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The ideal-MHD theory of axisymmetric modes with toroidal mode number $n = 0$ in tokamak plasmas is developed. These modes are resonant at the magnetic X-points of the tokamak divertor separatrix. As a consequence, current sheets form along the separatrix, which profoundly affect the stability of vertical plasma displacements. In particular, current sheets at the magnetic separatrix lead to stabilization of $n = 0$ modes, at least on the ideal-MHD time scale, adding an important ingredient to the mechanism of passive feedback stabilization.

Tokamak experiments adopt magnetic divertors [1] in order to access high plasma confinement (H-mode [2]) regimes, and to reduce the adverse effects of plasma-wall interactions. In divertor configurations, nested magnetic flux surfaces are bounded by a separatrix with one or more X-points. Closed flux surfaces inside the separatrix are separated from open ones, where plasma-edge interactions dominate. Divertors also create elongated plasmas, which are prone to an instability, initiated by a plasma axisymmetric mode with toroidal mode number $n = 0$, leading to Vertical Displacement Events (VDE), where the entire plasma shifts vertically until it touches the vacuum chamber. Uncontrolled VDEs must be avoided, as they lead to the abrupt termination of the tokamak discharge. Therefore, conducting structures are embedded in a tokamak device as a way to stabilize $n = 0$ modes, which, in the ideal magneto-hydro-dynamic (MHD) limit, would otherwise grow on Alfvén time scales. The passive stabilization mechanism is the development of image currents in these structures and on the wall. Active feedback stabilization, by means of currents in coils outside the vacuum chamber, is used to suppress the residual growth on the slower time scale associated with wall resistivity [3, 4].

While there is consensus that VDEs are triggered by $n = 0$ modes, there is no theory that connects this instability with the X-point magnetic topology. A generic $n = 0$ perturbation is singular at the X-point(s) of the divertor separatrix, where the poloidal component of the equilibrium magnetic field $\mathbf{B_{eq}}$ vanishes and $\mathbf{B_{eq}} \cdot \mathbf{k} = 0$, with $\mathbf{k}$ the perturbation wavevector. Consequently, axisymmetric current sheets localized along the separatrix are likely to form. Such process has been studied in the context of astrophysical plasmas [5, 6], as well as in connection with fundamental laboratory plasma experiments (e.g., Ref. [7]), and is well known to researchers working in magnetic fusion (e.g., Refs. [8 11]). However, the resonant interaction between $n = 0$ modes and the X-point(s) of the divertor separatrix has been overlooked by the tokamak community, in spite of the importance of divertors for the successful operation of fusion experiments. One notable exception is Ref. [11], where the stabilizing property of separatrix current sheets is noted in analogy with the stabilizing surface currents associated with external kinks. However, Ref. [11] falls short of developing the analytic theory of the X-point resonance for $n = 0$ modes. There is experimental evidence, not yet satisfactorily explained, of $n = 0$ perturbations and of current sheets at divertor X-points in tokamaks such as the Joint European Torus (JET) [12–15]. Also, current sheets are observed in numerical simulations of the vertical instability with advanced codes that treat correctly the X-point geometry, such as M3D-C$^1$, NIMROD, and JOREK [16–18]. However, an analytic understanding of why these current sheets form, and more importantly, the impact they have on the stability of vertical displacements, is not available.

The aim of this article is to remedy this situation, by providing a fully analytic, normal mode analysis of $n = 0$ modes within the framework of the ideal-MHD model. Our main finding is that, whenever the plasma extends to the magnetic separatrix, the frozen magnetic flux constraint on the resonant X-point field line gives rise to current sheets, which can lead to stabilization of vertical displacements on ideal-MHD time scales, adding an important ingredient to the mechanism of passive feedback stabilization. We also find ideal-MHD stable $n = 0$ modes that oscillate with a frequency below the poloidal Alfvén frequency, and as such are unaffected by continuum damping, which opens the possibility that these modes interact with energetic particles, leading to a new type of fast ion instability in tokamak plasmas [19].

Our analysis is based on the standard reduced ideal-MHD model [20]. The magnetic field is $\mathbf{B} = \mathbf{e}_\phi \times \nabla\psi + B_\phi \mathbf{e}_\phi$, where $\mathbf{e}_\phi$ is the unit vector along the ignorable toroidal direction, and $B_\phi$ is nearly constant. Toroidal curvature effects are neglected. The plasma flow is $\mathbf{v} = \mathbf{e}_\phi \times \nabla\varphi$. The magnetic flux function, $\psi$, and the stream function, $\varphi$, obey the dimensionless equations: $\partial_t\psi + [\varphi, \psi] = 0$; $\partial_t \nabla \cdot (\varrho \nabla\varphi) + [\varphi, U] = [\psi, J]$, where $[\chi, \eta] = \mathbf{e}_\phi \cdot \nabla\chi \times \nabla\eta$, $J = \nabla^2\psi$ is the current density, and $U = \nabla^2\varphi$ is the flow vorticity. Space and time are normalized as $\hat{r} = r/r_0$, where $r_0 = a\,b/[(a^2 + b^2)/2]^{1/2}$ is an equilibrium scale length, with $a$ and $b$ ($a < b$) the semi-axes of a convenient elliptical flux surface (see below), and $\hat{t} = t/\tau_A$, where ${\tau_A}^{-1} = B_p'/(4\pi\,\varrho_m)^{1/2}$ is the relevant Alfvén time, with $B_p'$ the on-axis radial derivative of the equilibrium poloidal field. The dimensionless fields are $\hat{\psi} = \psi/(B_p' r_0^2)$, $\hat{\varphi} = (\tau_A/r_0{}^2)\varphi$. The plasma density is $\hat{\varrho} = \varrho_m/\varrho_{m0}$, with $\varrho_{m0}$ its on-axis value, and the current density is $\hat{J} = (4\pi/cB_p')J_\phi$. In order to simplify the notation, over-hats are dropped.

At equilibrium, fields are stationary and flows are absent. We adopt a relatively simple, "straight tokamak" equilibrium [21, 22], which, nonetheless, is able to capture the essential physics of the X-point resonance. External currents are schematically represented as placed symmetrically along the vertical $y$-axis, at $y = \pm l$. The equilibrium current density, $J_{eq}$, is uniform up to an elliptical flux surface with semi-axes $a$ and $b$, and drops to zero beyond that surface. Since at equilibrium $[\psi_{eq}, J_{eq}] = 0$, the relevant solution is $J_{eq} = J_{eq}(\psi_{eq}) = 2H(\psi_b - \psi_{eq})$, where $H(x)$ is the unit step function, and $\nabla^2\psi_{eq} = J_{eq}(\psi_{eq})$. We introduce elliptical coordinates $(\mu, \theta)$, $x = A\sinh(\mu)\cos(\theta)$, $y = A\cosh(\mu)\sin(\theta)$, and $A = \sqrt{b^2 - a^2}$. The convenient elliptical flux surface corresponds to $\mu = \mu_b$, where $a = A\sinh\mu_b$, $b = A\cosh\mu_b$, and $\psi_{eq} = \psi_b = 1/2$.

We introduce the ellipticity parameter, $e_0 = (b^2 - a^2)/(b^2 + a^2)$. As shown in [21], the solution for the equilibrium flux, in the limit $e_0 \ll l^2/(a^2+b^2)$, is represented as follows. For $\mu \le \mu_b$ and $\psi_{eq} = \psi_{eq}^- \le \psi_b$, the solution of $\nabla^2\psi_{eq}^- = 2$ is $\psi_{eq}^-(x, y) = \psi_b \cdot (x^2/a^2 + y^2/b^2)$. For $\mu > \mu_b$ and $\psi_{eq} = \psi_{eq}^+ \ge \psi_b$, the solution of $\nabla^2\psi_{eq}^+ = 0$ such that the flux and its normal derivative are continuous across the elliptical surface is:

$$\psi_{eq}^+(\mu,\theta) = \psi_b + \alpha^2\left\{\mu - \mu_b + \frac{e_0}{2}\sinh\left[2(\mu-\mu_b)\right]\cos(2\theta)\right\} \tag{1}$$

where $\alpha^2 = (1 - e_0^2)^{-1/2}$. Magnetic flux surfaces, $\psi_{eq}(\mu,\theta) = \text{const}$, exhibit a magnetic separatrix for $\psi_{eq}(\mu,\theta) = \psi_X = \alpha^2\mu_b$, with X-points located at $\mu = \mu_X = 2\mu_b$ and $\theta = \theta_X = \pi/2, 3\pi/2$ (along the vertical y-axis). This solution is valid up to $|x|$ and $|y| \ll l$, including the separatrix.

Let $\psi(\mu,\theta,t) = \psi_{eq}(\mu,\theta) + \tilde{\psi}(\mu,\theta)\,e^{\gamma t}$ and $\varphi(\mu,\theta,t) = \tilde{\varphi}(\mu,\theta)\,e^{\gamma t}$, where the over-tilde denotes small perturbations. To first order in perturbed quantities,

$$\gamma\,\tilde{\psi} + [\tilde{\varphi}, \psi_{eq}] = 0, \tag{2}$$

$$\gamma\,\nabla\cdot(\varrho_{eq}\nabla\tilde{\varphi}) = \left[\tilde{\psi}, J_{eq}\right] + \left[\psi_{eq}, \tilde{J}\right]. \tag{3}$$

The case where the equilibrium plasma density also drops to zero at the elliptical flux surface is analyzed in Ref. [19], where agreement is found with the standard result of Ref. [23]. In this case, the plasma does not extend to the separatrix and the X-points have no impact on $n = 0$ modes. It is shown in [19] that a rigid-shift vertical displacement, $\tilde{\varphi} = \gamma\xi x$, with $\xi = const$, is the exact analytic solution of Eqs. (2)-(3). In elliptical coordinates,

$$\tilde{\varphi}(\mu,\theta) = \gamma\,\xi\,A\sinh\mu\cos\theta. \tag{4}$$

The solution for the perturbed flux is

$$\tilde{\psi}^-(\mu,\theta) = -\frac{\xi}{b}\frac{\cosh\mu}{\cosh\mu_b}\sin\theta, \tag{5}$$

and, in the vacuum region, if no wall is present, $\tilde{\psi}^+(\mu,\theta) = -(\xi/b)\,e^{-(\mu-\mu_b)}\sin\theta$. Thus, a perturbed current sheet on the elliptical boundary is found, $\tilde{J}(\mu,\theta) = \tilde{j}_b(\theta)\delta(\mu-\mu_b)$, where $\delta(x)$ is the delta function. A simple calculation [19] determines the function $\tilde{j}_b(\theta)$ and the mode growth rate. If the wall is present, passive feedback stabilization is obtained [19] by appropriate modification of the vacuum flux solution.

In this article, we consider the more relevant scenario, where the equilibrium plasma density is uniform, but extends to the magnetic separatrix, $\varrho = H(\psi_X - \psi_{eq})$, while keeping $J_{eq} = 2H(\psi_b - \psi_{eq})$. Perturbed currents are now allowed to flow along the separatrix, and correct treatment of the X-point resonance becomes essential. With reference to Fig. 1, we indicate with $\Omega$ the region inside the elliptical flux surface, with $\Delta$ the region between the elliptical surface and the separatrix, and with $V_\pm$ the vacuum regions.

The rigid-shift for the stream function, $\tilde{\varphi}$, given by Eq. (4), is still the exact and unique analytic solution of Eqs. (2)-(3), its validity now extending all the way to the separatrix. Consequently, the perturbed flux in region $\Omega$ is still given by Eq. (5), while, in region $\Delta$,

$$\tilde{\psi}_\Delta(\mu,\theta) = \frac{\xi}{b}\frac{\sinh(\mu-2\mu_b)}{\sinh\mu_b}\sin\theta \tag{6}$$

involves only one harmonic in the elliptical angle $\theta$, and satisfies the ideal-MHD constraint at the X-points, $\tilde{\psi}(\mu_X, \theta_X) = 0$. Moreover, $\tilde{\psi}$ is continuous at the elliptical surface, but its first derivative is discontinuous, and so a perturbed current sheet is still present there: the term $[\psi_{eq}, \tilde{J}]$ in Eq. (3) generates a delta function, $\delta(\mu - \mu_b)$, at the elliptical surface, which is exactly cancelled by the other delta-function term arising from $[\tilde{\psi}, J_{eq}]$.

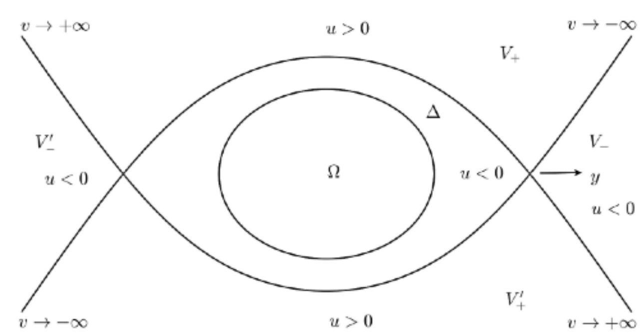


FIG. 1. Equilibrium magnetic structure. Note that the y-axis has been rotated by $90^0$.

Since the separatrix is not a $\mu = \text{const}$ surface, several $\theta$-harmonics couple into the vacuum solution. The latter must decay to zero at infinity and can be expressed as $\tilde{\psi}_{V_\pm}(\mu,\theta) = \sum c_m \exp(-m\mu)\sin(m\theta)$. Unfortunately, the analytic evaluation of the coefficients $c_m$ is an inherently intractable problem. This difficulty is resolved if $\tilde{\psi}$ in regions $\Delta$ and $V_\pm$ is expressed in flux coordinates

$$u = \alpha^{-2}[\psi^+_{eq}(\mu,\theta) - \psi_X] \tag{7}$$

$$v = \theta - \frac{\pi}{2} + \frac{e_0}{2}\cosh[\,2(\mu-\mu_b)\,]\sin(2\theta) \tag{8}$$

where $\psi^+_{eq}$ is given by Eq. (1). Since $\nabla^2 u = 0$ and $\partial_\theta u = -\partial_\mu v$, coordinates $(u,v)$ are orthogonal in regions $\Delta$ and $V_\pm$. The separatrix corresponds to $u = 0$ and, at the X-points, $v = 0, \pi$. On the elliptical boundary, $u = u_b = -\mu_b + (e_0/2)\sinh(2\mu_b)$ and $v = \theta - \pi/2 + (e_0/2)\sin(2\theta)$. With reference to Fig. 1, $u$ is negative in regions $\Delta$, $V_-$ and $V'_-$, and positive in regions $V_+$ and $V'_+$; $v$ ranges from $-\infty$ to $+\infty$. In region $\Delta$, $\nabla^2\tilde{\psi}_\Delta = 0$ has solution

$$\tilde{\psi}_\Delta(u,v) = \frac{\xi}{b}\sum_{m,odd}^{\infty}[\alpha_m\cosh(mu) + \beta_m\sinh(mu)]\cos(mv) \tag{9}$$

The summation involves odd integers only. Note that, at the X-points, $\tilde{\psi}_\Delta(u=0, v=0,\pi) \propto \sum \alpha_m = 0$.

The elliptical surface between regions $\Omega$ and $\Delta$ is special, since both $\mu = \mu_b$ and $u = u_b < 0$ are constant there. We exploit this fact to obtain explicit expressions for the coefficients $\alpha_m$ and $\beta_m$. Let us use $\tilde{\psi}_\Delta(u_b, v) = \tilde{\psi}_\Delta(\mu_b,\theta)$ and the identity $(\partial_u\tilde{\psi}_\Delta)_{u_b} = (\partial_\mu\tilde{\psi}_\Delta)_{\mu_b}(1 + e_0\cos 2\theta)^{-1}$. Also, set $\alpha_m = a_m + b_m$, $\beta_m = -a_m + b_m$. We find

$$a_m = -\frac{e^{mu_b}}{2m}\sum_{j=\pm 1}\left(\frac{b}{a} + j\right) J_{\frac{m-j}{2}}\left(\frac{me_0}{2}\right) \tag{10}$$

$$b_m = \frac{e^{-mu_b}}{2m}\sum_{j=\pm 1}\left(\frac{b}{a} - j\right) J_{\frac{m-j}{2}}\left(\frac{me_0}{2}\right) \tag{11}$$

where $J_\nu(x)$ are integer-order Bessel functions [24].

The perturbed flux is a periodic function of $v$ only in region $\Delta$. In the $V_\pm$ vacuum regions, suitable representations for $\tilde{\psi}$, solutions of $\nabla^2\tilde{\psi} = 0$, that decay at large values of $|u|$, are

$$\tilde{\psi}_{V_+}(u,v) = \frac{\xi}{b}\int_0^\infty dt\, e^{-tu}\alpha(t)\sin\left[\,t\left(\frac{\pi}{2} - v\right)\right] \tag{12}$$

$$\tilde{\psi}_{V_-}(u,v) = \frac{\xi}{b}\int_0^\infty dt\, e^{tu}\beta(t)\cos(tv) \tag{13}$$

Symmetry considerations yield similar representations in regions $V'_\pm$.

Flux continuity at the separatrix, $u = 0$, provides a relationship between the functions $\alpha(t)$ and $\beta(t)$. The perturbed flux, $\tilde{\psi}_{V_+}(0,v) = (\xi/b)\int_0^\infty dt\,\alpha(t)\sin\{\,t\,[(\pi/2) - v]\,\}$, takes on different forms for different interval values of $v$, namely: $\tilde{\psi}_{V_+}(0,v) = \tilde{\psi}_\Delta(0,v)$ for $0 \le v \le \pi$, $\tilde{\psi}_{V_+}(0,v) = \tilde{\psi}_{V_-}(0,v)$ for $-\infty < v \le 0$, and $\tilde{\psi}_{V_+}(0,v) = \tilde{\psi}_{V'_-}(0,v)$ for $\pi \le v < +\infty$. The inverse Fourier-Sine transform of these relations gives

$$\alpha(t) = \frac{2}{\pi t}\cos\left(\frac{\pi t}{2}\right)\left[v.p.\int_0^\infty dt'\,\frac{t'^2\beta(t')}{t^2 - t'^2} - \sum_{m,odd}^{\infty}\frac{m^2\alpha_m}{t^2 - m^2}\right] + \sin\left(\frac{\pi t}{2}\right)\beta(t), \tag{14}$$

where we used $\int_0^\infty \beta(t) = \sum \alpha_m = 0$.

One of the essential mathematical aspects of this solution is that the perturbed magnetic flux develops a singularity at the X-points. Indeed, let us consider

$$\tilde{\psi}_\Delta(0,v) = \frac{\xi}{b}\sum_{m,odd}^{\infty}\alpha_m\cos(mv) \tag{15}$$

approaching the upper X-point at $v = 0$ along the separatrix. The leading asymptotic behavior of the Fourier coefficients for large values of $m$ is $\alpha_m \sim b_m \sim p/m^{3/2}$, where $p = [\pi\sinh^2(\mu_b)\tanh(2\mu_b)]^{-1/2} = \alpha[(a^2 + b^2)/\pi a^2]^{1/2}e_0^{1/2}$ is a positive constant [24]. Thus, the Fourier spectrum for the perturbed flux along the separatrix never decays exponentially for $m \to \infty$, which is indicative of singular behavior. Indeed, consistently with $\alpha_m \sim p/m^{3/2}$, one obtains $\tilde{\psi}_\Delta(0,v) \sim -p(\pi v/2)^{1/2}$ and $(\partial_u\tilde{\psi}_\Delta)_{u=0} \sim (p/2)(\pi/2v)^{1/2}$ as $v \to 0^+$. This behavior can also be found by direct expansion of $\tilde{\psi}_\Delta$ for $u = 0$ and small $v$, using Eqs. (7) and (8).

The same type of behavior must hold for $(\partial_u\tilde{\psi}_{V_\pm})_{u=0}$ as $|v| \to 0$. Analysis of Eq. (14) shows that this requires

$$\beta(t) \sim \frac{q}{t^{3/2}} + o(t^{-3/2}) \quad \text{as} \quad t \to \infty \tag{16}$$

$$\alpha(t) \sim \frac{1}{t^{3/2}}\left[\left(\frac{p}{2} - q\right)\cos\left(\frac{\pi t}{2}\right) + \left(\frac{p}{2} + q\right)\sin\left(\frac{\pi t}{2}\right)\right] + o(t^{-3/2}) \quad \text{as} \quad t \to \infty. \tag{17}$$

Another mathematical aspect that is central to our analysis, and that leads to the determination of the constant parameter $q$ in Eqs. (16)-(17), is that the perturbed flux along the separatrix, $\tilde{\psi}_{V_+}(0,v)$, in the vicinity of the upper X-point at $v = 0$, can be written as the sum of even and odd functions of $v$, $\tilde{\psi}_{V_+}(0,v) = \tilde{\psi}_{even}(v) + \tilde{\psi}_{odd}(v)$, where the even and odd parts, related to the terms in Eq. (17) proportional to $\sin\pi t/2$ and $\cos\pi t/2$, respectively, can be considered as independent solutions, which can be discussed separately. Asymptotic analysis for small of $v$ reveals that, for the even mode, $q = p/2$, while for the odd mode, $q = -p/2$. Analogous conclusions are obtained if the X-point at $v = \pi$ is considered instead.

Having determined the perturbed flux in regions $\Delta$ and $V_\pm$ (the latter at least in the vicinity of the X-points), we are ready to evaluate the perturbed current density, which, in flux coordinates, is obtained from $\tilde{J}(u,v) = |\nabla u|^2(\partial_u^2 + \partial_v^2)\tilde{\psi}$. Clearly, $\tilde{J}$ vanishes everywhere except that at the separatrix. Let $\tilde{J}(u,v) = j_X(v)\delta(u)$. It follows that, for both even and odd modes, near the upper X-point, where $|\nabla u|^2(0,v) \sim |v|$,

$$j_X(v) \sim -\frac{2e_0}{ab}\sqrt{\frac{\pi}{2}}\left(\frac{p}{2}+q\right)\frac{\xi}{b}|v|^{1/2} \quad \text{as} \quad v \to 0. \quad (18)$$

A straightforward procedure determines the dispersion relation for the eigenfrequency $\gamma^2$. In regions $\Delta$ and $V_\pm$, Eq. (3) is written as $\gamma\nabla\cdot(\varrho\nabla\tilde{\varphi}) = \nabla\cdot(\tilde{J}\,\mathbf{e}_z\times\nabla\psi_{eq})$. We integrate with respect to $u$ over a narrow interval across the separatrix between regions $\Delta$ and $V_+$, in the vicinity of the upper X-point, where $v$ is positive and small:

$$\lim_{\delta u\to 0}\int_{-\delta u}^{\delta u} du\,|\nabla u|^{-2}\,\nabla\cdot(\tilde{J}\,\mathbf{e}_z\times\nabla\psi_{eq}) = \frac{dj_X}{dv}\,\frac{d\psi_{eq}}{du}\bigg|_{u=0} \quad (19)$$

$$\lim_{\delta u\to 0}\gamma\int_{-\delta u}^{\delta u} du\,|\nabla u|^{-2}\,\nabla\cdot(\varrho\nabla\tilde{\varphi}) = -\gamma(\partial_u\tilde{\varphi})_{u=0^-}$$
$$\sim \gamma^2\frac{\xi}{b}\frac{a^{1/2}b^{3/2}}{2e_0^{1/2}}v^{-1/2} \quad \text{as} \quad v\to 0^+, \quad (20)$$

where we have used Eq. (4) for the stream function. Note that a consistent behavior is found, as both terms in Eqs. (19) and (20) are proportional to $v^{-1/2}$. Balancing the two terms, we obtain

$$\gamma^2 = -2\sqrt{\frac{\pi a}{2b}}\left(q+\frac{p}{2}\right)\left(1-e_0^2\right)^{1/2} e_0^{3/2}\omega_A^2 \quad (21)$$

where $\omega_A = \tau_A^{-1}$ and dimensions have been reintroduced.

The following conclusions can be drawn at this stage. For the odd-parity solution, $q = -p/2$, the $n=0$ mode is neutrally stable with $\gamma = 0$ and no current sheet develops at the magnetic separatrix. This solution can be considered merely as a redefinition of the equilibrium, with the current-carrying plasma shifted vertically by a distance $\xi$ and the equilibrium current density modified by current sheets at the elliptical flux surface $\mu = \mu_b$.

For the even-parity solution, $q = +p/2$, a current sheet develops at the separatrix, and evidently this is sufficient to stabilize the $n=0$ mode, which in this case oscillates with a real frequency $\omega = \pm i\gamma \sim e_0\omega_A$ (on account of parameter $p \sim e_0^{1/2}$). Since $\omega_A$ is the Alfvén frequency based on the poloidal magnetic field and $e_0 < 1$, the mode frequency falls below the Alfvén continuum spectrum and therefore is unaffected by continuum damping. The even-parity mode can be destabilized by the resonant interaction with fast particle orbits [19] and becomes a possible candidate for the interpretation of finite-amplitude $n=0$ fluctuations recently observed in JET experiments [14, 15].

As we have seen, current sheets always form at the plasma boundary, where the density drops to zero. However, when the boundary coincides with the separatrix, the nature of the current sheet changes significantly. The following argument clarifies the situation. Suppose that ellipticity is small and the density drops to zero in region $\Delta$, at a magnetic flux surface, $u = u_c < 0$, between the elliptical surface at $u = u_b$ and the separatrix at $u = 0$. The "standard case", with $u_c = u_b$, and in the absence of the wall, is unstable [19]. Evidently, the nature of the current sheet that develops at the plasma boundary $u = u_c$ turns from destabilizing to stabilizing as $u_c$ approaches 0. Consider the perturbed magnetic flux on the boundary. From Eq. (9), $\tilde{\psi}_\Delta(u_c,v) = (\xi/b)\sum_{m,odd}^{\infty}\gamma_m(u_c)\cos(mv)$, where $\gamma_m(u_c) = \alpha_m\cosh mu_c + \beta_m\sinh mu_c$. The Fourier coefficients decay exponentially for large $m$, namely, $\gamma_m(u_c) \sim e^{-m/m_{max}}$ for $m > m_{max}(u_c) \sim |u_c|^{-1}$. Therefore, the Fourier series defining $\tilde{\psi}_\Delta(u_c,v)$ represents an analytic function of $v$, that becomes non-analytic as $u_c \to 0$. We also know that the coefficients $\alpha_m$, $\beta_m$, can be expressed in terms of Bessel functions of argument $me_0/2$. Therefore, for small ellipticity, such that $e_0 < m_{max}^{-1} \sim |u_c|$, the $m=1$ component dominates the Fourier series, i.e., $\tilde{\psi}_\Delta(u_c,v) \approx (\xi/b)\gamma_1(u_c)\cos v$. In the same limit, the current sheet at $u = u_c$ can be expressed as $\tilde{J}(u_c,v) = j_c(v)\delta(u-u_c)$, with $j_c \propto -\cos v$. Repeating the procedure that led to Eq. (21), but now with the boundary at $u_c$, we find that the sign of $-\gamma^2$ depends on the sign of the second derivative of $j_c(v)$ near $v = 0$. Thus, we reach the following conclusion. For a fixed value of $e_0 < 1$, the sign of $d^2j_c/dv^2$ at $v = 0$ is positive, resulting in an unstable $n = 0$ mode, as long as the plasma boundary is located in region $\Delta$ a finite distance from the separatrix, i.e., $|u_c| > e_0$. In the limit $|u_c| \to 0$, $j_c(v) \to j_X(v) \propto -|v|^{1/2}$, whose second derivative is negative near $v = 0$, and the $n = 0$ mode is stabilized. The change in the stability properties occurs for values of $|u_c| < e_0$, as more and more terms in the Fourier representation of $\tilde{\psi}_\Delta(u_c,v)$ become important.

The theory discussed in this article presents analogies with the physics of current sheet formation from the evolution of internal kink modes [8, 9], and with the magnetic island coalescence problem [10]. In these works, the ideal-MHD constraint causes magnetic flux to pile up near the X-points, leading to perturbed localized currents and a stabilizing effect in the ideal-MHD limit. For the island coalescence problem, it was found that a chain of magnetic islands becomes ideal-MHD unstable when the island width exceeds a critical threshold. In any case, flux pile up prevents any further nonlinear evolution for both unstable internal kinks and island coalescence, unless the ideal-MHD constraint is relaxed, e.g., by resistivity. In our problem, vertical displacements are found to be linearly stable in the ideal-MHD limit, when the mode resonance at the equilibrium X-points of the divertor separatrix is properly taken into account.

In conclusion, the resonant behavior of n=0 modes at divertor X-points is an ideal-MHD phenomenon. Therefore, it is reasonable that it must be treated first according to the ideal-MHD model. Future work will consider extended-MHD effects, but this article represents the starting point for future developments. We have found that, when the plasma density extends to the magnetic separatrix and $n = 0$ perturbations resonate at the magnetic X-points, vertical displacements are stable, at least on ideal-MHD time scales, without any need for passive stabilization elements. The stabilization mechanism is a direct consequence of the ideal-MHD flux-freezing constraint on the X-points, which generates current sheets localized along the magnetic separatrix, exerting a force capable of pushing back the plasma in its vertical motion. This also suggests that plasma electrical resistivity in a narrow boundary layer along the magnetic separatrix, in addition to wall resistivity, may have a profound impact on the stability of $n = 0$ vertical displacements.

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