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HOMOGENEOUS RIEMANNIAN MANIFOLDS WITH NON-TRIVIAL NULLITY

ANTONIO J. DI SCALA, CARLOS OLMOS, AND FRANCISCO VITTONE

Abstract. This is a substantially improved version of an earlier preprint of the authors with a similar title.

We develop a general theory for irreducible homogeneous spaces $M = G/H$, in relation to the nullity distribution ν of their curvature tensor. We construct natural invariant (different and increasing) distributions associated with the nullity, that give a deep insight of such spaces. In particular, there must exist an order-two transvection, not in the nullity, with null Jacobi operator. This fact was very important for finding out the first homogeneous examples with non-trivial nullity, i.e. where the nullity distribution is not parallel. Moreover, we construct irreducible examples of conullity $k = 3$, the smallest possible, in any dimension. None of our examples admit a quotient of finite volume. We also proved that H is trivial and G is solvable if $k = 3$.

Another of our main results is that the leaves, i.e. the integral manifolds, of the nullity are closed (we used a rather delicate argument). This implies that M is a Euclidean affine bundle over the quotient by the leaves of ν. Moreover, we prove that $\nu^\perp$ defines a metric connection on this bundle with transitive holonomy or, equivalently, $\nu^\perp$ is completely non-integrable (this is not in general true for an arbitrary autoparallel and flat invariant distribution).

We also found some general obstruction for the existence of non-trivial nullity: e.g., if G is reductive (in particular, if M is compact), or if G is two-step nilpotent.

1. Introduction

Given a Riemannian manifold M with curvature tensor R and a point $p \in M$, the *nullity subspace* ν_p of M at p is defined as the subset of T_pM consisting of those vectors that annihilate R, i.e.,

$$\nu_p = \{v \in T_pM \,:\, R_{\cdot,\cdot}\, v \equiv 0\}.$$

The concept of nullity of the curvature tensor was first introduced by Chern and Kuiper in [CK]. For a general Riemannian manifold, the dimension of the nullity subspace at a point p, called the *index of nullity* at p, might change from point to point. In the open and dense subset Ω of M where the index of nullity is locally constant, $q \mapsto \nu_q$ is an autoparallel distribution with flat (totally geodesic) integral manifolds (the so-called leaves of the nullity). Moreover, in the open subset of Ω where the index of nullity attains its minimum, the integral manifolds of ν are complete (cf. [M1]). The distribution ν of Ω is called the *nullity distribution* of M. We say that the nullity (distribution) ν is trivial if it is the tangent space to

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a local de Rham flat factor of Ω, i.e. ν is a parallel distribution with respect to the Levi-Civita connection or equivalently M locally splits off the leaves (i.e. the integral manifolds) of the nullity distribution. If $\nu = \{0\}$ we also say that ν is trivial (in this case there must exist no local de Rham flat factor of Ω).

Manifolds with positive index of nullity have been studied by different authors (see for example [M1], [M2], [G], [CM], [Ro1], [Ro2] and more recently, [S], [BVK] and [FZ]). The existence of non-trivial nullity has strong geometric implications. In particular, for the so-called CN2 manifolds, i.e., with codimension of the nullity at most two.

In this paper we study the nullity of an irreducible homogeneous Riemannian manifold $M = G/H$. Since the curvature tensor R is invariant under isometries, M has constant index of nullity and so ν is a well defined G-invariant autoparallel and flat distribution of M. No progress had been made in this direction, except for some special cases where the nullity turned out to be trivial, i.e. parallel (see, for instance, [CFS]). Moreover, it was not known whether an irreducible homogeneous space with non-trivial nullity could exist.

We develop a general structure theory for such spaces, in relation to the nullity. Our approach is geometric and based on general facts about Killing fields, in contrast with the usual Lie algebra approach, i.e. we deal with Killing vector fields rather than with left-invariant objects. We use the so-called Kostant connection in the canonical bundle $E = TM \oplus \Lambda^2(TM)$, and his method [K] for computing the holonomy in terms of the Nomizu operators ∇X, where $X \in \mathcal{K}(M)$ (the space of Killing fields).

A first question that naturally arises is whether the nullity foliation in a homogeneous Riemannian manifold is a homogeneous foliation, i.e. given by the orbits of an isometric group action. If M has no Euclidean de Rham factor, we proved that this is never the case. Moreover, no Killing field $X \neq 0$ of M can be always tangent to the nullity, see Proposition 3.19. Note that any single leaf of nullity is a homogeneous submanifold, but the subgroups of $I(M)$ that act on different leaves are in general only conjugate to each other.

For dealing with the nullity ν we have to study non-trivial geometric distributions that are naturally associated to ν (when it is non-trivial). Namely, the osculating distributions $\nu^{(1)}$, $\nu^{(2)}$ of ν, of first and second order. That is, if $\nu^{(0)} := \nu$, then $\nu^{(i+1)}$ is obtained by adding to $\nu^{(i)}$ the covariant derivative, in any direction, of fields that lie in $\nu^{(i)}$ $(i = 0, 1)$. Since such distributions are $I(M)$-invariant one has, in particular, that $\nu^{(1)} = \nu + \hat{\nu}$, where $\hat{\nu}$ is the so-called adapted distribution of ν. Namely, $\hat{\nu}_p$ is the linear span of $\{\nabla_{\nu_p} Z\}$, where Z is a Killing field of M induced by G. In our geometric construction it also appears another natural $I(M)$-invariant distribution $\mathcal{U}$, the so-called bounded distribution, obtained by adding to ν the directions of the Killing fields whose normal component is bounded on a given leaf of ν. The distribution $\mathcal{U}$ is G-invariant and ν is parallel in the directions of $\mathcal{U}$. Moreover, it is the largest G-invariant distribution with this property.

Our main results are summarized in the following theorems, which in particular relate the nullity to the so-called distribution of symmetry [BOR]. By making use of our construction, we were able to produce the first examples of irreducible homogeneous manifolds with non-trivial nullity. Moreover, we found, in any dimension, irreducible examples with codimension of the nullity equals to 3, the smallest

possible, and co-index of symmetry 2, the smallest possible (see Theorem 9.6). In, particular, in contrast to the compact case [BOR], there is no bound, in terms of the co-index of symmetry, for the dimension of the space.

In any locally irreducible homogeneous Riemannian manifold, the codimension of the nullity must be always at least 3. In fact, a CN2 space is semi-symmetric, and a semi-symmetric locally homogeneous space is locally symmetric due to a well known result of Z. I. Szabó (see [S], Proposition 5.1, and [BVK]). Thus, a CN2 homogeneous space is locally the product of a flat factor and a surface of (non zero) constant curvature. We also obtain this result as a by-product of our main constructions.

Theorem A. *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor. Assume that the nullity distribution ν is non-trivial and let k be its codimension. Let $\nu^{(1)}, \nu^{(2)}$, $\hat{\nu}$ and $\mathcal{U}$, be the osculating, of order 1 and 2, the adapted and the bounded G-invariant distributions associated to ν, respectively. Then*

(1) *$\nu^{(1)} = \nu + \hat{\nu}$ is autoparallel and flat, and $\mathcal{U}$ is integrable. Moreover, we have the following inclusions (in particular, $k \geq 3$):*

$$\{0\} \subsetneq \nu \subsetneq \nu^{(1)} \subsetneq \nu^{(2)} \subset \mathcal{U} \subsetneq TM.$$

Moreover, the integral manifolds of ν and $\nu^{(1)}$ are simply connected (and so isometric to a Euclidean space).

(2) *For any $p \in M$, $v \in \hat{\nu}_p$, there exists a transvection $Y \in \mathcal{K}^G(M)$ at p, with $Y_p = v$, and the Jacobi operator $R_{\cdot,v}v$ is null. Moreover, $[Y,[Y,\mathcal{K}(M)]] = 0$ and Y does not belong to the center of $\mathcal{K}^G(M)$ (the Killing fields induced by G). In particular, there exists such a transvection with $Y_p \notin \nu_p$.*

(3) *The representation of the isotropy H on $\nu_p^{\perp}$ is faithful ($p = [e]$). Moreover, $\dim H \leq \frac{1}{2}(k-2)(k-3)$ and so, if the equality holds and G is connected, $H \simeq \mathrm{SO}(k-2)$. In particular, if $k = 3$, $H = \{e\}$.*

(4) *If $k = 3$, then G is solvable. Moreover, there exist irreducible examples in any dimension where G is not unimodular, and so M does not admit a finite volume quotient.*

We should point out that we do not know any example of an irreducible simply connected homogeneous Riemannian manifold $M = G/H$, with a non-trivial nullity, such that either H is non-trivial or G is non-solvable (G connected). There seems to be no geometric reason for the non-existence of such a space.

By making use of some delicate arguments we were able to prove that the leaves of the nullity foliation of an irreducible simply connected homogeneous space M are closed (embedded) submanifolds. Then we prove that M is an affine bundle over the quotient of M by the leaves of the nullity foliation with an affine connection. Moreover, $\nu^{\perp}$ defines an affine connection on this bundle which has a transitive holonomy group, and so this distribution is completely non-integrable. The existence of a completely non-integrable geometric distribution on a Riemannian manifold

has usually strong implications, as e.g. the so-called homogeneous slice theorem used in [Th] and [HL].

Theorem B. *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor, where $G = I(M)^o$. Assume that M has a non-trivial nullity distribution ν. Then*

(1) *Any integral manifold $N(q)$ of the nullity distribution ν is a closed embedded submanifold of M (or equivalently, the Lie subgroup E^q of G that leaves $N(q)$ invariant is closed).*

(2) *M is the total space of a Euclidean affine bundle over the quotient $B = G/E^p$ of M by the leaves of nullity foliation with standard fiber $N(p) \simeq \mathbb{R}^\mu$ ($p = [e]$).*

(3) *$\nu^\perp$ defines an affine metric connection, in the sense of Ehresmann, on the affine bundle $M \to B$. Moreover, the holonomy group associated to $\nu^\perp$ is transitive (or equivalently, $\nu^\perp$ is completely non-integrable).*

Observe that part (3) above is not true for an arbitrary autoparallel and flat G-invariant distribution of a homogeneous Riemannian manifold $M = G/H$. In fact, in a symmetric space of the non-compact type, presented as a solvable Lie group $M = G$, the normal spaces of a foliation by parallel (geometric) horospheres define an autoparallel and flat G-invariant distribution $\mathcal{A}$, but $\mathcal{A}^\perp$ is integrable.

By making use of our main results, we found some obstructions for the existence of non-trivial nullity. In fact, the existence of transvections of order 2 imposes general restrictions on the presentation group G. Namely,

Proposition C. *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor. Then the nullity distribution is trivial, with any G-invariant metric, in any of the following cases:*

(a) If the Lie algebra of G is reductive, i.e. the direct sum of an abelian ideal and a semisimple ideal (in particular, if M is compact).

(b) If the Lie algebra $\mathfrak{g}$ of G is 2-step nilpotent.

Open questions. Let $M = G/H$ be a simply connected irreducible Riemannian homogeneous manifold with positive index of nullity.

- Is the isotropy group H always trivial?
- Is G necessarily solvable?

The reader interested in the proofs of Theorems A, B and Proposition C and that would like to avoid the preliminaries can find them in Section 8.

This is a substantially improved new version of an older preprint of the authors that circulated with a similar title.

2. Preliminaries and basic facts

Let $(M, \langle\,,\,\rangle)$ be a (connected) complete Riemannian manifold with Levi-Civita connection ∇. A vector field X of M is called a Killing field if

$$v \mapsto \nabla_v X \tag{2.0.1}$$

is a skew-symmetric endomorphisms of T_pM, for all $p \in M$. Such a condition is called the *Killing equation* and reflects the fact that the flow of X is by isometries.

Let $I(M)$ denote the Lie group of isometries of M. The Lie algebra $\mathrm{Lie}(I(M))$ of $I(M)$ is naturally identified with the Lie algebra $\mathcal{K}(M)$ of Killing fields of M. Namely, the map $z \overset{j}{\mapsto} z^*$ is a linear isomorphism from $\mathrm{Lie}(I(M))$ onto $\mathcal{K}(M)$ that satisfies

$$[x, y]^* = -[x^*, y^*], \tag{2.0.2}$$

where

$$z^*_q = z.q := \frac{\mathrm{d}}{\mathrm{d}t}_{|0} \mathrm{Exp}(tz)q.$$

In fact, let $f : I(M) \to M$ be the map $f(g) = g(p)$, $p \in M$ fixed. Then the right invariant vector field with initial condition $z \in T_e I(M)$ is f-related to the Killing field z^*. The vector field z^* is called the Killing field *induced* by $z \in \mathrm{Lie}(I(M))$.

Remark 2.1. If one should define the Lie algebra $\mathrm{Lie}(I(M))$ by using right-invariant vector fields instead of left-invariant vector fields, then the map j would be a Lie algebra isomorphism (see, for instance, A.2 in [BCO]).

If G acts by isometries on M and $z \in \mathfrak{g} = \mathrm{Lie}(G)$, then the field z^* is called a Killing field of M induced by G. We will denote the set of such vector fields by $\mathcal{K}^G(M)$. If the action of G on M is not effective, there could exist non-zero elements $z \in \mathfrak{g}$, such that the corresponding $z^* \equiv 0$.

Let $X \in \mathcal{K}(M)$. The initial conditions of X at $p \in M$ are given by the pair

$$(X)^p := (X_p, (\nabla X)_p) \in T_pM \oplus \Lambda^2(T_pM).$$

where $(\nabla X)_p$ denotes the skew-symmetric endomorphism defined by equation (2.0.1). These conditions completely determine the Killing field X, in the sense that two Killing fields with the same initial conditions at any fixed point p must coincide on M.

A Killing field X, besides the Killing equation, satisfies the following identity, for all $p \in M$, $u, v \in T_pM$

$$\nabla^2_{u,v} X = R_{u,X_p} v \qquad \textit{affine Killing equation}. \tag{2.1.1}$$

The affine Killing equation reflects the fact the flow of X preserves the Levi-Civita connection.

Equations (2.0.1) and (2.1.1) motivate the introduction of the so-called *Kostant connection* $\tilde{\nabla}$ on the vector bundle

$$E := TM \oplus \Lambda^2(TM)$$

(see [K, CO]). Here $\Lambda^2(T_pM)$ is, as usual, identified with the skew-symmetric endomorphisms of T_pM. The bundle E is called the *canonical bundle* and $\tilde{\nabla}$ is given by

$$\tilde{\nabla}_u(Z, B) = (\nabla_u Z - Bu, \nabla_u B - R_{u,Z_p}) \tag{2.1.2}$$

$u \in T_pM$, where (Z, B) is a section of E and R is the curvature tensor of M. The Killing fields of M are naturally identified with the parallel sections of E in the following way: (X, B) is a parallel section of E if and only if X is a Killing field of M and $B = \nabla X$.

If X is a Killing field, then the section $q \mapsto (X_q, (\nabla X)_q)$ is called the *canonical lift* of X to E.

The Kostant connection allows us to determine the initial conditions of a Killing field X at any $q \in M$ if we know the initial conditions $(X)^p$ at a fixed p. In fact, we must compute the parallel transport, in the Kostant connection, of $(X)^p$ along any curve from p to q (in particular, by using a geodesic).

From the affine Killing equation and the Bianchi identity one can determine the initial conditions at p of the bracket $[X, X']$ of any two Killing fields in terms of the initial conditions $(X)^p = (v, B)$, $(X')^p = (v', B')$ (see Lemma 2.4 of [R]). Namely,

$$([X, X'])^p = (B'v - Bv', R_{v,v'} - [B, B']). \tag{2.1.3}$$

This equation gives a useful formula for computing the curvature in terms of Killing fields X and Y:

$$R_{X_p,Y_p} = (\nabla[X, Y])_p + [(\nabla X)_p, (\nabla Y)_p]. \tag{2.1.4}$$

The well-known Koszul formula gives the Levi-Civita connection ∇ in terms of brackets of vector fields and scalar products. Since the Lie derivative of the metric tensor along any Killing vector field is zero, we have the following expression for ∇ in terms of Killing fields X, Y, Z (see (3.4) in p. 617 of [ORT])

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle. \tag{2.1.5}$$

Remark 2.2. Regarding Killing fields as sections of the canonical bundle E, one can easily prove the following fact: let Z^n be a sequence of Killing fields on M induced by G such that, for some $p \in M$, their initial conditions at p, $(Z^n)^p = (Z^n_p, (\nabla Z^n)_p)$ converge to (v, B). Then (v, B) is the initial condition at p of a Killing field Y induced by G. Moreover, $((Z^n)_q, (\nabla Z^n)_q) \to (Y_q, (\nabla Y)_q)$, for all $q \in M$.

2.1. **Parallel transport along integral curves of Killing fields.**

Let X be a Killing field and let ϕ_t be its associated flow. Observe that such a flow is always of the form $q \mapsto \mathrm{Exp}(tz)q$, for some z in the Lie algebra of the isometry group. Let $p \in M$ and let $c(t) = \phi_t(p)$ be the integral curve of X by p. Let τ_t denote the parallel transport along $c(t)$, form 0 to t. Then it is not difficult to show that $\tau_t^{-1} \circ \mathrm{d}\phi_t : T_pM \to T_pM$ is a 1-parameter subgroup of linear isometries. Moreover, (see e.g. Remark 2.3 of [OS]),

$$\tau_t^{-1} \circ \mathrm{d}_p\phi_t = \mathrm{e}^{t(\nabla X)_p}. \tag{2.2.1}$$

In fact, if $\gamma'(0) = v \in T_pM$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}|_0 \tau_t^{-1} \circ \mathrm{d}_p\phi_t(v) &= \tfrac{\mathrm{D}}{\partial t}|_0 \tfrac{\partial}{\partial s}|_0 \phi_t(\gamma(s)) = \tfrac{\mathrm{D}}{\partial s}|_0 \tfrac{\partial}{\partial t}|_0 \phi_t(\gamma(s)) \\ &= \tfrac{\mathrm{D}}{\mathrm{d}s}|_0 X_{\gamma(s)} = \nabla_v X. \end{aligned}$$

Remark 2.3. If H is a 1-dimensional Lie subgroup of a compact Lie group K, then the closure T of H is an abelian compact subgroup of K, i.e. a torus. From this it is not hard to see that there is a sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, which tends to $+\infty$ and such that

$$\tau_{t_n}^{-1} \circ \mathrm{d}_p\phi_{t_n} = \mathrm{e}^{t_n(\nabla X)_p}$$

tends to the identity transformation of T_pM (or to any other element of the closure of $\{\mathrm{e}^{t(\nabla X)_p} : t \in \mathbb{R}\}$).

Remark 2.4. Let X be a Killing field that belongs to the isotropy algebra at p, i.e., $X_p = 0$. Let, in the previous notation, ϕ_t be the flow associated to X and $c(t) = \phi_t(p) \equiv p$. Then from (2.2.1) it follows that $\mathrm{d}_p\phi_t = \mathrm{e}^{t(\nabla X)_p}$. So, via the isotropy representation at p, ϕ_t is identified with $\mathrm{e}^{t(\nabla X)_p}$.

2.2. **Holonomy of homogeneous spaces: Kostant's results.**

Let $M = G/H$ be a homogeneous Riemannian manifold and let $\mathcal{K}^G(M)$ be the space of Killing fields on M induced by G. Let $p \in M$ and let $\tilde{\mathfrak{h}}(p)$ be the Lie subalgebra of $\mathfrak{so}(T_pM)$ which is algebraically spanned by the set $\{(\nabla X)_p : X \in \mathcal{K}^G(M)\}$. Kostant proved in [K] that $\tilde{\mathfrak{h}}(p)$ contains the holonomy algebra $\mathfrak{hol}(p)$ of M at p and it is contained in its normalizer $\mathfrak{n}(p)$ of $\mathfrak{hol}(p)$ in $\mathfrak{so}(T_pM)$, i.e,

$$\mathfrak{hol}(p) \subset \tilde{\mathfrak{h}}(p) \subset \mathfrak{n}(p). \tag{2.4.1}$$

Moreover, if M is locally irreducible and it is not Ricci flat, he proved that $\tilde{\mathfrak{h}}(p)$ coincides with $\mathfrak{hol}(p)$ (for a modern treatment of this subject see the survey [CDO]). Since Alekseevskii and Kimelfeld proved in [AK] that a homogeneous and Ricci flat space must be flat, one has that $\tilde{\mathfrak{h}}(p) = \mathfrak{hol}(p)$ for a locally irreducible homogeneous Riemannian manifold.

On the other hand, for any locally irreducible Riemannian manifold, the normalizer of the holonomy algebra, inside the orthogonal algebra, properly contains the holonomy algebra if and only if the space is Kähler and Ricci flat (see e.g. [CDO], [BCO, Prop. 5.2.3]). This, together with Kostant result (2.4.1), implies that for a possibly reducible homogeneous space M,

$$\mathfrak{hol}(p) = \tilde{\mathfrak{h}}(p) \tag{2.4.2}$$

if M has no (local) Euclidean de Rham factor or M has a Euclidean de Rham factor of dimension 1.

Corollary 2.5. *Let $M = G/H$ be a homogeneous Riemannian manifold. Assume that there is a non-trivial subspace $\mathbb{V}$ of T_pM which is invariant by $(\nabla X)_p$, for all $X \in \mathcal{K}^G(M)$. Then $\mathbb{V}$ extends locally to a parallel distribution of M and so M locally splits.*

Proof. The subspace $\mathbb{V}$ is invariant by $\tilde{\mathfrak{h}}(p)$, and therefore by $\mathfrak{hol}(p)$ from (2.4.1). Then, by Remark 2.4, $\mathbb{V}$ locally extends to a parallel non-trivial distribution. So, de Rham decomposition theorem applies. But, for the sake of self-completeness, let us do a direct proof.

From Remark 2.4 $\mathbb{V}$ is invariant under the isotropy algebra. Since we are working locally we may assume that H is connected. Then $\mathbb{V}$ extends to a G-invariant distribution $\mathcal{D}$ on M. Moreover, for any $q \in M$, $\nabla_{\mathcal{D}_q} X \subset \mathcal{D}_q$, for all $X \in \mathcal{K}^G(M)$. Let ξ be a field on M that lies in $\mathcal{D}$ and let $X \in \mathcal{K}^G(M)$ be arbitrary. Since $\mathcal{D}$ is G-invariant then $[X, \xi]$ lies in $\mathcal{D}$. But $[X, \xi] = \nabla_X \xi - \nabla_\xi X$. Since $\nabla_\xi X$ lies in $\mathcal{D}$, then $\nabla_X \xi$ lies in $\mathcal{D}$. Then $\mathcal{D}$ is a non-trivial parallel distribution and M splits locally. □

2.3. **The index of symmetry.**

Let M be a homogeneous Riemannian manifold. A Killing field X on M is called a *transvection* at q if

$$(\nabla X)_q = 0.$$

If X is a transvection at q it follows, from (2.2.1), that $\gamma(t) := \phi_t(q)$ is a geodesic in M and $\mathrm{d}_q\phi_t$ gives the parallel transport along $\gamma(t)$.

We now introduce some basic definitions that were given in [ORT] (see also [BOR]).

The *Cartan subspace* at q is

$$\mathfrak{p}^q := \{X \in \mathcal{K}(M) : X \text{ is a transvection at } q\}, \tag{2.5.1}$$

the *symmetric isotropy algebra* at q is

$$\mathfrak{t}^q := [\mathfrak{p}^q, \mathfrak{p}^q],$$

and the *symmetric subspace* at q is

$$\mathfrak{s}_q := \mathfrak{p}^q.q = \{X_q \ : \ X \in \mathcal{K}(M),\ X \text{ is a transvection at } q\}.$$

It turns out that

$$\tilde{\mathfrak{g}}^q = \mathfrak{t}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra, the so-called *Cartan algebra* at q. (We have used the notation $\tilde{\mathfrak{g}}^q$ instead of the more natural $\mathfrak{g}^q$, as in the references, in order to be consistent with further notation).

Since we are assuming that M is homogeneous, all the previous objects are conjugate to each other by an isometry if we change the base point. In this way $\mathfrak{s}$ defines an $I(M)$-invariant distribution on M which is autoparallel. So it is well defined the so-called *index of symmetry*, $i_{\mathfrak{s}}(M)$, as the dimension over M of the distribution $\mathfrak{s}$.

The integral manifold $L(q)$ of $\mathfrak{s}$ through $q \in M$ is a totally geodesic submanifold of M, called the *leaf of symmetry* through q. The leaves of symmetry are globally symmetric spaces as it follows from Corollary 2.3 in [BOR].

The autoparallel subdistribution of $\mathfrak{s}$ associated to the flat local de Rham factors of the leaves of symmetry will be denoted by $\mathfrak{s}^0$. The set of associated transvections at q will be denoted by $\mathfrak{p}_0^q$, i.e, $\mathfrak{p}_0^q = \{X \in \mathfrak{p}^q \ : \ X_q \in \mathfrak{s}_q^0\}$. Observe that $\mathfrak{p}_0^q$ is the abelian part of $\mathfrak{p}^q$. That is to say,

$$\mathfrak{p}_0^q = \{X \in \mathfrak{p}^q \, : \, [X, \mathfrak{p}^q] = 0\}\,. \tag{2.5.2}$$

The *co-index of symmetry* is the codimension of the distribution of symmetry. If M is not locally symmetric, then its co-index of symmetry is at least 2. This was shown in [BOR] for the compact case and by Reggiani for the general case in [R, Theorem 2.2].

Remark 2.6. Observe that all the previous geometric objects have been defined using all the Killing fields of M, i.e, Killing fields induced by the whole isometry group $I(M)$. In general, if $M = G/H$, a transvection at q may not be a Killing field induced by the presentation group G if G is a proper subgroup of $I(M)^o$. For instance, if $S^3 = \mathrm{Spin}_3$, then no transvection of the sphere is a Killing field induced by the presentation group.

We now generalize Lemma 3.3 in [ORT] for the case where M is not necessarily compact.

Lemma 2.7. *Keeping the notations of this section, the Lie subgroup $\tilde{G}^q \subset I(M)$ whose Lie algebra is $\tilde{\mathfrak{g}}^q$, acts almost effectively on $L(q)$.*

Proof. We are going to use the following fact: for a Riemannian homogeneous space M, the Killing form B of $\mathrm{Lie}(I(M))$ is negative definite when restricted to $\mathrm{Lie}(I(M)_q)$ (the Lie algebra of the isotropy at q).

Consider the ideal $\mathfrak{h}$ of $\tilde{\mathfrak{g}}^q$ given by the elements X such that $X_{|L(q)} \equiv 0$. Then $\mathfrak{h} \subset \mathfrak{t}^q$. Since $[\mathfrak{t}^q, \mathfrak{p}^q] \subset \mathfrak{p}^q$ we get that

$$[\mathfrak{h}, \mathfrak{p}^q] = 0.$$

Therefore, if $X \in \mathfrak{h}$ and $Y, Z \in \mathfrak{p}^q$, then

$$B(X, [Y, Z]) = B([X, Y], Z) = 0.$$

So $B(X, \mathfrak{t}^q) = 0$, and then $X \equiv 0$. □

2.4. **The nullity of the curvature tensor.**

Let M be a Riemannian manifold. The nullity of the curvature tensor R at $p \in M$ is

$$\nu_p := \{v \in T_pM : R_{v,x} = 0, \forall x \in T_pM\}$$

or, equivalently, due to the identities of the curvature tensor,

$$\nu_p := \{v \in T_pM : R_{x,y}v = 0, \forall x, y \in T_pM\}.$$

The nullity ν defines a (differentiable) distribution in the open and dense subset Ω of M where the dimension $\dim(\nu_q)$ is locally constant. Moreover, as it is well-known, it is an autoparallel distribution (or equivalently, it is integrable with totally geodesic integral manifolds). For the sake of self-completeness let us show this fact. Let X, Y, Z, W be arbitrary vector fields in a connected component of Ω such that X and Y lie in ν. Then, a direct calculation shows that $(\nabla_Z R)_{W,X}Y = 0 = (\nabla_W R)_{X,Z}Y$. Then, by the second Bianchi identity, one has that

$$\begin{aligned} 0 &= (\nabla_X R)_{Z,W}Y \\ &= \nabla_X R_{Z,W}Y - R_{\nabla_X Z,W}Y - R_{Z,\nabla_X W}Y - R_{Z,W}\nabla_X Y \\ &= -R_{Z,W}\nabla_X Y \end{aligned}$$

which shows that $\nabla_X Y$ lies in ν.

Lemma 2.8. *Let M be a Riemannian manifold. Let $\gamma_v(t)$ be a geodesic everywhere tangent to ν, with $\gamma(0) = p$, $\gamma'(0) = v$. Denote by τ_t the parallel transport along $\gamma(t)$ from 0 to t. Let X be an arbitrary Killing field on M. Then*

(i) $X_{\gamma(t)} = \tau_t(X_p) + t\tau_t(\nabla_v X)$.
(ii) $\nabla_{\gamma'(t)}(\nabla X) = 0$, *i.e.,* ∇X *is parallel along* $\gamma(t)$, *or equivalently*

$$(\nabla X)_{\gamma(t)} = \tau_t((\nabla X)_p) := \tau_t \circ (\nabla X)_p \circ \tau_t^{-1}.$$

Proof. Since X is a Killing field then $X_{\gamma(t)}$ is a Jacobi field along $\gamma(t)$. Observe that the Jacobi operator $R_{\cdot,\gamma'(t)}\gamma'(t) = 0$. Then the Jacobi equation yields $\frac{\mathrm{D}^2}{\mathrm{d}t}X_{\gamma(t)} = 0$. This shows (i).

Part (ii) follows immediately from formula (2.1.1). □

Let now M be a homogeneous Riemannian manifold. Then the nullity distribution ν, being a geometric object, is invariant under any isometry $g \in I(M)$, i.e., $g_*(\nu) = \nu$. Thus, $\dim(\nu_q)$ does not depend on $q \in M$ and therefore ν is a (smooth) distribution in M.

We will denote by $N(x)$ the so-called *leaf of nullity* by x, i.e, the totally geodesic (maximal) integral manifold of ν that contains x. Then if $g \in I(M)$, we must have that $gN(x) = N(gx)$, for all $x \in M$.

As a first consequence of the results of Section 2.2 one can give a simple proof of Proposition C for the case where M is compact:

Proposition 2.9. *Let M be a compact homogeneous Riemannian manifold, which does not split off, locally, a flat de Rham factor. Then the distribution of nullity is trivial.*

Proof. Let $\gamma_v(t)$ be a non-constant geodesic tangent to the nullity distribution and let $X \in \mathcal{K}(M)$ be arbitrary. Since M is compact, then X is bounded and so, by Lemma 2.8, $\nabla_v X = 0$. By Corollary 2.5, since M is homogeneous, M splits locally the direction of v. A contradiction. □

2.5. **Homogeneous flat spaces.**

The main purpose of this section is to prove the well-known result that any connected transitive Lie subgroup G of the isometry group of $\mathbb{R}^n$ must contain a pure translation. Moreover we will prove that any isometry of $\mathbb{R}^n$ that commutes with all the elements of such a group G must be a translation. As it will become clear in sections 3.2, 6.1 and 7 (the following sections), the constructions and techniques presented in the proofs of the following lemmas (here) can be easily adapted to homogeneous spaces with non-trivial nullity.

Lemma 2.10. *Let G be a connected Lie subgroup of the full isometry group $I(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. Assume that G acts transitively on $\mathbb{R}^n$. Then G must contain a pure translation.*

Proof. We identify any element X of the Lie algebra $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ as a Killing field of the Euclidean space. Namely, if $X = B + v$, $B \in \mathfrak{so}(n)$ and $v \in \mathbb{R}^n$, then $X_q = Bq + v$. Observe that for any $q \in \mathbb{R}^n$ and any $w \in T_q\mathbb{R}^n$,

$$\nabla_w X = Bw,$$

and so ∇X is a parallel skew-symmetric tensor. Therefore, X is a transvection if and only if $B = 0$, i.e., X is a pure translation.

We shall see that there always exists a non-trivial transvection in the Lie algebra $\mathfrak{g}$ of G. In fact, let $X = B + v \in \mathfrak{g}$, $B \neq 0$ and let $w \in \mathbb{R}^n$ such that $Bw \neq 0$. For each $t \in \mathbb{R}$ put $q_t = tw$. Then $\|X_{q_t}\| \to \infty$, if $t \to +\infty$. Since G acts transitively on $\mathbb{R}^n$, for each $t \in \mathbb{R}$ we can choose an element $g_t \in G$ such that $g_t(q_t) = 0$. Set $X^t = (g_t)_*(X) \in \mathfrak{g}$. Then $X^t_0 = t\mathrm{d}g_t(Bw) + \mathrm{d}g_t(v)$ and $\nabla X^t = \mathrm{d}g_t B \mathrm{d}(g_t)^{-1}$. Let now $t_k \to +\infty$ be such that the sequence (of constant norm) $\mathrm{d}g_{t_k}(Bw)$ converges to some $0 \neq u \in \mathbb{R}^n$. Then $\frac{1}{t_k} X^{t_k}$ converges to the transvection associated to u. □

We will now introduce some technical results and fix some notation that we will need for the proof of our last lemma. Let G be a transitive Lie group of isometries of $\mathbb{R}^n$ with Lie algebra $\mathfrak{g}$ and let $\mathfrak{a} \neq \{0\}$ be the ideal of $\mathfrak{g}$ that consists of all the transvections in $\mathfrak{g}$. Let

$$\mathbb{V} = \{X_0 : X \in \mathfrak{a}\}.$$

One has, from the fact that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, that $B\mathbb{V} \subset \mathbb{V}$, where $X = B + v \in \mathfrak{g}$ is arbitrary. Therefore, one also has that $B\mathbb{V}^\perp \subset \mathbb{V}^\perp$.

Consider the G-invariant distribution $\mathcal{D}^\perp$ on $\mathbb{R}^n$ defined by $\mathbb{V}^\perp$ such that $\mathcal{D}^\perp_0 = \mathbb{V}^\perp$. Let us identify $\mathbb{V}^\perp$ with the integral manifold of $\mathcal{D}^\perp$ by 0. Since G acts transitively on $\mathbb{R}^n$, for any $w \in \mathbb{V}^\perp$ there exists $X \in \mathfrak{g}$ of the form $X = B + w$, where $B \neq 0$. Then $X_0 \in \mathbb{V}^\perp$ and so $X_{|\mathbb{V}^\perp}$ is always tangent to $\mathbb{V}^\perp$. Therefore, the Lie subgroup G' of G that leaves $\mathbb{V}^\perp$ invariant is transitive on this subspace.

Let $X = B+v \in \mathfrak{g}$ be arbitrary. Let us see that the restriction $B_{|\mathbb{V}^\perp} = 0$. In fact, if $B_{|\mathbb{V}^\perp} \neq 0$, the same limit argument used in the proof of Lemma 2.10 for $g_t \in G'$ would lead to the construction of a non-trivial transvection of $\mathbb{R}^n$ in a direction perpendicular to $\mathbb{V}$. This is a contradiction that proves that

$$B_{|\mathbb{V}^\perp} = 0.$$

Observe that all the integral curves of $X \in \mathfrak{a}$ are geodesics (i.e., lines) that lie in the G-invariant distribution $\mathcal{D}$ defined by $\mathbb{V}$. If $X \in \mathfrak{g}$ is such that $X_0 \in \mathbb{V}^\perp$, then the integral line of X with initial condition 0 is also a geodesic.

Lemma 2.11. *Let G be a connected Lie subgroup of $I(\mathbb{R}^n)$ which acts transitively on $\mathbb{R}^n$. Let $g \in I(\mathbb{R}^n)$ be an isometry that commutes with all the elements of G. Then g is a translation.*

Proof. We will keep the notations of the previous paragraphs. Observe first that $g \in I(\mathbb{R}^n)$ commutes with all the elements of G if and only if $g_*(X) = X$ for all $X = B + z \in \mathfrak{g}$, the Lie algebra of G.

Let us write $g(x) = Ux + u$, where $U \in \mathrm{O}(n)$, $u \in \mathbb{R}^n$. Then, taking $X \in \mathfrak{a}$, one has that U leaves $\mathbb{V}$ invariant and $U_{|\mathbb{V}}$ is the identity. Then, for an arbitrary $X = B + z$, one has that $UBU^{-1} = B$, since $B_{|\mathbb{V}^\perp} = 0$. Let us assume that $z \in \mathbb{V}^\perp$. Then

$$\begin{aligned} g_*(X)_q &= \mathrm{d}g(X_{g^{-1}q}) = U(B(U^{-1}(q-u)) + z) \\ &= B(q-u) + Uz = X_q = Bq + z \end{aligned} \tag{2.11.1}$$

and so

$$(U - Id)z = Bu.$$

But the left hand side belongs to $\mathbb{V}^\perp$ and the right one to $\mathbb{V}$. Then $Uz = z$, for all $z \in \mathbb{V}^\perp$, since there is a Killing field induced by G in any direction. Then $U = Id$, since U acts trivially on $\mathbb{V}$, and g is a translation. □

As a consequence of the results of this section one obtains the well-known
Fact: any homogeneous flat Riemannian manifold is a product of a torus and a Euclidean space (cf. [AK]).

Let $\Gamma \subset I(\mathbb{R}^n)$ be a discrete subgroup that acts properly discontinuously and commutes with G. Then from Lemma 2.11 Γ consists of translations. So the quotient space of $\mathbb{R}^n$ by Γ is a G-homogeneous space which is the Riemannian product of a flat torus by a Euclidean space. This implies that any flat homogeneous Riemannian manifold is such a product and so it is isometric to an abelian Lie group with an invariant metric (the presentation group G may be non-abelian).

Before concluding this section, let us recall the well-known fact that a Lie group of isometries that acts simply transitively on $\mathbb{R}^n$ must be (2-step) solvable (see e.g. [Al]). In fact, this can also be obtained using the techniques developed in this section.

3. The nullity of homogeneous spaces

3.1. The osculating distributions of the nullity.

Let $M = G/H$ be a Riemannian homogeneous manifold and assume that its nullity distribution ν is non-trivial. We also assume that ν is not parallel. Otherwise, M would split off, locally, a flat factor.

Let us consider the *osculating distribution* $\nu^{(1)}$ associated to the nullity distribution. Namely, if $C^\infty(\nu)$ are the tangent fields of M that lie in ν,

$$\nu^{(1)}_q = \nu_q + \operatorname{span}\{\nabla_w X : X \in C^\infty(\nu), w \in T_qM\}.$$

Then $\nu^{(1)}$ is a G-invariant distribution that properly contains ν, since ν is non-parallel. It is not hard to see that one only needs to consider $\nabla_w X$, for X in some family of fields that lie in ν and such that X_q span ν_q.

The osculating distribution can be defined for any G-invariant distribution $\mathcal{H}$ and $\mathcal{H}^{(1)}$ is also a G-invariant distribution. The osculating distribution of order k is defined as

$$\mathcal{H}^k = (\mathcal{H}^{k-1})^{(1)},$$

where $\mathcal{H}^{(0)} := \mathcal{H}$.

Lemma 3.1. *Let $\mathcal{H}$ be a G-invariant distribution of M. Then*

$$\mathcal{H}^{(1)}_q = \mathcal{H}_q + \operatorname{span}\{\nabla_v Z : Z \in \mathcal{K}^G(M), v \in \mathcal{H}_q\}.$$

Proof. Since $\mathcal{H}$ is G-invariant, the flow of any Killing field Z, induced by G, leaves $\mathcal{H}$ invariant. Then, differentiating the flow, one obtains that $[Z, X]$ lies in $\mathcal{H}$, if X lies in $\mathcal{H}$. So, if X lies in $\mathcal{H}$,

$$\nabla_{Z_q} X = \nabla_{X_q} Z + [Z, X]_q \sim \nabla_{X_q} Z \pmod{\mathcal{H}_q}.$$

Since M is homogeneous, there are Killing fields in any arbitrary direction. This proves the lemma. □

We will show later that $\nu^{(1)}$ is an autoparallel and flat distribution (that properly contains ν). Moreover, we will show that $\nu^{(2)}$ is contained in a (natural) proper G-invariant integrable distribution.

3.2. Homogeneous geodesics tangent to the nullity.

Let $M = G/H$ be a presentation of a homogeneous Riemannian manifold, where G is a connected Lie group which acts on M by isometries. Fix $p \in M$ and let $N(p)$ be the leaf of nullity by p.

If X is a Killing field that is tangent to $N(p)$ at p, then $X_{|N(p)}$ must be always tangent to $N(p)$. This follows from the fact that X is projectable to the quotient of M by the integral manifolds of ν (see Section 7).

Let, for $p \in M$,

$$E^p = \{g \in G : gN(p) = N(p)\}.$$

Then E^p is a Lie subgroup of G which acts smoothly and transitively on $N(p)$ (this action may be non-effective).

The Killing fields of M induced by E^p are those Killing fields induced by G that are tangent to $N(p)$ at p (or equivalently, are always tangent to $N(p)$). They form a Lie subalgebra of $\mathcal{K}(M)$ which we denote by $\mathfrak{e}^p \simeq \operatorname{Lie}(E^p)$.

Observe that the totally geodesic submanifold $N(p)$ of M is extrinsically homogeneous and flat. Moreover, from Section 2.5, it is globally flat. Recall that for

any Killing field Z of M, ∇Z is parallel along $N(p)$ (see Lemma 2.8). So, any transvection $X \in \mathcal{K}^G(M)$ at p must be also a transvection at all $q \in N(p)$. If in addition it is tangent to $N(p)$, it is called an *extrinsic transvection* of $N(p)$.

Lemma 3.2. *Assume that $M = G/H$ has non-trivial nullity distribution ν and does not split off, locally, a Euclidean factor. Then any leaf of nullity $N(p)$ is simply connected and so isometric to a Euclidean space. Moreover, the pullback $i^*(TM)$ over the inclusion $i : N(p) \to M$ is globally flat.*

Proof. Assume that $N(p)$ is not simply connected. We have, from Section 2.5, that $N(p)$ has a non-trivial closed geodesic γ_v (in fact, this is a general fact about homogeneous spaces since any geodesic loop must be a closed geodesic). From Lemma 2.8 any Killing field X must be parallel along γ_v and thus $\nabla_v X = 0$. Then, by Corollary 2.5, $\mathbb{R}v$ extends locally to a parallel distribution and so M locally splits off a line. A contradiction which proves the first assertion. The second assertion follows from the fact that $R_{u,v} = 0$, if $u, v \in \nu_q$, and the first part. □

With the same arguments as in Section 2.5, by considering isometries $g_t \in E^p$ we have the following results

Lemma 3.3. *If there exists $X \in \mathfrak{e}^p$ such that $X_{|N(p)}$ is not an intrinsic transvection of $N(p)$, then there exists $0 \neq Y \in \mathfrak{e}^p$ such that it is an extrinsic transvection at any point of $N(p)$ (i.e. $\mathfrak{p}^p \cap \mathfrak{e}^p \neq \{0\}$).*

Lemma 3.4. *Let $\mathcal{D}$ be the parallel distribution of $N(p)$ which is given by the directions of the extrinsic transvections and let $\mathcal{D}^\perp$ be its complementary perpendicular distribution on $N(p)$. Then, for any $q \in N(p)$, it holds:*

a) If $v \in \mathcal{D}_q$, then the geodesic $\gamma_v(t)$ is the integral curve of an extrinsic transvection $X \in \mathfrak{e}^p$ of $N(p)$.

b) If $v \in \mathcal{D}_q^\perp$, then the geodesic $\gamma_v(t)$ is the integral curve of some $X \in \mathfrak{e}^p$.

Let $\hat{\nu}$ be the G-invariant distribution of M defined by

$$\hat{\nu}_q := \operatorname{span}\{\nabla_w Z : Z \in \mathcal{K}^G(M), w \in \nu_q\}. \tag{3.4.1}$$

Then, from Lemma 3.1,

$$\nu^{(1)} = \nu + \hat{\nu}, \tag{3.4.2}$$

where $\nu^{(1)}$ is the osculating distribution. Note that $\nu \subsetneq \nu^{(1)}$, since ν is not a parallel distribution.

Definition 3.5. The distribution $\hat{\nu}$ will be called the *adapted distribution* of ν.

Proposition 3.6 (Existence of transvections)**.** *Let $M = G/H$ be a homogeneous Riemannian manifold, which does not split off (locally) a flat factor and with a non-trivial nullity distribution ν. Then, for any $y \in \hat{\nu}_p$, there exists a transvection $Y \in \mathcal{K}^G(M)$ such that $Y_p = y$.*

Proof. Let $Z \in \mathcal{K}^G$ and let $v \in \nu_p$ that either belongs to $\mathcal{D}_p$ or to $\mathcal{D}_p^\perp$ (we keep the notation of Lemma 3.4). Then, by the above mentioned lemma, there exits $X \in \mathfrak{e}^p$ such that $\gamma_v(t) = \phi_t(p)$, where ϕ_t is the flow associated to X.

Let $Z \in \mathcal{K}^G(M)$ with $\nabla_v Z \neq 0$. Then, from Lemma 2.8,

$$Z_{\gamma_v(t)} = \tau_t(Z_p + t\nabla_v Z),$$

where τ_t is the parallel transport along $\gamma(t)$, and

$$\nabla_{\gamma_v'(t)}(\nabla Z) = 0,$$

or equivalently

$$(\nabla Z)_{\gamma_v(t)} = \tau_t((\nabla Z)_p) = \tau_t \circ (\nabla Z)_p \circ \tau_t^{-1}.$$

Let us consider the family

$$Z^t := (\phi_{-t})_*(Z) \qquad (t \in \mathbb{R})$$

of Killing fields induced by G. Let us compute their initial conditions at p. First recall that from (2.2.1), $\tau_t^{-1} \circ \mathrm{d}_p\phi_t = \mathrm{e}^{t(\nabla X)_p}$ and so, since $(\mathrm{d}_p\phi_t)^{-1} = \mathrm{d}_{\gamma_v(t)}\phi_{-t}$,

$$\mathrm{d}_{\gamma_v(t)}\phi_{-t} \circ \tau_t = \mathrm{e}^{-t(\nabla X)_p}$$

$$\begin{aligned} Z^t_p &= \mathrm{d}_{\gamma_v(t)}\phi_{-t}(Z_{\gamma(t)}) = \mathrm{d}_{\gamma_v(t)}\phi_{-t}(\tau_t(Z_p + t\nabla_v Z)) \\ &= \mathrm{e}^{-t(\nabla X)_p} Z_p + t\mathrm{e}^{-t(\nabla X)_p}\nabla_v Z. \end{aligned}$$

$$\begin{aligned} (\nabla Z^t)_p &= \mathrm{d}_{\gamma_v(t)}\phi_{-t}((\nabla Z)_{\gamma(t)}) = \mathrm{d}_{\gamma_v(t)}\phi_{-t}(\tau_t((\nabla Z)_p)) \\ &= \mathrm{e}^{-t(\nabla X)_p}((\nabla Z)_p) = \mathrm{e}^{-t(\nabla X)_p} \circ (\nabla Z)_p \circ \mathrm{e}^{t(\nabla X)_p}. \end{aligned}$$

Consider now the family $\frac{1}{t}Z^t$, $t \neq 0$. They are also Killing fields induced by G and, by Remark 2.3, we can choose a sequence of real numbers $\{t_n\} \to +\infty$ such that $\mathrm{e}^{-t_n(\nabla X)_p}$ tends to the identity transformation of T_pM. Then $\frac{1}{t_n}Z^{t_n}$ converges to a Killing field Y with initial conditions (see Remark 2.2)

$$(Y)^p = (\nabla_v Z, 0).$$

Since $\nu_p = \mathcal{D}_p \oplus \mathcal{D}_p^\perp$, and the sum of two transvections at p is a transvection at p, we finish the proof. □

Remark 3.7. Since $\nu \subsetneq \nu^{(1)} = \nu + \hat{\nu}$, there must exist $Z \in \mathcal{K}^G(M)$ and $v \in \nu_p$ such that v either belongs to $\mathcal{D}_p$ or to $\mathcal{D}_p^\perp$ and $\nabla_v Z \notin \nu_p$. Then, from Proposition 3.6, there exists a non-trivial transvection Y at p such that $Y_p = \nabla_v Z \notin \nu_p$.

Definition 3.8. A transvection Y at p in the direction of $\nabla_v Z \in \hat{\nu}_p$, $Z \in \mathcal{K}^G(M)$, $v \in \nu_p$, will be called an *adapted transvection* to the vector v. (See Proposition 3.6).

3.3. **Curvature of adapted transvections.**

Proposition 3.9. *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor. Assume that M has a non-trivial nullity distribution and let Y be an adapted transvection to the vector $v \in \nu_p$. Then, for any $U \in \mathcal{K}^G(M)$ that is bounded along $\gamma_v(t)$, one has that*

$$R_{U_p, Y_p} = 0.$$

Proof. From Lemma 3.2 the integral manifold $N(p)$ of the nullity ν is complete and simply connected, and hence isometric to $\mathbb{R}^m$. Moreover, the pull-back $i_*(TM)$ is globally flat. Let $\gamma_v(t)$ be a geodesic in $N(p)$, where $v \in T_pN(p) = \nu_p$. Write $v = v_1 + v_2$, where $v_1 \in \mathcal{D}_p$, $v_2 \in \mathcal{D}_p^\perp$ (see Lemma 3.4). Then the parallel transport τ_t along γ_v, from 0 to t, can be achieved as the composition of the parallel transport $\bar{\tau}_t^t \circ \bar{\tau}_t$ where $\bar{\tau}_t$ is the parallel transport along the geodesic $\gamma_{v_2}(t)$, from 0 to t, and $\bar{\tau}_t^t$ is the parallel transport along the geodesic $\gamma^t(s) = \gamma_{v_2}(t) + \gamma_{v_1}(s)$, from $s = 0$ to $s = t$ (we identify $N(p) \simeq \mathbb{R}^m$). Observe, that for any t, there is a transvection H^t such that its associated flow $s \mapsto \psi_s^t$, satisfies $\gamma^t(s) = \psi_s^t(\gamma_{v_2}(t))$.

The geodesic γ_{v_2} is homogeneous, i.e. $\gamma_{v_2}(t) = \phi_t(p)$, where ϕ_t is the flow associated to a Killing field $X \in \mathfrak{e}^p$, with $X_p = v_2$.

Then, from formula (2.2.1),

$$\bar{\tau}_t = \mathrm{d}_p\phi_t \circ \mathrm{e}^{-t(\nabla X)_p},$$

and

$$\bar{\tau}_t^t = \mathrm{d}_{\gamma_{v_2}(t)}\psi_t^t = \mathrm{d}_{\phi_t(p)}\psi_t^t$$

Then

$$\tau_t = \mathrm{d}_{\phi_t(p)}\psi_t^t \circ \mathrm{d}_p\phi_t \circ \mathrm{e}^{-t(\nabla X)_p}.$$

If $g^t = \psi_t^t \circ \phi_t$, then g^t is an isometry and

$$\tau_t = d_p g^t \circ \mathrm{e}^{-t(\nabla X)_p}. \tag{3.9.1}$$

Let $Z \in \mathcal{K}^G(M) \simeq \mathfrak{g}$ be such that Z is not bounded along $\gamma_v(t)$, or equivalently, by Lemma 2.8, $w = \nabla_v Z \neq 0$. Let $Y \in \mathcal{K}^G(M)$ be a transvection adapted to v whose initial conditions at p are

$$(Y)^p = (w, 0).$$

Let $U \in \mathcal{K}^G(M)$ be bounded along $\gamma_v(t)$, or equivalently, $\nabla_v U = 0$. We will determine the initial conditions of the bracket $[U, Z]$ at a point $\gamma_v(t)$, for an arbitrary t. More precisely, we are interested in the second component, $(\nabla[U, Z])_{\gamma_v(t)}$. Recall, from Lemma 2.8 (ii), that for any $\hat{Z} \in \mathcal{K}^G(M)$, $\nabla\hat{Z}$ is parallel along γ_v. From (2.1.3), one has that

$$(\nabla[U, Z])_{\gamma_v(t)} = R_{U_{\gamma_v(t)}, Z_{\gamma_v(t)}} - [(\nabla U)_{\gamma_v(t)}, (\nabla Z)_{\gamma_v(t)}]. \tag{3.9.2}$$

Since ∇U and ∇Z are parallel along γ_v, so is the bracket $[\nabla U, \nabla Z]$. But $\nabla[U, Z]$ is parallel as well, and hence, from (3.9.2),

$$R_{U_{\gamma_v(t)}, Z_{\gamma_v(t)}} \tag{3.9.3}$$

must be parallel along γ_v.

One has, from Lemma 2.8, that $U_{\gamma_v(t)} = \tau_t(U_p)$ and that $Z_{\gamma_v(t)} = \tau_t(Z_p) + t\tau_t w$, where $w = \nabla_v Z$. So, replacing in (3.9.3), one obtains that

$$R_{\tau_t(U_p), \tau_t(Z_p)} + tR_{\tau_t(U_p), \tau_t(w)} \tag{3.9.4}$$

must be parallel along $\gamma_v(t)$. In particular, this expression must be bounded along $\gamma_v(t)$.

Since M is homogeneous, both curvature operators $R_{\tau_t(U_p),\tau_t(Z_p)}$ and $R_{\tau_t(U_p),\tau_t(w)}$ are bounded by the supremum

$$\sup\{\, \|R_{x,y}\| \,:\, x,y \in T_pM,\ \|x\|,\|y\| \leq C\} < \infty$$

for a suitable constant C. This implies that $\|R_{\tau_t(U_p),\tau_t(w)}\|$ should tend to 0 as t tends to infinity.

Then, recalling (3.9.1),

$$\|R_{\tau_t(U_p),\tau_t(w)}\| = \|R_{\mathrm{d}_p g^t(a(t)),\mathrm{d}_p g^t(b(t))}\| \tag{3.9.5}$$

$$= \|d_p g^t \circ R_{a(t),b(t)} \circ (d_p g^t)^{-1}\| \tag{3.9.6}$$

$$= \|R_{a(t),b(t)}\|, \tag{3.9.7}$$

where $a(t) = \mathrm{e}^{-t(\nabla X)_p}(U_p)$, $b(t) = \mathrm{e}^{-t(\nabla X)_p}(w)$.

By Remark 2.3 one can take a sequence $\{t_n\}$ tending to infinity such that $\mathrm{e}^{-t_n(\nabla X)_p}$ tends to the identity of T_pM. Then one concludes that $\|R_{a(t_n),b(t_n)}\|$ tends to $\|R_{U_p,w}\|$. Since $\|R_{\tau_{t_n}(U_p),\tau_{t_n}(w)}\| = \|R_{a(t_n),b(t_n)}\|$ must tend to 0 as $t \to \infty$, we conclude that $R_{U_p,w} = 0$. □

Remark 3.10. In the proof of Proposition 3.9 we only used that the projection of U to $\nu^\perp$ is bounded. Then: $R_{U_p,Y_p} = 0$ *if Y is an adapted transvection at p and U belongs to the bounded algebra* $\mathfrak{u}^p$ (see Definition 5.1 and its preceding paragraph).

Lemma 3.11. *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and with a non-trivial nullity distribution. Let $0 \neq Y$ be a transvection adapted to $v \in \nu_p$. Then*

- *(i) If Z is any Killing field of M, then $[Y,Z]$ is bounded along γ_v (or equivalently, $\nabla_v[Y,Z] = 0$).*
- *(ii) If U is any Killing field of M which is bounded along γ_v, then $[Y,U]$ is a transvection at p (i.e., $(\nabla[Y,U])_p = 0$).*
- *(iii) If $\bar{Y}$ is any transvection at p, then $[Y,\bar{Y}] = 0$.*
- *(iv) $[Y,[Y,[Y,\mathcal{K}(M)]]] = 0$, or equivalently, identifying Killing fields with elements of the isometry algebra, $\mathrm{ad}_Y^3 = 0$, in the Lie algebra of the full isometry group of M.*

Proof. Let $Z \in \mathcal{K}(M)$. Then, from (2.1.3), $(\nabla[Y,Z])_p = R_{Y_p,Z_p}$. So,

$$\nabla_v[Y,Z] = R_{Y_p,Z_p}v = 0,$$

since v is in the nullity. Then $[Y,Z]$ is bounded along γ_v. This proves (i).

To see (ii) observe that $(\nabla[Y,U])_p = R_{Y_p,U_p} = 0$ by Proposition 3.9. Thus, $[Y,U]$ is a transvection at p.

Let now $\bar{Y}$ be a transvection at p. In particular $\nabla_v\bar{Y} = 0$ and so $\bar{Y}$ is bounded along γ_v. Then part (ii) applies and $(\nabla[Y,\bar{Y}])_p = 0$. Since $Y,\bar{Y}$ are both transvections at p, by (2.1.3), $[Y,\bar{Y}]_p = 0$. Then $([Y,\bar{Y}])^p = (0,0)$ and so $[Y,\bar{Y}]$ vanishes identically on M. This proves (iii).

Finally if $Z \in \mathcal{K}(M)$ is arbitrary, then $[Y, Z]$ is bounded by part (i). Then applying (ii) $[Y, [Y, Z]]$ is a transvection at p. Then, by part (iii),

$$[Y, [Y, [Y, Z]] = 0.$$

□

We improve part (iv) of Lemma 3.11. Namely,

Theorem 3.12. *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and with a non-trivial nullity distribution. Let $0 \neq Y$ be a transvection adapted to $v \in \nu_p$. Then $[Y, [Y, \mathcal{K}(M)]] = 0$, or equivalently, identifying Killing fields with elements of the isometry algebra, $\mathrm{ad}_Y^2 = 0$, in the Lie algebra of the full isometry group of M. Moreover, $[Y, \mathcal{K}^G(M)] \neq 0$.*

Proof. We will not regard, as before, Killing fields along γ_v, but along the geodesic $\beta(t) = \phi_t(p)$, where ϕ_t is the flow associated to Y. Since Y is a transvection at p, $d_p\phi_t$ coincides with the parallel transport along $\beta(t)$. Then if ψ is any field in M, $[Y, \psi]_{\beta(t)}$ is the covariant derivative $\frac{\mathrm{D}}{\mathrm{d}t}\psi_{\beta(t)}$. Let us apply this for $\psi = Z \in \mathcal{K}(M)$. Keep in mind that $Z_{\beta(t)}$ is a Jacobi field along β. So, from Lemma 3.11, (iv)

$$\frac{\mathrm{D}^3}{\mathrm{d}t^3} Z_{\beta(t)} = 0. \tag{3.12.1}$$

Now in general the curvature tensor R is invariant under isometries and $d_p\phi_t$ coincides with the parallel transport along $\beta(t)$. This implies that the Jacobi operator $R_{\cdot,\beta'(t)}\beta'(t)$ diagonalizes in a parallel basis with constant distinct eigenvalues $\lambda_0 = 0, \lambda_1, \cdots, \lambda_r$ (as in symmetric spaces (see [BOR])).

Let $V^0(t), V^1(t), \cdots, V^r(t)$ be the eigenspaces of the Jacobi operator $R_{\cdot,\beta'(t)}\beta'(t)$ associated to $0, \lambda_1, \cdots, \lambda_r$, respectively. Any of such subspaces must be parallel along $\beta(t)$. Then the orthogonal projection $Z^i(t)$ of $Z_{\beta(t)}$ to $V^i(t)$ is of one of the following types, according with the sign of λ_i.

(a) $Z^0(t) = a(t) + tb(t)$, where $a(t), b(t) \in V^0(t)$ are parallel fields along $\beta(t)$.

(b) If $\lambda_i > 0$, $Z^i(t) = \cos(\sqrt{\lambda_i}t)a(t) + \sin(\sqrt{\lambda_i}t)b(t)$, where $a(t), b(t) \in V^i(t)$ are parallel fields along $\beta(t)$.

(c) If $\lambda_i < 0$, $Z^i(t) = \cosh(\sqrt{-\lambda_i}t)a(t) + \sinh(\sqrt{-\lambda_i}t)b(t)$, where $a(t), b(t) \in V^i(t)$ are parallel fields along $\beta(t)$.

But (3.12.1) implies that $Z^i(t) = 0$, for $i = 1, \ldots, r$ hence $Z_{\beta(t)} = Z^0(t)$ and so

$$\frac{\mathrm{D}^2}{\mathrm{d}t^2} Z_{\beta(t)} = 0. \tag{3.12.2}$$

Then $[Y, [Y, Z]]_{\beta(t)} \equiv 0$. In particular, $[Y, [Y, Z]]_p = 0$. But, by Lemma 3.11 (i) and (ii), $(\nabla[Y, [Y, Z]])_p = 0$. Then $[Y, [Y, Z]] = 0$. This proves that

$$[Y, [Y, \mathcal{K}(M)]] = 0.$$

It only remains to show that $[Y, \mathcal{K}^G(M)] \neq 0$. Assume, on the contrary, that $[Y, Z] = 0$, for all $Z \in \mathcal{K}^G(M)$. Since Y is a transvection at $\beta(t)$, for all t, the covariant derivative along $\beta(t)$ of $Z_{\beta(t)}$, as we have seen, coincides with $[Y, Z]_{\beta(t)} = 0$ as assumed. Then $Z_{\beta(t)}$ is parallel along $\beta(t)$ hence $\nabla_{\beta'(0)}Z = \nabla_{Y_p}Z = 0$. Since Z is arbitrary in $\mathcal{K}^G(M)$ we conclude, from Corollary 2.5, that M splits locally the direction of Y_p. A contradiction. □

In the proof of the above theorem it was shown that the adapted transvection Y has null Jacobi operator along $\beta(t)$ or equivalently at p. Indeed being M homogeneous, there is a Killing field in any direction, and we conclude that the Jacobi operator has only one eigenvalue $\lambda_0 = 0$. From Remark 3.7 we may assume that $Y_p \notin \nu_p$. Then we have the following result that will be very useful for finding irreducible homogeneous Riemannian manifolds with non-trivial nullity distribution.

Corollary 3.13. *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and with a non-trivial nullity distribution ν. Then any adapted transvection Y to $v \in \nu_p$ has a null Jacobi operator $R_{\cdot,Y_p}Y_p$. Moreover, there exists such an adapted transvection Y with $Y_p \notin \nu_p$.*

Any transvection at p belongs, by definition, to the Cartan subspace $\mathfrak{p}^p$ at p (see (2.5.1)). Those with trivial Jacobi operator must lie in the abelian part $\mathfrak{p}_0^p$ of the Cartan subspace. Namely,

Corollary 3.14. *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and with a non-trivial nullity distribution. Then any transvection Y adapted to $v \in \nu_p$ belongs to the abelian part $\mathfrak{p}_0^p$ of the Cartan subspace at p. In particular, the distribution of symmetry $\mathfrak{s}$ of M is non-trivial and so the index of symmetry of M is positive.*

3.4. **Transvections with null Jacobi operator.**

We have the following result for homogeneous spaces that have transvections with null Jacobi operator (not depending on the existence of a non-trivial nullity).

Proposition 3.15. *Let $M = G/H$ be a homogeneous Riemannian manifold where $G = I(M)^o$ and such that M does not split off, locally, a flat factor. Let $\mathfrak{a}_0^p$ be the set of transvections Y at $p \in M$ with null Jacobi operator $R_{\cdot,Y_p}Y_p$. Then*

(i) If $0 \neq Y \in \mathfrak{a}_0^p$, then $[Y,[Y,\mathcal{K}(M)]] = 0$ and $[Y,\mathcal{K}^G(M)] \neq 0$.
(ii) The set $\mathfrak{a}_0^p$ is a vector space. Moreover, it is an abelian Lie algebra and the distribution $q \to \mathfrak{a}_0^q.q$ is autoparallel and flat.

Proof. Let $Y \in \mathfrak{a}_0^p$. Let ϕ_t the flow associated to Y. Then $\gamma(t) = \phi_t(p)$ is the geodesic with initial condition $v = Y_p$. Let $Z \in \mathcal{K}(M)$. Then, from the Jacobi equation, $Z_{\gamma(t)} = u(t) + tw(t)$, where $u(t), w(t)$ are the parallel transports along $\gamma(t)$ of Z_p and $\nabla_v Z$, respectively. Since $[Y,Z] = \frac{d}{dt}_{|0}(\phi_{-t})_*(Z)$, by formula (2.2.1), we obtain that

$$[Y,Z]_{\gamma(t)} = \nabla_{\gamma'(t)}Z_{\gamma(t)} = w(t),$$

i.e., $[Y,Z]_{\gamma(t)}$ is parallel along $\gamma(t)$. So,

$$[Y,[Y,Z]]_{\gamma(t)} = 0.$$

Then $U = [Y,[Y,Z]]$ belongs to the double isotropy algebra at v. In particular U belongs to the isotropy algebra $\mathcal{K}(M)_p \subset \mathfrak{so}(T_pM)$ (via the isotropy representation at p). Moreover, if ψ_s is the flow associated to U, $\mathrm{d}_p\psi_s(v) = v$, for all s. Or, equivalently,

$$\psi_s(\gamma(t)) = \gamma(t).$$

Since isometries map transvections into transvections, from the above equalities we obtain that $(\psi_s)_*(Y) = Y$ and therefore, differentiating the flow, $-[U,Y] = [Y,U] = 0$.

Let B be the Killing form of $\mathcal{K}(M)$. Then $B_{|\mathcal{K}(M)_p\times\mathcal{K}(M)_p}$ is negative definite (see the beginning of the proof of Lemma 2.7).

$$B(U,U) = B([Y,[Y,Z]],U) = -B([Y,Z],[Y,U]) = 0.$$

Then $U = 0$, since $U \in \mathcal{K}(M)_p$. This proves the first assertion of (i).

If $[Y, \mathcal{K}^G(M)] = 0$, then $g_*(Y) = Y$, for all $g \in G$. Then, since G acts transitively on M, Y is a transvection at any point and so a parallel field. Then M locally splits off a line. A contradiction. This finishes the proof of (i).

Since $\mathfrak{a}_0^p$ is invariant by scalar multiplications, one has to show that if $X, Y \in \mathfrak{a}_0^p$ then $Z = X + Y \in \mathfrak{a}_0^p$ (observe that the Jacobi null condition is not a linear condition). In the notation of Section 2.3, we have that $\mathfrak{a}_0^p \subset \mathfrak{p}_0^p$, the abelian part of the Cartan subspace at p. Observe that $[X, Y] = 0$, since this field is zero when restricted to the symmetry leaf $L(p)$ (see Lemma 2.7). Then $[X,[Y,\cdot]] = [Y,[X,\cdot]]$. Taking this and the fact that $[X,[X,\cdot]] = 0 = [Y,[Y,\cdot]]$ into account, we obtain that

$$[Z,[Z,[Z,\cdot]] = 0.$$

The same argument used in the proof of Theorem 3.12 shows that the Jacobi operator $R_{\cdot,Z_p}Z_p$ is zero (see also the paragraph just after this theorem). Then $Z \in \mathfrak{a}_0^p$. So $\mathfrak{a}_0^p$ is a vector subspace of the abelian Lie algebra $\mathfrak{p}_0^p$. If $L(p)$ is the leaf of symmetry by p, then its Euclidean factor is given by $A.p$, where A is the abelian connected Lie group of isometries with Lie algebra $\mathfrak{p}_0^p$. If A' is the connected Lie group associated to $\mathfrak{a}_0^p$, then $A'.p$ is totally geodesic in $A.p$. Since $A.p$ is totally geodesic in M, then $A'.p$ is totally geodesic in M and it is an integral manifold of the distribution $q \to \mathfrak{a}_0^q.q$. This concludes the proof. □

Observe, from Proposition 3.15 and Theorem 3.12, that in any direction of $\hat{\nu}_p$ there exists a transvection X with $\mathrm{ad}_X^2 = 0$ and $\mathrm{ad}_X \neq 0$.

3.5. **Applications to semisimple and nilpotent homogeneous spaces.**

Corollary 3.16. *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor. Assume that Lie algebra $\mathfrak{g}$ of G is reductive. Then M has a trivial nullity.*

Proof. Assume that the nullity ν is non-trivial and let $p = [e] \in M$. We will regard $\mathfrak{g}$ as the Lie algebra $\mathcal{K}^G$ of Killing fields induced by G. Then, by Proposition 3.6 and Theorem 3.12, there exists an adapted transvection $0 \neq Y \in \mathfrak{g}$ of M at p, such that $\mathrm{ad}_Y^2 = 0$ and $\mathrm{ad}_Y \neq 0$. Let us decompose $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k} \oplus \mathfrak{i}$ into the direct sum the ideals which are abelian, of the compact type, and of the noncompact type, respectively.

Let us write $Y = Y^0 + Y^1 + Y^2$, where $Y^0 \in \mathfrak{a}$, $Y^1 \in \mathfrak{k}$, and $Y^2 \in \mathfrak{i}$. Since $\mathrm{ad}_Y^2 = 0$, then $\mathrm{ad}_{Y^1}^2 = 0$. Hence $Y^1 = 0$, since $\mathfrak{k}$ is of the compact type.

On the one hand, since $\langle \nabla_{\mathfrak{g}} Y, \mathfrak{a}\rangle_p = 0$, we obtain, from equation (2.1.5), that

$$\langle [Y,\mathfrak{g}], \mathfrak{a}\rangle_p = 0. \tag{3.16.1}$$

On the other hand, since $\mathrm{ad}_Y \neq 0$, $Y^2 \neq 0$. Hence Y^2 is a 2-step nilpotent element of $\mathfrak{i}$. Then, by the Jacobson-Morozov theorem [OVG], Y^2 belongs to a $\mathfrak{sl}(2)$-triple, in $\mathfrak{i}$, $\{Y^2, Z, W\}$ such that $[W, Y^2] = 2Y^2$, $[W, Z] = -2Z$ and $[Y^2, Z] = W$.

Since Y is a transvection at p, by equation (2.1.5), we have that

$$\begin{aligned} 0 &= \langle \nabla_Y Y, W\rangle_p = \langle [Y,W], Y\rangle_p = \langle [Y,W], Y^2\rangle_p \text{ by (3.16.1)} \\ &= \langle [Y^2, W], Y^2\rangle_p = -2\|Y_p^2\|^2. \end{aligned}$$

Then $Y_p^2 = 0$ and so Y^2 belongs to the isotropy algebra $\mathfrak{h} = \mathrm{Lie}(H)$. Let $(\ ,\ )$ be an $\mathrm{Ad}(H)$-invariant inner product in $\mathfrak{g}$. Then ad_{Y^2} is skew-symmetric and so $\mathrm{ad}_{Y^2} = 0$, since $\mathrm{ad}_{Y^2}^2 = 0$. Then $Y^2 \in \mathfrak{a}$. A contradiction. □

The same argument of the above corollary shows that there are no transvections of order 2 in a homogeneous Riemannian manifold $M = G/H$ with $\mathfrak{g}$ reductive.

Corollary 3.17. *Let $M = G$ be a 2-step nilpotent Lie group with a left invariant metric which does not split off a local flat factor. Then the nullity distribution of M is trivial.*

Proof. We proceed by contradiction assuming that the nullity distribution ν is non-trivial. Let Y be the transvection at $p \in M$ given by Theorem 3.12 such that $[Y, \mathfrak{g}] \neq 0$ i.e. Y does not belong to the center $\mathfrak{c}$ of $\mathfrak{g}$. From (2.1.5),

$$0 = 2\langle \nabla_{\mathfrak{c}} Y, \mathfrak{g}\rangle_p = \langle [Y, \mathfrak{g}], \mathfrak{c}\rangle_p \ .$$

Since $[Y, \mathfrak{g}] \subset \mathfrak{c}$ due to the fact that G is 2-step nilpotent we get that $[Y, \mathfrak{g}]_p = 0$ hence $[Y, \mathfrak{g}] = 0$. A contradiction. □

Remark 3.18. The above corollary also follows from well-known facts about the Ricci tensor and the de Rham factor of a 2-step nilpotent Lie group with a left invariant metric [E, Proposition (2.5) and Proposition (2.7)] .

3.6. **The nullity is not a homogeneous foliation.**

We finish this section by showing that the nullity foliation is far from being a homogeneous foliation (i.e. given by the orbits of an isometric group action).

Observe, from the affine Killing equation (2.1.1), that a Killing field X lies in the nullity distribution if and only if the Nomizu tensor ∇X is parallel. Below we show that a Killing field $X \neq 0$ that lies in the nullity must be tangent to the (local) Euclidean de Rham factor. We were unable to find this result in the literature.

Proposition 3.19. *Let M be a homogeneous Riemannian manifold without Euclidean (local) de Rham factor. Assume that the nullity distribution ν of M is non-trivial. Then there exists no Killing field of M, not identically zero, such that X is always tangent to ν.*

Proof. We may assume that M is simply connected. Let $M = M_1 \times \cdots \times M_r$ be the de Rham decomposition of M, where $M_1, \cdots, M_r$ are irreducible Riemannian manifolds. The nullity ν of M is the direct sum of the nullities ν_i of M_i, $i = 1, \cdots, r$. Let $X \in \mathcal{K}(M)$ be always tangent to ν. The projection X_i of X to any given factor M_i belongs to $\mathcal{K}(M_i)$. If M_i has a trivial nullity, then $X_i = 0$. So we may assume that $\nu_i \neq 0$. Assume that $X_i \neq 0$. From formula (2.1.1) we have that ∇X_i is a parallel skew-symmetric $(1,1)$-tensor of M_i. If $\ker(\nabla X_i) = TM$, then X_i is a parallel field and so M_i splits off a line. A contradiction. Let $p \in M$. Then $(\nabla X_i)_p$ has a complex eigenvalue $\lambda \notin \mathbb{R}$. Let $\{0\} \neq \mathbb{V} \subset T_pM$ be the $(\nabla X_i)_p$-invariant subspace associated to λ and $\bar{\lambda}$. Since ∇X_i is a parallel tensor, then $\mathbb{V}$ extends to a parallel distribution of M_i and since M_i is irreducible, then $\mathbb{V} = T_pM$. In this

case, eventually by rescaling X, we have that $J = \nabla X_i$ is a Kähler structure on M_i and so M_i is a Kähler (homogeneous) manifold. Then the field $\xi = JX_i$ lies in the nullity distribution ν_i. Moreover, as it is standard to check, $\nabla^2\xi = 0$. So ξ satisfies the affine Killing equation $\nabla^2_{u,v}\xi = R_{u,\xi}v$ (see 2.1.1). If ϕ_t is the flow associated to ξ, for any given t, ϕ_t is a homothetic transformation of M associated to the constant e^{ta}, where $A = aId$ is the symmetric part of $\nabla\xi$ (cf. [KN], Lemma 1, pg. 242). In fact, this symmetric part A, from the affine Killing equation, must be parallel. Moreover, since M_i is irreducible, A has only one (constant) eigenvalue. In our particular case $\nabla\xi = J\nabla X_i = -Id$, and so $a = -1$. But, in a homogeneous non-flat irreducible space, any homothetic transformation is an isometry. This is a general fact for a complete Riemannian manifolds (see [KN], Theorem 3.6, pg. 242). For the sake of self-completeness, we will show this in our homogeneous context. In fact, $\mathrm{d}_p\phi_t : T_pM \to T_{\phi_t(p)}M$ is a homothetic map. Namely,

$$\langle \mathrm{d}_p\phi_t(v), \mathrm{d}_p\phi_t(v)\rangle = \mathrm{e}^{-2t}\langle v, w\rangle.$$

One has, since ϕ_t preserves the Levi-Civita connection, that $\mathrm{d}_p\phi_t$ maps the curvature tensor R^p of M_i at p into the curvature tensor $R^{\phi_t(p)}$ at $\phi_t(p)$. Then, by a standard calculation,

$$\mathrm{e}^{-2t}\|R^{\phi_t(p)}\| = \|R^p\|.$$

This is a contradiction, since M_i is homogeneous and non-flat. Then $X_i = 0$, for any $i = 1, \cdots, r$. Then $X = 0$. □

The same proof works assuming M to be complete not necessarily homogeneous.

4. Symmetry and nullity

Let $M = G/H$ be a homogeneous locally irreducible Riemannian manifold with a non-trivial distribution of symmetry $\mathfrak{s}$. Recall that $\mathfrak{s}$ is not contained in the nullity distribution ν, see Remark 3.7. Since both distributions ν and $\mathfrak{s}$ are G-invariant their sum

$$\tilde{\nu} = \nu + \mathfrak{s} \tag{4.0.1}$$

has constant rank, and hence $\tilde{\nu}$ is a distribution on M. Observe that the above sum could be non-direct.

Lemma 4.1. *The distribution $\tilde{\nu}$ is autoparallel. Moreover, if $\tilde{N}(p)$ is an integral manifold of $\tilde{\nu}$ then the restrictions $\mathfrak{s}_{|\tilde{N}(p)}$, $\nu_{|\tilde{N}(p)}$ are parallel distributions of $\tilde{N}(p)$.*

Proof. Let $Y \in \mathfrak{p}^p$, the Cartan subspace at p (see (2.5.1)), and let $c(t)$ be a curve contained in the leaf of nullity $N(p)$ joining p and an arbitrary point $q \in N(p)$. From the affine Killing equation (2.1.1) one has that ∇Y is parallel along $c(t)$. This implies that $(\nabla Y)_q = 0$ for all $q \in N(p)$ hence $Y_q \in \mathfrak{s}_q$. Since p is arbitrary, we get

$$\nabla_\nu \mathfrak{s} \subset \mathfrak{s}.$$

Let ϕ_t be the flow associated to Y. Since ν is G-invariant and, by equation (2.2.1), $d_p\phi_t$ gives the parallel transport along (the geodesic) $\phi_t(p)$, we must have that ν is parallel along the leaf of symmetry $L(p)$ at p. Since p is arbitrary we conclude that

$$\nabla_\mathfrak{s} \nu \subset \nu.$$

Then, since ν and $\mathfrak{s}$ are both autoparallel, we conclude that $\tilde{\nu}$ is autoparallel. □

Let now $\mathfrak{s}^0$ be the flat part of the distribution of symmetry (see equation (2.5.2)) and consider the distribution

$$\tilde{\nu}^0 = \mathfrak{s}^0 + \nu \tag{4.1.1}$$

which is not in general a direct sum.

We have the following lemma.

Lemma 4.2. *The nullity distribution is properly contained in the $I(M)$-invariant distribution $\tilde{\nu}^0$, which is autoparallel and flat.*

Proof. First observe that, from Corollary 3.14, there is a transvection $Y \in \mathfrak{s}^0$ which does not lie in ν. So ν is properly contained in $\tilde{\nu}^0$. Since both $\mathfrak{s}^0$ and ν are $I(M)$-invariant, so is $\tilde{\nu}^0$.

By the above Lemma we have that locally $\tilde{N}(p) = L(p) \times W$ as Riemannian product, where W is a Riemannian manifold. Now $\mathfrak{s}^0$ is the flat parallel distribution tangent to the whole flat de Rham factor of any leaf of symmetry $L(q) \subset \tilde{N}(q)$. So we conclude that the restriction $\mathfrak{s}^0_{|\tilde{N}(p)}$ is parallel.

Then

$$\tilde{\nu}^0_{|\tilde{N}(p)} = \mathfrak{s}^0_{|\tilde{N}(p)} + \nu_{|\tilde{N}(p)}$$

is a parallel distribution of $\tilde{N}(p)$. This implies that $\tilde{\nu}^0$ is an autoparallel distribution of M. Moreover, it must be flat, since $\mathfrak{s}^0$ and ν are parallel and flat distributions of $\tilde{N}(p)$. □

5. The osculating distributions and the isotropy

Here we give the details of the proofs of part (1) and (3) of Theorem A. Without loss of generality, keeping the notation of this theorem, we may assume that the presentation group G is connected.

We already showed that

$$\nu^{(1)} = \nu + \hat{\nu} \tag{5.0.1}$$

and that $\nu \subsetneq \nu^{(1)} = \nu + \hat{\nu}$, see equation (3.4.2). Moreover, $\hat{\nu} \subset \mathfrak{s}^0$ by Corollary 3.14. By using that $\hat{\nu}$ is G-invariant and making the same arguments as in Section 4 one has that $\nu^{(1)}$ is an autoparallel and flat, and so a proper, distribution of M. One can also prove this fact by using that $R_{X,Y} = 0$, if X, Y are transvections that belong to $\mathfrak{p}^p_0$ (and $\hat{\nu}_p \subset \mathfrak{p}^p_0.p = \mathfrak{s}^p_0$).

The inclusion $\nu^{(1)} \subsetneq \nu^{(2)}$ is proper. In fact, if $\nu^{(1)} = \nu^{(2)}$, $\nu^{(1)}$ would be a parallel distribution and, since it is flat, M would have a Euclidean de Rham factor, which contradicts the assumptions of Theorem A.

The osculating distribution $\nu^{(2)}$ is G-invariant and by (5.0.1) one has that

$$\nu^{(2)} = \nu^{(1)} + \hat{\nu}^{(1)}.$$

Then, from Lemma 3.1, and taking into account that $\hat{\nu} \subset \nu^{(1)}$, it follows that

$$\nu^{(2)}_p = \nu^{(1)}_p + \operatorname{span}\{\nabla_v Z : Z \in \mathcal{K}^G(M), v \in \hat{\nu}_p\}. \tag{5.0.2}$$

But, if X is a transvection with $X_p = v \in \hat{\nu}_p$, $\nabla_v Z = [X, Z]_p + \nabla_{Z_p} X = [X, Z]_p$. Then

(5.0.3)
$$\nu_p^{(2)} = \nu^{(1)} + \operatorname{span}\{[X,Z]_p : Z \in \mathcal{K}^G(M), X \text{ is a transvection at } p \text{ with } X_p \in \hat{\nu}_p\}$$

Let X be a transvection at p, with $X_p \in \hat{\nu}_p$, and let $Z \in \mathcal{K}^G(M)$ be arbitrary. Then, from (2.1.3), the initial conditions at p of $[X,Z]$ are

$$([X,Z])^p = ([X,Z]_p, (\nabla[X,Z])_p) = (\nabla_{X_p} Z, R_{X_p,Z_p}). \tag{5.0.4}$$

Observe that

$$\nabla_{\nu_p}[X,Z] = R_{X_p,Z_p}\nu_p = \{0\}. \tag{5.0.5}$$

For $p \in M$, define the subspace $\mathfrak{u}^p$ of $\mathcal{K}^G(M)$ as

$$\mathfrak{u}^p = \{U \in \mathcal{K}^G(M) \,:\, \nabla_{\nu_p} U \subset \nu_p\}.$$

From Lemma 2.8 it follows that for any $q \in N(p)$, $\mathfrak{u}^p = \mathfrak{u}^q$. Moreover, from formula (2.1.4) it follows that $\mathfrak{u}^p$ is a Lie subalgebra of $\mathcal{K}^G(M)$. Observe that $U \in \mathfrak{u}^p$ if and only if the normal component to ν of $U_{|N(p)}$ is parallel with respect to the normal connection of $N(p)$. Notice that such normal component is parallel with respect the normal connection if and only if it is parallel as a section of the pullback bundle $i_*(TM)$ endowed with the Levi-Civita connection, since TN is a parallel sub-bundle of the pullback $i_*(TM)$, (cf. Lemma 3.2).

If g is an isometry, then $\mathfrak{u}^{g(p)} = g_*(\mathfrak{u}^p)$.

Definition 5.1. The Lie algebra $\mathfrak{u}^p$ is called the *bounded algebra at p.* The G-invariant distribution $\mathcal{U}$ defined by $\mathcal{U}_p = \mathfrak{u}^p.\, p$ is called the *bounded distribution.*

Observation. The Lie algebra $\mathfrak{u}^p$ contains:

- Any transvection at p.
- Any element in the linear span of
$$\{[X,Z] : Z \in \mathcal{K}^G(M), X \text{ a transvection at } p \text{ with } X_p \in \hat{\nu}_p\}.$$
- Any Killing field which is tangent to $N(p)$ at p (and so always tangent to $N(p)$). In particular, any Killing field in the isotropy algebra at p.
- The bounded distribution does not depend on the presentation group G of M. This follows from the fact the bounded algebra contains the isotropy algebra.

From the above properties we have that

$$\nu_p^{(2)} \subset \mathfrak{u}^p.\, p. \tag{5.1.1}$$

Let $\bar{G}^p \subset G$ be the Lie subgroup associated to the Lie subalgebra $\mathfrak{u}^p \subset \mathfrak{g} = \operatorname{Lie}(G) \simeq \mathcal{K}^G(M)$.

Then, from the previous observation, $\bar{G}^p$ contains the isotropy subgroup G_p. In fact, since M is simply connected, and G is assumed to be connected, then G_p is connected. Note that $\bar{G}^{gp} = g\bar{G}^p g^{-1}$.

We have the following result, whose proof is standard, from the fact that $G_p \subset \bar{G}^p$.

Lemma 5.2. *$\mathcal{U}$ is an integrable distribution with integral manifolds $\bar{G}^q \cdot q$, $q \in M$.*

Lemma 5.3. *$\mathcal{U}$ is a proper distribution of M (or equivalently, since the bounded algebra $\mathfrak{u}^p$ contains the isotropy algebra at p, any bounded algebra is a proper subalgebra of $\mathfrak{g}$).*

Proof. Suppose on the contrary that $\mathcal{U} = TM$. Then for any $p \in M$, $\bar{G}^p \cdot p = M$. Since $\mathfrak{u}^p = \mathrm{Lie}(\bar{G}^p)$, we would have that

$$\nabla_{\nu_p} X \subset \nu_p \,,$$

for any Killing field induced by $\bar{G}^p$. Then, from Corollary 2.5, ν is a parallel distribution. A contradiction. □

Summarizing, one has the following G-invariant distributions:

$$\{0\} \neq \nu \subsetneq \nu^{(1)} \subsetneq \nu^{(2)} \subset \mathcal{U} \subsetneq TM, \tag{5.3.1}$$

where ν and $\nu^{(1)}$ are autoparallel and flat, and $\mathcal{U}$ is integrable and G-invariant.

In the examples of last section we have that $\nu^{(2)} = \mathcal{U}$ and this distribution is not autoparallel, see Remark 9.3.

Remark 5.4. Observe that (5.3.1) implies that the codimension k of the nullity must be at least 3. Moreover, if $k = 3$ then $\operatorname{codim} \nu^{(1)} = 2$, $\nu^{(2)} = \mathcal{U}$, and $\operatorname{codim} \mathcal{U} = 1$.

Theorem 5.5. *Let $M^n = G/H$ be a simply connected irreducible homogeneous manifold with nullity distribution ν of codimension k (G is not assumed to be connected). Then*

(i) The representation ρ of H on $\nu_p^\perp$ is faithful ($p = [e]$).
(ii) $\dim H \leq \frac{1}{2}(k-2)(k-3)$. In particular, if $k = 3$, the isotropy is trivial (if G is connected).

Proof. Assume that $\ker(\rho)$ is a non-trivial normal subgroup of H. Let $\mathbb{V}$ be the set of fixed vectors of $\ker(\rho)$. Then $\nu_p^\perp \subset \mathbb{V} \subsetneq T_pM$. Then, by [BCO], Theorem 9.1.2, $\mathbb{V}$ extends to an autoparallel G-invariant distribution $\mathcal{D}$ which contains the foliation $\nu^\perp$. Observe that for any $q \in M$, $v \in \mathcal{D}_q$, $w \in \nu_q$ $R_{v,w} = 0$. Then by Proposition 3.3 in [Di], M splits. A contradiction. This proves (i).

From (5.3.1), H leaves any of the following subspaces invariant: $\nu_p \subsetneq \nu_p^{(1)} \subsetneq \mathcal{U}_p \subsetneq T_pM$. Let us consider the representation ρ of H on the orthogonal complement $\nu_p^\perp$. Then there exist three non-trivial H-invariant mutually perpendicular subspaces $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ of dimensions d_1, d_2 and d_3 respectively such that

$$\nu_p^\perp = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3,$$

where $d_1 \leq d_2 \leq d_3$, and $d_1 + d_2 + d_3 = k$.

Then, by making use of part (i),

$$\begin{aligned} \dim H &= \dim \rho(H) \leq \dim \mathrm{SO}(d_1) + \dim \mathrm{SO}(d_2) + \dim \mathrm{SO}(d_3) \\ &\leq \dim \mathrm{SO}(k-2) = \frac{1}{2}(k-2)(k-3), \end{aligned}$$

where the last inequality is standard to show. This proves (ii). □

We finish this section by proving that $\nu^{(1)}$ is a parallel distribution when restricted to any leaf of the bounded distribution.

Lemma 5.6. *Let X, U be vector fields of M such that X lies in $\nu^{(1)}$ and U lies in $\mathcal{U}$. Then $\nabla_U X$ lies in $\nu^{(1)}$ (and so $\nu^{(1)}$, restricted to any leaf of $\mathcal{U}$, is a parallel and flat distribution).*

Proof. Recall that $\nu^{(1)} = \nu + \hat{\nu}$. We may assume that either X belongs to ν or X belongs to $\hat{\nu}$.

(a) Let $p \in M$ be arbitrary, let X belong to ν and let Z be a Killing field that belongs to the bounded algebra $\mathfrak{u}^p$, with $Z_p = U_p$. Since ν is G-invariant $[Z, X]$ lies in ν. Then $\nabla_{Z_p} X \in \nu_p$ if and only if $\nabla_{X_p} Z \in \nu_p$, which follows from the definition of the bounded algebra.

(b) Let X belong to $\hat{\nu}$. From the definition of $\hat{\nu}$, the fields of the form $\nabla_W Z$ span $\hat{\nu}$, where $Z \in \mathcal{K}^G(M)$ and W is a vector field of M that lies in ν. So we may assume that $X = \nabla_W Z$. Let U be a vector field of M that lies in $\mathcal{U}$. From (2.1.1) $\nabla^2_{U,W} Z = R_{U,Z} W = 0$. So,

$$0 = \nabla^2_{U,W} Z = \nabla_U \nabla_W Z - \nabla_{\nabla_U W} Z.$$

Observe, from part (a), that $\nabla_U W$ lies in ν. Then $\nabla_{\nabla_U W} Z$ lies in $\hat{\nu}$ and hence $\nabla_U \nabla_W Z$ lies in $\hat{\nu}$.

□

6. Homogeneous spaces with co-nullity 3

Let $M^n = G/H$ be a simply connected Riemannian manifold with non-trivial nullity distribution ν of codimension 3. Then, by Theorem 5.5, $H = \{e\}$ and so $M = G$, with a left invariant metric. Then the autoparallel and G-invariant distribution $\nu^{(1)}$ has codimension 2, and the integrable G-invariant distribution $\mathcal{U} \supset \nu^{(1)}$ has codimension 1.

Let $p \in M$ be fixed. Then there exist Lie subgroups $H_1 \subset H_2$ of G such that $H_1 \cdot p \subset H_2 \cdot p$ are the integral manifolds by p of $\nu^{(1)}$ and $\mathcal{U}$, respectively (or, equivalently $\mathrm{Lie}(H_2) = \mathfrak{u}^p$).

Lemma 6.1. *We keep the assumptions and notation of this section. Then*

(i) $H_1 \cdot p$ is isometric to $\mathbb{R}^{n-2}$.
(ii) $H_2 \cdot p$ is intrinsically flat.

Proof. That $H_1.p$ is flat was proved at the beginning of Section 5. By Corollary 3.13, and part (ii) of Proposition 3.15, the Jacobi operator in any vector tangent to $\nu^{(1)}$ is null. So the proof of Lemma 3.2 also shows that the integral manifolds of $\nu^{(1)}$ are simply connected. This proves (i).

Observe, since $k = 3$, that $\nu^{(1)}$ is an autoparallel and flat sub-distribution of codimension 1 of $\mathcal{U}$. From Lemma 5.6 it follows that $\nu^{(1)}$, restricted to any leaf $S = H_2 \cdot p$ of $\mathcal{U}$ is a parallel and flat distribution of codimension 1. Then S is flat, which proves (ii). □

6.1. **Excluding the Levi factors if $\mathbf{k = 3}$.**

Let M be a homogeneous simply connected Riemannian manifold without Euclidean de Rham factor. Assume that M has a non-trivial nullity distribution of codimension 3. By Theorem 5.5, M has no isotropy and so $M = G$, where $G = I(M)^o$ is endowed with a left invariant metric.

Let $\mathcal{U}$ be the bounded distribution, which has codimension 1 (see Remark 5.4). The integral manifold by p of $\mathcal{U}$ is given by $H_2 \cdot p$, where the Lie algebra of H_2 is the bounded algebra $\mathfrak{u}^p$. Since G acts freely $\mathfrak{u}^p$ has codimension 1 in the Lie algebra $\mathfrak{g} \simeq \mathcal{K}(M)$ of G. Recall, from Lemma 6.1 (ii), that $H_2 \cdot p$ is intrinsically flat. Then, since there is no isotropy, $\mathfrak{u}^p$ is solvable (see Section 2.5).

Assume that the Levi decomposition of $\mathfrak{g}$ has a non-trivial Levi factor. Namely,

$$\mathfrak{g} = \mathfrak{h} \ltimes s,$$

where $\mathfrak{h}$ is a semisimple Lie algebra and s is the (solvable) radical of $\mathfrak{g}$. Observe that the intersection $\mathfrak{h} \cap \mathfrak{u}^p$ has codimension 1 in $\mathfrak{h}$. Moreover, since $\mathfrak{h}$ is semisimple, the projection of $\mathfrak{u}^p$ to $\mathfrak{h}$ cannot be onto. This implies, since the codimension of $\mathfrak{u}^p$ in $\mathfrak{g}$ is 1, that

$$\mathfrak{u}^p = (\mathfrak{h} \cap \mathfrak{u}^p) \ltimes \mathfrak{s}. \tag{6.1.1}$$

Assume that $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ is a direct sum of simple ideals. With the same arguments as before the projection $\mathfrak{h}_i'$ of $\mathfrak{u}^p$ to $\mathfrak{h}_i$ has codimension 1 in $\mathfrak{h}_i$ and coincides with $\mathfrak{h}_i \cap \mathfrak{u}^p$, $i = 1, \cdots, r$. We must have $\mathfrak{u}^p = (\mathfrak{h}_1' \oplus \cdots \oplus \mathfrak{h}_r') \ltimes \mathfrak{s}$, which implies, $r = 1$, since $\mathfrak{u}^p$ has codimension 1.

If $\mathfrak{h}$ is of the compact type, then $\mathfrak{h} \cap \mathfrak{u}^p$ is solvable and so abelian. Since $\mathfrak{h}$ is simple, this intersection must be properly contained in $\mathfrak{h}$ and so of codimension 1. Then $\mathfrak{h} \cap \mathfrak{u}^p$ is in the center of $\mathfrak{h}$ since each $\mathrm{ad}(x)$ is skew-symmetric w.r.t. the Killing form. A contradiction that shows that $\mathfrak{h}$ is of the non-compact type.

Let $\mathfrak{h} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{h}$. Then $\mathfrak{k}$ has an $\mathrm{ad}_{\mathfrak{k}}$-invariant positive definite inner product $\langle\, , \rangle$. The intersection $\mathfrak{k} \cap \mathfrak{u}^p$ is solvable hence abelian and has codimension at most 1. Since each $\mathrm{ad}(x)$, $x \in \mathfrak{k}$ is skew-symmetric w.r.t. $\langle\, , \rangle$ we conclude that $\mathfrak{k}$ is abelian. So, in any case, $\mathfrak{h} = \mathfrak{sl}_2$.

We have shown, if $\mathfrak{g}$ is not solvable, that

$$\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathfrak{s},$$

where $\mathfrak{s}$ is the radical of $\mathfrak{g}$. Let $\mathfrak{b} := \mathfrak{sl}_2 \cap \mathfrak{u}^p$, which is solvable since $\mathfrak{u}^p$ is so. As previously observed,

$$\mathfrak{u}^p = \mathfrak{b} \ltimes \mathfrak{s}.$$

Since $\mathfrak{b}$ is solvable, there exist, as it is well-known, a basis A, B, C of $\mathfrak{sl}_2$, such that A, B span $\mathfrak{b}$ and

$$[A, B] = 2B, \quad [A, C] = -2C, \quad [B, C] = A \tag{6.1.2}$$

i.e. A, B, C is a so-called $\mathfrak{sl}_2$-triple.

We will identify any element $v \in \mathfrak{g}$ with the Killing field $q \mapsto v.q$ of M. This identification is a Lie algebra anti-isomorphism. With this identification, after replacing A by $-A$, we have the same relations of (6.1.2) for A, B, C.

Lemma 6.2. *We keep the assumptions and notation of this section. Then*

(i) $\nabla_{\nu_p} B = \{0\}$.
(ii) $\nabla_{\nu_p} A = \{0\}$.

(iii) $\nabla_{\nu_p} C = \{0\}$.

Proof. From equation 2.1.4 we have that

$$2\nabla B = \nabla[A,B] = R_{A,B} - [\nabla A, \nabla B].$$

Then

$$2(\nabla B)_{|\nu_p} = (\nabla[A,B])_{|\nu_p} = -[\nabla A, \nabla B]_{|\nu_p} \tag{6.2.1}$$

$$= -[(\nabla A)_{|\nu_p}, (\nabla B)_{|\nu_p}]. \tag{6.2.2}$$

where last equality follows from the fact that $A, B \in \mathfrak{u}^p$ and so $\nabla_{\nu_p} A, \nabla_{\nu_p} B \subset \nu_p$. But the skew-symmetric endomorphism $-[(\nabla A)_{|\nu_p}, (\nabla B)_{|\nu_p}]$ of ν_p is perpendicular to $(\nabla B)_{|\nu_p}$ (with the usual inner product). Then $\nabla_{\nu_p} B = \{0\}$ which proves (i).

From 2.1.4 we have that

$$\nabla_{\nu_p} A = \nabla_{\nu_p}[B,C] = -(\nabla_{\nabla_{\nu_p} C} B - \nabla_{\nabla_{\nu_p} B} C) \tag{6.2.3}$$

$$= -\nabla_{\nabla_{\nu_p} C} B, \tag{6.2.4}$$

where the last equality is due to (i).

On the one hand, $-\nabla_{\nabla C_{\nu_p}} B$ must be perpendicular to $\ker(\nabla B)_p \supset \nu_p$. On the other hand, $\nabla_{\nu_p} A \subset \nu_p$. Then, from 6.2.3, we obtain that $\nabla_{\nu_p} A = \{0\}$, and so (ii).

Recall that $[A,C] = -2C$, and let $v \in \nu_p$ be arbitrary. Then, by equation 2.1.4

$$-2\nabla_v C = \nabla_v[A,C] = -(\nabla_{\nabla_v C} A - \nabla_{\nabla_v A} C) \tag{6.2.5}$$

$$= -\nabla_{\nabla_v C} A, \tag{6.2.6}$$

where the last equality is due to (ii). Since $(\nabla A)_p$ is skew-symmetric the last term of the above equality is perpendicular to $\nabla_v C$. But the first term of this equality is proportional to $\nabla_v C$. Then $\nabla_v C = 0$, which proves (iii). □

Lemma 6.2 implies that C belongs to the bounded algebra $\mathfrak{u}^p$. But $\mathfrak{g}$ is linearly spanned by C and $\mathfrak{u}^p$. Then

$$\mathfrak{g} = \mathfrak{u}^p.$$

This contradicts Lemma 5.3. Then $\mathfrak{g}$ has no Levi factor and so we obtain the following result:

Theorem 6.3. *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without Euclidean de Rham factor. Assume that the nullity distribution of M is non-trivial and of codimension $k = 3$. Then $H = \{e\}$ and G is solvable.*

7. The leaves of ν are closed and $\nu^\perp$ is completely non-integrable

Lemma 7.1. *Let $M = G/H$ be a (non-simply connected) homogeneous Riemannian manifold and let $\mathcal{D}^0$ be the parallel distribution of M associated to its local Euclidean de Rham factor of M (G connected). Let $\bar{F}(p)$ be the closure of a (maximal) integral manifold $F(p)$ by p of $\mathcal{D}^0$. Then there is a closed abelian normal subgroup A of $I(M)$ (not depending on p and non-necessarily contained in G) such that $\bar{F}(p) = A \cdot p$, for all $p \in M$. In particular, $\bar{F}(p)$ is a flat (embedded) homogeneous submanifold of M.*

Proof. Let $\tilde{M}$ be the universal cover of M and write it as $\tilde{M} = \mathbb{R}^k \times M_1$, where $\mathbb{R}^k$ is the Euclidean de Rham factor. Let $\tilde{G} \subset I(\tilde{M})$ be the (connected) lift of G. Let Γ be the deck transformations of $\tilde{M}$. Then Γ commutes with $\tilde{G}$. Let Γ^0 be the image of the projection of Γ to $I(\mathbb{R}^k)$. Let $\tilde{G}^0$ and $\tilde{G}^1$ be the images of the projections of $\tilde{G}$ into $I(\mathbb{R}^k)$ and $I(M_1)$, respectively. Then Γ^0 commutes with the transitive group $\tilde{G}^0$ of isometries. Then the elements of Γ^0 are translations (see Section 2.5). Let $\tilde{T} \simeq \mathbb{R}^k$ be group of translations of $\mathbb{R}^k$. Then $\tilde{T} \times \tilde{G}^1$ acts transitively on $\tilde{M}$ and commutes with Γ. Then $T \times \tilde{G}^1$ projects to a transitive group of isometries of M. Let $T \subset I(M)$ be the projection of $\tilde{T}$. Observe, since $\tilde{T}$ is a normal subgroup of $I(\tilde{M})$, that T is a normal subgroup of $I(M)$. Let A be the closure of T in $I(M)$. Then A is a normal subgroup of $I(M)$ and $\bar{F}(p) = A \cdot p$. □

Remark 7.2. Observe that in the assumptions and notation of Lemma 7.1 it follows that the family of closures of the integral manifolds of $\mathcal{D}^0$ is an $I(M)$-invariant foliation of M.

Let $M = G/H$ be a simply connected irreducible homogeneous Riemannian manifold with a non-trivial nullity distribution ν, where $G = I(M)^o$ (the identity component of the isometry group of M). Let, for $p \in M$, E^p be the Lie subgroup of G that leaves invariant the integral manifold $N(p)$ of ν by p (see Section 3.2). Observe that since ν is G-invariant, $G_p = (E^p)_p$. We may assume that $p = [e]$ so that $G_p = H$. Then, since $N(p)$ and H are connected, E^p is connected. Let $\bar{E}^p$ be the closure of E^p in $G = I(M)^o$ and let $\bar{N}(p)$ be the closure of $N(p)$ in M. Then

$$\bar{N}(p) = \bar{E}^p \cdot p.$$

Since $\nu_p \subset T_p\bar{N}(p)$, then $\bar{\nu} = \nu_{|\bar{N}(p)}$ is a distribution of $\bar{N}(p)$. Since E^p is a normal subgroup of $\bar{E}^p$, then the integral manifolds of the autoparallel distribution $\bar{\nu}$ are given by

$$E^p \cdot x \qquad (x \in \bar{N}(p)). \tag{7.2.1}$$

Lemma 7.3. *$\bar{\nu}$ is contained in the nullity distribution of $\bar{N}(p)$.*

Proof. Let us first show that $\bar{\nu}$ is in the nullity of the second fundamental form α of $\bar{N}(p)$. Let X be a Killing field of M induced by $\bar{E}^p$ and let $\gamma_v(t) = \mathrm{Exp}(tu)p$ be a homogeneous geodesic in $N(p)$, $u \in \mathrm{Lie}(E^p)$ with $u.p = v$. The proof of Proposition 3.6 shows that there is transvection on the direction of $\nabla_v X$. Moreover, from the construction of such a transvection, one has that $\nabla_v X \in T_p\bar{N}(p))$. Since the Killing field X, induced by $\bar{E}^p$, is arbitrary, we conclude that ν_p belongs to the nullity of α at p (and the same is true for any $q \in \bar{N}(p)$). Then, from the Gauss equation, one obtains that $\bar{\nu}$ is contained in the nullity distribution of $\bar{N}(p)$. □

Remark 7.4. Observe that the proof of Lemma 7.3 applies also to prove the following fact: let G' be a Lie subgroup of G that contains E^p and let $S = G' \cdot p$. Then $\nu_{|S}$ is contained in the nullity distribution of the Riemannian manifold S.

Theorem 7.5. *Let $M = G/H$ be a simply connected irreducible homogeneous Riemannian manifold with non-trivial nullity distribution ν, where $G = I(M)^o$. Then any (maximal) integral manifold of ν is a closed (embedded) submanifold of M.*

Proof. Applying Proposition 3.19 (and the comment below it) any Killing field, induced by $\bar{E}^p$, that lies in the nullity of $\bar{N}(p)$ must be tangent to $\mathcal{D}^0$, the distribution associated to the local Euclidean de Rham factor of $\bar{N}(p)$. Since the Killing

fields induced by E^p lie in ν, which is included in the nullity of $\bar{N}(p)$, then $N(p)$ is included in the integral manifold $F(p)$ of $\mathcal{D}^0$. By Lemma 7.1 the closure $\bar{F}(p)$ is a flat embedded submanifold of M. Observe that $\bar{F}(p) = T^r \times \mathbb{R}^s$, since it is flat and homogeneous (see Section 2.5).

By Remark 7.2 there is a maximal (connected) Lie subgroup S of $\bar{E}^p$ that contains E^p and such that $S \cdot p = \bar{F}(p)$. Then E^p is a normal subgroup of S. Then the orbits $N(y) = E^p \cdot y$ are parallel totally geodesic submanifolds of $\bar{F}(p)$, $y \in \bar{F}(p)$ (see Lemma 7.6). Let us consider the intersection $T_pN(p) \cap T_pT^r$. If this intersection is different from $\{0\}$, then $N(p)$ has a bounded geodesic γ_v. Then any Killing field of M is bounded along γ_v and hence, by Lemma 2.8, parallel along γ_v. So, for any $X \in \mathcal{K}^G(M)$, $\mathbb{R}v$ is invariant under $(\nabla X)_p$. Then, by Corollary 2.5, M splits off the direction of v. A contradiction. So $T_pN(p) \cap T_pT^r = \{0\}$. In this case, as it is not hard to see, $N(p)$ is a closed submanifold of $\bar{F}(p)$. Then $N(p)$ is closed or, equivalently, E^p is a closed subgroup of $I(M)$. □

Lemma 7.6. *Let G be a connected subgroup of $I(\mathbb{R}^n)$ which acts transitively on $\mathbb{R}^n$. Let G' be connected normal subgroup of G. Then the orbits of G' are parallel affine subspaces of $\mathbb{R}^n$.*

Proof. It is well-known that any (connected) Lie subgroup of $I(\mathbb{R}^n)$ has a totally geodesic orbit (see e.g. Theorem 3.5, pg. 100 in [AVS]). Then, since G' is a normal subgroup of G, all orbits of G' are affine subspaces of $\mathbb{R}^n$. Let d be the distance between the affine subspaces $G' \cdot x$ and $G' \cdot y$. We may assume that $d(x, y) = d$. So for any $g'x \in G' \cdot x$, $d(g'x, g'y) = d$. Then any point of $G' \cdot y$ is at a distance d to some point in $G' \cdot x$. Then the affine subspace $G' \cdot x$ must be parallel to the affine subspace $G' \cdot y$. □

7.1. **The affine bundle and the connection given by $\nu^\perp$.**

We keep the assumptions and notation of this section. From Theorem 7.5 we have that $N(p)$ is closed, or equivalently, E^p is a closed subgroup of $G = I(M)^o$ ($p = [e]$). Then $M = G/H$ is the total space of a fiber bundle over $B = G/E^p$ with standard fiber $E^p/H = N(p) \simeq \mathbb{R}^k$ (with a Euclidean affine structure, see Lemma 3.2). The projection of M onto B will be denoted by π. Observe that B is the quotient space $M/\mathcal{N}$ of M by the leaves of the nullity foliation $\mathcal{N} = \{N(q) : q \in M\}$. Since the elements of $\mathcal{N}$ are, in a natural way, Euclidean affine spaces, one has that M is an Euclidean affine bundle (and so an affine combination of local sections of M is a local section). Observe that G leaves $\mathcal{N}$ invariant and, for any $g \in G$, g is an isometry between $\pi^{-1}(\pi(q)) = N(q)$ and $\pi^{-1}(\pi(gq)) = N(gq)$. Moreover, Lemma 3.19 implies that G acts almost effectively on $M/\mathcal{N} = B$.

Let us consider the natural affine connection on $M \xrightarrow{\pi} B$ given by the distribution $\nu^\perp$. In fact, a perpendicular variation of totally geodesic manifolds is by isometries. So, the local horizontal lift of curves in B gives rise to local isometries between the involved fibers. From this particular situation, it is well-known, and standard to show, that any piece-wise differentiable curve $c : [0, 1] \to B$ can be lifted to a (unique) horizontal curve $\tilde{c}_u : [0, 1] \to M$ with $\tilde{c}_u(0) = u$, for any $u \in \pi^{-1}(c(0))$. Then there is a well defined parallel transport $\tau_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$, which is an isometry, given by $\tau_c(u) = \tilde{c}_u(1)$. Then $\nu^\perp$ is an affine connection.

For each $b \in B$, let $\Phi(b) \subset I(\pi^{-1}(b))$ denote the holonomy group of $\nu^\perp$ at p (holonomy groups are conjugated by parallel transport). Note that B is simply

connected, since M is simply connected and the fibers are connected. Then the holonomy groups $\Phi(b)$ are connected.

Let us consider, for $q \in M$, the holonomy subbundle $\mathrm{Hol}(q)$. Namely, $\mathrm{Hol}(q)$ consists of all the elements of M that can be reached from q by a horizontal curve. The holonomy subbundles foliate M . Moreover, any holonomy subbundle intersects any given fiber $\pi^{-1}(b)$ in an orbit of the holonomy group $\Phi(b)$.

The holonomy subbundles, despite what happens in a principal bundle, may have different dimensions depending on the dimensions of the orbits of the holonomy group. But in our case the holonomy subbundles have all the same dimension (and so their tangent spaces define a smooth distribution), since, for any $g \in G$, $u \in M$,

$$\mathrm{Hol}(gq) = g\mathrm{Hol}(q) \quad \text{and} \quad \Phi(\pi(gq)) = g\Phi(\pi(q))g^{-1}. \tag{7.6.1}$$

In particular, if $g \in E^q$ then $\Phi(\pi(q)) = g\Phi(\pi(q))g^{-1}$.

This implies that $\tilde{E}^q := \{g_{|N(q)} : g \in E^q\}$ is included in the normalizer of $\Phi(\pi(q))$ in $I(N(q))$. Then $L = \Phi(\pi(q)).\tilde{E}^q$ is a Lie group of isometries, which is transitive on $N(q) \simeq \mathbb{R}^k$, and $\Phi(\pi(q))$ is a normal subgroup of L. Then, by Lemma 7.6, we have that:

$(*)$ *The orbits of* $\Phi(\pi(q))$ *are parallel affine subspaces of* $N(q)$.

Observe that the above property implies that $\Phi(\pi(q))$ *acts polarly* on $N(q)$.

Let $\mathcal{Y}$ be the distribution of M defined by the normal spaces of the holonomy subbundles. Namely,

$$\mathcal{Y}_q = (T_q\mathrm{Hol}(q))^\perp \subset \nu_q. \tag{7.6.2}$$

Then $\mathcal{Y} \subset \nu$ and, from $(*)$, $\mathcal{Y}_{|N(q)}$ is a parallel (i.e. constant) foliation of $N(q)$, for all $q \in M$ (which is perpendicular to the holonomy orbits). Moreover, since all the orbits of $\Phi(\pi(q))$ are principal orbits:

$(**)$ *any* $w \in \mathcal{Y}_q$ *is a fixed vector of the isotropy* $\Phi(\pi(q))_q$.

Let $p \in M$ be fixed and let $\xi \in \mathcal{Y}_p$. Then ξ induces a normal vector field of $\mathrm{Hol}(p)$ in the following way: if $q \in \mathrm{Hol}(p)$, choose $\tilde{c}_p : [0,1] \to M$ be a horizontal piece-wise differentiable curve with $\tilde{c}_p(0) = p$, $\tilde{c}_p(1) = q$. Let $c = \pi \circ \tilde{c}_p$, then define

$$\tilde{\xi}(q) = \mathrm{d}\tau_c(\xi).$$

From $(**)$ one obtains that $\tilde{\xi}$ is well defined (and it is standard to show that it is smooth). Let us show that $\tilde{\xi}$ is a parallel normal vector field. Observe that

$$T_p\mathrm{Hol}(p) = \nu_p^\perp \oplus T_p(\Phi(\pi(p)) \cdot p).$$

From the construction of $\tilde{\xi}$, taking into account that $\Phi(\pi(p))$ acts polarly on $N(p)$, one obtains that

$$\nabla^\perp_v \tilde{\xi} = 0, \quad \text{for all } v \in T_p(\Phi(\pi(p)) \cdot p).$$

Let now $u \in \nu_p^\perp$ and let $\tilde{c}_p(s)$ be a horizontal curve with $\tilde{c}_p'(0) = u$ and let $c(t) = \pi(\tilde{c}_p(t))$. Let $\gamma_\xi(t)$ be a geodesic (i.e. a line) in $N(p)$ with $\gamma_\xi'(0) = \xi$. Let τ^s be the parallel transport along $c(s)$, form 0 to s. Let us consider

$$f(s,t) = \tau^s(\gamma_\xi(t)),$$

which is variation of geodesics that are tangent to ν. Then

$$J_\xi(t) := \tfrac{\partial f}{\partial s}_{|(0,t)}$$

is a Jacobi field along the geodesic $\gamma_\xi(t)$.

Note that $s \mapsto f(s,t_0)$ is a horizontal curve, i.e. tangent to $\nu^\perp$, and so $J^\xi(t)$ is a horizontal field along $\gamma_\xi(t)$.

Since $N(p)$ is totally geodesic in M, $\nu^\perp_{|N(p)}$ is a parallel subbundle of the pull-back of TM, via the inclusion of $N(p)$ in M.

Then

$$J'_\xi(t) := \tfrac{\mathrm{D}}{\mathrm{d}t} J_\xi(t), \tag{7.6.3}$$

as well as $J_\xi(t)$, are horizontal fields along $\gamma_\xi(t)$.

Observe that

$$\begin{aligned} J'_\xi(0) &= \tfrac{\mathrm{D}}{\partial t}_{|(0,0)} \tfrac{\partial}{\partial s} f(s,t) = \tfrac{\mathrm{D}}{\partial s}_{|(0,0)} \tfrac{\partial}{\partial t} f(s,t) \\ &= \tfrac{\mathrm{D}}{\mathrm{d}s}_{|0} \mathrm{d}\tau^s(\xi) = \tfrac{\mathrm{D}}{\mathrm{d}s}_{|0} \tilde{\xi}(\tilde{c}_p(s)) \\ &= \nabla_u \tilde{\xi} \in \nu_p^\perp. \end{aligned}$$

Since $u \in \nu_p^\perp$ is arbitrary, then $\tilde{\xi}$ is also $\nabla^\perp$-parallel in the horizontal directions and so

$$\nabla^\perp \tilde{\xi} = 0.$$

Observe, since $\xi \in (T_p(\mathrm{Hol}(p)))^\perp$ is arbitrary, that:

$(***)$ *The normal bundle of* $\mathrm{Hol}(p)$ *is globally flat.*

Let A be the shape operator of $\mathrm{Hol}(p)$. It is standard to show, since $\nabla^\perp \tilde{\xi} = 0$, that $J'_\xi(0) = -A_\xi(u)$. So, the Jacobi field $J_\xi(t)$ has the following initial conditions:

$$\begin{aligned} J_\xi(0) &= u, \\ J'_\xi(0) &= -A_\xi(u). \end{aligned}$$

Since $J'_\xi(0) \in \nu_p^\perp$, this shows that any shape operator of $\mathrm{Hol}(p)$ leaves $\nu_p^\perp$ invariant.

Note that $J_\xi(t) = \tilde{c}'_{\gamma_\xi(t)}(0)$, where $\tilde{c}_{\gamma_\xi}(t)$ is the horizontal lift of $c(t) = \pi(\tilde{c}_p(t))$. Then

$$\mathrm{d}\pi(J_\xi(t)) = c'(0) = \mathrm{d}\pi(u). \tag{7.6.4}$$

Assume that $\lambda \neq 0$ is an eigenvalue of A_ξ and let $u \neq 0$ be an eigenvector of A_ξ associated to λ. Since $\gamma_\xi(t)$ lies in $N(p)$, then

$$J_\xi(t) = (1 - t\lambda)\hat{u}(t),$$

where $\hat{u}(t)$ is the parallel transport of u along $\gamma_\xi(t)$. Then $J(1/\lambda) = 0$, which contradicts (7.6.4). This shows that

$$A_{\xi|\nu_p^\perp} = 0 = A_{\tilde{\xi}(p)|\nu_p^\perp}.$$

From its construction, $\tilde{\xi}$ is constant along the holonomy orbit $\Phi(\pi(p)) \cdot p$ (see $(*)$ and $(**)$). Then

$$A_{\xi|T_p(\Phi(\pi(p))\cdot p)} = 0$$

and so

$$A_\xi = 0.$$

Since p and $\xi \in (T_p(\mathrm{Hol}(p)))^\perp = \mathcal{Y}_p$ are arbitrary, we conclude that any holonomy subbundle $\mathrm{Hol}(q)$ is a totally geodesic submanifold of M. Observe that the distribution $\mathcal{Y}$ is autoparallel, since $\mathcal{Y} \subset \nu$ and is parallel inside the leaves of the nullity (see the paragraph below (7.6.2)).

Since the holonomy subbundles are totally geodesic and its perpendicular distribution $\mathcal{Y}$ is autoparallel, we conclude that $\mathcal{Y}$ is a parallel distribution. This is a contradiction, since M is irreducible, unless $\mathcal{Y} = 0$. Then $\mathrm{Hol}(p) = M$ and so $\Phi(p)$ is transitive on $N(p)$.

By summarizing our main results in the section we obtain:

Theorem 7.7. *Let $M = G/H$ be a simply connected irreducible homogeneous Riemannian manifold with a non-trivial nullity distribution ν. Then the quotient space B of M by the leaves of the nullity is a manifold and M is a Euclidean affine fiber bundle over B, with standard fiber isometric to $\mathbb{R}^k$. Moreover, $\nu^\perp$ defines a metric affine connection on M with a transitive holonomy group (and so $\nu^\perp$ is completely non-integrable).*

For Euclidean or spherical submanifolds, the complete non-integrability of the distribution perpendicular to the relative nullity (i.e. the nullity of the second fundamental form) was proved in [V] (this is not true in hyperbolic space).

8. The proof of the main results

The main theorems stated in the introduction were proved throughout the paper or are direct consequences of previous results. We sum them up here.

Proof of Theorem A. The first part of (1) was proved throughout Section 5 and concluding with equation (5.3.1). The fact that the integral manifolds of ν are simply connected was proved in Lemma 3.2. By Corollary 3.13, and part (ii) of Proposition 3.15, the Jacobi operator in any vector tangent to $\nu^{(1)}$ is null. So the proof of Lemma 3.2 also shows that the integral manifolds of $\nu^{(1)}$ are simply connected.

Part (2). The existence of an adapted transvection Y, see Definition 3.8, with $Y_p = v \notin \nu_p$ was proved in Proposition 3.6. The fact that the Jacobi operator $R_{,v}v$ is null was proved in Theorem 3.12 and stated in Corollary 3.13. Theorem 3.12 also states that $[Y, [Y, \mathcal{K}(M)]] = 0$ and that Y does not belong to the center of $\mathcal{K}^G(M)$. For an arbitrary $v \in \hat{\nu}_p$ the existence of a transvection Y with the stated properties follows from Proposition 3.15, since the adapted transvections at p span $\hat{\nu}_p$.

Part (3) is Theorem 5.5.

The first part of (4) is Theorem 6.3. In Section 9 we construct non-trivial examples, in any dimension, with $k = 3$ and G not unimodular. For the non-unimodularity see Remark 9.9. □

Proof of Theorem B. It follows from Theorem 7.5 and Theorem 7.7. □

Proof of Proposition C. Part (a) is Corollary 3.16. If M is compact see Proposition 2.9 for a direct proof. Part (b) is Corollary 3.17. □

9. Examples with nullity of codimension 3 and co-index of symmetry 2

In this section we construct examples of irreducible Riemannian homogeneous spaces with nullity of codimension 3 and co-index of symmetry 2 in any dimension greater or equal to 4. As explained in the introduction such examples are optimal since neither the nullity can be greater nor the co-index of symmetry can be smaller due to Reggiani's Theorem [R].

Let $G = \mathbb{R}^d \rtimes \mathbb{R}$ be the semidirect product of the abelian groups where $\mathbb{R}$ acts on $\mathbb{R}^d$ as $\mathrm{Exp}(sA) = \mathrm{e}^{sA}$, $s \in \mathbb{R}$ and A is defined as

$$A = \begin{bmatrix} aM_{d-1} & ae_1 \\ -ae_1^t & a \end{bmatrix}$$

where $M_{d-1} = (m_{ij})$ is $(d-1)\times(d-1)$ skew-symmetric with $m_{ij} = 1$ for $i < j$, e_1 is the first canonical column of $\mathbb{R}^{d-1}$, e_1^t its transpose, and the constant a is chosen so $1 = trace(AA^t)$, i.e. $a^2 = \frac{1}{3+(n-2)(n-3)}$, where $n = d+1$.

Observe that G is simply connected. We can regard G as the connected Lie subgroup of $\mathrm{GL}(d+1,\mathbb{R})$ whose Lie algebra is generated by the following $d+1$ matrices:

$$E_i := \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

where $i = 1, \ldots, d$ and $e_1, \cdots, e_d$ are the canonical columns of $\mathbb{R}^d$.

Let g be the left invariant metric on $G \subset \mathrm{GL}(d+1,\mathbb{R})$ whose restriction to the tangent space at the identity T_eG is given by

$$g(X,Y) = \mathrm{trace}(XY^t),$$

where Y^t indicates the transpose matrix. Observe that $E_1, \cdots, E_d, \mathbf{A}$ is an orthonormal base of T_eG.

Let $\tilde{E}_1, \cdots, \tilde{E}_d, \tilde{\mathbf{A}}$ be the right invariant vector fields of G such that

$$(\tilde{E}_1)_e = E_1, \cdots, (\tilde{E}_d)_e = E_d, (\tilde{\mathbf{A}})_e = \mathbf{A}.$$

Observe that the flow associated to any of these fields is given by the left multiplication by $\mathrm{Exp}(tu)$, where u is the evaluation of the field at e. So $\tilde{E}_1, \cdots, \tilde{E}_d, \tilde{\mathbf{A}}$ are Killing fields of G, since their associated flows are by isometries. For the sake of simplifying the notation we will identify, when there is no possible confusion,

$$\tilde{E}_1 \simeq E_1, \cdots, \tilde{E}_d \simeq E_d, \tilde{\mathbf{A}} \simeq \mathbf{A}. \tag{9.0.1}$$

Lemma 9.1. *Let ∇ be the Levi-Civita connection of (G,g) and R its curvature tensor. Then the following identities hold at $e \in G$:*

i) $\nabla E_1 = \nabla E_2 = \cdots = \nabla E_{d-1} = 0$, *i.e.* $E_1, \cdots, E_{d-1}$ *are transvections at* $e \in G$.

ii) $\nabla E_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}$.

iii) $\nabla \mathbf{A} = \begin{bmatrix} -aM_{d-1} & -ae_1 & 0 \\ ae_1^t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

where the matrices have to be regarded as (skew-symmetric) linear endomorphisms of T_eG in the orthonormal basis $E_1, \cdots, E_d$, $\mathbf{A}$.

Proof. The proof is a computation by using equation (2.1.5) and the fact that bracket at e of any two right invariant fields is minus the bracket at e of the corresponding left invariant fields (see (2.0.2)). To show (i) notice that equation (2.1.5) gives $\langle \nabla_{E_i} E_j, E_k \rangle = 0$ for any $i,j,k \in 1, \cdots, d$. Equation (2.1.5) gives

$$2\langle \nabla_{\mathbf{A}} E_j, E_k \rangle = \langle [\mathbf{A}, E_j], E_k \rangle + \langle [\mathbf{A}, E_k], E_j \rangle.$$

So, for $1 \leq j \leq d-1$, we get

$$2\langle \nabla_{\mathbf{A}} E_j, E_k\rangle = \langle Ae_j, e_k\rangle + \langle Ae_k, e_j\rangle = am_{jk} + am_{kj} = 0$$

which shows (i).
To show (ii) observe that from the definition of A, if $1 < k < d$:

$$2\langle \nabla_{\mathbf{A}} E_d, E_k\rangle = \langle Ae_d, e_k\rangle + \langle Ae_k, e_d\rangle = \langle ae_1 + ae_d, e_k\rangle + \langle Ae_k, e_d\rangle = 0 = 2\langle C^a E_d, E_k\rangle$$

and

$$2\langle \nabla_{\mathbf{A}} E_d, E_d\rangle = \langle Ae_d, e_d\rangle + \langle Ae_d, e_d\rangle = 2a = 2\langle C^a \mathbf{A}, E_d\rangle\,,$$

where $C^a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}$. This shows (ii).

Finally,

$$2\langle \nabla_{E_i}\mathbf{A}, E_j\rangle = \langle [E_i, \mathbf{A}], E_j\rangle + \langle [\mathbf{A}, E_j], E_i\rangle = 2\langle \nabla_{E_i}\mathbf{A}, E_j\rangle = -\langle Ae_i, e_j\rangle + \langle Ae_j, e_i\rangle.$$

So, if $i, j \in \{1, \cdots, d-1\}$, we have

$$2\langle \nabla_{E_i}\mathbf{A}, E_j\rangle = -am_{ij} + am_{ji} = -2am_{ij} = 2\langle D^a E_i, E_j\rangle,$$

where $D^a = \begin{bmatrix} -aM_{d-1} & -ae_1 & 0 \\ ae_1^t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now for $1 < j < d$ we get

$$\begin{aligned} 2\langle \nabla_{E_d}\mathbf{A}, E_j\rangle &= -\langle Ae_d, e_j\rangle + \langle Ae_j, e_d\rangle = -\langle ae_1 + ae_d, e_j\rangle + \langle Ae_j, e_d\rangle = 0 \\ &= 2\langle D^a E_d, E_j\rangle. \end{aligned}$$

If $j = 1$, then

$$2\langle \nabla_{E_d}\mathbf{A}, E_1\rangle = -\langle Ae_d, e_1\rangle + \langle Ae_1, e_d\rangle = -2a = 2\langle D^a E_d, E_1\rangle.$$

The last case to check is

$$2\langle \nabla_{\mathbf{A}}\mathbf{A}, E_j\rangle = 0 = 2\langle D^a A, E_j\rangle.$$

Thus, since $\nabla\mathbf{A}$ is skew-symmetric, we have proved (iii). □

Lemma 9.2. *The nullity ν_e of R at $e \in G$ is generated by $E_2, \cdots, E_{d-1}$.*

Proof. We are going to compute R_{X,E_i} for $i = 1, \cdots, d-1$ by using formula (2.1.4). Let X be an arbitrary Killing field of (G, g). We regard $E_1, \cdots, E_d, \mathbf{A}$, from the identification 9.0.1, as Killing fields of (G, g). We have that

$$R_{X,E_i} = \nabla[X, E_i],$$

since $\nabla E_i = 0$ by Lemma 9.1, part (i). So

$$R_{X,E_i} = \nabla[\sum_{j=1}^{d}\langle X, E_j\rangle E_j + \langle X, \mathbf{A}\rangle\mathbf{A}, E_i] = \langle X, \mathbf{A}\rangle\nabla[\mathbf{A}, E_i] = \langle X, \mathbf{A}\rangle R_{\mathbf{A},E_i}\,.$$

By formula (2.1.4) $R_{\mathbf{A},E_i} = \nabla[\mathbf{A}, E_i]$ then

$$R_{\mathbf{A},E_i} = \nabla\left(\sum_{k=1}^{d-1} am_{ki}E_k - a\delta_{1i}E_d\right)$$

and by Lemma 9.1 we get $R_{\mathbf{A},E_i} = -a\delta_{1i}\nabla E_d$. Then $E_2, \cdots, E_{d-1}$ belongs to the nullity ν_e of R. Since $R_{\mathbf{A},E_1} = -a\nabla E_d$, Lemma 9.1 (ii) implies that

$$\nu_e = span\{E_2, \cdots, E_d\}$$

because of $R_{\mathbf{A}E_1} \neq 0$ and $\ker(\nabla E_d) = span\{E_1, \cdots, E_{d-1}\}$. □

Remark 9.3. Observe that in these examples the Killing vector field Y defined as

$$Y := E_1 - \sum_{k=3}^{d-1} E_k$$

is a transvection as that given by part (ii) of Theorem A. Indeed, by part (iii) of Lemma 9.1, $Y = \frac{1}{a}\nabla_{E_2}\mathbf{A}$. Hence $Y_e \in \hat{\nu}_e$ and $Y_e \notin \nu_e$, and the Jacobi operator $R_{\cdot,Y}Y$ is null:

$$R_{X,Y}Y = R_{X,E_1}E_1 = \langle X, \mathbf{A}\rangle R_{\mathbf{A},E_1}E_1 = \langle X, \mathbf{A}\rangle(-a)\nabla_{E_1}E_d = 0\,.$$

Actually, in our examples, $\nu_e^{(1)} = span\{E_1, E_2, \cdots, E_{d-1}\}$ and

$$\nu_e^{(2)} = \mathcal{U}_e = span\{E_1, E_2, \cdots, E_d\}.$$

Lemma 9.4. *There is no (non-trivial) $\nabla\mathbf{A}$-invariant proper subspace of ν_e.*

Proof. From part (iii) of Lemma 9.1 and Lemma 9.2, any non-trivial invariant subspace of $\nabla\mathbf{A}$ in ν_e determines a non-trivial invariant subspace of the matrix M_{d-1} which is contained in the subspace of $\mathbb{R}^{d-1}$ generated by the vectors $e_2, \cdots, e_{d-1}$. But this is not possible by the lemma in Appendix. □

To show that (G, g) is an irreducible Riemannian manifold we use the following result:

Lemma 9.5. *Let $M = G/H$ be a homogeneous Riemannian manifold whose nullity distribution has codimension 3. If M is locally reducible, then the flat local de Rham factor is non-trivial.*

Proof. If there is no local flat factor, then there must be an irreducible local factor whose nullity has codimension 1. This implies that all sectional curvatures are zero (since any plane of the tangent space does intersect the nullity subspace). Hence this factor must be flat. A contradiction. □

Theorem 9.6. *For each $n \geq 4$ the n-dimensional simply connected homogeneous Riemannian manifold (G, g), constructed in this section, is irreducible. Its nullity distribution has codimension 3 and its co-index of symmetry is 2. Moreover its Ricci tensor has four eigenvalues: zero with multiplicity $n-3$, $-a^2$ and $a^2\left(\frac{-1\pm\sqrt{5}}{2}\right)$. So (G, g) has sectional curvatures of both signs and its scalar curvature is $-2a^2 = \frac{-2}{3+(n-2)(n-3)}$.*

Proof. That the index of symmetry $i_{\mathfrak{s}}(G)$ is $n-2$ follows from $i)$ in Lemma 9.1. Let us show that our examples are irreducible Riemannian manifolds. In fact, if (G, g) is reducible, by Lemma 9.5 there is a non-trivial flat de Rham factor E of (G, g). Observe that the parallel G-invariant distribution $\mathcal{E}$ of (G, g), associated to the factor E, is contained in the nullity distribution ν. Moreover $[\mathbf{A}, \mathcal{E}] \subset \mathcal{E}$ since $\mathcal{E}$ is is invariant by isometries. Let Z be a parallel vector field of (G, g) that lies in $\mathcal{E}$. Then $([\mathbf{A}, Z])_e = (\nabla_Z\mathbf{A})_e - (\nabla_{\mathbf{A}}Z)_e = (\nabla_Z\mathbf{A})_e \in \mathcal{E}_e \subset \nu_e$. Since the parallel vector fields of (G, g) generate $\mathcal{E}$, it follows that $\nabla\mathbf{A}$ leaves invariant the

non-trivial subspace $\mathcal{E}_e$ of ν_e. This contradicts Lemma 9.4. A direct computation using Lemma 9.1 and formula (2.1.4) shows that the Ricci tensor restricted to $\nu_e^\perp = \mathrm{span}\{E_1, E_d, \mathbf{A}\}$ is given by the matrix $\begin{bmatrix} 0 & a^2 & 0 \\ a^2 & -a^2 & 0 \\ 0 & 0 & -a^2 \end{bmatrix}$ (with respect to the basis $E_1, E_d, \mathbf{A}$). □

Remark 9.7. For $d = 3$, by changing the matrix A, it is possible to show the existence of 1-parameter family (G_λ, g_λ) of 4-dimensional non-homothetic irreducible homogeneous metrics with nullity of dimension 1.

Remark 9.8. With the same ideas, by modifying the matrix A, it is possible to construct examples with $k > 3$.

Remark 9.9. Observe that $\mathrm{trace}(\mathrm{ad}(\mathbf{A})) \neq 0$ i.e. our solvable groups are not unimodular hence they do not admit finite volume quotients [Mi76, Remark, Lemma 6.2.].

10. Appendix: invariant subspaces of the skew-symmetric matrix M

Let $M = (m_{ij})$ be the real $d \times d$, $(d > 2)$, skew-symmetric matrix with $m_{ij} = 1$ if $i < j$. Let $\mathbb{W} \subset \mathbb{R}^d$ be the subspace generated by the canonical vectors $e_2, \cdots, e_d$ i.e. the orthogonal complement of the first canonical vector e_1. The goal of this appendix is to prove the following:

Lemma 10.1. *There are no M-invariant, non-trivial, subspaces of $\mathbb{R}^d$ which are contained in $\mathbb{W}$.*

Proof. By contradiction assume that there is a non-trivial M-invariant subspace $\mathbb{U} \subset \mathbb{W}$. We have to consider two cases $dim(\mathbb{U}) = 1$ or $dim(\mathbb{U}) = 2$.

Case $dim(\mathbb{U}) = 1$. Since M is skew-symmetric we have that $\mathbb{U} \subset \ker(M)$. Let $(a_1, \cdots, a_d) \neq 0 \in \mathbb{U} \subset \ker(M)$. Then

$$M.a = (\sum_{k=2}^{d} a_k, \cdots, -\sum_{k=1}^{i-1} a_k + \sum_{k=i+1}^{d} a_k, \cdots, -\sum_{k=1}^{d-1} a_k)\,. \tag{10.1.1}$$

So subtracting two consecutive components we obtain:

$$\begin{cases} 0 = a_2 + a_1, \\ 0 = a_3 + a_2 \\ \quad \vdots \\ 0 = a_d + a_{d-1}. \end{cases}$$

Since $a \in \mathbb{U} \subset \mathbb{W}$ we have $a_1 = 0$ hence $a = 0$. A contradiction.

Case $dim(\mathbb{U}) = 2$. In this case $\mathbb{U}$ is spanned by two vectors $a, b := M.a$. Since M is skew-symmetric there is $r \neq 0 \in \mathbb{R}$ such that $M.b = r.a$. We can assume $r \neq 1$ since otherwise the vector $a + b$ is M-invariant which was excluded in Case 1. By using equation 10.1.1 we get

$$\begin{cases} b_1 - b_2 = a_1 + a_2 \\ ra_1 - ra_2 = b_1 + b_2 \end{cases}$$

By using $a, b \in \mathbb{U} \subset \mathbb{W}$ we get that $a_1 = b_1 = 0$ and so $\begin{cases} -b_2 = a_2 \\ -ra_2 = b_2 \end{cases}$ hence $a_2 = b_2 = 0$ due to $r \neq 1$. Now equation 10.1.1 gives us for $i = 1, \cdots, d-1$

$$\begin{cases} b_i - b_{i+1} = a_i + a_{i+1} \\ ra_i - ra_{i+1} = b_i + b_{i+1} \end{cases}.$$

So if we assume, as inductive hypothesis, that $a_1 = a_2 = \cdots = a_i = b_1 = b_2 = \cdots = b_i = 0$ we obtain for a_{i+1}, b_{i+1}:

$$\begin{cases} -b_{i+1} = a_{i+1} \\ -ra_{i+1} = b_{i+1} \end{cases}.$$

Hence $a_{i+1}, b_{i+1} = 0$ due to $r \neq 1$. Then $a = b = 0$. A contradiction.

□

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