Some Curves and the Lengths of their Arcs

Amelia Carolina Sparavigna

Department of Applied Science and Technology

Politecnico di Torino

Here we consider some problems from the Finkel's solution book, concerning the length of curves. The curves are Cissoid of Diocles, Conchoid of Nicomedes, Lemniscate of Bernoulli, Versiera of Agnesi, Limaçon, Quadratrix, Spiral of Archimedes, Reciprocal or Hyperbolic spiral, the Lituus, Logarithmic spiral, Curve of Pursuit, a curve on the cone and the Loxodrome. The Versiera will be discussed in detail and the link of its name to the Versine function.

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Here we consider some of the problems propose in the Finkel's solution book, having the full title: A mathematical solution book containing systematic solutions of many of the most difficult problems, Taken from the Leading Authors on Arithmetic and Algebra, Many Problems and Solutions from Geometry, Trigonometry and Calculus, Many Problems and Solutions from the Leading Mathematical Journals of the United States, and Many Original Problems and Solutions. with Notes and Explanations by Benjamin Franklin Finkel, 1888. A discussion on the Versiera and its history is proposed (in Italian).

Cissoid of Diocles

Conchoid of Nicomedes

Lemniscate of Bernoulli

Versiera of Agnesi and the Versine

Limaçon - Quadratrix

Spiral of Archimedes

Reciprocal or Hyperbolic spiral

Lituus

Logarithmic spiral

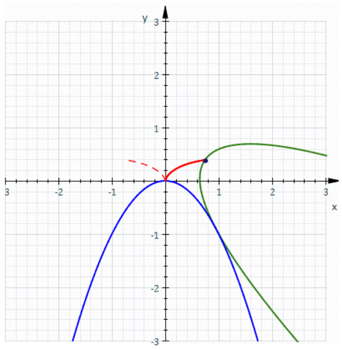
Curve of Pursuit

On the cone

Loxodrome

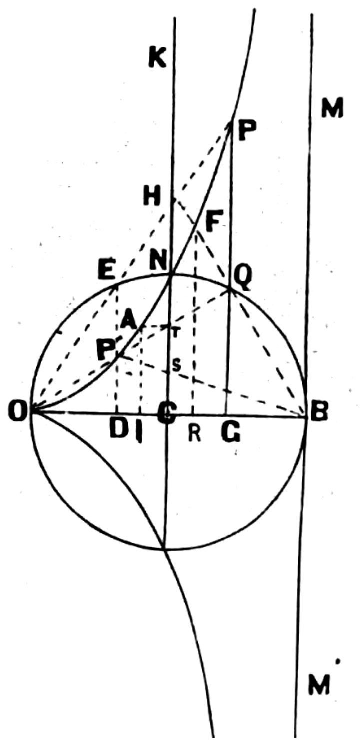
Cissoid of Diocles

The Cissoid of Diocles is the curve generated by the vertex of a parabola rolling on an equal parabola.



"A pair of parabolas face each other symmetrically: one on top and one on the bottom. Then the top parabola is rolled without slipping along the bottom one, and its successive positions are shown in the animation. Then the path traced by the vertex of the top parabola as it rolls is a roulette shown in red, which is cissoid of Diocles." Sam Derbyshire - en:File:RouletteAnim2.gif - "Animation of a roulette of one parabola against another, producing a Cissoid of Diocles. Selfmade with MuPAD".

Available at en.wikipedia.org/ wiki/Cissoid_of_Diocles#/media/File: RouletteAnim2.gif



$y^2 = \frac{x^3}{2a - x}$ is the equation of the cissoid referred do rectangular axes.

$\rho = 2a \sin\theta \tan\theta$ is the polar equation of the curve.

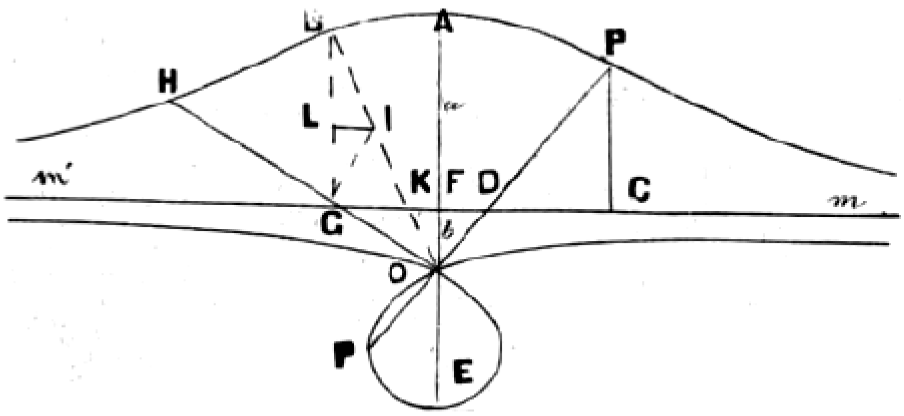
To find the length of an arc OAP of the cissoid.

Formula.—$s = OAP =$

$$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \left(\frac{(3a - x)\sqrt{x}}{\sqrt{(2a - x)^3}}\right)^2}\, dx = a \int \sqrt{\frac{8a - 3x}{(2a - x)^3}}\, dx = a \left\{ \sqrt{\frac{8a - 3x}{2a - x}} - 2 + 3 \log \left[\frac{\sqrt{2a}(\sqrt{3} + 2)}{\sqrt{3}\sqrt{2a - x} + \sqrt{8a - 3x}} \right] \right\}$$

The word "cissoid" comes from the Greek κισσοειδής kissoeidēs "ivy shaped" from κισσός kissos "ivy" and -οειδής -oeidēs "having the likeness of". The curve is named for Diocles who studied it in the 2nd century BCE.

Conchoid of Nicomedes



This curve is described by the equation:

$\rho = b \sec \theta \pm a$

The secant function (usually abbreviated as sec) is the reciprocal function of the cosine function.

Prob. XLV. To find the length of an arc of conchoid.

Formula $s = \int \sqrt{1 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int \sqrt{1 + \tan^2\theta\, sec^2\theta}\; d\theta$

The length of a line given by a polar equation r(θ) can be obtained by integration. let us consider . The length L of the line from A to B, where $\theta_A = a$, $\theta_B = b$ is given by:

$$L = \int_a^b \sqrt{[r(\theta)]^2 + \left[\frac{dr(\theta)}{d\theta}\right]^2}\, d\theta = \int_{\theta_A}^{\theta_B} \sqrt{[r(\theta)]^2 + \left[\frac{dr(\theta)}{d\theta}\right]^2}\, d\theta$$

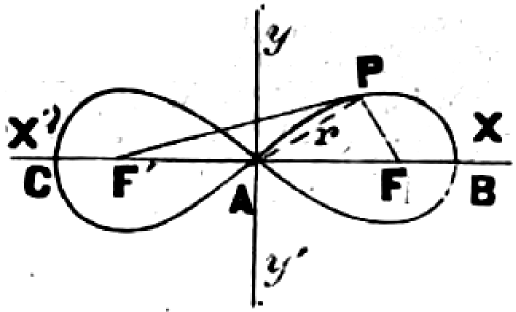
Lemniscate of Bernoulli

This curve is what a Cassinian becomes when $m=a$. A Cassinian is a Cassini Oval, having rectangular equation:

$$[y^2+(a+x)^2][y^2+(a-x)^2]=m^4 \quad .$$

In polar coordinates: $r^4-2a^2r^2\cos 2\theta+a^2-m^4=0$. Then, the Lemniscate is:

$$(x{+}^2+y^2)^2=2a^2(x^2-y^2) \quad , \quad r^2=2a^2\cos 2\theta$$



Problem. To find the length of the arc of the Lemniscate.

Formula.—$s=\int\sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$=\int\sqrt{r^2+\frac{a^4}{r^2}\left(1-\frac{r^4}{a^4}\right)}\,d\theta=\int\frac{a^2}{r}d\theta=\int_o^a-\frac{a^2dr}{\sqrt{(a^4-r^4)}}=$$

$$-a^2\int_o^a\left[\frac{1}{a^2}+\frac{1}{2}\cdot\frac{r^4}{a^6}+\frac{1}{2}\cdot\frac{3}{4}\frac{r^8}{a^{10}}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{r^{12}}{a^{14}}+\&c.\right]dr=a^2\left\{\frac{r}{a^2}+\frac{1}{2}\cdot\frac{1}{5}\frac{r^5}{a^6}\right.$$

$$\left.+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}\cdot\frac{r^9}{a^{10}}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}\frac{r^{13}}{a^{14}}+\&c.\right\}$$

. When $r=a$, $s=a\left[1+\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}+\&c\right]$ = arc BPA. ∴ The entire length of the curve is $4a\left[1+\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}+\&c\right]$

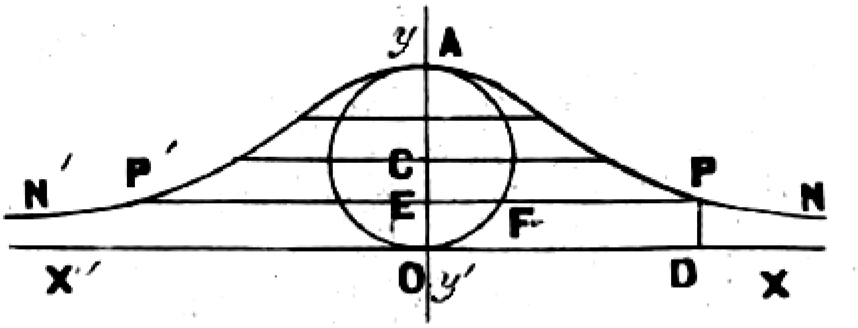
See details of calculation given by Makoto Kato (2012), Arc length formula for the lemniscate,https://math.stackexchange.com/q/189889.

The word Lemiscate comes from the Latin "lemniscatus" meaning "decorated with ribbons", from the Greek ληµνίσκος meaning "ribbons", or which alternatively may refer to the wool from which the ribbons were made.

The Versiera of Agnesi and the Versine

The equation of it referred to rectangular coordinates is:

$$x^2 y = 4a^2(2a - y)$$



Let P be any point of the curve, $PD = y$, the ordinate of the point P and $OD = x$, the abscissa. The definition of the curve requires:

$$EP : EF :: AO : EO \quad \text{or} \quad x : EF :: 2a : y$$

$$EF = \sqrt{(AE \times EO)} = \sqrt{(2a - y)y} \quad , \quad x : \sqrt{(2a - y)y} :: 2a : y$$

And then, the equation given above.

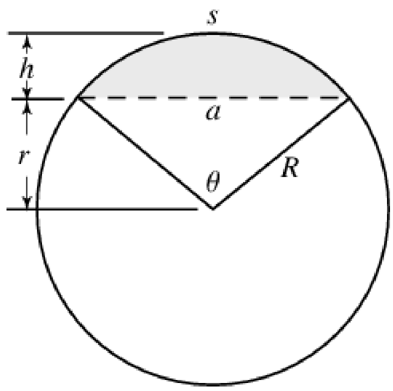
The polar equation is: $r(r^2 - r^2 \sin^2\theta + 4a^2)\sin\theta = 8a^3$

Problem - To find the length of an arc of the Versiera.

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx = \int \sqrt{1 + \frac{4x^8}{(x^2 + 4a^2)^4}} \; dx$$

In the previous discussion, we have used the length of a chord. Here formulas from

https://mathworld.wolfram.com/CircularSegment.html



The length of the chord is:

$a=2R\sin(\theta/2)$ $a=2\,r\,\tan(\theta/2)$ $a=2\sqrt{R^2-r^2}$ $a=2\sqrt{h(2R-h)}$

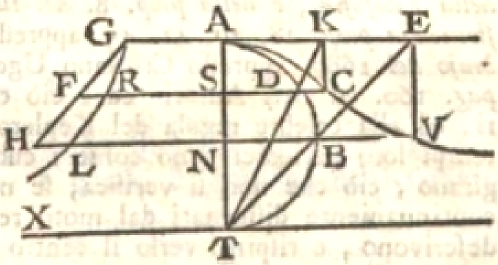
Chi ha proposto, per la prima volta, la Versiera e come è nato il suo nome?

"II LORIA, al Cap. XII del I volume, a pp. 93-99, sotto il titolo Versiera, visiera, pseudo-versiera, scrive: « ... il nome (versiera) ora riferito s'incontra per la prima volta nelle Note al Trattato del Galileo del moto naturalmente accelerato del P. GUIDO GRANDI (1718), ove si legge che il nome versiera (latino versoria) deriva dalle parole sinus versus e che la curva fu ottenuta la prima volta dal GRANDI nell'opera intitolata Quadratura Circuli et Hyperbolae » (I ediz. 1703, II ediz. 1710). Il capitolo è interessante e contiene anche molte indicazioni bibliografiche dalle quali risulta che la curva venne attribuita all'AGNESI in Italia e fuori. ... Guido GRANDI, nella sua Quadratura Circuli et Hyperbolae, ... dà la sua prima definizione della versiera alla quale seguono proprietà interessanti, ne indichiamo alcune. 1) La porzione di piano compresa fra la curva e il suo asintoto, estesa indefinitamente dalle due bande, equivale a] quadruplo del cerchio generatore della curva stessa; 2) Rotando detta porzione di piano attorno all'asintoto si ottiene un solido equivalente al doppio di quello generato nella stessa rotazione dal cerchio generatore. Trova poi l'equazione della curva in coordinate cartesiane ortogonali, generalizzando a curve delle quali la versiera è un caso particolare. Applica la versiera a considerazioni di ottica." ... "Ma veniamo al pregevole lavoro Instituzioni Analitiche di MARIA GAETANA AGNESI, pubblicato a Milano, nella Regia Ducal Corte nel 1748. Tratta della Versiera nel I volume in due punti: alle pp. 380-81 dandone l'equazione in coordinate cartesiane $xy^2=a^2(a-x)$ dove a è il diametro del cerchio generatore, e ciò era già stato fatto; alle pp. 392-93 dando una costruzione della versiera che non è altro che la seconda costruzione del GRANDI. E null'altro trovo su questa curva nel libro". Estratti da: Luigi Tenca, La versiera di. . . Guido Grandi. Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. 12 (1957), n.3, p. 458–460. Zanichelli, link http://www.bdim.eu/item?id=BUMI_1957_3_12_3_458_0

Corollario VI.

Quando poi le forze fuſſero reciproche de' quadrati delle diſtanze, ſarebbe la ſcala A G F H un' iperbola quadratica fra gli ſteſſi aſintoti; e la ſcala delle velocità A C V ſarebbe quella curva, che io deſcrivo nel mio libro delle quadrature alla prop. 4. nata da' ſeni verſi, che da me ſuole chiamarſi la *Verſiera*, in latino però *Verſoria:* dimanierachè le velocità S C, N V ſarebbero in ragione compoſta della ſudduplicata de' ſpazj ſcorſi A S, A N direttamente, e della ſudduplicata de' ſpazj che reſtano fino al termine T, cioè di N T, S T reciprocamente.

12 Il che però non ſi potendo dimoſtrare dalle coſe da me nel luogo citato circa le proprietà di queſta curva propoſte; ſtimo bene, atteſa l' utilità, che può ricavarſi in Meccanica da queſta Curva, il darne ora queſta facile deſcrizione, ricavandone ciò che fa al noſtro propoſito. Sia dunque il mezzo cerchio A D B T, e nel punto eſtremo A del diametro lo tocchi la retta A E, a cui dall' altro termine del diametro T ſi conducano le rette T K, T E, ſeganti la periferia in D, B, ed ordinate le D S, B N nel ſemicircolo, ſi compiſcano i rettangoli K A S C, E A N V. La curva che paſſa pe' punti A, C, V così determinati, è la noſtra *Verſiera*, ed è evidente eſſere i quadrati S C, N V eguali a' quadrati A K, A E; ma il quadrato A K al quadrato A E ha ragione compoſta del quadrato A K al quadrato A T, e di queſto al quadrato A E; delle quali ragioni la prima è quella del quadrato S D al quadrato S T, ovvero della retta A S alla S T; la ſeconda è quella del quadrato T N al quadrato N B, ovvero della T N alla A N; pertanto ſarà il qua-



Rr 2 drato

Estratto alle Note di Grandi alle OPERE DI GALILEO GALILEI DIVISE IN QUATTRO TOMI, in questa nuova Edizione accresciute di molte cose inedite. TOMO TERZO · Volume 3 · 1744

"Ma perché chiamarla la "versiera di Agnesi", dal momento che la curva non è stata scoperta da Gaetana Agnesi, e che lei mai ha pensato di attribuirsela? La matematica milanese non ne porta colpa, si è limitata ad affermare – nelle sue Istituzioni analitiche, a p. 318 – che questa curva prende il nome di versiera, perché così aveva letto in una nota del matematico Guido Grandi, che aveva collaborato alla prima edizione fiorentina delle opere del Galilei, ... Allora com'è nato l'equivoco? La spiegazione è assai semplice se si pensa al successo dell'opera dell'Agnesi: per chi leggeva per la prima volta quel nome – e non era a conoscenza della "nota" del Grandi – la versiera era "la versiera di Agnesi". ... E l'Agnesi scrive: «curva… che dicesi la Versiera». Tutto qui. ... Trascuriamo il fatto che in realtà l'equazione fu già studiata da Fermat sessant'anni prima e consideriamo invece quanto scrisse Guido Grandi nella sua nota ..." da La storia vera della "versiera" di M.G. Agnesi, https://nusquamia.wordpress.com/2019/08/05/la-storia-vera-della-versiera-di-m-g-agnesi/

In questo testo si trova una discussione del termine versiera. Dopo la discussione del lavoro di Grandi, si arriva alla distinzione tra *sinus rectus* e *sinus versus*. "Vediamo invece che cosa sia il sinus versus. Non è un'"insenatura contraria", o "nemica", ma una funzione trigonometrica, meno conosciuta delle funzioni trigonometriche abituali seno, coseno, tangente, cotangente: ma tutt'altro che un'entità misteriosa; tant'è che in italiano ha un nome, quello di "senoverso". Viene indicato come sinv θ, o senv θ, o versen θ. Dato un angolo θ, il senoverso di θ è: senv θ = 1 – cos θ".

Il termine che appare più vicino alla "versiera", come intesa da Grandi, si ritroverebbe ad essere il "versine" Inglese.

"The versine or versed sine is a trigonometric function found in some of the earliest (Vedic Aryabhatia I) trigonometric tables. The versine of an angle is 1 minus its cosine. There are several related functions, most notably the coversine and haversine. The latter, half a versine, is of particular importance in the haversine formula of navigation." Da en.wikipedia.org/wiki/Versine

$$\text{versin}(\theta) = 1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$$

C'è anche una discussione in Inglese sulla Versiera, proposta da Evelyn Lamb, May 28, 2018 in blogs.scientificamerican.com/roots-of-unity/a-few-of-my-favorite-spaces-the-witch-of-agnesi/ - Viene detto. "Grandi had given the curve the name versiera in the first place, saying he adapted it from the Latin word versoria, which in turn is derived from the word for "turn." In fact, in some ways 2018 is the 300th birthday not just of Agnesi but of the witch as well; Grandi had discussed the curve in 1703, but he didn't christen it versiera until 1718."

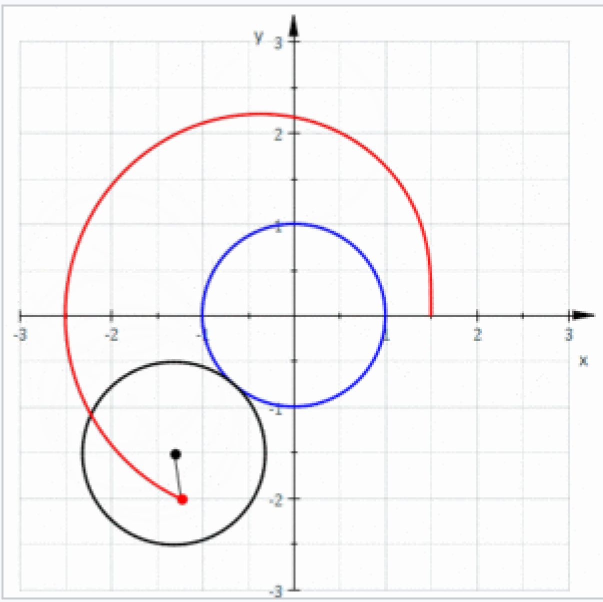
Verſoria magnetica non congruunt cum ve-
ro meridiano in toto aliquo meridiano .

Il termine versoria appare usato in navigazione come dalla "Philosphia Magnetica" di Niccolò Cabeo, 1629.

Limaçon

The name of the curve comes from the Latin limax meaning 'a snail'.

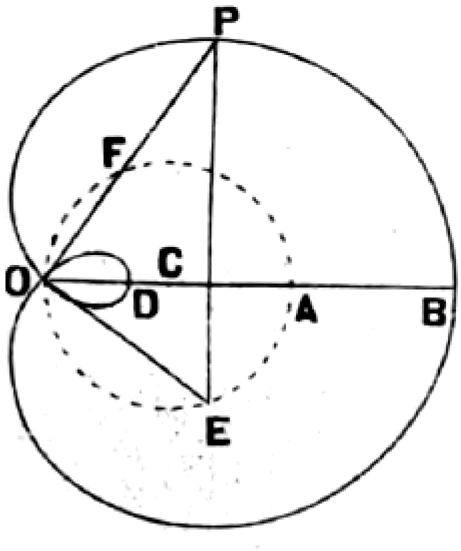
In geometry, a limaçon or limacon, also known as a limaçon of Pascal, is defined as a roulette formed by the path of a point fixed to a circle when that circle rolls around the outside of a circle of equal radius. It can also be defined as the roulette formed when a circle rolls around a circle with half its radius so that the smaller circle is inside the larger circle. Thus, they belong to the family of curves called centered trochoids; more specifically, they are epitrochoids. The cardioid is the special case in which the point generating the roulette lies on the rolling circle; the resulting curve has a cusp.



Construction of the limaçon $r = 2 + \cos(\pi - \theta)$ with polar coordinates' origin at $(x, y)=(1/2, 0)$

Construction of the limaçon with polar coordinates' origin at (x, y)=(1/2, 0) - Author Sam Derbyshire - http://en.wikipedia.org/wiki/Image:EpitrochoidIn1.gif

Depending on the position of the point generating the curve, it may have inner and outer loops (giving the family its name), it may be heart-shaped, or it may be oval.



Equations:

$(x^2+y^2-ax)^2=b^2(x^2+y^2)$, $r=a\cos\theta\pm b$ with $a=OA$, $b=PF$

Prob. LI. To find the length of an arc of the Limaçon.

Formula.—$s=$

$$\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=$$

$$\int\sqrt{(a\cos\theta+b)^2+a^2\sin^2\theta}\,d\theta=\int\sqrt{(a^2+b^2+2ab\cos\theta)}\,d\theta=$$

$$\int\sqrt{\left\{(a+b)^2\cos^2\frac{\theta}{2}+(a-b)^2\sin^2\frac{\theta}{2}\right\}}\,d\theta.$$ $\therefore$ The rectification of the Limaçon depends on that of an ellipse whose semi-axes are $(a+b)$ and $(a-b.)$

When $a=b$, the curve is the *cardioid*, the polar equation of which is $r=a(1+\cos\theta)$, and $s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=$

$$a\int\sqrt{2+2\cos\theta}\,d\theta=\pm2a\int\cos\tfrac{1}{2}\theta\,d\theta=2a\int_0^\pi\cos\tfrac{1}{2}\theta\,d\theta-$$

$$2a\int_\pi^{2\pi}\cos\tfrac{1}{2}\theta\,d\theta=8a=$$ the entire length of the cardioid.

Quadratrix

1. The Quadratrix is the locus of the intersection, P, of the radius, OD, and the ordinate QN, when these move uniformally, so that $ON:OA::\angle BOD:\frac{1}{2}\pi$.

2. $y=x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)$ is the rectangular equation of the curve, in which $a=OA$, $x=ON$, and $y=IN$.

3. The curve effects the quadrature of the circle, for $OC:OB::OB:\text{arc } ADB$.

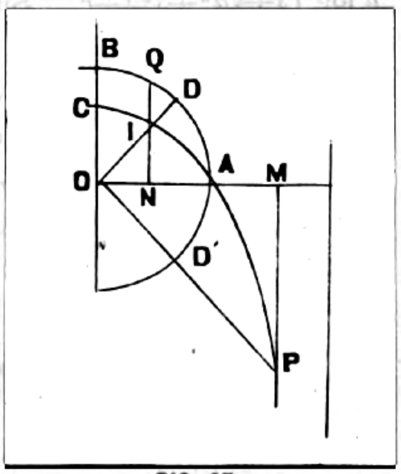


FIG. 27.

Prob. LIII. To find the area enclosed above the x—axis.

Formula.—$A=\int y\,dx=\int x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)dx=4a^2\pi^{-1}\log 2.$

NOTE.—This curve was invented by Dinostratus, in 370 B. C.

"The quadratrix or trisectrix of Hippias (also quadratrix of Dinostratus) is a curve, which is created by a uniform motion. It is one of the oldest examples for a kinematic curve, that is a curve created through motion. Its discovery is attributed to the Greek sophist Hippias of Elis, who used it around 420 BC in an attempt to solve the angle trisection problem (hence trisectrix). Later around 350 BC Dinostratus used it in an attempt to solve the problem of squaring the circle (hence quadratrix). ... The quadratrix is mentioned in the works of Proclus (412–485), Pappus of Alexandria (3rd and 4th centuries) and Iamblichus (c. 240 – c. 325). Proclus names Hippias as the inventor of a curve called quadratrix and describes somewhere else how Hippias has applied the curve on the trisection problem. Pappus only mentions how a curve named quadratrix was used by Dinostratus, Nicomedes and others to square the circle. He neither mentions Hippias nor attributes the invention of the quadratrix to a particular person. Iamblichus just writes in a single line, that a curve called a quadratrix was used by Nicomedes to square the circle. Though based on Proclus' name for the curve it is conceivable that Hippias himself used it for squaring the circle or some other curvilinear figure, most historians of mathematics assume that Hippias invented the curve, but used it only for the trisection of angles. Its use for squaring the circle only occurred decades later and was due to mathematicians like Dinostratus and Nicomedes. This interpretation of the historical sources goes back to the German mathematician and historian Moritz Cantor." https://en.wikipedia.org/wiki/Quadratrix_of_Hippias

Spiral of Archimedes

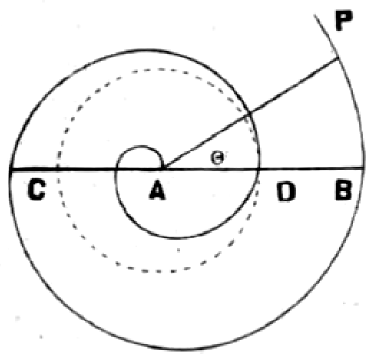
This spiral is the locus of a point revolving about and receding from a fixed point so that the ratio of the radius vector to the angle through which it has moved from the polar axis, is constant.

$$r = \frac{a}{\theta}$$

Archimedes described such a spiral in his book On Spirals. Conon of Samos was a friend of his and Pappus states that this spiral was discovered by Conon.

Prob. LXIII. To find the length of the spiral of Archimedes.

Formula.—$s = \int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int\sqrt{(r^2+a^2)}\,d\theta = a\int\sqrt{(1+\theta^2)}\,d\theta = \frac{1}{2}a\theta\sqrt{(1+\theta^2)} + \frac{1}{2}a\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}$, which is the length of the curve measured from the origin.



$s = a\pi\sqrt{[1+(2\pi)^2]} + \frac{1}{2}a\log\left\{2\pi+\sqrt{[1+(2\pi)^2]}\right\}$ is the length of the curve made by one revolution of the generating point.

Prob. LXIV. To find the area of the spiral of Archimedes.

Formula.—$A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}a^2\int\theta^2 d\theta = \frac{1}{6}a^2\theta^3 = \frac{r^3}{6a}$, the area measured from the origin.

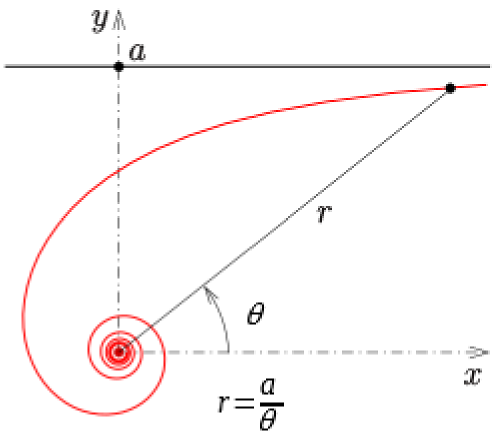
Spiral: From Middle French spirale, from Medieval Latin spiralis, from Latin spira, from Ancient Greek σπεῖρα (speíra, “wreath, coil, twist”).

Reciprocal or Hyperbolic spiral

The curve is the locus of a point revolving around and receding from a fixed point so that it polar equation is:

$$r=\frac{a}{\theta}\quad .$$

Because it can be generated by a circle inversion of an Archimedean spiral, it is called reciproke spiral, too. Pierre Varignon first studied the curve in 1704. Later Johann Bernoulli and Roger Cotes worked on the curve as well.



Problem: To find the length of the Hyperbolic spiral.

$$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{a}{\theta^2}\int\sqrt{(1+\theta^2)}d\theta$$

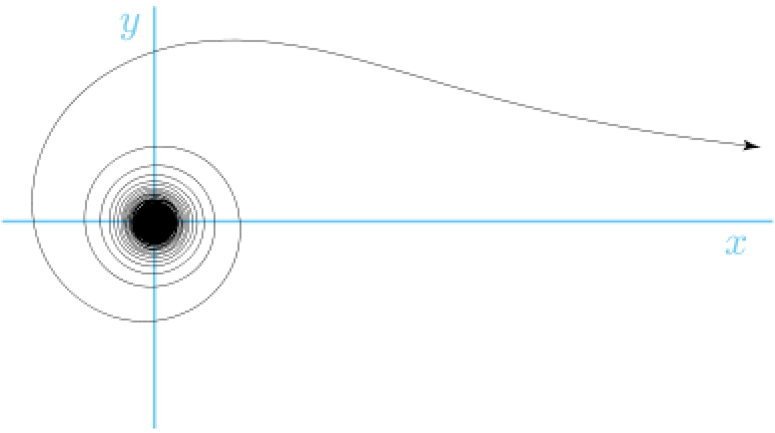
$$s=\theta\sqrt{1+\theta^2}+\log[\theta+\sqrt{1+\theta^2}]-\theta^{-1}(1+\theta^2)^{\frac{3}{2}}$$

$$s=\log[\theta+\sqrt{1+\theta^2}]-\theta^{-1}\sqrt{1+\theta^2}$$

Lituus

In mathematics, a lituus is a spiral with polar equation: $r=\frac{a}{\sqrt{\theta}}$. Thus, the angle θ is inversely proportional to the square of the radius r.

The curve was named for the ancient Roman lituus by Roger Cotes in a collection of papers entitled Harmonia Mensurarum (1722), which was published six years after his death. Roger Cotes (1682 – 1716) was an English mathematician, known for working closely with Isaac Newton by proofreading the second edition of his famous book, the Principia, before publication. He also invented the quadrature formulas known as Newton–Cotes formulas, and made a geometric argument that can be interpreted as a logarithmic version of Euler's formula.



Prob. LXVII. To find the length of the Lituus.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{1}{2}a\theta^{-\frac{3}{2}}\int\sqrt{(1+4\theta^2)}\,d.\theta=$

$\left[-\frac{1}{3}a\left\{\theta^{-\frac{1}{2}}(1+\theta^2)^{\frac{3}{2}}\right\}-\frac{5}{16}\theta^{\frac{3}{2}}\left(\frac{2}{3}-\frac{1}{7}\theta^2-\frac{1}{44}\theta^4-\frac{1}{120}\theta^6-\&c.\right)\right]_{\theta'}^{\theta''}.$

Prob. LXVIII. To find the area of the Lituus.

Formula.—$A=\frac{1}{2}\int r^2\,d\theta=\frac{a^2}{2}\int\frac{d\theta}{\theta}=\frac{1}{2}a^2\log\theta.$

Logarithmic spiral

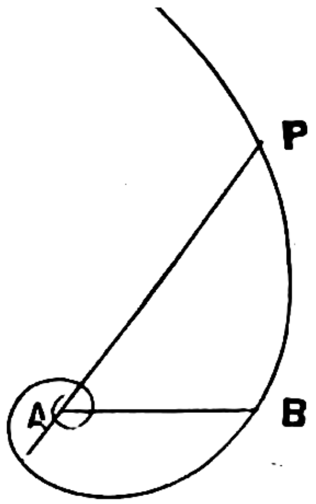
The Logarithmic Spiral is the locus generated by a point revolving around and receding from a fixed point in such a manner that the radius vector increases in a geometrical ratio, while the variable angle increases in an arithmetical ratio.

The polar equation of the logarithmic spiral: $r = a^{\theta}$

Problem: To find the length of the logarithmic spiral.

$$\text{Formula:} \quad s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)}\, d\theta = \int \sqrt{r^2 + \left(\frac{r^2}{m^2}\right)}\, d\theta$$

m is the modulus of the system of logarithms.



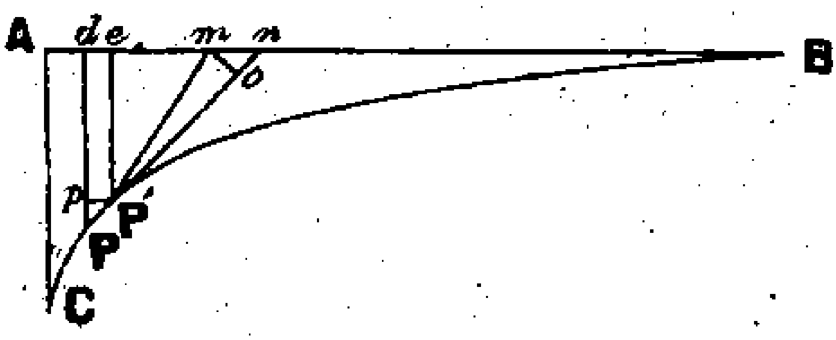
Prob. LXX. To find the area of the Logarithmic Spiral.

Formula.—$A = \frac{1}{2}\int r^2 d\theta = \frac{m}{2}\int r dr = \frac{1}{4} m r^2$. Since $m = 1$, in the Naperian System of Logarithms, $A = \frac{1}{4} r^2$, *i. e.*, the area is $\frac{1}{4}$ of the square of the radius vector.

"The first to describe a logarithmic spiral was Albrecht Dürer (1525) who called it an "eternal line" ("ewige lini"). More than a century later, the curve was discussed by Descartes (1638), and later extensively investigated by Jacob Bernoulli, who called it Spira mirabilis, "the marvelous spiral"." en.wikipedia.org/wiki/Logarithmic_spiral

Curve of Pursuit

A fox is 80 rods north of a hound and runs directly east 350 rods before being overtaken. How far will the hound run before catching the fox if he runs towards the fox all the time, and upon a level plain?



Let C and A be the position of the hound and fox at the start, P and m corresponding positions of the hound and fox any time during the chase, and P' and n their positions the next instant, B the point where the hound catches the fox. $CPP'B$ is the curve described by the hound.

Join m and P and n and P' ; they are tangents to the curve at P and P' . mo is perpendicular to $P'n$, and $P'p$ is perpendicular to Pd . Pd and $P'e$ are perpendicular to AB .

Let $AC=a$, $AB=b$, $Am=x$, $Bd=y$, $Pm=w$, $arc\,CP=s$, $curve\ CPB=s_1$ and r the ratio of the hound's rate to the fox's rate. We have that $mn=dx$, $ed=P'p=dy$, $PP'=ds$, $no-PP'=dw$, $s=rx$ (constant speed of hound on the curve and the fox on the straight line), $ds=r\,dx$, therefore: $dx/ds=1/r$. From the similar right triangles PpP' and mon :

$$PP':mn::pP':no \quad \text{then} \quad ds:dx::dy:no \text{ , that is: } no=\frac{dx\,dy}{ds}=\frac{dy}{r}$$

$$\frac{dy}{r}-ds=dw \quad , \quad dy-r^2dx=r\,dw \text{ . After integration: } y-r^2x=rw+C \text{ .}$$

When $x=0,\ y=0$ we have that $w=a$, then $C=-ra$

$$y-r^2x=rw+C=rw-ra$$

When $x=b,\ y=b,\ w=0$, we have:

$$b-r^2b=ar \quad , \quad r^2b-ra=b \quad , \quad r^2-\frac{a}{b}r=1$$

$$r^2-\frac{a}{b}r+\frac{a^2}{4b^2}=\frac{a^2+4b^2}{4b^2} \quad , \quad r-\frac{a}{2b}=\frac{1}{2b}\sqrt{a^2+4b^2} \quad , \quad 2rb-a=\sqrt{a^2+4b^2}$$

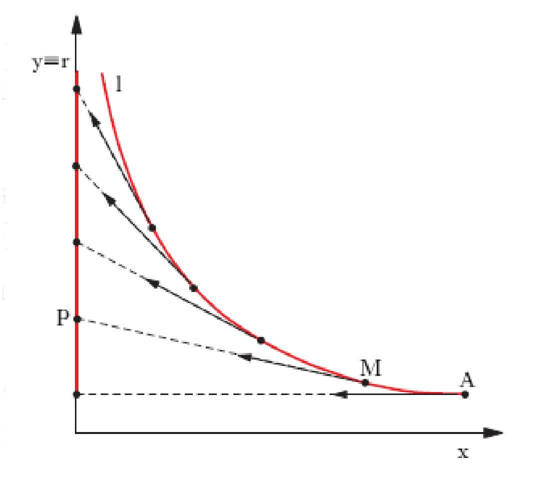
$$rb=\frac{1}{2}(a+\sqrt{a^2+4b^2})=s_1 \text{ since } s=rx \text{ and then } s_1=rb \ .$$

$$s_1=\frac{1}{2}(a+\sqrt{a^2+4b^2})=392.2783 \ rods \ .$$

This is the distance the hound runs to catch the fox. The path of the hound is known as the "curve of pursuit". It is also know as a radiodrome.

"In geometry, a radiodrome is the pursuit curve followed by a point that is pursuing another linearly-moving point. The term is derived from the Greek words ῥᾴδιος, rhā́idios, 'easier' and δρόμος, drómos, 'running'." en.wikipedia.org/wiki/Radiodrome

Da https://www.treccani.it/enciclopedia/curva-di-inseguimento/



Si chiama curva d'inseguimento (vedi figura), relativa alla *retta r* che è descritta con moto uniforme dal punto P, la traiettoria l di un punto M che si muove con *velocità scalare costante* inseguendo il punto P. L'inseguimento avviene in modo che la tangente in M alla curva l passi per il punto P. La curva d'inseguimento è detta anche *curva di caccia o curva del cane*, in quanto è la traiettoria del 'cane' M che 'caccia la lepre' P (o come nel libro di Finkel la volpe). Assunta la *retta r come asse y* di un riferimento cartesiano, e detto *k* il rapporto delle velocità scalari dei due punti M e P, l'equazione delle curva d'inseguimento è:

$$y=b+\frac{x^{1-k}-a^{-2k}x^{1+k}}{4(1-k)}$$

quando $k \neq 1$, e invece

$$y=b+\frac{a\ln x}{2\sqrt{2\ln a}}-\frac{x^2\sqrt{2\ln a}}{4^a}$$

quando $k=1$; le due costanti *a* e *b* sono le coordinate della posizione iniziale A del punto mobile M.

https://mathworld.wolfram.com/ApolloniusPursuitProblem.html

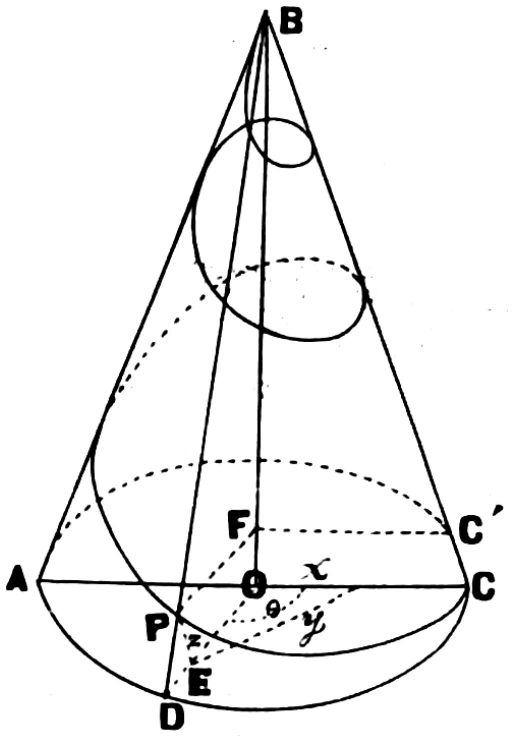
Apollonius Pursuit Problem: "Given a ship with a known constant direction and speed v, what course should be taken by a chase ship in pursuit (traveling at speed V) in order to intercept the other ship in as short a time as possible? The problem can be solved by finding all points which can be simultaneously reached by both ships, which is an Apollonius circle with $\mu=v/V$. If the circle cuts the path of the pursued ship, the intersection is the point towards which the pursuit ship should steer. If the circle does not cut the path, then it cannot be caught".

en.wikipedia.org/wiki/Circles_of_Apollonius

"The circles of Apollonius are any of several sets of circles associated with Apollonius of Perga, a renowned Greek geometer. Most of these circles are found in planar Euclidean geometry, but analogs have been defined on other surfaces; for example, counterparts on the surface of a sphere can be defined through stereographic projection".

On the cone

What is the length of a thread winding spirally round a cone, whose radius is R and altitude a, the thread passing round n times and intersecting the slant height at equal distance apart?



Let P be any point of the thread (x, y, z), coordinates of the point. Angle $PFC' = DOC = \theta$. $BO = a$, the altitude, $DO = R$ the radius of the base and r the radius of the cone at the point P. Equations of the thread are:

$$x = r\cos\theta \quad (1)$$
$$y = r\sin\theta \quad (2)$$
$$z = \left(\frac{a}{2\pi n}\right)\theta \quad (3)$$

Triangles DEF and DOB are similar:

$$r = \frac{R}{a}(a - z) = R\left(1 - \frac{\theta}{2\pi n}\right) \quad (4)$$

The distance between P and its consecutive position is $ds=\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dz$.

$$s=\int ds=\int \sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dz \quad .$$

Substituting the value of r into (1) and (29 and differentiating:

$$dx=-\frac{R}{2\pi n}\left[\cos\theta+(2\pi n-\theta)\sin\theta\right]d\theta$$

$$dy=-\frac{R}{2\pi n}\left[\sin\theta-(2\pi n-\theta)\cos\theta\right]d\theta$$

$$dz=\frac{a}{2\pi n}\,d\theta$$

Therefore:

$$s=\int_0^{2\pi n}\frac{a\,d\theta}{2\pi n}\sqrt{1+\frac{R^2}{a^2}[\cos\theta+(2\pi n-\theta)\sin\theta]^2+\frac{R^2}{a^2}[\sin\theta-(2\pi n-\theta)\cos\theta]^2}$$

Then:

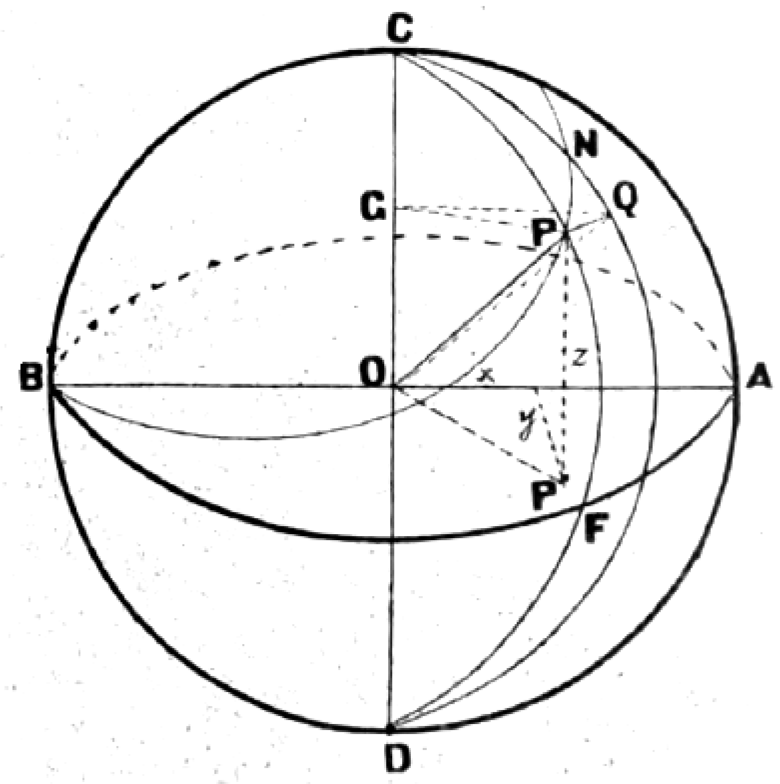
$$\int_0^{2\pi n}\frac{a}{2\pi n}\sqrt{\left[1+\frac{R^2}{a^2}+\frac{R^2}{a^2}(2\pi n-\theta)^2\right]}d\theta=\int_0^{2\pi n}\frac{1}{2\pi n}\sqrt{[a^2+R^2+R^2(2\pi n-\theta)^2]}d\theta=\frac{1}{2\pi n}\left\{-\frac{R(2\pi n-\theta)}{2}\sqrt{\left[\frac{a^2+R^2}{R^2}+(2\pi n-\theta)^2\right]}-R\left(\frac{a^2+R^2}{R^2}\right)\log_e\left[(2\pi n-\theta)+\sqrt{\left((2\pi n-\theta)^2+\frac{a^2+R^2}{R^2}\right)}\right]\right\}_0^{2\pi n}=\tfrac{1}{2}\sqrt{(a^2+R^2+4\pi^2n^2R^2)}+\frac{a^2+R^2}{4\pi nR}\log_e\left[\frac{2\pi nR+\sqrt{(a^2+R^2+4\pi^2n^2R^2)}}{\sqrt{(a^2+R^2)}}\right]$$

$$=\tfrac{1}{2}\sqrt{(h^2+4\pi^2n^2R^2)}+\frac{h^2}{4\pi nR}\log_e\left[\frac{2\pi nR+\sqrt{(h^2+4\pi^2n^2R^2)}}{h}\right],$$

where $h=\sqrt{(a^2+R^2)}$, the slant height.

Loxodrome

The word loxodrome comes from Ancient Greek λοξός loxós: "oblique" + δρόμος drómos: "running" (from δραμεῖν drameîn: "to run"). The loxodrome is also known as the rhumb line. The word rhumb may come from Spanish or Portuguese rumbo/rumo ("course" or "direction") and Greek ῥόμβος rhómbos, from rhémbein.



A ship starts on the equator and travels due north-east at all time; how far has it traveled when its longitude, for the first time, is the same as that of the point of departure?

B is the starting point. BPN is its course. P its position at any time. N is the position at the next instant. PN is an element of the curve, which is known as the Loxodrome or Rhumb line. Let θ the longitude of the point P, ϕ the latitude. Let (x, y, z) the rectangular coordinate of P. Let $\varphi = \pi/4$ the constant angle PNQ. r is the radius of the sphere.

$$ds^2 = r^2 d\phi^2 + r^2 \cos^2\phi \, d\theta^2$$

$$PQ = r\cos\phi\, d\theta \quad , \quad NQ = r\, d\phi \quad , \quad PQ/NQ = \tan\varphi = \frac{\cos\phi\, d\theta}{d\phi}$$

$$\cos\phi\, d\theta = \tan\varphi\, d\phi$$

$$ds^2 = r^2 d\phi^2 + r^2\tan^2\varphi\, d\phi^2 \quad , \quad ds = r\sqrt{1+\tan^2\varphi} = \frac{r}{\cos\varphi} d\phi$$

$$s = \frac{r}{\cos\varphi}\int_{\phi_2}^{\phi_1} d\phi = \frac{r}{\cos\varphi}(\phi_1 - \phi_2)$$

We know that
https://en.wikipedia.org/wiki/List_of_integrals_of_trigonometric_functions

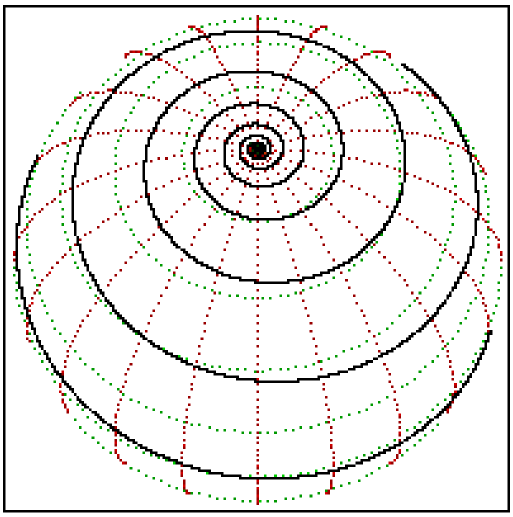
$$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

Therefore:

$$\theta = \tan\varphi \int \frac{d\phi}{\cos\phi} = \tan\varphi \ln[\tan(\pi/4 + \phi/2)]$$

$$e^{\theta\cot\varphi} = \tan(\pi/4 + \phi/2) \quad \text{then:} \quad \phi = 2\tan^{-1}(e^{\theta\cot\varphi}) - \pi/2$$

When $\theta = 2\pi$, $\phi = \pi/4$, $\phi = 2\tan^{-1}(e^{2\pi}) - \pi/2 = 89°47'9''.6 \approx \pi/2$



Loxodrome

Image Courtesy:
Karl Bednarik, Peter Steinberg.