^2(jx)\|_2^2,$$

and the proof of (3.7) readily follows from (3.6).

The last part of the statement comes out by slightly modifying the proof of Lemma 5.4 by which we infer that the problem

$$\inf_{p\in P^{per}_{\alpha,\beta}}\int_{\Omega}p(x,y)\,\sin^4(jx)\,dx\,dy$$

admits the solution

$$p_j(x,y)=\alpha\chi_{S_j}(x,y)+\beta\chi_{\Omega\setminus S_j}(x,y)\qquad\text{for a.e. }(x,y)\in\Omega\,,$$

where $S_j=\{(x,y)\in\Omega\,:\,\sin^4(jx)\leqslant t_j\}$ for $t_j>0$ such that $|S_j|=\frac{\beta-1}{\beta-\alpha}\,|\Omega|$.

6. Appendix

The aim of this section is to give further information on the eigenvalues $\lambda_h(p)$ ($h\in\mathbb{N}_+$) of (1.1); in particular, we compare problem (1.1) with the more studied Dirichlet and Neumann type problems and we derive a Weyl-type asymptotic law for $\lambda_h(p)$.

We begin by writing the above mentioned problems in our rectangular domain Ω; the Dirichlet problem reads

$$\begin{cases}\Delta^2u=\lambda\,p(x,y)u & \text{in }\Omega\\ u(0,y)=u_x(0,y)=u(\pi,y)=u_x(\pi,y)=0 & \text{for }y\in(-\ell,\ell)\\ u(x,\pm\ell)=u_y(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\,,\end{cases}\tag{6.1}$$

with weak form

$$\int_\Omega \Delta u \Delta v = \lambda \int_\Omega puv \qquad \forall v \in H_0^2.$$

The Neumann type problem reads

$$(6.2)\qquad \begin{cases} \Delta^2 u = \lambda\, p(x,y)u & \text{in } \Omega \\ u_{xx}(0,y)+\sigma u_{yy}(0,y) = u_{xxx}(0,y)+(2-\sigma)u_{yyx}(0,y)=0 & \text{for } y\in(-\ell,\ell) \\ u_{xx}(\pi,y)+\sigma u_{yy}(\pi,y) = u_{xxx}(\pi,y)+(2-\sigma)u_{yyx}(\pi,y)=0 & \text{for } y\in(-\ell,\ell) \\ u_{yy}(x,\pm\ell)+\sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell)+(2-\sigma)u_{xxy}(x,\pm\ell)=0 & \text{for } x\in(0,\pi) \\ u_{xy}(0,\pm\ell) = u_{xy}(\pi,\pm\ell)=0 \end{cases}$$

with weak form

$$(u,v)_{H_*^2} = \lambda \int_\Omega p(x,y)\, uv\, dx\, dy \qquad \forall v \in H^2\,,$$

where the scalar product $(\cdot,\cdot)_{H_*^2}$ is defined in Section 2. It's worth mentioning that the corner conditions in (6.2) make consistent the two formulations of the problem (classical and weak), while they are unnecessary when dealing with problem (1.1). Indeed, $u\in H_*^2\cap C^2(\overline{\Omega})$ clearly satisfies $u(0,y) = u_y(0,y) = u_{yx}(0,y) = u_{xy}(0,y) = u_{yy}(0,y) = 0$ for all $y\in[-\ell,\ell]$ and similarly it happens for $x=\pi$. We refer to [23, Sections 4] for the derivation of the boundary conditions in (1.1) and to [14] for those in (6.2), see also [32] for the formulation of (6.2) in a more general setting.

By exploiting the inclusions $H_0^2\subset H_*^2\subset H^2$, we derive

Proposition 6.1. *Let $p\in P_{\alpha,\beta}$ and let $\lambda_h(p)$ be the eigenvalues of* (1.1)*. Furthermore, denote with $\Lambda_h^{Dir}(p)$ and $\Lambda_h^{Neu}(p)$ the divergent sequences of eigenvalues of* (6.1) *and* (6.2)*. There holds:*

$$(6.3)\qquad \Lambda_h^{Neu}(p)\leqslant \lambda_h(p)\leqslant \Lambda_h^{Dir}(p) \qquad \textit{for all } h\in\mathbb{N}_+$$

and

$$\lambda_h(p)\sim \frac{h^2\,16\pi^2}{\big(\int_\Omega \sqrt{p}\,dxdy\big)^2} \qquad \textit{as}\quad h\to+\infty.$$

We refer to [12] for similar comparisons and a sharper asymptotic analysis in the Neumann case for the homogeneous plate.

Proof. The proof is based on the variational characterization of the eigenvalues (2.2) and on some general results presented in [24].

To prove (6.3) we observe that, by density arguments, it follows that $\|\Delta u\|_2^2 = \|u\|_{H_*^2}^2$ for all $u\in H_0^2(\Omega)$; in this way we may write both the variational representation of Dirichlet and Neumann eigenvalues in (6.1) and (6.2) as

$$\Lambda_h^{Dir}(p) = \inf_{\substack{W_h\subset H_0^2\\ \dim W_h = h}} \sup_{u\in W_h\setminus\{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2} \quad\text{and}\quad \Lambda_h^{Neu}(p) = \inf_{\substack{W_h\subset H^2\\ \dim W_h = h}} \sup_{u\in W_h\setminus\{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2}.$$

Observing that $H_0^2\subset H_*^2\subset H^2$, we infer

$$\inf_{\substack{W_h\subset H^2\\ \dim W_h = h}} \sup_{u\in W_h\setminus\{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2} \leqslant \inf_{\substack{W_h\subset H_*^2\\ \dim W_h = h}} \sup_{u\in W_h\setminus\{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2} \leqslant \inf_{\substack{W_h\subset H_0^2\\ \dim W_h = h}} \sup_{u\in W_h\setminus\{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2},$$

implying inequality (6.3).

Finally, the asymptotic law for $\lambda_h(p)$ follows from [24, Theorem 4.1, 4.2], where the authors prove

$$\Lambda_h^{Dir}(p)\sim 16\pi^2 \frac{h^2}{\big(\int_\Omega\sqrt{p}\,dxdy\big)^2} \quad\text{and}\quad \Lambda_h^{Neu}(p)\sim 16\pi^2 \frac{h^2}{\big(\int_\Omega\sqrt{p}\,dxdy\big)^2} \qquad \textit{as}\quad h\to+\infty,$$

implying the same asymptotic behavior for $\lambda_h(p)$. □

The estimate (6.3) confirms the general principle that, fixed $h \in \mathbb{N}_+$, any additional constraint increases the eigenvalue and, therefore, the vibration frequency. We point out that the Dirichlet problem represents the most constrained situation, while the Neumann the most free. Problem (1.1) has intermediate boundary conditions, reflecting the trend given by (6.3).

Acknowledgments. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and are partially supported by the INDAM-GNAMPA 2019 grant: "Analisi spettrale per operatori ellittici con condizioni di Steklov o parzialmente incernierate" and by the PRIN project 201758MTR2: "Direct and inverse problems for partial differential equations: theoretical aspects and applications" (Italy).

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