Cardano Formula and Some Figurate Numbers

DOI: 10.5281/zenodo.4663050

Cardano Formula and Some Figurate Numbers

Amelia Carolina Sparavigna
Department of Applied Science and Technology, Politecnico di Torino

Here we apply the Cardano formula to obtain the sequence of the positive integers from the sequences of integer numbers described by the cubic function of n. Examples will be proposed concerning figurate numbers.

Keywords: Groupoid Representations, Integer Sequences, OEIS, On-Line Encyclopedia of Integer Sequences, Cardano formula.

Torino, 5 April 2021.

In a previous discussion [1], we have studied the sequence of the Stella Octangula numbers:

$$s_n = 2n^3 - n = n(2n^2 - 1)$$

(OEIS, On-Line Encyclopedia of Integer Sequences, A007588: 1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, 5474, 6735, 8176, 9809, 11646, 13699, 15980, ... [2]. In [1]. we have also proposed a generalised sum: $s_n \oplus s_m = s_{n+m}$, with the form (see Refs. [3-5] for discussions):

$$s_n \oplus s_m = s_n + s_m + 3(n+s_n)^{2/3}(m+s_m)^{1/3} + 3(n+s_n)^{1/3}(m+s_m)^{2/3}$$

In this formula, we have a sequence $(n+s_n) = n + 2n^3 - n = 2n^3$, that is : 2, 16, 54, 128, 250, 432, 686, 1024, 1458, 2000, 2662, 3456, 4394, 5488, 6750, 8192, 9826, 11664, 13718, 16000, ... (OEIS A033431).

Then, the generalized sum can be written as: $s_n \oplus s_m = s_n + s_m + 6(n^2 m + n m^2)$.

DOI: 10.5281/zenodo.4663050

In [1], we concluded observing that a Stella Octangula number is suitable for being solved immediately by means of the Cardano formula:

$$n^3 - \frac{n}{2} - \frac{S_n}{2} = 0 \quad , \quad n^3 + p\,n + q = 0$$

Cardano formula for root n has the form:

$$n = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}}$$

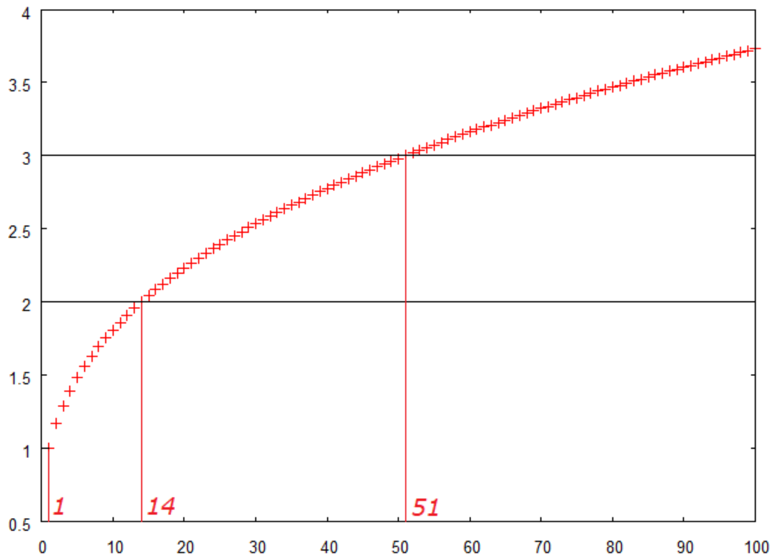

Fig.1 - Stella Octangula Numbers and Cardano formula.

The sequence of the positive integers can be obtained by means of Cardano formula from the sequence of Stella Octangula numbers.

We can use the same approach for integer sequences described by a cubic function of n :

$$an^3 + b\,n^2 + c\,n + d = an^3 + b\,n^2 + c\,n + (d' - I_n) = 0 \quad (*)$$

DOI: 10.5281/zenodo.4663050

Then, the integer sequence of the positive integers can be obtained by means of a generalized Cardano formula, from a given sequence of I_n numbers, of the form described by (*).

The solution is the following:

$$n = \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

The generalization of the Cardano formula has been provided by Eric Schechter, Vanderbilt University, https://math.vanderbilt.edu/schectex/courses/cubic/ . We can use a numerical approach to find the root as given above. Here in the following Fig.1, the result of numerical calculus, for the first few numbers of the Stella Octangula sequence.

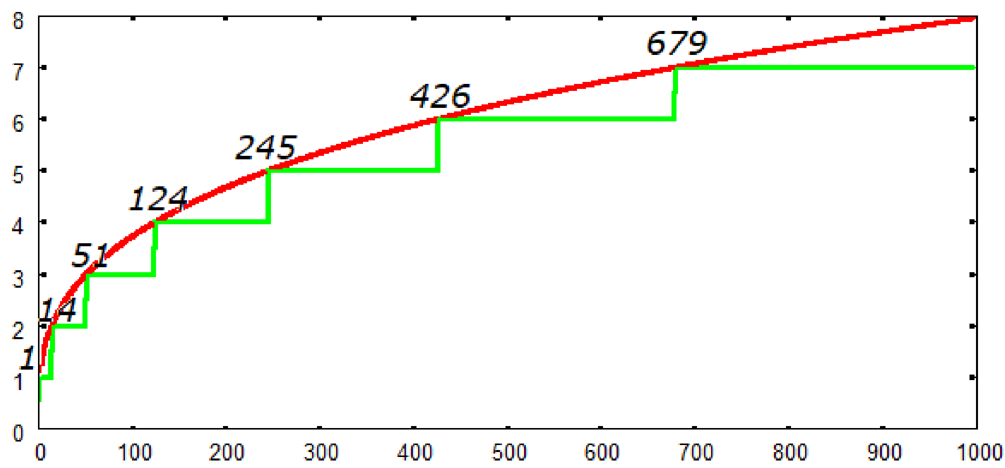

Fig.1 : The root of the generalized Cardano formula for Stella Octangula.

Let us continue showing other examples. We consider sequences of integers, described by cubic functions, being also figurate numbers.

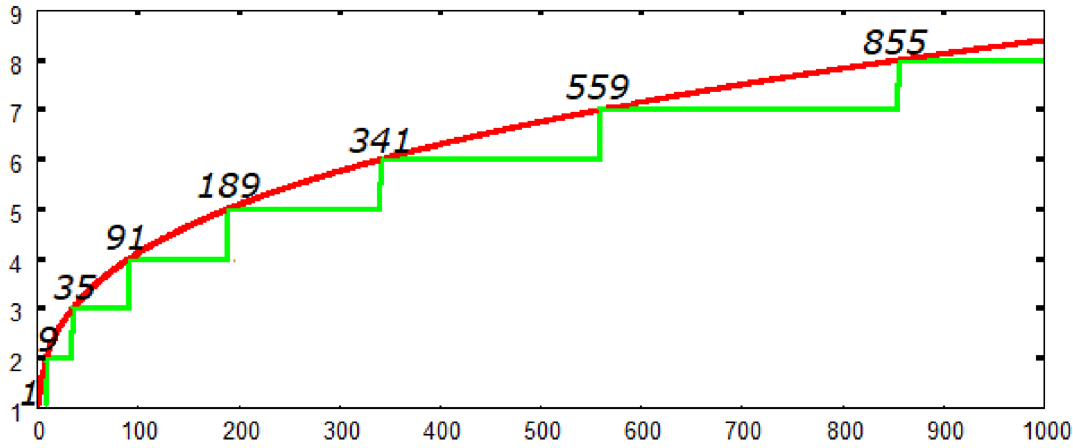

Fig. 2 - Centered Cube Numbers: $(2n-1)(n^2-n+1)$

https://mathworld.wolfram.com/CenteredCubeNumber.html : 1, 9, 35, 91, 189, 341, 559, 855, 1241, 1729, 2331, 3059, 3925, ...

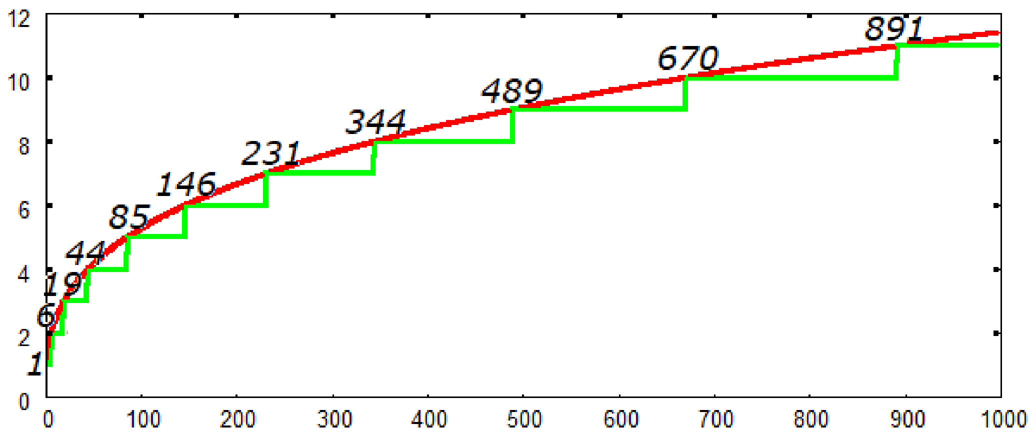

Fig.3 - Octahedral Numbers: $\frac{1}{3}n(2n^2+1)$

https://mathworld.wolfram.com/OctahedralNumber.html : 1, 6, 19, 44, 85, 146, 231, 344, 489, 670, 891, 1156, ...

DOI: 10.5281/zenodo.4663050

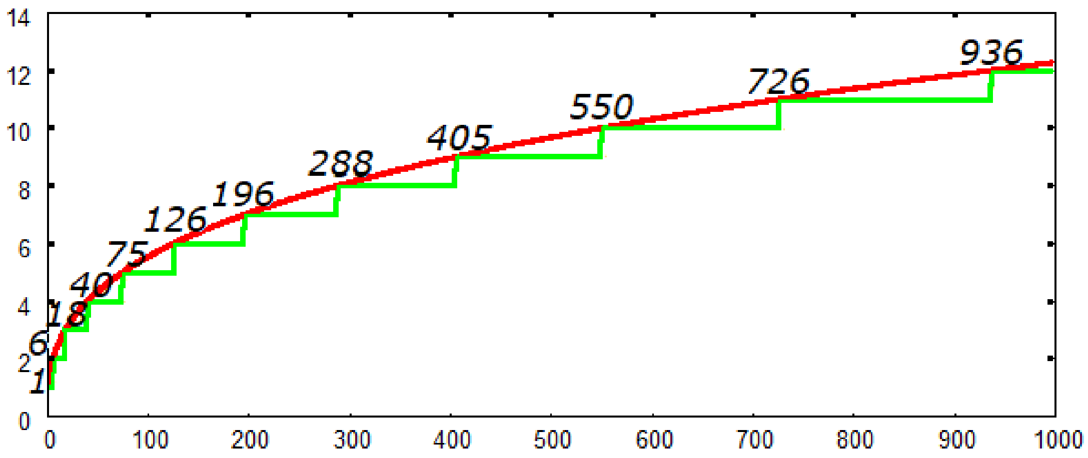

Fig.4 - Pentagonal Pyramidal Numbers: $\frac{1}{2}n^2(n+1)$

https://mathworld.wolfram.com/PentagonalPyramidalNumber.html : 1, 6, 18, 40, 75, 126, 196, 288, 405, 550, 726, 936, ...

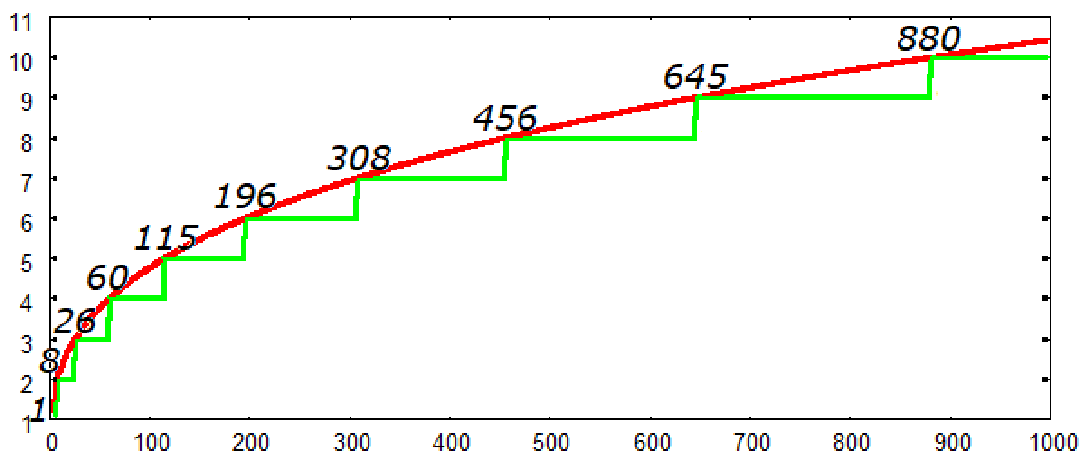

Fig.5 - Heptagonal Pyramidal Numbers: $\frac{1}{6}n(n+1)(4n-1)$

https://mathworld.wolfram.com/HeptagonalPyramidalNumber.html : 1, 8, 26, 60, 115, 196, 308, 456, 645, 880, ...

DOI: 10.5281/zenodo.4663050

References

[1] Sparavigna, Amelia Carolina. (2021, April 4). Generalized Sum of Stella Octangula Numbers. Zenodo. http://doi.org/10.5281/zenodo.4662348

[2] https://oeis.org/A007588

[3] Amelia Carolina Sparavigna, 2019. Composition Operations of Generalized Entropies Applied to the Study of Numbers, International Journal of Sciences, vol. 8(04), pages 87-92, DOI: 10.18483/ijSci.2044

[4] Amelia Carolina Sparavigna, 2019. Some Groupoids and their Representations by Means of Integer Sequences, International Journal of Sciences, vol. 8(10), pages 1-5, DOI: 10.18483/ijSci.2188

[5] Sparavigna, Amelia Carolina. (2021, April 4). Binary operations inspired by generalized entropies applied to figurate numbers. Zenodo. http://doi.org/10.5281/zenodo.4661828