A note on the natural density of product sets

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A note on the natural density of product sets

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Abstract

Given two sets of natural numbers $\mathcal{A}$ and $\mathcal{B}$ of natural density 1, we prove that their product set $\mathcal{A} \cdot \mathcal{B} := \{ab : a \in \mathcal{A},\, b \in \mathcal{B}\}$ also has natural density 1. On the other hand, for any $\varepsilon > 0$, we show there are sets $\mathcal{A}$ of density $> 1 - \varepsilon$ for which the product set $\mathcal{A} \cdot \mathcal{A}$ has density $< \varepsilon$. This answers two questions of Hegyvári, Hennecart and Pach.

1. *Introduction*

Given two sets of natural numbers $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \cdot \mathcal{B} := \{ab : a \in \mathcal{A},\, b \in \mathcal{B}\}$ be their *product set*. Also, for any positive integer k, let $\mathcal{A}^k$ denote the k-fold product $\mathcal{A} \cdots \mathcal{A}$.

The problem of studying the cardinality of product sets has long been of interest in mathematics. The classic *multiplication table problem* due to Erdős [2, 3] asks for bounds on the cardinality M_n of the $n \times n$ multiplication table, that is, of the set $\{1, \ldots, n\}^2$. Erdős showed that $M_n = o(n^2)$ and Ford [5], following earlier results of Tenenbaum [11], determined the exact order of magnitude of M_n. More recently [7], the second author of the present paper provided uniform bounds for $\#(\{1, \ldots, n_1\} \cdots \{1, \ldots, n_s\})$ holding for a wide range of $n_1, \ldots, n_s \in \mathbb{N}$.

For more general sets $\mathcal{A}$, the problem of estimating $\#(\mathcal{A} \cap [1, x])^2$ was studied by Cilleruelo, Ramana, and Ramaré [1]. For example, they studied this problem when $\mathcal{A}$ is the set of shifted primes, the set of sums of two squares, and the set of shifted sums of two squares. Moreover, they computed the (almost sure) asymptotic behavior for $\#\mathcal{A}^2$ when $\mathcal{A}$ is a random subset of $\{1, \ldots, n\}$ that contains each element of $\{1, \ldots, n\}$ independently with probability $\delta \in (0, 1)$. The third author of the present paper [10] extended this last result to the product of arbitrarily many sets, and Mastrostefano [9] gave a necessary and sufficient condition for having $\#\mathcal{A}^2 \sim (\#\mathcal{A})^2/2$ almost surely.

Hegyvári, Hennecart and Pach [6] considered the analogous problem for infinite sets of natural numbers. In this context, the role of the cardinality is played by the *natural density* $\mathbf{d}(\mathcal{A})$ of a set $\mathcal{A}$, defined as usual by

$$\mathbf{d}(\mathcal{A}) = \lim_{x \to \infty} \frac{\#\mathcal{A} \cap [1, x]}{x}.$$

They asked the following questions ([6, Questions 3 and 2], respectively):

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QUESTION 1. If $\mathcal{A}$ is a set of natural numbers of density 1, is it true that $\mathcal{A}^2$ also has density 1?

QUESTION 2. Is it true that $\inf_{\mathcal{A}\subset\mathbb{N}:\ \mathbf{d}(\mathcal{A})=\alpha} \mathbf{d}(\mathcal{A}^2) = 0$ for any $\alpha \in [0, 1)$, or at least for $\alpha \in [0, \alpha_0)$ for some $\alpha_0 \in (0, 1)$?

Clearly, Question 1 has an affirmative answer if $1 \in \mathcal{A}$, and Hegyvári, Hennecart and Pach showed that it also suffices that $\mathcal{A}$ contains an infinite subset of mutually coprime integers $a_1 < a_2 < \cdots$ such that $\sum_{i=1}^{\infty} a_i^{-1} = +\infty$. Here, we show that the answer is 'yes' in full generality.

THEOREM 1. *Let* $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. *If* $\mathbf{d}(\mathcal{A}) = \mathbf{d}(\mathcal{B}) = 1$, *then* $\mathbf{d}(\mathcal{A}\cdot\mathcal{B}) = 1$.

COROLLARY. *If* $\mathcal{A} \subset \mathbb{N}$ *is such that* $\mathbf{d}(\mathcal{A}) = 1$, *then* $\mathbf{d}(\mathcal{A}^k) = 1$ *for each* $k = 2, 3, \ldots$

REMARK. In fact, the case $\mathcal{A} = \mathcal{B}$ of Theorem 1 implies easily the general case. Indeed, if $\mathbf{d}(\mathcal{A}) = \mathbf{d}(\mathcal{B}) = 1$, then $\mathbf{d}(\mathcal{A}\cap\mathcal{B}) = 1$. In addition, if $(\mathcal{A}\cap\mathcal{B})^2$ has density 1, then so does $\mathcal{A}\cdot\mathcal{B}$.

As it will be clear from the proof, the difference in the density of $\mathbf{d}(\mathcal{A}^2)$ with respect to Erdős's multiplication table problem lies in the fact that many elements of $\mathcal{A}^2$ come from very 'unbalanced' products, meaning products ab such that the sizes of a and b are completely different.

Let us now turn to Question 2. We will answer it in a strong form that shows, among other things, that the condition that $\mathbf{d}(\mathcal{A}) = 1$ in Theorem 1 cannot be relaxed.

THEOREM 2. *For* $\alpha \in [0, 1]$, *we have*

$$\inf_{\mathcal{A}\subseteq\mathbb{N}:\ \mathbf{d}(\mathcal{A})=\alpha} \mathbf{d}(\mathcal{A}^2) = \begin{cases} 0 & \text{if } \alpha < 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

2. *Preliminaries*

Notation. We employ Landau's notation $f = O(g)$ and Vinogradov's notation $f \ll g$ both to mean that $|f| \leqslant C|g|$ for a some constant $C > 0$. Moreover, we write $f \asymp g$ to mean that $f \ll g$ and $g \ll f$. The notation $f = o(g)$ as $x \to a$ (respectively, $f \sim g$ as $x \to a$) means that $\lim_{x\to a} f(x)/g(x) = 0$ (respectively, $= 1$). Given an integer n, we write $P^-(n)$ and $P^+(n)$ for its smallest and largest prime factors, respectively, with the convention that $P^-(1) = \infty$ and $P^+(1) = 1$. If $P^+(n) \leqslant y$, we say that n is *y-smooth*, and if $P^-(n) > y$, we say that it is *y-rough*. As usual, we let $\Phi(x, y)$ denote the number of y-rough numbers in $[1, x]$. Given any integer n, we may write it uniquely as $n = ab$ with $P^+(a) \leqslant y < P^-(b)$. We then call a and b the *y-smooth and y-rough part* of n, respectively. Finally, we let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity.

We need some standard lemmas. We give their proofs for the sake of completeness.

LEMMA 2.1. *For* $x \geqslant y > 1$, *we have* $\Phi(x, y) \ll x/\log y$.

Proof. This follows for example from [8, Theorem 14.2] with $f(n) = 1_{P^-(n)>y}$. □

LEMMA 2.2. *Uniformly for* $x \geqslant y^2 \geqslant 1$ *and* $u \geqslant 1$, *we have*

$$\#\{n \leqslant x : \exists d|n \text{ such that } P^+(d) \leqslant y^{1/u} \text{ and } d > y\} \ll x \cdot (e^{-u} + y^{-1/3}).$$

Proof. Without loss of generality, $u \geqslant 4$. Let $\mathcal{B}$ denote the set of $n \in \mathbb{Z} \cap [1, x]$ that have a $y^{1/u}$-smooth divisor $d > y$. Given $n \in \mathcal{B}$, let $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ be the sequence of prime factors of n of size $\leqslant y^{1/u}$ listed in increasing order and according to their multiplicity. By our assumption on n, we must have $p_1 \cdots p_k > y$. Let j be the smallest integer such that $p_1 \cdots p_j > y$. We must have $j \geqslant 5$ because all factors p_i are $\leqslant y^{1/u} \leqslant y^{1/4}$. We then set $a = p_1 \cdots p_{j-2}$, $p = p_{j-1}$, and $b = n/(ap)$, so that $a > y/(p_{j-1}p_j) \geqslant \sqrt{y}$, $ap \leqslant y$, and $P^+(a) \leqslant p \leqslant P^-(b)$. Consequently,

$$\#\mathcal{B} \leqslant \sum_{p \leqslant y^{1/u}} \sum_{\substack{P^+(a) \leqslant p \\ \sqrt{y} < a \leqslant y/p}} \sum_{\substack{b \leqslant x/(ap) \\ P^-(b) \geqslant p}} 1 \ll \sum_{p \leqslant y^{1/u}} \sum_{\substack{P^+(a) \leqslant p \\ a > \sqrt{y}}} \frac{x}{ap \log p} \tag{1}$$

by Lemma 2.1. If we let $\varepsilon_p = \min\{2/3, 2/\log p\}$, then Rankin's trick implies

$$\frac{\#\mathcal{B}}{x} \ll \sum_{p \leqslant y^{1/u}} \sum_{P^+(a) \leqslant p} \frac{(a/\sqrt{y})^{\varepsilon_p}}{ap \log p} = \sum_{p \leqslant y^{1/u}} \frac{y^{-\varepsilon_p/2}}{p \log p} \sum_{P^+(a) \leqslant p} \frac{1}{a^{1-\varepsilon_p}}.$$

The sum over a equals $\prod_{q \leqslant p}(1 - q^{-1+\varepsilon_p})^{-1}$ with q denoting a prime number. Since $q^{\varepsilon_p} = 1 + O(\log q/\log p)$ for $q \leqslant p$, Mertens' estimates [8, Theorem 3.4] imply that the sum over a is $\ll \log p$. We conclude that

$$\frac{\#\mathcal{B}}{x} \ll y^{-1/3} + \sum_{100 < p \leqslant y^{1/u}} \frac{e^{-\log y/\log p}}{p} \leqslant y^{-1/3} + \sum_{j \geqslant 1} \sum_{y^{1/(u(j+1))} < p \leqslant y^{1/(uj)}} \frac{e^{-ju}}{p}$$
$$\ll y^{-1/3} + \sum_{j \geqslant 1} e^{-ju} \ll y^{-1/3} + e^{-u}$$

using Mertens' estimates once again. This completes the proof. □

LEMMA 2.3. *Let* $y \geqslant 2$ *and* $\lambda \in [0, 1.99]$, *and set* $Q(\lambda) = \lambda \log \lambda - \lambda + 1$ *for* $\lambda > 0$ *and* $Q(0) = 0$. *If* $0 \leqslant \lambda \leqslant 1$, *then*

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \leqslant \lambda \log \log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)},$$

whereas if $1 \leqslant \lambda \leqslant 1.99$, *then*

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \geqslant \lambda \log \log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)}.$$

Proof. The result is trivial if $\lambda = 0$ by Mertens' estimates [8, Theorem 3.4], so assume $\lambda > 0$. If $0 < \lambda \leqslant 1$, then

$$\sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \leqslant \lambda \log \log y}} \frac{1}{m} \leqslant \sum_{P^+(m) \leqslant y} \frac{\lambda^{\Omega(m) - \lambda \log \log y}}{m} = (\log y)^{-\lambda \log \lambda} \prod_{p \leqslant y} \left(1 - \frac{\lambda}{p}\right)^{-1}$$
$$\asymp (\log y)^{-Q(\lambda)} \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right)^{-1}$$

where we used Mertens' estimates once again. Similarly, if $1 \leqslant \lambda \leqslant 1.99$, then

$$\sum_{\substack{P^+(m)\leqslant y \\ \Omega(m)\geqslant \lambda \log\log y}} \frac{1}{m} \leqslant \sum_{P^+(m)\leqslant y} \frac{\lambda^{\Omega(m)-\lambda\log\log y}}{m} \asymp (\log y)^{-Q(\lambda)} \prod_{p\leqslant y}\left(1-\frac{1}{p}\right)^{-1}.$$

This completes the proof. □

LEMMA 2.4. *Let $\mathcal{P}$ be a set of primes such that $\sum_{p\in\mathcal{P}} 1/p < \infty$. Then*

$$\mathbf{d}(\{n \in \mathbb{N} : p|n \ \Rightarrow p \notin \mathcal{P}\}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right).$$

Proof. The number of integers $n \leqslant x$ with a prime divisor $p > \log x$ from $\mathcal{P}$ is

$$\leqslant \sum_{p>\log x,\, p\in\mathcal{P}} \frac{x}{p} = o(x) \qquad \text{as} \quad x \to \infty,$$

because $\sum_{p\in\mathcal{P}} 1/p$ converges. Hence, if we write $\mathcal{P}' = \mathcal{P} \cap [1, \log x]$, then

$$\#\{n \leqslant x : p|n \ \Rightarrow p \notin \mathcal{P}\} = \#\{n \leqslant x : p|n \ \Rightarrow p \notin \mathcal{P}'\} + o(x) = x \prod_{p\in\mathcal{P}'}\left(1-\frac{1}{p}\right) + o(x)$$

from the inclusion–exclusion principle that has $\leqslant 2^{\#\mathcal{P}'} \leqslant 2^{\log x} = o(x)$ steps (for example, see [8, Theorem 2.1]). Since $\prod_{p\in\mathcal{P}\setminus\mathcal{P}'}(1-1/p) \sim 1$ by our assumption that $\sum_{p\in\mathcal{P}} 1/p < \infty$, the proof is complete. □

3. *Proof of Theorem 1*

Assume x is sufficiently large and let $y = y(x)$ and $u = u(x)$ to be chosen later, with $y, u \to +\infty$ slowly as $x \to +\infty$. In particular, $y \leqslant \sqrt{x}$. In the following, for the sake of notation, we will often omit the dependence on x, y, u.

With a small abuse of notation, given an integer n, let n_{smooth} denote its $y^{1/u}$-smooth part and let n_{rough} denote its $y^{1/u}$-rough part. We then set

$$\mathcal{N} = \{n \leqslant x : n_{\text{smooth}} \leqslant y\}.$$

By Lemma 2.2, we have $\#\mathcal{N} \sim x$ as $x \to \infty$. Therefore, in order to prove Theorem 1, it is enough to show that

$$\#\mathcal{C} = o(x), \quad \text{where} \quad \mathcal{C} := \mathcal{N} \setminus (\mathcal{A} \cdot \mathcal{B}).$$

Let $n \in \mathcal{C}$. Since $n = n_{\text{smooth}} \cdot n_{\text{rough}}$, we must have that either $n_{\text{smooth}} \notin \mathcal{A}$ or $n_{\text{rough}} \notin \mathcal{B}$. Consequently,

$$\#\mathcal{C} \leqslant S_1 + S_2$$

with

$$S_1 := \#\{n \in \mathcal{N} : n_{\text{smooth}} \notin \mathcal{A}\} \quad \text{and} \quad S_2 := \#\{n \in \mathcal{N} : n_{\text{rough}} \notin \mathcal{B}\}.$$

Let us first bound S_1. Letting $m = n_{\text{smooth}}$, we have

$$S_1 \leqslant \sum_{m \leqslant y,\, m \notin \mathcal{A}} \Phi(x/m, y^{1/u}) \ll \frac{ux}{\log y} \sum_{m \leqslant y,\, m \notin \mathcal{A}} \frac{1}{m}$$

by Lemma 2.1. Since we have assumed that $\mathbf{d}(\mathcal{A}) = 1$, we must have that $\mathbf{d}(\mathbb{N} \setminus \mathcal{A}) = 0$ and thus

$$\alpha(t) := \frac{1}{\log t} \sum_{m \leqslant t,\, m \notin \mathcal{A}} \frac{1}{m} \;\to\; 0 \qquad \text{as} \quad t \to \infty.$$

Hence, setting $u = u(y) := \alpha(y)^{-1/2}$, we have $u \to +\infty$ and $S_1 = o(x)$ as $x \to +\infty$.

Let us now bound S_2. Writing $m' = n_{\text{rough}}$, we have

$$S_2 \leqslant \sum_{m \leqslant y} \#\{m' \leqslant x/m : m' \notin \mathcal{B}\}.$$

By hypothesis, we have $\mathbf{d}(\mathcal{B}) = 1$, so that $\mathbf{d}(\mathbb{N} \setminus \mathcal{B}) = 0$. Thus

$$\beta(t) := \sup_{s \geqslant t} \frac{\#((\mathbb{N} \setminus \mathcal{B}) \cap [1, s])}{s} \;\to\; 0 \qquad \text{as} \quad t \to \infty.$$

Hence, setting $y := \min(x^{1/2}, \exp(\beta(x^{1/2})^{-1/2}))$, we have $y \to +\infty$ as $x \to +\infty$ and

$$S_2 \leqslant \sum_{d \leqslant y} \beta(x/d) \cdot \frac{x}{d} \leqslant x\beta(x/y) \sum_{d \leqslant y} \frac{1}{d} \ll x\beta(x^{1/2}) \log y \leqslant x\beta(x^{1/2})^{1/2} = o(x).$$

In conclusion, $\#\mathcal{C} = o(x)$, as desired.

Remark. The proof of Theorem 1 can be made quantitative. For example, if one has $\#\{n \leqslant x : n \notin \mathcal{A}\}, \#\{n \leqslant x : n \notin \mathcal{B}\} \ll x(\log x)^{-a}$ for some fixed $0 < a < 1$, then taking $y = \exp((\log x)^{\frac{a}{1+a}})$ and $u = \log\log x$ in the above argument yields

$$\#\{n \leqslant x : n \notin \mathcal{A} \cdot \mathcal{B}\} \ll xe^{-u} + \frac{xu}{(\log y)^a} + \frac{x \log y}{(\log x)^a} \ll x(\log x)^{-\frac{a^2}{1+a} + o(1)}.$$

An interesting question is to determine the optimal exponent of $\log x$ in this upper bound.

4. *Proof of Theorem* 2

The case $\alpha = 1$ follows from Theorem 1, whereas for the case $\alpha = 0$ one can just observe that $\mathbf{d}(\emptyset) = \mathbf{d}(\emptyset^2) = 0$. We may thus assume $\alpha \in (0, 1)$. Given any $\varepsilon > 0$, we need to construct a set $\mathcal{A}$ of density α such that the density of $\mathcal{A}^2$ exists and is smaller than ε.

Let $k \in \mathbb{N}$, $y \geqslant 1$ and a set of primes $\mathcal{P} \subset (y, +\infty)$ with $\sum_{p \in \mathcal{P}} 1/p < \infty$ to be chosen later. Using the notation $\Omega_y(n) = \sum_{p^a | n,\, p \leqslant y} 1$, let us consider the sets

$$\mathcal{B}_{y,k,\mathcal{P}} := \big\{n \in \mathbb{N} : \Omega_y(n) \geqslant k,\ (n, p) = 1\ \forall p \in \mathcal{P}\big\}.$$

The key property these sets have is that $\mathcal{B}_{y,k,\mathcal{P}}^2 = \mathcal{B}_{y,2k,\mathcal{P}}$.

Now, using Lemma 2.4 twice (once, with $\mathcal{P}_{\text{Lemma 2.4}} = \mathcal{P} \cup \{p \leqslant y\}$ and once with $\mathcal{P}_{\text{Lemma 2.4}} = \{p \leqslant y\}$), we find that

$$\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(m)\leqslant y\\ \Omega(m)\geqslant k}}\frac{1}{m} = \mathbf{d}(\mathcal{B}_{y,k,\emptyset})\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right).$$

Similarly,

$$\mathbf{d}(\mathcal{B}^2_{y,k,\mathcal{P}}) = \mathbf{d}(\mathcal{B}_{y,2k,\mathcal{P}}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\mathbf{d}(\mathcal{B}_{y,2k,\emptyset}).$$

Now, take $y := \exp(\exp(4k/3))$, so that $k = \frac{3}{4}\log\log y$. For any fixed $\varepsilon > 0$, Lemma 2.3 implies that if k is sufficiently large in terms of α and ε, then $\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$ and $\mathbf{d}(\mathcal{B}_{y,2k,\emptyset}) < \varepsilon$. Let us fix for the remainder of the proof such a choice of k. We then construct $\mathcal{P}$ in the following way: we take $p_1 > y$ to be the smallest prime such that $(1-1/p_1)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$, $p_2 > p_1$ the smallest prime such that $(1-1/p_1)(1-1/p_2)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$ and so on. Taking $\mathcal{P} := \{p_1, p_2, \dots\}$ we clearly have $\mathbf{d}(\mathcal{B}_{y,k,\emptyset})\prod_{p\in\mathcal{P}}(1-1/p) = \alpha$. Thus, $\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \alpha$ and $\mathbf{d}(\mathcal{B}^2_{y,k,\mathcal{P}}) < \varepsilon$, as desired.

REMARK. If $\mathbf{d}(\mathcal{A}^2)$ in Theorem 2 is replaced by the upper density $\overline{\mathbf{d}}(\mathcal{A}^2)$, then one could just take $\mathcal{A}$ to be any density α subset of $\{n \in \mathbb{N} : \Omega_y(n) \geqslant \frac{3}{4}\log\log y\}$ for y large enough. However, in general there is no guarantee that $\mathcal{A}^2$ has asymptotic density. For this reason, in order to prove Theorem 2, it is more convenient to construct explicit suitable sets $\mathcal{A}$.

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